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Weighted (LB)-spaces of Holomorphic Functions and the Dual Density Conditions

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Abstract. We consider weighted inductive limits of spaces of holomorphic functions which are defined as countable unions of weighted Banach spaces of type H^∞. We study the problem of projective description and analyze when these spaces have the dual density conditions of Bierstedt and Bonet.

Espacios (LB) ponderados de funciones holomorfas

Resumen. Consideramos límites inductivos ponderados de espacios de funciones holomorfas que están definidos como la unión numerable de espacios ponderados de Banach de tipo H^∞. Estudiamos el problema de la descripción proyectiva y analizamos cuando estos espacios tienen la condición de densidad dual de Bierstedt y Bonet.

1 Introduction

Countable locally convex inductive limits of spaces of holomorphic functions arise in linear partial differential equations, convolution equations, distribution theory and representation of distributions as boundary values of holomorphic functions, complex analysis in one and several variables and spectral theory and the holomorphic calculus. The problem of *projective description* for weighted (LB)-spaces of holomorphic functions on G is to find out, under which conditions one of the following assertions holds

(1) $\mathcal{V}_0H(G) = H\overline{V}_0(G)$ algebraically and topologically resp.

(2) $\mathcal{V}H(G) = H\overline{V}(G)$ algebraically and topologically.

A positive answer is important because in that case it is possible to describe the topology of the weighted (LB)-space of holomorphic functions with help of the system $(\|\cdot\|_{\overline{v}})_{\overline{v}\in\overline{V}}$ of weighted sup-seminorms.

The main result of Bierstedt, Meise and Summers [11] yields that projective description holds algebraically and topologically if $\mathcal{V} = (v_n)_{n\in\mathbb{N}}$ has condition

$$(S) \quad \forall n \in \mathbb{N}\ \exists m > n \text{ such that } v_m/v_n \text{ vanishes at } \infty \text{ on } G.$$

Positive results for weighted (LB)-spaces of holomorphic functions on $\mathbb{D}$ can also be found in [7] and [23]. There condition (S) is not needed, but each weight has to be normal in the sense of Shields and Williams, see [22].

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In general the answer to the problem of projective description is negative, see [13, 14, 15]. Taylor as well as Bierstedt, Meise and Summers ([29] or [30] Proposition 2 resp. [11]) showed that $H\overline{V}(G)$ and $\mathcal{V}H(G)$ coincide algebraically and have the same bounded subsets. $\mathcal{V}H(G)$ always is a regular inductive limit.

In particular, inductive limits arise in connection with distributions and the Paley-Wiener-Schwartz theorems. Some important examples are listed below:

(1) This example is taken from [11]. We consider the space $W = \mathcal{E} = \mathcal{E}(\mathbb{R}^N)$ of all infinitely often differentiable functions on $\mathbb{R}^N$, endowed with its usual (F)-space topology. A suitable version of the Paley-Wiener-Schwartz theorem yields that the Fourier transform is a topological isomorphism from the space $W' = \mathcal{E}'$ of all distributions with compact support, endowed with the strong topology $\beta(\mathcal{E}', \mathcal{E})$ onto $\hat{W}' = \mathcal{V}H(\mathbb{C}^N)$, where $\mathcal{V} = (v_n)_{n\in\mathbb{N}}$ is defined by

$$v_n(z) = \prod_{j=1}^{N} \frac{\exp(-n|\mathrm{Im}(z_j)|}{(1+|z_j|)^n}, \qquad z = (z_1, \ldots, z_N) \in \mathbb{C}^N.$$

In this case we have $\mathcal{V}H(\mathbb{C}^N) = \mathcal{V}_0H(\mathbb{C}^N) = H\overline{V}_0(\mathbb{C}^N) = H\overline{V}(\mathbb{C}^N)$.

(2) Let $\mathcal{V} = (v_n)_{n\in\mathbb{N}}$ be given by

$$v_n(z) = (1 - |z|^2)^n, \qquad z \in \mathbb{D}, \quad n \in \mathbb{N}.$$

The inductive limit $\mathcal{V}H(G)$ is often denoted by $\mathcal{A}^{-\infty}$ and is a space of Bergman type. We refer to [18].

(3) The following example was studied in [25]. Given a bounded convex domain $G \subset \mathbb{C}$ as well as $d_G(z) = \inf_{t\in\partial G} |z - t|$, $z \in G$. Then we define $\mathcal{V} = (v_n)_{n\in\mathbb{N}}$ by

$$v_n(z) := (\min(1, (d_G(z))^n)), \qquad z \in G, \quad n \in \mathbb{N}.$$

In [25] $\mathcal{V}H(G)$ is denoted by $\mathcal{A}^{-\infty}(G)$.

The dual density conditions were introduced by Bierstedt and Bonet in [4]. In case of weighted (LB)-spaces of continuous functions they showed that projective description is equivalent to $\mathcal{V}C(X)$ having the dual density condition. It remained open what happens in the case of holomorphic functions. In this article we show that the result of Bierstedt and Bonet remains true in the framework of the class $\mathcal{W}$ of radial weights on the unit disk which was introduced by Bierstedt and Bonet in [7].

This article is organized as follows. Section 2 gives the necessary notations and definitions. In Section 3 we study when the projective hulls satisfy the dual density condition(s) and when they have metrizable bounded sets. Section 4 is devoted to the class $\mathcal{W}$ of radial weights on the unit disk. In this framework we prove that projective description is equivalent to $\mathcal{V}H(\mathbb{D})$ having the dual density condition.

2 Definitions and Notations

Our notation on locally convex (l.c.) spaces is standard; see for example Jarchow [19], Köthe [20], Meise, Vogt [24] and Pérez Carreras, Bonet [26]. For a locally convex space E, E^* denotes the space of all linear functionals on E while E' is the topological dual and E'_b the strong dual. If E is a locally convex space, $\mathcal{U}_0(E)$ and $\mathcal{B}(E)$ stand for the families of all absolutely convex 0-neighborhoods and absolutely convex bounded sets in E, respectively.

A locally convex space E is called (DF)-*space* if the following conditions are satisfied:

(1) E has a fundamental sequence of bounded subsets.

(2) For every sequence $(U_n)_{n\in\mathbb{N}}$ of closed, absolutely convex 0-neighborhoods in E such that $U := \cap_{n\in\mathbb{N}} U_n$ absorbs each bounded subset of E, U is a 0-neighborhood in E.

(DF)-spaces were introduced by Grothendieck in [17]. Main examples of (DF)-spaces are strong duals of Fréchet spaces and (LB)-spaces. Conversely, the strong dual of a (DF)-space is always a Fréchet space.

A locally convex space E is said to satisfy the *countable neighborhood property* (c.n.p.) if for every sequence $(U_n)_{n\in\mathbb{N}}$ of 0-neighborhoods in E there are numbers $a_n > 0$ such that

$$U := \bigcap_{n\in\mathbb{N}} a_n U_n$$

is a 0-neighborhood in E. The c.n.p. was given implicitly by Schwartz (see [28] on p. 95) and has got its name by Floret in [16]. By [26] Corollary 8.3.3 and Proposition 8.3.5 each (DF)-space has the countable neighborhood property.

The dual density conditions were introduced for locally convex spaces and thoroughly studied in the case of (DF)-spaces by Bierstedt and Bonet in [4].

A locally convex space E is said to satisfy the *dual density condition* (DDC) (resp. the *strong dual density condition* $(SDDC)$) if the following holds: for every function $\lambda\colon \mathcal{B}(E) \to \mathbb{R}_+\backslash\{0\}$ and for every $A \in \mathcal{B}(E)$ there are a finite subset $\mathcal{B}$ of $\mathcal{B}(E)$ and $\mathcal{U} \in \mathcal{U}(E)$ such that

$$A \cap U \subset \overline{\Gamma}\left(\bigcup_{B\in\mathcal{B}} \lambda(B)B\right) \quad \left(\text{resp. } \Gamma\left(\bigcup_{B\in\mathcal{B}} \lambda(B)B\right)\right),$$

where Γ (resp. $\overline{\Gamma}$) denotes the absolutely convex hull (resp. the closed absolutely convex hull).

In [4] Bierstedt and Bonet showed that the dual density conditions are equivalent in E if E is the strong dual of a Fréchet space. Moreover, a (DF)-space E has the dual density condition if and only if the bounded sets in E are metrizable (see [4] 2. Remarks and 5. Theorem). Finally a Fréchet space E satisfies the density condition if and only if its strong dual has (DDC) (or equivalent $(SDDC)$).

