RAC$\mathbb{SAM}$
Rev. R. Acad. Cien. Serie A. Mat.
VOL. **99** (2), 2005, pp. 235–241
Matemática Aplicada / Applied Mathematics
Comunicación Preliminar / Preliminary Communication

Generalized Lubrification Models Blow-up and Global Existence Result

J. Emile Rakotoson, J. Michel Rakotoson and Cédric Verbeke

Abstract. We study a general mathematical model linked with various physical models. Especially, we focus on those models established by King or Spencer-Davis-Voorhees related to thin films extending the lubrication model studied by Bernis-Friedman. According to the initial data, we prove that, either, blow up or global existence can be obtained.

Sobre un modelo general de lubricación, explosión y existencia global

Resumen. Se considera un modelo general de lubricación que engloba varios modelos físicos. En particular nos centramos en los modelos propuestos por King y Spencer, Davis y Voorhees, donde se generalizan los modelos de lubricación estudiados por Bernis y Friedman. Dependiendo de los datos iniciales, se demuestra la explosión de soluciones o su existencia global.

1 Introduction

We are interested in the following class of equations:

$$(KSDV)_{\alpha\gamma\beta}\begin{cases} h_t = \left(hh_{xx} + \left(\dfrac{1}{2} - \alpha\right)h_x^2\right)_{xx} - \gamma\left(\dfrac{h_x^3}{h}\right)_x + \beta h_x^6, \\ h(t)\text{-periodic on }]-1,+1[, \quad t \geqslant 0, \\ h(0) = h_0, \end{cases}$$

with $\alpha \in \mathbb{R}$, $\gamma \geqslant 0$, $\beta \geqslant 0$.

For $\beta = 0$, the model was established by J. R. King [7]. Here, we show that if the initial data $h_0 \geqslant 0$, $\gamma \geqslant (2/3)\,\alpha$ then any admissible weak local solution h is necessarily nonnegative. Moreover, there is *no global* weak solution on $\mathbb{R}_+$ of $(KSDV)_{\alpha\gamma 0}$ and the blow up time must occur before $T_0 = (2\overline{h}_0)/(\overline{h_0^2} - (\overline{h}_0)^2)$ provided that h_0 is non constant.

On the other hand, if $h_0 \leqslant 0$ then we have a value $\alpha_c \leqslant 5/6$ such that if $\alpha > \alpha_c$, for all $T > 0$, there is a *non positive* global weak solution h on $[0,T]$ being in particular in $L^2\Big(0,T;H^2_{\rm per}(-1,+1)\Big)$. If $\alpha \leqslant \alpha_c$, we show that there is a weak solution, when γ is greater than $(1-\alpha)^2$. Moreover, adapting the energy method used by Bernis [1] in the case $\alpha = 1$ and $\gamma = \beta = 0$, we can show that all weak solution has a finite speed propagation.

Presentado por J. I. Díaz.
Recibido: 7 de octubre de 2005. *Aceptado*: 16 de noviembre de 2005.
Palabras clave / Keywords: Thin films, surface evolution, blow-up, global existence, higher order equations.
Mathematics Subject Classifications: 35G30, 35B60, 76A20

The case $\beta \geqslant 0$, $\alpha = \gamma = 0$, the model was established by Spencer-Davis-Voorhees [11].

If $\beta > 0$, $h_0 \leqslant 0$, we show that there exists a global solution $h \leqslant 0$ provided that h_0 belongs to a class of functions.

If $h_0 \geqslant 0$, blow-up should occur. We show this fact under the Dirichlet boundary conditions for the weak solution of $(KSDV)_{00\beta}$. The blow time $T_{\max} \leqslant \dfrac{1}{\lambda_1^3\beta} \ln\left(1 - \dfrac{\lambda_1\beta}{\overline{h_0\varphi_1}}\right) = t_*$, assuming $\overline{h_0\varphi_1} > \lambda_1\beta$, $\displaystyle\lim_{t\to T_{\max}} \int_0^t |h(\sigma)|^6_{L^1(-1,+1)}\, d\sigma = +\infty$.