In the sequel G denotes an open subset of $\mathbb{C}^N$, $N \geq 1$. The space $H(G)$ of all holomorphic functions on G will usually be endowed with the topology *co* of uniform convergence on the compact subsets of G. For a decreasing sequence $\mathcal{V} = (v_n)_{n\in\mathbb{N}}$ of strictly positive continuous functions (weights) on G we define

$$Hv_n(G) := \{f \in H(G);\ \|f\|_n := \sup_{z\in G} v_n(z)|f(z)| < \infty\},$$
$$H(v_n)_0(G) := \{f \in H(G);\ v_n|f| \text{ vanishes at } \infty \text{ on } G\},$$
$$\mathcal{V}H(G) := \text{ind}_n Hv_n(G) \text{ and } \mathcal{V}_0H(G) := \text{ind}_n H(v_n)_0(G).$$

B_n (resp. $B_{n,0}$) denotes the closed unit ball of $Hv_n(G)$ (resp. $H(v_n)_0(G)$). By $\overline{B_n}$ and $\overline{B_{n,0}}$ we denote the *co*-closures of the corresponding sets. Note that $\overline{B_n} = B_n$. The system $\overline{V}$ of weights was introduced in [11] as

$$\overline{V} := \{\overline{v} : G \to]0,\infty[; \overline{v} \text{ continuous}, \forall k\ \exists r_k > 0 :\ \overline{v} \leq \inf_k r_k v_k \text{ on } G\}.$$

The corresponding weighted spaces for $\overline{V}$ are called *projective hulls* and are given by

$$H\overline{V}(G) := \{f \in H(G); \|f\|_{\overline{v}} := \sup_{z\in G} \overline{v}(z)|f(z)| < \infty\ \forall \overline{v} \in \overline{V}\},$$
$$H\overline{V}_0(G) := \{f \in H(G);\ \overline{v}|f| \text{ vanishes at } \infty \text{ on } G\ \forall \overline{v} \in \overline{V}\}.$$

The system $(C_{\overline{v}})_{\overline{v}\in\overline{V}}$ (resp. $(C_{\overline{v},0})_{\overline{v}\in\overline{V}}$) , where

$$C_{\overline{v}} := \{f \in H\overline{V}(G); \|f\|_{\overline{v}} \leq 1\} \text{ and } C_{\overline{v},0} := \{f \in H\overline{V}_0(G); \|f\|_{\overline{v}} \leq 1\},$$

gives a 0-neighborhood base of $H\overline{V}(G)$ (resp. $H\overline{V}_0(G)$). We write $\overline{C_{\overline{v}}}$ and $\overline{C_{\overline{v},0}}$ to refer to the *co*-closure.

An important tool to handle weighted spaces of holomorphic functions are the so called *associated growth conditions* mentioned by Andersen and Duncan in [1] and studied thoroughly by Bierstedt, Bonet and Taskinen in [9]. Let v be a weight on G. Its *associated growth* condition is defined by

$$\widetilde{v}(z) := \sup\{|g(z)|;\ g \in H(G),\ |g| \leq v\}, \qquad z \in G.$$

A weight v on a balanced domain $G \subset \mathbb{C}^N$, $N \geq 1$, is said to be *radial* if $v(z) = v(\lambda z)$ holds for every $\lambda \in \mathbb{C}$ such that $|\lambda| = 1$. On a balanced open subset G of $\mathbb{C}^N$, $N \geq 1$, each $f \in H(G)$ has a Taylor series representation about zero,

$$f(z) = \sum_{k=0}^{\infty} p_k(z), \qquad z \in G,$$

where p_k is a k-homogeneous polynomial ($k = 0, 1, \ldots$). The series converges to f uniformly on each compact subset of G. The *Cesàro means* of the partial sums of the Taylor series of f are denoted by $S_n(f)$ ($n = 0, 1, \ldots$); that is,

$$[S_n(f)](z) = \frac{1}{n+1} \sum_{l=0}^{\infty} \left(\sum_{k=0}^{l} p_k(z) \right), \qquad z \in G.$$

Each $S_n(f)$ is a polynomial (of degree $\leq n$) and $S_n(f) \to f$ uniformly on every compact subset of G ($f \in H(G)$ arbitrary).

3 The dual density conditions in projective hulls

In this section we study when $H\overline{V}(G)$ (resp. $H\overline{V}_0(G)$) has the dual density conditions. Moreover we show that $H\overline{V}(G)$ has (DDC) if and only if it satisfies $(SDDC)$. Finally we analyze when these spaces have metrizable bounded sets. First we need some auxiliary results.

Lemma 1 *Let E be a l.c. space with the fundamental sequence $(B_n)_{n\in\mathbb{N}}$ of bounded subsets. E has (DDC) (resp. $(SDDC)$) if and only if for every sequence $(\lambda_j)_{j\in\mathbb{N}}$ of strictly positive numbers and for every $n \in \mathbb{N}$ there are $m > n$ and $U \in \mathcal{U}(E)$ such that*

$$B_n \cap U \subset \overline{\sum_{j=1}^{m} \lambda_j B_j} \left(\textit{resp. } B_n \cap U \subset \sum_{j=1}^{m} \lambda_j B_j \right) \tag{1}$$

PROOF. If E has (DDC), for every sequence $(\lambda_j)_{j\in\mathbb{N}}$ of strictly positive numbers and for every $n \in \mathbb{N}$ there are $m > n$ and $U \in \mathcal{U}(E)$ such that $B_n \cap U \subset \overline{\Gamma}\left(\cup_{j=1}^{m} \lambda_j B_j\right) \subset \overline{\sum_{j=1}^{m} \lambda_j B_j}$.
Conversely, we fix a sequence $(\mu_j)_{j\in\mathbb{N}}$ of strictly positive numbers and $n \in \mathbb{N}$. Put $\lambda_j := \frac{\mu_j}{2^j}$ for every $j \in \mathbb{N}$ and apply (1). Then there are $m > n$ and $U \in \mathcal{U}(E)$ with

$$B_n \cap U \subset \overline{\sum_{j=1}^{m} \lambda_j B_j} = \overline{\sum_{j=1}^{m} \frac{\mu_j}{2^j} B_j} \subset \overline{\Gamma}\left(\cup_{j=1}^{m} \mu_j B_j\right).$$

Hence, E has the dual density condition.

For the strong dual density condition the proof is analogous. ■

Lemma 2 *$H\overline{V}(G)$ has (DDC) if and only if it satisfies $(SDDC)$.*

PROOF. The previous lemma and the fact that $H\overline{V}(G)$ and the regular inductive limit $\mathcal{V}H(G)$ have the same bounded sets imply that $H\overline{V}(G)$ has the dual density condition if and only if for every sequence $(\lambda_j)_{j\in\mathbb{N}}$ of strictly positive numbers and for every $n \in \mathbb{N}$ we can find $m > n$ and $\overline{v} \in \overline{V}$ such that $B_n \cap C_{\overline{v}} \subset \overline{\sum_{j=1}^m \lambda_j B_j}$. $\sum_{j=1}^m \lambda_j B_j$ is *co*-compact, hence closed in $H\overline{V}(G)$. We conclude $\overline{\sum_{j=1}^m \lambda_j B_j} = \sum_{j=1}^m \lambda_j B_j$. An application of Lemma 1 yields the claim. ■

Proposition 1 *Let G be a balanced open subset of $\mathbb{C}^N$, $N \geq 1$. Moreover we assume that $\mathcal{V} = (v_n)_{n\in\mathbb{N}}$ is a decreasing sequence of strictly positive continuous and radial functions on G such that $H(v_1)_0(G)$ contains the polynomials. If $H\overline{V}(G)$ (resp. $H\overline{V}_0(G)$) has the dual density condition, then the following condition holds:*

$(*)$ *for every sequence $(\lambda_j)_{j\in\mathbb{N}}$ of strictly positive numbers and for every $n \in \mathbb{N}$ there are $m > n$ and $\overline{v} \in \overline{V}$ such that*

$$\left(\min\left(\frac{1}{v_n}, \frac{1}{\overline{v}}\right)\right)^{\sim} \leq \sum_{j=1}^{m} \frac{\lambda_j}{v_j} \text{ on } G.$$

PROOF. By definition, for every sequence $(\lambda_j)_{j\in\mathbb{N}}$ of strictly positive numbers and every $n \in \mathbb{N}$ there are $m > n$ and $\overline{v} \in \overline{V}$ such that

$$B_n \cap C_{\overline{v}} \subset \sum_{j=1}^{m} \lambda_j B_j \quad \left(\text{resp. } B_{n,0} \cap C_{\overline{v},0} \subset \overline{\sum_{j=1}^{m} \lambda_j B_{j,0}}\right). \tag{2}$$

It is enough to show that (2) implies $(*)$. We fix $f \in H(G)$ such that $|f| \leq \left(\min\left(\frac{1}{v_n}, \frac{1}{\overline{v}}\right)\right)^{\sim}$ on G; hence $|f| \leq \frac{1}{v_n}$ and $|f| \leq \frac{1}{\overline{v}}$ on G. W.l.o.g. we can choose $\overline{v} \in \overline{V}$ strictly positive, continuous and radial (see [8]).