2 Notations - Functional spaces - Useful Lemmas

The norm in the Lebesgue space $L^p(-1,+1)$ is denoted by $|\cdot|_p$ or $|\cdot|_{L^p(-1,+1)}$. We define the usual Sobolev spaces $H^m(-1,+1) = W^{m,2}(-1,+1) = \Big\{\varphi \in L^2(-1,+1) : \varphi^{(j)} \in L^2(-1,+1) \text{ for } j = 0,\dots,m\Big\}$, for $m \geqslant 0$. In particular, we will need the following closed subspaces, for $m \geqslant 1$:

$$H^m_{\mathrm{per}}(-1,+1) = \Big\{\varphi \in H^m(-1,+1) : \varphi^{(j)}(1) = \varphi^{(j)}(-1),\ j = 0,\dots,m-1\Big\}.$$

Here and elsewhere, $\varphi^{(j)} = \varphi_x^{(j)}$ denotes the derivative of order j. In particular, $\varphi_x = \varphi'$, $\varphi_{xx} = \varphi^{(2)} = \varphi$", ... We denote by $\overline{\varphi}$ the average of φ over $(-1,+1)$, i.e.

$$\frac{1}{2}\int_{-1}^{+1} \varphi(x)\,dx = \overline{\varphi}, \qquad \varphi_+ = \max(\varphi,0), \qquad \varphi_- = -\min(\varphi,0).$$

The norm on $H^m_{\mathrm{per}}(-1,+1)$ is the same as for $H^m(-1,+1)$ or any equivalent norm as

$$|\varphi|_{H^m_{\mathrm{per}}(-1,+1)} = \left|\int_{-1}^{+1} \varphi(x)\,dx\right| + \left(\int_{-1}^{+1} \left|\varphi^{(m)}(x)\right|^2 dx\right)^{\frac{1}{2}}.$$

The dual space of a Banach space V is denoted by V'.

The other spaces we use, are classical functional spaces, like $L^p(0,T;V)$ where V is a Banach space.

The following lemmas has been already used in the previous papers (see for instance [4] or Bernis [1] for Lemma 2).

Lemma 1 *Let* $w \geq 0,\ w \in H^2_{\mathrm{per}}(-1,+1)$ *then for any* $s \geq 0$, $s \neq 1$,

$$\int_{-1}^{+1} \frac{w_x^4}{w^{2-s}}\,dx \leq \frac{9}{(1-s)^2}\int_{-1}^{+1} w^s w_{xx}^2\,dx.$$

Lemma 2 *Let* $\varphi \geq 0,\ \varphi \in H^2_{\mathrm{per}}(-1,+1)$.
Then $\forall \alpha \in \mathbb{R}$, $\forall \epsilon > 0$,

$$(\alpha - 1)\int_{-1}^{+1} (\varphi+\epsilon)^{-\alpha}|\varphi_x|^3\,dx = 2\int_{-1}^{+1} (\varphi+\epsilon)^{1-\alpha}|\varphi_x|\varphi_{xx}\,dx.$$

3 Main results

3.1 Case $\beta = 0$

To study this case we first introduce the notion of weak solution

Definition 1 (admissible weak solution for $(KSDV)_{\alpha\gamma 0}$) *A function $h \in L^2\big(0,T;H^2_{\mathrm{per}}(-1,+1)\big)$ is called an admissible weak solution of $(KSDV)_{\alpha\gamma 0}$ if*

1. $\gamma \dfrac{h_x^3}{h} \in L^1\big((0,T)\times(-1,+1)\big)$,
2. $h_t \in L^2\Big(0,T;\big(H^2_{\mathrm{per}}(-1,+1)\big)'\Big)$

and $\forall \varphi \in H^2_{\mathrm{per}}(-1,+1)$, we have in $\mathcal{D}'(0,T)$:

$$\frac{d}{dt}\int_{-1}^{+1}\varphi(x)h(t,x)\,dx = \int_{-1}^{+1}\left(hh_{xx+}+\left(\frac{1}{2}-\alpha\right)h_x^2\right)\varphi_{xx}(x)\,dx + \gamma\int_{-1}^{+1}\frac{h_x^3}{h}\varphi_x\,dx$$

and $h(0)=h_0$ (given in $L^1(-1,+1)$).