Now, we consider the sequence $(S_k f)_{k\in\mathbb{N}}$ of the Cesàro means (of the partial sums) of the Taylor series about 0. We obtain $|S_k f| \leq 1/v_n$ and $|S_k f| \leq 1/\overline{v}$ on G (see [8, Proposition 1.2.(c)]). Moreover, each polynomial $S_k f$ is an element of $H\overline{V}(G)$ (resp. $H\overline{V}_0(G)$) and hence of $B_n \cap C_{\overline{v}}$ (resp. $B_{n,0} \cap C_{\overline{v},0}$). (2) yields $S_k f \in \overline{\sum_{j=1}^m \lambda_j B_j} = \sum_{j=1}^m \lambda_j B_j$ (resp. $S_k f \in \overline{\sum_{j=1}^m \lambda_j B_{j,0}} \subset \sum_{j=1}^m \lambda_j B_j$) for every $k \in \mathbb{N}$. Thus, each $S_k f$ can be written as $S_k f = \sum_{j=1}^m \lambda_j g_j$ where $g_j \in B_j$ for every $j \in \{1, \ldots, m\}$. We get

$$|S_k f| \leq \sum_{j=1}^{m} \frac{\lambda_j}{v_j} \text{ on } G \text{ for every } k \in \mathbb{N}.$$

Since $S_k f \to f$ pointwise, we obtain that $|f| \leq \sum_{j=1}^m \lambda_j / v_j$ on G. Taking the supremum over all f we get $(*)$. ■

There are sequences $\mathcal{V} = (v_n)_{n\in\mathbb{N}}$ of strictly positive continuous functions on an open subset G of $\mathbb{C}^N$, $N \geq 1$, with condition $(*)$ such that $\mathcal{V}H(G)$ and $H\overline{V}(G)$ do not coincide topologically:

In [14] Bonet and Taskinen constructed a sequence $\mathcal{V} = (v_n)_{n\in\mathbb{N}}$ of strictly positive continuous functions on an open set $G \subset \mathbb{C}^2$ with condition

(M) $\forall n \in \mathbb{N}\ \forall Y$ not relatively compact in X $\exists m = m(n, Y) > n$ such that: $\inf_{y\in Y} \frac{v_m(y)}{v_n(y)} = 0$ such that $H\overline{V}(G)$ and $\mathcal{V}H(G)$ do not coincide. We know that (M) implies

(V_2) for every sequence $(\lambda_j)_{j\in\mathbb{N}}$ of strictly positive numbers and for every $n \in \mathbb{N}$ there are $m > n$ and $\overline{v} \in \overline{V}$ with

$$\min\left(\frac{1}{v_n}, \frac{1}{\overline{v}}\right) \leq \sum_{j=1}^{m} \frac{\lambda_j}{v_j} \text{ on } G$$

(see [10] and [2, Proposition I.2.4]). $(*)$ obviously follows from (V_2). Note that $(*)$ differs from (V_2) by using associated growth conditions.

By [4, Theorem 1.5] a (DF)-space has the dual density condition if and only if it has metrizable bounded sets. The last property can be characterized for $H\overline{V}(G)$. We need some preparations.

Proposition 2 *Let E be a l.c. space with the c.n.p. and a fundamental sequence $(B_n)_{n\in\mathbb{N}}$ of bounded sets. Then E has metrizable bounded sets if and only if there is $U \in \mathcal{U}(E)$ such that for every $n \in \mathbb{N}$ and every $V \in \mathcal{U}(E)$ there is $a > 0$ with*

$$B_n \cap aU \subset V. \tag{3}$$

PROOF. First we assume that condition (3) is satisfied. Since $\left(\frac{1}{m}U\right)_{m\in\mathbb{N}}$ is a 0-neighborhood base in B_n, $n \in \mathbb{N}$ arbitrary, with respect to the topology induced by E, this yields the claim.

Conversely, we assume that E has metrizable bounded sets. We fix $n \in \mathbb{N}$ and choose a decreasing sequence $(V_{n,k})_{k\in\mathbb{N}}$ of absolutely convex 0-neighborhoods in E such that for every $V \in \mathcal{U}(E)$ there is $k \in \mathbb{N}$ with $B_n \cap V_{n,k} \subset V$. We apply the c.n.p. to find numbers $a_{n,k} > 0$ such that $U := \bigcap_{n,k\in\mathbb{N}} a_{n,k} V_{n,k}$ is a 0-neighborhood with all the desired properties. ■

Lemma 3 *$H\overline{V}(G)$ has the c.n.p.*

PROOF. By [12, Theorem 1.6], $C\overline{V}(G)$ is a (DF)-space, hence satisfies the c.n.p. (see [26, Proposition 8.3.5]). Since $H\overline{V}(G)$ is a topological subspace of $C\overline{V}(G)$, $H\overline{V}(G)$ also has the c.n.p. ■

Proposition 3 *Consider the following assertions.*

(a) $H\overline{V}(G)$ (resp. $H\overline{V}_0(G)$) has metrizable bounded sets.

(b) $\mathcal{V} = (v_n)_{n\in\mathbb{N}}$ satisfies the following condition: there is $\overline{v} \in \overline{V}$ such that for every $\overline{w} \in \overline{V}$ there is $a > 0$ with

$$\left(\min\left(\frac{1}{v_n}, \frac{a}{\overline{v}}\right)\right)^{\sim} \leq \frac{1}{\overline{w}} \text{ on } G. \tag{4}$$

Then (a) implies (b). If we assume in addition that $G \subset \mathbb{C}^N$, $N \geq 1$, is balanced and $\mathcal{V} = (v_n)_{n\in\mathbb{N}}$ is a decreasing sequence of strictly positive continuous and radial functions on G such that $H(v_1)_0(G)$ contains the polynomials, then (a) follows from (b).

PROOF. By Proposition 2, $H\overline{V}(G)$ (resp. $H\overline{V}_0(G)$) has metrizable bounded sets if and only if there is $\overline{v} \in \overline{V}$ such that for every $n \in \mathbb{N}$ and every $\overline{w} \in \overline{V}$ there is $a > 0$ with

$$B_n \cap aC_{\overline{v}} \subset C_{\overline{w}} \quad (\text{resp. } B_{n,0} \cap aC_{\overline{v},0} \subset C_{\overline{w},0})\,. \tag{5}$$

To finish the proof we have to show the equivalence of (4) and (5).

(4) $\Longrightarrow$ (5): We fix $f \in B_n \cap aC_{\overline{v}}$ (resp. $B_{n,0} \cap aC_{\overline{v},0}$). Hence f is an element of $H\overline{V}(G)$ (resp. $H\overline{V}_0(G)$) with $|f| \leq \left(\min\left(\frac{1}{v_n}, \frac{a}{\overline{v}}\right)\right)^{\sim} \leq \frac{1}{\overline{w}}$ on G. Thus, f belongs to $C_{\overline{w}}$ (resp. $C_{\overline{w},0}$).

(5) $\Longrightarrow$ (4): Under the additional assumptions we fix $f \in H(G)$ such that $|f| \leq \left(\min\left(\frac{1}{v_n}, \frac{a}{\overline{v}}\right)\right)^{\sim}$ on G. This implies $|f| \leq \frac{1}{v_n}$ and $|f| \leq \frac{a}{\overline{v}}$ on G.
W.l.o.g. we can find $\overline{v} \in \overline{V}$ strictly positive, continuous and radial. From this point on the proof works analogously to the proof of Proposition 1. ■

4 The dual density conditions for inductive limits

This section is devoted to the class $\mathcal{W}$ of radial weights on the unit disk. We study when the weighted inductive limits of holomorphic functions have the dual density conditions and analyze the connection between the dual density conditions and the problem of projective description.

First of all we introduce the class $\mathcal{W}$ of radial weights on the unit disk. Let $\mathcal{W}$ be a class of strictly positive continuous radial weights v on the unit disc $\mathbb{D}$ which satisfy $\lim_{r\to 1-} v(r) = 0$ and for which the restriction of v to $[0, 1)$ is non-increasing. We suppose that the class $\mathcal{W}$ is stable under finite minima and under multiplication by positive scalars.

Next, we assume that there is a sequence $R_n\colon H(\mathbb{D}) \to H(\mathbb{D})$, $n \in \mathbb{N}$, of linear operators which are continuous for the compact open topology and such that the range of each R_n is a finite dimensional subspace of the polynomials. It is also assumed that $R_n R_m = R_{\min(n,m)}$ holds for each n, m with $n \neq m$ and that for each polynomial p there is n such that $R_n p = p$, from which it follows that $R_m p = p$ for each $m \geq n$. Moreover, we suppose that there is $c > 0$ such that $\sup_{|z|=r} |R_n p(z)| \leq c \sup_{|z|=r} |p(z)|$ for each n, each $r \in (0, 1)$ and each polynomial p.