Theorem 1 *Let $h_0 \geq 0$, $h_0 \in L^2(-1,+1)$, non constant and $\gamma \geq (2/3)\,\alpha$. For any local admissible weak solution $\big([0,T_*),h\big)$ we necessarily have:*

$$T_* < T = \frac{2\overline{h}_0}{\overline{h_0^2}-(\overline{h_0})^2},$$

with $\overline{h_0^2} = \dfrac{1}{2}\displaystyle\int_{-1}^{+1} h_0^2(x)\,dx$.

Lemma 3 *Let I be an open interval, Φ in $C^m(\mathbb{R})$, $m \geq 1$ such that $\Phi^{(k)}(0)=0$ for $k=0,\ldots,m-1$. For any $v \in H^m(I)$, we have $\Phi(v_+) \in H^m(I)$.*

In particular for $m=2$,

$$\frac{d^2}{dx^2}\Phi(v_+)\Phi\text{''}(v_+)v_{+x}^2 + \Phi'(v_+)v_{xx}, \qquad \textit{a.e. and in } \mathcal{D}'(\mathcal{I}).$$

Corollary 1 (of Th. 1) *Let $H_0 > 1$, $H_0 \in C[-1,+1]$, non constant. Let $\widetilde{T}_0 = \dfrac{2(\overline{H_0-1})}{\overline{(H_0-1)^2}+1-\overline{H}_0}$. There is no global weak solution defined on $[0,\widetilde{T}_0]$ for the problem:*

$$\begin{cases} \dfrac{\partial H}{\partial t} = \dfrac{\partial^2}{\partial X^2}\left[(H-1)H_{XX}+\frac{1}{2}H_X^2\right] \quad \textit{in } \big(H^2_{\mathrm{per}}(-1,+1)\big)', \\ H(t)\textit{-periodic}, \\ H(0)=H_0. \end{cases}$$

In particular, there is no function H, such that $H \in C(\overline{Q_{\widetilde{T}_0}})$, H_X^2, H_{XX} in $L^2(Q_{\widetilde{T}_0})$ satisfying the above problem.

3.2 Case $h_0 \leqslant 0$: Existence of global solutions for $\beta = 0$

In order to study $(KSDV)_{\alpha\gamma 0}$, we make the change of function $v=-h$, thus we have to study :

$$(K_{\alpha\gamma})\begin{cases} \partial_t v = -\left(vv_{xx}+\left(\dfrac{1}{2}-\alpha\right)v_x^2\right)_{xx} + \gamma\left(\dfrac{v_x^3}{v}\right)_x, \\ v(t)\text{-periodic}, \\ v(0)=v_0 \geqslant 0. \end{cases}$$

Here $\alpha \in \mathbb{R}$, $\gamma \geqslant 0$.

We will need the following Sobolev constant: $$c_b = \inf_{\substack{w \geqslant 0 \\ w \in H^2_{\text{per}}(-1,+1)}} \left[\frac{\displaystyle\int_{-1}^{+1} w_{xx}^2 \, dx}{\displaystyle\int_{-1}^{+1} \frac{w_x^4}{w^2} \, dx} \right]^{\frac{1}{2}}.$$

Notice that $\alpha_c = 1 - \dfrac{1}{2} c_b$.
According to the estimate given in Lemma 1, we have $c_b > 1/3$, $\alpha_c < 5/6$.

Theorem 2 *Let $v_0 \in L^2(-1,+1)$, $v_0 \geq 0$, $T > 0$, $\gamma \geq 0$ if $\alpha > \alpha_c$ and $\gamma > (1-\alpha)^2_*$, if $\alpha \leq \alpha_c$. Then, there exists at least one weak solution $u \geq 0$, of $(K_{\alpha\gamma})$ in the sense that $u \in L^2\Big(0,T; H^2_{\text{per}}(-1,+1)\Big) \cap C\Big([0,T]; L^2(-1,+1)\text{-weak}\Big)$, $\forall \varphi \in H^2_{\text{per}}(-1,+1)$ and in $\mathcal{D}'(0,T)$:*

$$\frac{d}{dt} \int_{-1}^{+1} \varphi(x) u(t,x)\, dx + \int_{-1}^{+1} \Big(u u_{xx} + \left(\frac{1}{2} - \alpha\right) u_x^2\Big) \varphi_{xx}\, dx + \gamma \int_{-1}^{+1} \frac{u_x^3}{u} \varphi_x \, dx = 0,$$

$u(0) = v_0$. Moreover, $u_t \in L^{\frac{4}{3}}\Big(0,T; \big(H^2_{\text{per}}(-1,+1)\big)'\Big)$.