Finally, setting $R_0 := 0$, and putting $r_n := 1 - 2^{-n}$, $n \in \mathbb{N} \cup \{0\}$, we assume that the following conditions are satisfied by the class $\mathcal{W}$:

(P1) There is $C \geq 1$ such that for each $v \in \mathcal{W}$ and for each polynomial p:

$$\frac{1}{C} \sup_n \left(\sup_{|z|=r_n} |(R_{n+2} - R_{n-1})p(z)| \right) v(r_n) \leq \|p\|_v$$
$$\leq C \sup_n \left(\sup_{|z|=r_n} |(R_{n+1} - R_n)p(z)| \right) v(r_n).$$

(P2) For each $v \in \mathcal{W}$ there is $D(v) \geq 1$ such that for each sequence $(p_n)_{n\in\mathbb{N}}$ of polynomials of which only finitely many are non-zero:

$$\sup_{z\in\mathbb{D}} \left| \sum_{n=1}^{\infty} (R_{n+1} - R_n) p_n(z) \right| v(z) \leq D(v) \sup_k \left(\sup_{|z|=r_k} |p_k(z)| \right) v(r_k).$$

The *main example* for $\mathcal{W}$ is the set of all the strictly positive continuous radial weights v on $\mathbb{D}$ which satisfy $\lim_{r\to 1-} v(r) = 0$, are non-increasing on $[0, 1)$, and such that there are $\varepsilon_0 > 0$ and $k(0) \in \mathbb{N}$ with the following conditions:

(L1) $\inf_k \dfrac{v(r_{k+1}}{v(r_k)} \geq \varepsilon_0,$

(L2) $\limsup_{k\to\infty} \dfrac{v(r_{k+k(0)})}{v(r_k)} < 1 - \varepsilon_0.$

In this case, R_n is the *convolution with the de la Vallée Poussin kernel*, i.e. for a holomorphic function f on $\mathbb{D}$, $f(z) = \sum_{k=0}^{\infty} a_k z^k$, we have

$$R_n f(z) := \sum_{k=0}^{2^n} a_k z^k + \sum_{k=2^n+1}^{2^{n+1}} \frac{2^{n+1} - k}{2^n} a_k z^k.$$

In fact, R_n is nothing but the arithmetic mean of the partial sums of index $2^n, \ldots, 2^{n+1} - 1$ of the Taylor series of f. The conditions $(L1)$ and $(L2)$ form a uniform version of the conditions introduced by W. Lusky in [21, 22], and they also appear in the sequence space representations for weighted (LB)-spaces given by

Mattila, Saksman, Taskinen [23]. Bierstedt and Bonet showed that $(L1)$ and $(L2)$ imply the conditions $(P1)$ and $(P2)$ (see [7]).

In this section we show that under certain assumptions the dual density condition for $H\overline{V}(\mathbb{D})$, $H\overline{V}_0(\mathbb{D})$, $\mathcal{V}H(\mathbb{D})$ and $\mathcal{V}_0H(\mathbb{D})$ is satisfied if and only if $\mathcal{V}$ enjoys condition $(*)$ of Proposition 1. This is equivalent to the topological equality $\mathcal{V}H(\mathbb{D}) = H\overline{V}(\mathbb{D})$. This is similar to the result of Bierstedt and Bonet, [5], in the setting of continuous functions.

In the sequel we assume w.l.o.g. that $v_n \geq 2v_{n+1}$ on G. First we need some auxiliary results.

Lemma 4 *Let G be a balanced open subset of $\mathbb{C}^N$, $N \geq 1$, and $\mathcal{V} = (v_n)_{n\in\mathbb{N}}$ be a decreasing sequence of strictly positive continuous and radial functions on G such that $H(v_1)_0(G)$ contains the polynomials and such that $\mathcal{V}_0H(G)$ is a topological subspace of $H\overline{V}_0(G)$. Then $(C_n)_{n\in\mathbb{N}}$, $C_n := B_n \cap \mathcal{V}_0H(G)$ for every $n \in \mathbb{N}$, yields a fundamental sequence of bounded sets in $\mathcal{V}_0H(G)$.*

PROOF. Obviously, each bounded set B in $\mathcal{V}_0H(G)$ is also bounded in $H\overline{V}(G)$. Hence there is $n \in \mathbb{N}$ with $B \subset C_n$. On the other hand, each C_n is bounded in $H\overline{V}(G)$. Each C_n is contained in $\mathcal{V}_0H(G)$. Since $\mathcal{V}_0H(G)$ is a topological subspace of $H\overline{V}_0(G)$, C_n is bounded in $\mathcal{V}_0H(G)$. ■

Proposition 4 *Let G and $\mathcal{V}$ be as in the previous lemma. The space of all polynomials $\mathcal{P}$ (endowed with the topology induced by $\mathcal{V}_0H(G)$) satisfies the dual density condition if and only if $\mathcal{V}_0H(G)$ has the dual density condition.*

PROOF. First, we show that $\mathcal{P}$ is large in $\mathcal{V}_0H(G)$, i.e. each bounded set in $\mathcal{V}_0H(G)$ is contained in the closure of a bounded set in $\mathcal{P}$ with respect to $\mathcal{V}_0H(G)$. For this we fix $n \in \mathbb{N}$ and $f \in C_n$. The sequence $(S_kf)_{k\in\mathbb{N}}$ converges to f in $H\overline{V}_0(G)$ (see [8, Section 1]), hence in $\mathcal{V}_0H(G)$ by assumption. Moreover, each polynomial S_kf belongs to $B_n \cap \mathcal{V}_0H(G) = C_n$. We conclude $C_n \subset \overline{\mathcal{P} \cap B_n}^{\mathcal{V}_0H(G)}$. It remains to show that $\mathcal{P}$ has the dual density condition if and only if $\mathcal{V}_0H(G)$ also satisfies this property. Since $\mathcal{V}_0H(G)$ is a (DF)-space and $\mathcal{P}$ is large in $\mathcal{V}_0H(G)$, from [26, 8.3.24] we know that $\mathcal{P}$ is a (DF)-space. An application of [4, Remark 1.10.(b)] yields that $\mathcal{P}$ has the dual density condition if and only if $\mathcal{V}_0H(G)$ also satisfies this property. ■

In the sequel we always assume that the sequence $\mathcal{V}$ of weights is contained in the class $\mathcal{W}$. In particular, note that if $\mathcal{V} = (v_n)_{n\in\mathbb{N}} \subset \mathcal{W}$ is a decreasing sequence of strictly positive continuous functions on $\mathbb{D}$ such that $H(v_1)_0(\mathbb{D})$ contains the polynomials, then $\mathcal{P}$ has the dual density condition if and only if $\mathcal{V}_0H(\mathbb{D})$ also enjoys this property (see [7]).

Theorem 1 *Let $\mathcal{V} = (v_n)_{n\in\mathbb{N}} \subset \mathcal{W}$ be a decreasing sequence of strictly positive continuous functions on $\mathbb{D}$ such that $H(v_1)_0(\mathbb{D})$ contains the polynomials. Then the following are equivalent:*

(a) $\mathcal{V}_0H(\mathbb{D})$ has the dual density condition.

(b) $\mathcal{V}H(\mathbb{D})$ enjoys the dual density condition.

(c) $H\overline{V}_0(\mathbb{D})$ satisfies the dual density condition.

(d) $\mathcal{V} = (v_n)_{n\in\mathbb{N}}$ satisfies condition $(*)$.

PROOF. By [6, Proposition 4] we have $\mathcal{V}H(\mathbb{D}) = (\mathcal{V}_0H(\mathbb{D})'_b)'_i$.

(a) $\Longrightarrow$ (b): Let $\mathcal{V}_0H(\mathbb{D})$ have the dual density condition. Then each bounded subset of $\mathcal{V}_0H(\mathbb{D})$ is metrizable by [4, Theorem 1.5.(a)]. [3, Theorem 1.4] yields that $\mathcal{V}_0H(\mathbb{D})'_b$ has the density condition and is distinguished. Thus, $\mathcal{V}H(\mathbb{D}) = (\mathcal{V}_0H(\mathbb{D})'_b)'_i = \mathcal{V}H(\mathbb{D})''_{bb}$, and $\mathcal{V}H(\mathbb{D})$ has the dual density condition.

(b) $\Longrightarrow$ (a): Let $\mathcal{V}H(\mathbb{D})$ have the dual density condition. Since $\mathcal{V}H(\mathbb{D}) = (\mathcal{V}_0H(\mathbb{D})'_b)'_i$ holds, by [27, Corollary 2], $\mathcal{V}_0H(\mathbb{D})'_b$ has the density condition. An application of [3, Theorem 1.4 and Theorem 1.5.(b)] implies that $\mathcal{V}_0H(\mathbb{D})$ enjoys the dual density condition.

(a) $\Longleftrightarrow$ (c): By [7, Theorem 1], $\mathcal{V}_0H(\mathbb{D})$ is a dense topological subspace of $H\overline{V}_0(\mathbb{D})$. In particular $H\overline{V}_0(\mathbb{D})$ is a (DF)-space (see [19, Theorem 12.4.8 (d)]). Using [4, 1.10.(a) and (b)] we obtain the desired equivalence.

(c) $\Longrightarrow$ (d): This follows immediately from Proposition 1.

(d) $\Longrightarrow$ (c): We assume that (d) is satisfied. By Lemma 1 we have to show:

For every sequence $(\mu_k)_{k\in\mathbb{N}}$ of strictly positive numbers and every $n \in \mathbb{N}$ there are $m > n$ and $\overline{v} \in \overline{V}$ with $B_{n,0} \cap C_{\overline{v},0} \subset \overline{\sum_{k=1}^{m} \mu_k B_{k,0}}$.

Since (a) and (c) are equivalent, it suffices to consider only polynomials. We fix $n \in \mathbb{N}$ as well as a sequence $(\mu_k)_{k\in\mathbb{N}}$ of strictly positive numbers. The assumption (d) yields $m > n$ and $\overline{v} \in \overline{V}$. Now put

$$\lambda_k := \frac{\mu_k}{(2c^2 + \max_{1\le i\le m} D_i)Cm} \qquad \text{for every } \ k \in \mathbb{N},$$

where c is the constant from the estimate before condition $(P1)$, C the constant from $(P1)$ and $D_i = D(v_i)$ for every $1 \le i \le m$ in the sense of $(P2)$.

We fix $p \in (B_{n,0} \cap C_{\overline{v},0}) \cap \mathcal{P}$. Then $|p| \le \min\left(\frac{1}{v_n}, \frac{1}{\overline{v}}\right)^{\sim}$ on $\mathbb{D}$ and (d) implies $\left(\min\left(\frac{1}{v_n}, \frac{1}{\overline{v}}\right)\right)^{\sim} \le \sum_{k=1}^{m} \frac{\lambda_k}{v_k} \le \max\left(\frac{m\lambda_1}{v_1}, \dots, \frac{m\lambda_m}{v_m}\right)$ on $\mathbb{D}$.

Putting $w := \min\left(\frac{v_1}{m\lambda_1}, \dots, \frac{v_m}{m\lambda_m}\right)$, w is an element of the class $\mathcal{W}$ and $w|p| \le 1$ on $\mathbb{D}$.

We write $\kappa_i := \frac{1}{m\lambda_i}$ for every $i \in \{1, \dots, m\}$, hence $w = \min(\kappa_1 v_1, \dots, \kappa_m v_m)$.
We have $p = \sum_{n=0}^{\infty}(R_{n+1} - R_n)p = R_1 p + \sum_{n=1}^{\infty}(R_{n+1} - R_n)p$, and the sum is finite. We first treat the term $R_1 p$.

By the condition before (P1) and the estimate on $w|p|$, we get

$$w(r_1) \sup_{|z|=r_1} |R_1 p(z)| \le c w(r_1) \sup_{|z|=r_1} |p(z)| \le c.$$

We select $i \in \{1, \dots, m\}$ with $w(r_1) = \kappa_i v_i(r_1)$. From the second inequality in (P1), applied to the polynomial $R_1 p$ and v_i, and once more the condition before (P1), we conclude

$$\begin{aligned}
\sup_{z\in\mathbb{D}} v_i(z)|R_1 p(z)| &\le C \sup_n v_i(r_n) \left(\sup_{|z|=r_n} |(R_{n+1} - R_n) R_1 p(z)| \right) \\
&= C v_i(r_1) \sup_{|z|=r_1} |(R_2 - R_1) R_1 p(z)| \\
&= C(\kappa_i)^{-1} w(r_1) \sup_{|z|=r_1} |(R_2 - R_1) R_1 p(z)| \\
&\le 2cC(\kappa_i)^{-1} w(r_1) \sup_{|z|=r_1} |R_1 p(z)| \\
&\le 2c^2 C(\kappa_i)^{-1}.
\end{aligned}$$

This implies one of the following facts: $R_1 p \in 2Cc^2 m\lambda_1 B_{1,0}, \dots, R_1 p \in 2Cc^2 m\lambda_m B_{m,0}$.

To treat the other term $p - R_1 p = \sum_{n=1}^{\infty}(R_{n+1} - R_n)p$, we first apply the first inequality in (P1) for w and the estimate for $w|p|$ to get

$$w(r_n) \left(\sup_{|z|=r_n} |(R_{n+2} - R_{n-1}) p(z)| \right) \le C \tag{6}$$

for every $n \in \mathbb{N}$. We can write $\mathbb{N}$ as a disjoint union $\cup_{i=1}^{m} J_i$ such that

$$w(r_j) = \kappa_i v_i(r_j) \text{ for } j \in J_i, \ 1 \le i \le m.$$

Now put $g_i = \sum_{n\in J_i}(R_{n+1}-R_n)p$ for $i \in \{1,\dots,m\}$, which is a polynomial; clearly $p - R_1p = \sum_{i=1}^m g_i$. We fix $i \in \{1,\dots,m\}$ and let $p_n^i := (R_{n+2}-R_{n-1})p$ for $n \in J_i$ and $p_n^i := 0$ otherwise. The properties of the sequence $(R_n)_{n\in\mathbb{N}}$ imply

$$g_i = \sum_{n\in J_i}(R_{n+1}-R_n)(R_{n+2}-R_{n-1})p = \sum_{n=1}^{\infty}(R_{n+1}-R_n)p_n^i,$$

and all the sums are finite; hence

$$\sup_{z\in\mathbb{D}} v_i(z)|g_i(z)| = \sup_{z\in\mathbb{D}} v_i(z)\left|\sum_{n=1}^{\infty}(R_{n+1}-R_n)p_n^i\right|.$$

Since only a finite number of the p_n^i are non-zero and all the weights belong to the class $\mathcal{W}$, we can apply (P2) and the estimate (6) to conclude

$$\begin{aligned}
\sup_{z\in\mathbb{D}} v_i(z)|g_i(z)| &\le D_i \sup_n \left(\sup_{|z|=r_n} |p_n^i(z)|\right) v_i(r_n)\\
&\le D_i \sup_{n\in J_i} \left(\sup_{|z|=r_n} |p_n^i(z)|\right) v_i(r_n)\\
&= D_i \sup_{n\in J_i} \left(\sup_{|z|=r_n} |(R_{n+2}-R_{n-1})p(z)|\right) v_i(r_n)\\
&\le D_i(\kappa_i)^{-1} \sup_{n\in J_i} \left(\sup_{|z|=r_n} |(R_{n+2}-R_{n-1})p(z)|\right) w(r_n)\\
&\le D_i(\kappa_i)^{-1}C.
\end{aligned}$$

This yields $g_i \in m\lambda_i D_i C B_{i,0}$ for every $i \in \{1,\dots,m\}$. Thus we have one of the following inclusions

$$p = R_1p + \sum_{i=1}^m g_i \in (2c^2 + D_1)C\lambda_1 m B_{1,0} + \sum_{i=2}^m D_i C\lambda_i m B_{i,0} \subset \sum_{i=1}^m \mu_i B_{i,0}, \dots,$$

$$p = R_1p + \sum_{i=1}^m g_i \in \sum_{i=1}^{m-1} D_i C\lambda_i m B_{i,0} + (2c^2 + D_m)C\lambda_m B_{m,0} \subset \sum_{i=1}^m \mu_i B_{i,0}.$$

Hence $H\overline{V}_0(\mathbb{D})$ has the (strong) dual density condition. ■

Theorem 2 *Let $\mathcal{V} = (v_n)_{n\in\mathbb{N}} \subset \mathcal{W}$ be a decreasing sequence of strictly positive continuous functions on $\mathbb{D}$. The following are equivalent:*

(a) $H\overline{V}(\mathbb{D})$ enjoys the dual density condition (or equivalently the strong dual density condition).

(b) The sequence $\mathcal{V} = (v_n)_{n\in\mathbb{N}}$ satisfies condition $(*)$.

PROOF. (a) $\Longrightarrow$ (b): This is Proposition 1.

(b) $\Longrightarrow$ (a): This proof is very similar to the one of the implication (d) $\Longrightarrow$ (c) in Theorem 1. Since there are some technical changes, we give the full proof here. We assume that (b) is true. By Lemma 1 we have to show:

> for every sequence $(\mu_k)_{k\in\mathbb{N}}$ of strictly positive numbers and every $n \in \mathbb{N}$ there are $m > n$ and $\overline{v} \in \overline{V}$ with $B_n \cap C_{\overline{v}} \subset \sum_{k=1}^m \mu_k B_k$.