3.2.1 Finite speed of propagation in the case where $\beta = 0$

Theorem 3 *Assume that $\alpha \in \mathbb{R}$, $\gamma > (1-\alpha)^2$. Then any weak solution of problem $(K_{\alpha\gamma})$ (i.e. $(KSDV)_{\alpha\gamma 0}$) possesses the finite speed of propagation property.*

According to Bernis [1, 2] one has:

$$(PP) \begin{cases} \text{Let } u_0 \text{ be in } H^1_{\text{per}}(-1,+1),\ u_0 \not\equiv 0 \text{ such that} \\ u_0 = 0 \text{ on an open subinternal } \omega =]b - r_0, b + r_0[\text{ of }]-1,+1[. \end{cases}$$

Definition 2 *Let $u\colon \overline{Q}_T \to \mathbb{R}$ be a function such that $u(0) = u_0$ in $]-1,+1[$. We say that u has a finite speed propagation if all ω satisfying (PP) there is a number $T_* \in]0,T[$ and two continuous functions $b_-(t)$, $b_+(t)$ such that $b_-(t) < b_+(t)$ in $(0,T^*)$, $b_-(0) = b - r_0$, $b_+(0) = b + r_0$ and $u(t) = 0$ in $\big(b_-(t), b_+(t)\big)$ for all $t \in (0,T^*)$.*

Proposition 1 *Let $v_0 \geq 0$ be in $L^2(-1,+1)$. For all $\alpha < 1$, there exists a function $u^\epsilon \in L^2\big(0,T; H^2_{\text{per}}(-1,+1)\big) \cap C\big([0,T]; L^2(-1,+1)\text{-weak}\big)$ satisfying: $\forall \psi \in H^2_{\text{per}}(-1,+1)$, $u^\epsilon \geq 0, u^\epsilon(0) = v_0$*

$$\frac{d}{dt} \int_{-1}^{+1} u^\epsilon \psi \, dx + \int_{-1}^{+1} u_x^\epsilon u_{xx}^\epsilon \psi_x \, dx + \int_{-1}^{+1} u^\epsilon u_{xx}^\epsilon \psi_{xx}\, dx - 2\overline{\alpha} \int_{-1}^{+1} \frac{u^\epsilon u_x^\epsilon}{u^\epsilon + \epsilon} u_{xx}^\epsilon \psi_x \, dx + \gamma \int_{-1}^{+1} \frac{(u_x^\epsilon)^3}{u^\epsilon + \epsilon} \psi_x \, dx = 0 \qquad \text{in } \mathcal{D}'(0,T)$$

This is equivalent to:

$$\begin{cases} u_t^\epsilon = -\big(u^\epsilon u_{xx}^\epsilon + \frac{1}{2}(u_x^\epsilon)^2\big)_{xx} + \alpha\big((u_x^\epsilon)^2\big)_{xx} + \gamma \left(\frac{(u_x^\epsilon)^3}{u^\epsilon + \epsilon}\right)_x + 2\overline{\alpha}\epsilon \left(\frac{u_{xx}^\epsilon u_x^\epsilon}{u^\epsilon + \epsilon}\right)_x, \\ \text{in } \Big(H^2_{\text{per}}(-1,+1)\Big)', \\ u^\epsilon \geq 0,\ u^\epsilon(0) = v_0. \end{cases}$$

4 Case $\beta > 0$, $\alpha = \gamma = 0$

4.1 Local existence and regularity

We start this section with the existence and uniqueness of a maximal solution.