We fix $n \in \mathbb{N}$ and a sequence $(\mu_k)_{k\in\mathbb{N}}$ of strictly positive numbers and we apply (b) to find $m > n$ and $\overline{v} \in \overline{V}$. Now put

$$\lambda_k := \frac{\mu_k}{C(2c^2 + \max_{1\le j\le m} D_j)Cm} \qquad \text{for every } k \in \mathbb{N},$$

where c is the constant from the estimate before condition $(P1)$, C the constant from $P1)$ and $D_j = D(v_j)$ for every $1 \le j \le m$ in the sense of $(P2)$.

We fix $f \in (B_n \cap C_{\overline{v}}) \cap \mathcal{P}$. Then $|f| \le \min\left(\frac{1}{v_n}, \frac{1}{\overline{v}}\right)^{\sim}$ on $\mathbb{D}$ and (b) implies $\left(\min\left(\frac{1}{v_n}, \frac{1}{\overline{v}}\right)\right)^{\sim} \le \sum_{k=1}^{m} \frac{\lambda_k}{v_k} \le \max\left(\frac{m\lambda_1}{v_1}, \dots, \frac{m\lambda_m}{v_m}\right)$ on $\mathbb{D}$.

Putting $w := \min\left(\frac{v_1}{m\lambda_1}, \dots, \frac{v_m}{m\lambda_m}\right)$, w is an element of the class $\mathcal{W}$.

We write $\kappa_i := \frac{1}{m\lambda_i}$ for every $i \in \{1, \dots, m\}$, hence $w = \min(\kappa_1 v_1, \dots, \kappa_m v_m)$.

Using the Cesàro means (of the partial sums) of the Taylor series of f about 0 we obtain a sequence $(S_k f)_{k\in\mathbb{N}}$ of polynomials with $w|S_k f| \le w|f| \le 1$ for every $k \in \mathbb{N}$ and $S_k f \to f$ in $(H(\mathbb{D}), co)$.

We have $S_k f = \sum_{n=0}^{\infty}(R_{n+1} - R_n)S_k f = R_1 S_k f + \sum_{n=1}^{\infty}(R_{n+1} - R_n)S_k f$, and the sum is finite. We first treat the term $R_1 S_k f$.

By the condition before (P1) and the estimate on $w|S_k f|$, we get

$$w(r_1) \sup_{|z|=r_1} |R_1 S_k f(z)| \le c w(r_1) \sup_{|z|=r_1} |S_k f(z)| \le c.$$

We select $i \in \{1, \dots, m\}$ with $w(r_1) = \kappa_i v_i(r_1)$. From the second inequality in (P1), applied to the polynomial $R_1 p$ and v_i, and once more the condition before (P1), we conclude

$$\sup_{z\in\mathbb{D}} v_i(z)|R_1 S_k f(z)| \le 2c^2 C(\kappa_i)^{-1}.$$

This implies one of the following facts: $R_1 S_k f \in 2Cc^2 m\lambda_1 B_1, \dots, R_1 S_k f \in 2Cc^2 m\lambda_m B_m$.

To treat the other term $S_k f - R_1 S_k f = \sum_{n=1}^{\infty}(R_{n+1} - R_n)S_k f$, we first apply the first inequality in (P1) for w and the estimate for $w|S_k f|$ to get

$$w(r_n)\left(\sup_{|z|=r_n} |(R_{n+2} - R_{n-1})S_k f(z)|\right) \le C. \tag{7}$$

We can write $\mathbb{N}$ as a disjoint union $\cup_{i=1}^{m} J_i$ such that

$$w(r_j) = \kappa_i v_i(r_j) \text{ for } j \in J_i.$$

Now put $g_i = \sum_{n\in J_i}(R_{n+1} - R_n)S_k f$ for $i \in \{1, \dots, m\}$, which is a polynomial; clearly $S_k f - R_1 S_k f = \sum_{i=1}^{m} g_i$. We fix $i \in \{1, \dots, m\}$ and let $(S_k f)_n^i := (R_{n+2} - R_{n-1})S_k f$ for $n \in J_i$ and $(S_k f)_n^i := 0$ otherwise. The properties of the sequence $(R_n)_{n\in\mathbb{N}}$ imply

$$g_i = \sum_{n\in J_i}(R_{n+1} - R_n)(R_{n+2} - R_{n-1})S_k f = \sum_{n=1}^{\infty}(R_{n+1} - R_n)(S_k f)_n^i,$$

and all the sums are finite; hence

$$\sup_{z\in\mathbb{D}} v_i(z)|g_i(z)| = \sup_{z\in\mathbb{D}} v_i(z)\left|\sum_{n=1}^{\infty}(R_{n+1} - R_n)p_n^i\right|.$$

Since only a finite number of the $(S_k f)_n^i$ are non-zero and all the weights belong to the class $\mathcal{W}$, we can apply (P2) and the estimate (6) to conclude as before

$$\sup_{z\in\mathbb{D}} v_i(z)|g_i(z)| \le D_i(\kappa_i)^{-1}C.$$

This yields $g_i \in m\lambda_i D_i C B_i$ for every $i \in \{1, \dots, m\}$. Thus one of the following inclusions holds

$S_k f = R_1 S_k f + \sum_{i=1}^m g_i \in (2c^2 + D_1)C\lambda_1 m B_1 + \sum_{i=2}^m D_i C\lambda_i m B_i \subset \sum_{i=1}^m \mu_i B_i, \dots,$

$S_k f = R_1 S_k f + \sum_{i=1}^m g_i \in \sum_{i=1}^{m-1} D_i C\lambda_i m B_i + (2c^2 + D_m)C\lambda_m m B_m \subset \sum_{i=1}^m \mu_i B_i.$

Finally, each $S_k f$ belongs to $\sum_{i=1}^m \mu_i B_i$. Every B_i is co-compact and $S_k f \to f$ with respect to co. Now, we can conclude that f is an element fo $\sum_{i=1}^m \mu_i B_i$. The claim follows. ■

Theorem 3 (Bierstedt, Bonet [7, Theorem 7]) *Let $\mathcal{V} = (v_n)_{n\in\mathbb{N}}$ be a decreasing sequence of strictly positive continuous functions on $\mathbb{D}$ in the class $\mathcal{W}$. Then the topological equality $\mathcal{V}H(\mathbb{D}) = H\overline{V}(\mathbb{D})$ holds if and only if for every sequence $(\lambda_j)_{j\in\mathbb{N}}$ of strictly positive numbers there is $\overline{v} \in \overline{V}$ such that for every $n \in \mathbb{N}$ and every $M > 0$ there is $m > n$ with*

$$(+) \qquad \left(\min\left(\frac{M}{v_n}, \frac{1}{\overline{v}}\right)\right)^{\sim} \leq \sum_{j=1}^m \frac{\lambda_j}{v_j}.$$

In the following we want to show the equivalence of the condition $(*)$ and $(+)$. To do this we first prove that under certain assumptions $H\overline{V}(\mathbb{D})$ is a (DF)-space. We need some auxiliary results.

Lemma 5 (Bierstedt, Bonet, [5, Lemma A in Section 5]) *Let E be a locally convex space with a fundamental sequence $(B_n)_{n\in\mathbb{N}}$ of absolutely convex bounded sets and $\mathcal{U}_0$ a fixed basis of absolutely convex 0-neighborhoods in E. Then E is a (DF)-space if and only if, for every sequence $(\lambda_n)_{n\in\mathbb{N}}$ of positive numbers and for every sequence $(W_n)_{n\in\mathbb{N}} \subset \mathcal{U}_0$, the intersection $\bigcap_{n\in\mathbb{N}}(W_n + \sum_{k=1}^n \lambda_k B_k)$ is a 0-neighborhood in E.*

Lemma 6 *Let $\mathcal{V} = (v_n)_{n\in\mathbb{N}}$ be a decreasing sequence of strictly positive continuous functions on $\mathbb{D}$ in the class $\mathcal{W}$ such that there is a fundamental system of weights in $\overline{V}$ contained in $\mathcal{W}$. Then, for every $n \in \mathbb{N}$, every $\overline{w}_n \in \overline{V} \cap \mathcal{W}$ and every sequence $(\lambda_k)_{k\in\mathbb{N}}$ of strictly positive numbers*

$$\frac{1}{2}B \subset \left(C_{\overline{w}_n} + \sum_{k=1}^n \lambda_k B_k\right)$$

holds, where $B := \{f \in H\overline{V}(\mathbb{D});\ |f| \leq \frac{1}{C(D_1+2c^2)\overline{w}_n} + \sum_{k=1}^n \frac{\lambda_k}{C(D_{k+1}+2c^2)} \frac{1}{2^{k+1}v_k}$ on $\mathbb{D}\}$ and c is the constant in the estimate before $(P1)$, C the constant in $(P2)$, $D_1 = D(\overline{w}_n)$, $D_{i+1} = D(v_i)$ for every $1 \leq i \leq n$ in the sense of $(P2)$.