Theorem 4 *There exists one and only one maximal solution* $\big([0, T_{\max}), h\big)$ *to the problem*

$$\begin{cases} h_t = \big(hh_{xx} + \frac{1}{2}h_x^2\big)_{xx} + \beta h_x^{(6)} & \textit{in }]0, T_{\max}[\times] - 1, +1[, \\ h \in L^2\Big(0, T; H^4_{\text{per}}(-1, +1)\Big) \cap C\Big([0, T]; H^1_{\text{per}}(-1, +1)\Big), \\ h(0) = h_0 \in H^1_{\text{per}}(-1, +1). \end{cases}$$

If $h_0 \in H^k_{\text{per}}(-1, +1)$*, then* $h \in L^\infty\Big(0, T; H^k_{\text{per}}(-1, +1)\Big)$.

Moreover, if $h_0 \in C^\infty_{\text{per}}[-1, +1]$*, then* $h \in L^\infty\Big(0, T; C^\infty_{\text{per}}[-1, +1]\Big),\ \forall T < T_{\max}$.

Lemma 4 *There exists only one maximal solution* $\Big([0, T_{*m}(h_0)), h\Big)$ *in the space*

$$L^2\Big(0, T; H^3_{\text{per}}(-1, +1)\Big) \cap C\Big([0, T]; L^2(-1, +1)\Big)$$

and satisfies:

$$\frac{d}{dt}\int_{-1}^{+1} h(t)\varphi\, dx = \int_{-1}^{+1} \big(hh_{xx} + \frac{1}{2}h_x^2\big)\varphi_{xx} - \beta\int_{-1}^{+1} h_x^{(3)}\varphi_x^{(3)}\, dx, \qquad \forall\varphi \in H^3_{\text{per}}(-1, +1)$$

in $\mathcal{D}'(0, T)$*, and* $h(0) \in L^2(-1, +1)$.

4.2 Blow-up for Dirichlet boundary

Theorem 5 *There exist at least one local solution* $\big([0, T^1_*], h\big)$*:*

$$h \in L^2\big(0, T; V\big) \cap C\big([0, T]; L^2(-1, +1)\big),\ \forall T < T^1_*.$$

$\forall\varphi \in V$*, in* $\mathcal{D}'(0, T^1_*)$*:*

$$\begin{cases} \dfrac{d}{dt}\displaystyle\int_{-1}^{+1} h\varphi\, dx = \int_{-1}^{+1} \Big(hh_{xx} + \frac{1}{2}h_x^2\Big)\varphi_{xx}\, dx - \beta\int_{-1}^{+1} h_x^{(3)}\varphi_x^{(3)}\, dx, \\ h(0) = h_0 \in L^2(-1, +1). \end{cases}$$

In particular, if $h_0 \in H^1(-1, +1)$ *and we consider:*

$$V = \Big\{\varphi \in H^3(-1, +1) : \varphi = \varphi_{xx} = 0\Big\},$$

then there is a unique maximal solution $\Big(\big[0, T^2_*(h_0)\big), h\Big)$ *satisfying* $(KSDW)_{00\beta}$ *in the weak variational sense given above. Moreover*

$$h \in L^2\big(0, T; H^4(-1, +1)\big) \cap C\big([0, T], H^1(-1, +1)\big), \quad \forall T < T^2_*(h_0).$$

Corollary 2 (of Theorem 5) *Let* $h_0 \in H^1(-1,+1),\ V = \Big\{\varphi \in H^3(-1,+1) : \varphi = \varphi_{xx} = 0\Big\}$. *Let* $\varphi_1 = b\cos\left(\frac{\pi}{2}x\right)$, *such that* $\int_{-1}^{+1}\varphi_1(x)\,dx = 1$, $\lambda_1 = \left(\frac{\pi}{2}\right)^2$. *Assume that* $\overline{h_0\varphi_1} > \lambda_1\beta$. *Then the weak maximal solution* $\Big([0,T_{\max}),h\Big)$ *blows up at*

$$T_{\max} \le t_* = -\frac{1}{\lambda_1^3\beta}\ln\left(1 - \frac{\lambda_1\beta}{\overline{h_0\varphi_1}}\right).$$

Moreover $\lim_{t\to T_{max}} |h(t)|_{L^2(-1,+1)} = +\infty$.