PROOF. This proof is very similar to the one of the implication (d) $\Longrightarrow$ (c) in Theorem 1. Since there are some technical changes, we give the full proof here. We fix $n \in \mathbb{N}$, $\overline{w}_n \in \overline{V} \cap \mathcal{W}$ and a sequence $(\lambda_k)_{k\in\mathbb{N}}$ of strictly positive numbers and choose $f \in \frac{1}{2}B$. We have $f \in H\overline{V}(\mathbb{D})$ and

$$\begin{aligned} |f| &\leq \frac{1}{2}\left(\frac{1}{C(D_1+2c^2)}\frac{1}{\overline{w}_n} + \sum_{k=1}^n \frac{\lambda_k}{C(D_{k+1}+2c^2)}\frac{1}{2^{k+1}v_k}\right) \\ &\leq \max\left(\frac{1}{C(D_1+2c^2)}\frac{1}{\overline{w}_n}, \frac{\lambda_1}{C(D_2+2c^2)}\frac{1}{v_1}, \dots, \frac{1}{C(D_{n+1}+2c^2)}\frac{1}{v_n}\right) \end{aligned}$$

Put $u := \min\left(C(D_1+2c^2)\overline{w}_n, \frac{C(D_2+2c^2)}{\lambda_1}v_1, \dots, \frac{C(D_{n+1}+2c^2)}{\lambda_n}v_n\right)$. u belongs to the class $\mathcal{W}$ and $u|f| \leq 1$ on $\mathbb{D}$. We write $\kappa_1 := C(D_1+2c^2)$, $\kappa_i := \frac{C(D_i+2c^2)}{\lambda_{i-1}}$ for $2 \leq i \leq n+1$, $u_1 := \overline{w}_n$, $u_i = v_{i-1}$ for $2 \leq i \leq n+1$.

Using the Cesàro means of the partial sums of the Taylor series of f about 0 we get a sequence $(S_k f)_{k\in\mathbb{N}}$ of polynomials with $u|S_k f| \leq u|f| \leq 1$ for every $k \in \mathbb{N}$ and $S_k f \to f$ in $(H(\mathbb{D}), co)$. By assumption $S_k f$ is an element of $H\overline{V}(\mathbb{D})$.

We have $S_k f = \sum_{n=0}^{\infty}(R_{n+1} - R_n)S_k f = R_1 S_k f + \sum_{n=1}^{\infty}(R_{n+1} - R_n)S_k f$ and the sum is finite. We first treat the term $R_1 S_k f$.

By the condition before (P1) and the estimate on $u|S_k f|$, we get

$$u(r_1) \sup_{|z|=r_1} |R_1 S_k f(z)| \leq c u(r_1) \sup_{|z|=r_1} |S_k f(z)| \leq c.$$

We select $i \in \{1, \dots, n+1\}$ with $u(r_1) = \kappa_i u_i(r_1)$. From the second inequality in (P1), applied to the polynomial $R_1 S_k f$ and u_i, and once more the condition before (P1), we conclude as before

$$\sup_{z \in \mathbb{D}} u_i(z) |R_1 S_k f(z)| \leq 2c^2 C(\kappa_i)^{-1}.$$

This implies one of the following facts: $R_1 S_k f \in \frac{2c^2 C}{C(D_2+2c^2)} \lambda_1 B_1, \dots, R_1 S_k f \in \frac{2c^2 C}{C(D_{n+1}+2c^2)} \lambda_n B_n$.

To treat the other term $S_k f - R_1 S_k f = \sum_{n=1}^{\infty}(R_{n+1} - R_n)S_k f$, we first apply the first inequality in (P1) for u and the estimate for $u|S_k f|$ to get

$$u(r_n) \left(\sup_{|z|=r_n} |(R_{n+2} - R_{n-1})S_k f(z)| \right) \leq C \tag{8}$$

for every $n \in \mathbb{N}$. We can write $\mathbb{N}$ as a disjoint union $\cup_{i=1}^{n+1} J_i$ such that

$$u(r_j) = \kappa_i u_i(r_j) \text{ for every } j \in J_i,\ i \in \{1, \dots, n+1\}.$$

Now put, for $i \in \{1, \dots, n+1\}$, $g_i = \sum_{n \in J_i}(R_{n+1} - R_n)S_k f$, which is a polynomial; clearly $S_k f - R_1 S_k f = \sum_{i=1}^{n+1} g_i$. We fix $i \in \{1, \dots, n+1\}$ and let $(S_k f)_n^i := (R_{n+2} - R_{n-1})S_k f$ for $n \in J_i$ and $(S_k f)_n^i := 0$ otherwise. The properties of the sequence $(R_n)_{n \in \mathbb{N}}$ imply

$$g_i = \sum_{n \in J_i}(R_{n+1} - R_n)(R_{n+2} - R_{n-1})S_k f = \sum_{n=1}^{\infty}(R_{n+1} - R_n)(S_k f)_n^i,$$

and all the sums are finite; hence

$$\sup_{z \in \mathbb{D}} u_i(z) |g_i(z)| = \sup_{z \in \mathbb{D}} u_i(z) \left| \sum_{n=1}^{\infty}(R_{n+1} - R_n)(S_k f)_n^i \right|.$$

Since only a finite number of the $S_k f_n^i$ are non-zero and all the weights belong to the class $\mathcal{W}$, we can apply (P2) and the estimate (6) to conclude as before

$$\sup_{z \in \mathbb{D}} u_i(z) |g_i(z)| \leq D_i(\kappa_i)^{-1} C.$$

We obtain $g_1 \in \frac{CD_1}{C(D_1+2c^2)} C_{\overline{w}_n}$ and $g_{i+1} \in \frac{CD_{i+1}\lambda_i}{C(D_{i+1}+2c^2)} B_i$ for every $i \in \{1, \dots, n\}$. Hence one of the following inclusions holds

$S_k f \in \frac{C(D_1+2c^2)}{C(D_1+2c^2)} C_{\overline{w}_n} + \frac{D_2 C}{C(D_2+2c^2)} \lambda_1 B_1 + \dots + \frac{D_{n+1} C}{C(D_{n+1}+2c^2)} \lambda_n B_n \subset C_{\overline{w}_n} + \sum_{k=1}^{n} \lambda_k B_k, \dots,$

$S_k f \in \frac{CD_1}{C(D_1+2c^2)} C_{\overline{w}_n} + \dots + \frac{C(D_{n+1}+2c^2)}{C(D_{n+1}+2c^2)} \lambda_n B_n \subset C_{\overline{w}_n} + \sum_{k=1}^{n} \lambda_k B_k.$

Finally we obtain $S_k f \in C_{\overline{w}_n} + \sum_{k=1}^{n} \lambda_k B_k$. $C_{\overline{w}_n}$ and B_i are co-compact for every $i \in \{1, \dots, n\}$ and $S_k \to f$ in $(H(\mathbb{D}), co)$. Thus, f is an element of $C_{\overline{w}_n} + \sum_{k=1}^{n} \lambda_k B_k$. ∎

The proof of the following proposition is analogous to the proof of [5] Proposition B in Section 4 with a slight change. We replace the "Ernst-Schnettler-trick" by the previous proposition.

Proposition 5 *Let $\mathcal{V} = (v_n)_{n\in\mathbb{N}}$ be a decreasing sequence of strictly positive continuous functions on $\mathbb{D}$ in the class $\mathcal{W}$ such that a fundamental system of weights in $\overline{V}$ is contained in $\mathcal{W}$. Then $H\overline{V}(\mathbb{D})$ is a (DF)-space.*

PROOF. We fix a sequence $(\lambda_k)_{k\in\mathbb{N}}$ of strictly positive numbers and a sequence $(\overline{w}_n)_{n\in\mathbb{N}}$ of weights in $\overline{V} \cap \mathcal{W}$. By Lemma 5 we have to show that

$$W := \bigcap_{n\in\mathbb{N}} \left(C_{\overline{w}_n} + \sum_{k=1}^{n} \lambda_k B_k \right)$$

is a 0-neighborhood in $H\overline{V}(\mathbb{D})$. By Lemma 6 we have $\frac{1}{2}B \subset C_{\overline{w}_n} + \sum_{k=1}^{n} \lambda_k B_k$. Now, we put

$$\begin{aligned} a_n &:= \frac{1}{2}\left(\frac{1}{C(D_1+2c^2)}\frac{1}{\overline{w}_n} + \sum_{k=1}^{n} \frac{\lambda_k}{C(D_{k+1}+2c^2)}\frac{1}{2^{k+1}v_k} \right) \\ \overline{v}_n &:= \frac{1}{a_n},\ n\in\mathbb{N},\ \text{and } \overline{v} := \sup_{n\in\mathbb{N}} \overline{v}_n. \end{aligned}$$

We get

$$\overline{v}_n \le 2\inf\left\{ \frac{C(D_2+2c^2)2^2}{\lambda_1}v_1, \ldots, \frac{C(D_{n+1}+2c^2)2^{n+1}}{\lambda_n}v_n, C(D_1+2c^2)\overline{w}_n \right\} \tag{9}$$

for every $n \in \mathbb{N}$. Hence $\overline{v}_n : G \to \mathbb{R}_+$ belongs to the Nachbin system $\overline{\overline{V}} = \overline{\overline{V}}(\mathcal{V})$ associated with $\mathcal{V}$ on G with respect to the discrete topology. (9) yields $\overline{w} \in \overline{\overline{V}}$ such that

$$\{f \in H\overline{V}(\mathbb{D}); \overline{w}|f| \le 1 \text{ on } G\} \subset C_{\overline{w}_n} + \sum_{k=1}^{n} \lambda_k B_k$$

for $n \in \mathbb{N}$. By [10] section 4.2, there is $\overline{v} \in \overline{V}$ with $\overline{w} \le \overline{v}$ and $\{f \in H\overline{V}(\mathbb{D});\ \overline{v}|f| \le 1 \text{ on } \mathbb{D}\}$ is a 0-neighborhood in $H\overline{V}(\mathbb{D})$ and contained in W. The claim follows. ■

Proposition 6 *Let $\mathcal{V} = (v_n)_{n\in\mathbb{N}}$ be a decreasing sequence of strictly positive continuous functions on $\mathbb{D}$ in the class $\mathcal{W}$ such that a fundamental system of weights in $\overline{V}$ is contained in $\mathcal{W}$. Then the conditions $(+)$ and $(*)$ are equivalent.*

PROOF. Obviously, $(+)$ implies $(*)$.
$(*) \Longrightarrow (+)$: By Theorem 3 $(*)$ yields that $H\overline{V}(\mathbb{D})$ has the dual density condition. Moreover $H\overline{V}(\mathbb{D})$ is a (DF)-space by Proposition 5. We obtain the topological equality $\mathcal{V}H(\mathbb{D}) = H\overline{V}(\mathbb{D})$ (see [8]) and finally $(+)$ (see Theorem 3). ■

Corollary 1 *Let $\mathcal{V} = (v_n)_{n\in\mathbb{N}}$ be a decreasing sequence of strictly positive continuous functions on $\mathbb{D}$ in the class $\mathcal{W}$ such that a fundamental system of weights in $\overline{V}$ is contained in $\mathcal{W}$. Consider the following assertions:*

(a) $H\overline{V}(\mathbb{D}) = \mathcal{V}H(\mathbb{D})$ holds algebraically and topologically.

(b) $H\overline{V}(\mathbb{D})$ has the dual density condition (or equivalently the strong dual density condition).

(c) $H\overline{V}(\mathbb{D})$ has metrizable bounded subsets.

(d) $H\overline{V}_0(\mathbb{D})$ enjoys the dual density condition.

(e) $\mathcal{V}_0 H(\mathbb{D})$ *satisfies the dual density condition.*

The assertions (a) to (d) are equivalent. If we assume in addition that $H(v_1)_0(\mathbb{D})$ *contains the polynomials, then all conditions are equivalent.*

We finish this article by providing examples which satisfy the assumption that a fundamental system of weights in $\overline{V}$ is contained in $\mathcal{W}$.

Proposition 7 *Let* $\mathcal{V} = (v_n)_{n\in\mathbb{N}}$ *be a decreasing sequence of strictly positive continuous functions on* $\mathbb{D}$. *We assume that there are* $\varepsilon_0 > 0$ *and* $k_0 \in \mathbb{N}$ *such that the following conditions are satisfied:*

($L1$*)* $\inf_k \dfrac{v_n(r_{k+1})}{v_n(r_k)} \geq \varepsilon_0$ *for every* $n \in \mathbb{N}$.

($\overline{L2}$*) There is* $k_1 \in \mathbb{N}$ *with*

$$v_n(r_{k+k_0}) < (1-\varepsilon_0)v_n(r_k)$$

for every $k \geq k_1$ *and* $n \in \mathbb{N}$.

Then there is a fundamental system of weights in $\overline{V}$ *contained in* $\mathcal{W}$.

PROOF. We fix $\overline{v} \in \overline{W}$. There is $\overline{w} \in \overline{W}$ with $\overline{v} \leq \overline{w}$ such that there is a sequence $(\beta_n)_{n\in\mathbb{N}}$ of positive numbers such that for every $r > 0$ there is $k(r) \in \mathbb{N}$ with

$$\overline{w}(z) = \min_{1\leq n\leq k(r)} \beta_n v_n(z) \tag{10}$$

for every $z \in \mathbb{D}$ with $|z| \leq r$.

We have to show that there are $\varepsilon_2 > 0$ and $k_2 \in \mathbb{N}$ such that

($L1$) $\inf_k \frac{\overline{w}(r_{k+1})}{\overline{w}(r_k)} \geq \varepsilon_2$.

($L2$) $\limsup_{k\to\infty} \frac{\overline{w}(r_{k+k_2})}{\overline{w}(r_k)} < 1-\varepsilon_2$.

We choose $\varepsilon_2 = \varepsilon_0$ and $k_2 = k_0$ and prove ($L1$). For a fixed $k \in \mathbb{N}$ it remains to show

$$\varepsilon_2 \overline{w}(r_k) \leq \overline{w}(r_{k+1}).$$

We select $0 < r_{k+1} < s < 1$ and get by (10)

$$\overline{w}(z) = \min_{1\leq n\leq k(s)} \beta_n v_n(z)$$

for $z \in \mathbb{D}$ with $|z| \leq s$. We distinguish the following cases:

(i) We have $\overline{w}(r_k) = \beta_j v_j(r_k)$ and $\overline{w}(r_{k+1}) = \beta_j v_j(r_{k+1})$. This yields $\varepsilon_2 \overline{w}(r_k) = \varepsilon_2 \beta_j v_j(r_k) \leq \beta_j v_j(r_{k+1}) = \overline{w}(r_{k+1})$.

(ii) We assume $\overline{w}(r_k) = \beta_j v_j(r_k)$ and $\overline{w}(r_{k+1}) = \beta_l v_l(r_{k+1})$. It follows
$\varepsilon_2 \overline{w}(r_k) = \varepsilon_2 \beta_j v_j(r_k) \leq \varepsilon_2 \beta_l v_l(r_k) \leq \beta_l v_l(r_{k+1}) = \overline{w}(r_{k+1})$

Thus $\inf_k \frac{\overline{w}(r_k)}{\overline{w}(r_{k+1})} \geq \varepsilon_2$.

It remains to show ($L2$). As before we choose $\varepsilon_2 = \varepsilon_0$ and $k_2 = k_0$. We have to prove that there is $N_0 \in \mathbb{N}$ such that for every $k \geq N_0$

$$\overline{w}(r_{k+k_2}) < (1-\varepsilon_2)\overline{w}(r_k)$$

holds. There is $k_1 \in \mathbb{N}$ such that for every $k \geq k_1$ and for every $n \in \mathbb{N}$ we have

$$v_n(r_{k+k_2}) < (1-\varepsilon_2)v_n(r_k).$$

We fix $k \geq k_1$ and select $0 < k + k_2 < s < 1$. Hence by (10)

$$\overline{w}(z) = \min_{1 \leq n \leq k(s)} \beta_n v_n(z)$$

for every $z \in \mathbb{D}$ with $|z| \leq s$.

We put $N_0 := k_1$ and distinguish the following cases:

(i) We have $\overline{w}(r_{k+k_2}) = \beta_j v_j(r_{k+k_2})$ and $\overline{w}(r_k) = \beta_j v_j(r_k)$. This implies $\overline{w}(r_{k+k_2}) = \beta_j v_j(r_{k+k_2}) < (1-\varepsilon_2)\beta_j v_j(r_k) = (1-\varepsilon_2)\overline{w}(r_k)$.

(ii) We assume $\overline{w}(r_{k+k_2}) = \beta_j v_j(r_{k+k_2})$ and $\overline{w}(r_k) = \beta_l v_l(r_k)$. Hence $\overline{w}(r_{k+k_2}) = \beta_j v_j(r_{k+k_2}) \leq \beta_l v_l(r_{k+k_2}) < (1-\varepsilon_2)\beta_l v_l(r_k) = (1-\varepsilon_2)\overline{w}(r_k)$.

The claim follows. ■

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