4.3 Global existence on $\mathbb{R}_+$ for $h_0 \leqslant 0$ and $\beta > 0$.

We want to prove the existence of a global solution on $\mathbb{R}_+$, $h_0 \leqslant 0$ (h_0 small). We introduce the function $v = -h$. The equation becomes :

$$(SDV_{0\beta})\begin{cases}\partial_t v = -(vv_{xx})_{xx} - \dfrac{1}{2}(v_x^2)_{xx} + \beta v_x^6,\\ v(t) \text{ is periodic,}\\ v(0) = v_0 \geqslant 0.\end{cases}$$

Theorem 6 *If* $c^2M(\beta,v_0)|v_0 - \overline{v}_0|_{L^2} \le (\overline{v}_0)^2$ *then the problem* $(SDV_{0\beta})$ *admits a unique nonnegative solution in the space*

$$L^2\big(0,T;H^3_{\text{per}}(-1,+1)\big)\cap C\big([0,T];H^1_{\text{per}}(-1,+1)\big).$$

References

[1] Bernis, F. (1996). Finite speed of propagation and continuity of the interface for thin viscous flows. *Adv. in Diff. Equa.* 1, **3**, 337–368.

[2] Bernis, F. (1996). Finite speed of propagation for thin viscous flow when $2 \leqslant n \leqslant 3$. *Eq. aux Dérivées partielles, Comptes Rendus de l'Académie des Sciences*, **322**, Série 1, 1169–1174.

[3] Bernis, F. and Friedman, A. (1990). Higher order nonlinear degenerate parabolic equations. *Journal of Differential Equations*, **83**, 179–206.

[4] Bertozzi, A. L. and Pugh, M. (1996). The lubrication approximation for thin Viscous film : regularity and long-time behavior of weak solutions. *Com. Pure and Appl. Math*, **XLIX**, 85–123.

[5] Bertozzi, A. L. and Pugh, M. (2000). Finite time blow-up of solution of some long-wave unstable thin film equation. *Indiana university Journal*, **49** (4), 1323–1366.

[6] Boutat, M., D'Angelo, Y., Hilout, S. and Lods, V. (2004). Existence and finite-time blow-up for the solution to a thin-film surface evolution problem. *Ass. Analysis*, **38**, 93–128.

[7] King, J. R. (2001). Two generalizations of the thin film equation. *Math. and Computer modelling*, **34**, 737–756.

[8] Rakotoson, J. M. and Temam, R. (2001). An optimal compactness theorem, and application to parabolic elliptic equations, *AML*, **14**, 303–306.

[9] Rossi, R. and Savaré, G. (2003). Tightness, Integral Equicontinuity and Compactness for Evolution Problems In Banach Spaces. *Anna. Scuola Norm. Sup. Pisa Cl Sci*, **5**, Vol II, 395–431.

[10] Simon, J. (1987). Compact sets in $L^p(0,T;B)''$. *Ann. Mat. Pura Appl.*, **146**, 65–96.

[11] Spencer, B. J., Davis, S. H. and Voorhees, P. W. (1992). Morphological instability in epitaxially-strained dislocation-free solid films : nonlinear evolution. *Appl. Math, Technical report* **9201**, Dept. of Engineering Sc. and Appl Math, McCormick School of Eng. and Appl. Sc. Northwestern, University Evanston, IL 60208.

[12] Spencer, B. J., Davis, S. H. and Voorhees, P. W. (1991) *Phys. Rev. Letters.* **67**, 5696.

[13] Temam, R. (1984). *Navier-Stokes Equations, Theory and Numerical Analysis*, 3rd Rev. Ed. North-Holland, Amsterdam.

J. Emile Rakotoson
Département de Mathématiques
Université de Fianarantsoa
Cité des Professeurs de l'Université
Logement N^{o}05 Ambonifahidrano
301 FIANARANTSOA
- MADAGASCAR
rako@math.univ-poitiers.fr

J. Michel Rakotoson
Laboratoire de Mathématiques
Université de Poitiers
SP2MI - Boulevard Marie et Pierre
Curie, Téléport 2
BP30179 86962
FUTUROSCOPE CHASSENEUIL
Cedex - FRANCE

Cédric Verbeke
Laboratoire de Mathématiques
Université de Poitiers
SP2MI - Boulevard Marie et Pierre
Curie, Téléport 2
BP30179 86962
FUTUROSCOPE CHASSENEUIL
Cedex - FRANCE