RAC$\mathbb{SAM}$
Rev. R. Acad. Cien. Serie A. Mat.
VOL. **99** (1), 2005, pp. 119–124
Matemática Aplicada / Applied Mathematics

Existence of solutions for a degenerate nonlinear evolution equation

A. Sanih Bonfoh

Abstract. We consider a model of generalized Cahn-Hilliard equations with a logarithmic free energy and a degenerate mobility, and obtain a result on the existence of solutions

Existencia de soluciones para una ecuación no lineal degenerada

Resumen. En este trabajo se considera un modelo de ecuaciones Cahn-Hilliard generalizadas con energía libre logarítmica y movilidad degenerada, y se obtiene un resultado sobre la existencia de soluciones

1 Introduction

We set $\Omega = \prod_{i=1}^{n}]0, L_i[,\ \ L_i > 0,\ i = 1, \dots, n,\ n = 2$ or 3, and consider the following system:

$$\begin{cases} \frac{\partial\rho}{\partial t} - a\cdot\nabla\frac{\partial\rho}{\partial t} - \operatorname{div}(A\nabla\frac{\partial\rho}{\partial t}) = -\nabla\cdot\ J, \\ J = -B(\rho)\nabla w, \\ w - b.\nabla w - \operatorname{div}(C\nabla w) = -\alpha\Delta\rho + f'(\rho) + \beta\frac{\partial\rho}{\partial t} + c.\nabla\frac{\partial\rho}{\partial t} - \operatorname{div}(D\nabla\frac{\partial\rho}{\partial t}), \\ \rho|_{t=0} = \rho_0, \\ \rho \text{ and } J \text{ are } \Omega-\text{periodic;} \end{cases} \tag{1}$$

where $\alpha, \beta > 0$, a, b, c are vectors in $\mathbb{R}^n$, ρ is the order parameter, w is the chemical potential, A, C, D are three n-dimensional symmetric and positive definite matrices with constant coefficients, and $B(\rho)$ is a degenerate symmetric and positive matrix (B is called the mobility tensor). For the sake of simplicity, we assume that $B(\rho) = \kappa(\rho)I$, I being the identity matrix and κ the function defined by:

$$\kappa(\rho) = (1-\rho^2)\overline{\kappa}(\rho), \tag{2}$$

where $\overline{\kappa} : [-1, 1] \to \mathbb{R}$ satisfies:

$$\overline{\kappa} \in \mathcal{C}^1(\mathbb{R}), \quad 0 < \overline{\kappa}_0 \le \overline{\kappa}(s) \le \overline{\kappa}_1, \ \ \forall s \in \mathbb{R}. \tag{3}$$

The free energy $f : [-1, 1] \to \mathbb{R}$ is given by:

$$\begin{cases} f(s) = \frac{1}{2}(1-s^2) + \frac{\theta}{2}[(1+s)\ln(\frac{1+s}{2}) + (1-s)\ln(\frac{1-s}{2})], \quad s \in]-1, 1[, \\ f(-1) = f(1) = 0, \end{cases} \tag{4}$$

Presentado por Díaz Díaz el 3 de noviembre de 2004.
Recibido: 4 de octubre de 2004. *Revisado*: 31 de enero de 2005. *Aceptado*: 2 de febrero de 2005.
Palabras clave / Keywords: generalized Cahn-Hilliard equation, degenerate mobility, logarithmic potential, existence of solutions
Mathematics Subject Classifications: 35A05, 35B45

with $0 < \theta < 1$.

The above problem is based on constitutive equations proposed by M. Gurtin in [8]. This system is a generalization of the Cahn-Hilliard equation, which describes very important qualitative features of two-phase systems, namely the transport of atoms between unit cells (see [3] and [4]). In his derivations, M. Gurtin takes into account the work of the internal microforces, the anisotropy and also the deformations of the material that are neglected here (see [7] and [10] where full nonlinear partial differential equations have been derived). Most of the mathematical literature on the Cahn-Hilliard equation (and also generalized Cahn-Hilliard equations) has concentrated on a polynomial nonlinearity and/or a constant mobility (see for instance [2], [10] and [11]). Some results concerning the Cahn-Hilliard equation with a logarithmic potential and/or a non-constant mobility can be found in [1], [5] and [6]. We propose in this paper to study the existence of solutions of the above problem.

For the mathematical setting of the problem, we denote by $\|\cdot\|$ and $(\cdot,\cdot)$ the usual norm and scalar product in $L^2(\Omega)$ (and also in $L^2(\Omega)^n$). For each $\rho \in L^1(\Omega)$, $m(\rho)$ stands for the average of ρ, that is, $m(\rho) = \frac{1}{|\Omega|}\int_\Omega \rho(x)\,dx$. For a space X, we denote by $\dot{X}$ the space $\{q \in X,\ m(q) = 0\}$, and by X' the dual space of X. We define $L_1 q = q - a.\nabla q - \operatorname{div}(A\nabla q)$, $L_2 q = q - b \cdot \nabla q - \operatorname{div}(C\nabla q)$ and $Nq = -\Delta q$, for all $q \in H^2_{per}(\Omega)$. The operator N is linear, self-adjoint, strictly positive with compact inverse N^{-1} on $\dot{H}^2_{per}(\Omega)$. We set $\Omega_T = \Omega \times]0,T[$, $\overline{q} = q - m(q)$; and denote by $L^2(\Omega_T)$ both spaces $L^2(0,T;L^2(\Omega))$ and $L^2(0,T;L^2(\Omega)^n)$. We endow $(\dot{H}^1_{per}(\Omega))'$ with the norm $\|\cdot\|_{-1}$ defined by $\|q\|_{-1} = \|N^{-\frac{1}{2}}q\|, \forall q \in (\dot{H}^1_{per}(\Omega))'$. There exist two positive constants c_1 and c_2 such that $c_1\|\overline{q}\|_{-1} \leq \|\overline{q}\| \leq c_2\|\nabla q\|,\ \forall q \in H^1_{per}(\Omega)$.

If $b = 0$, then the linear operator L_2 is self-adjoint, strictly positive with compact inverse L_2^{-1} on $H^2_{per}(\Omega)$. We can then introduce the following weak formulation of the problem:
Find $(\rho, J) : [0,T] \to H^1_{per}(\Omega) \times L^2(\Omega)^n$, such that $\rho(0) = \rho_0$, and for a.e. $t \in [0,T]$, $\forall T > 0$,

$$\frac{d}{dt}(L_1\rho, q) = (J, \nabla q), \quad \forall q \in H^1_{per}(\Omega), \tag{5}$$

$$(J, q) = (\kappa(\rho)\nabla L_2^{-1}(-\alpha\Delta\rho + f'(\rho) + \beta\frac{\partial\rho}{\partial t} + c.\nabla\frac{\partial\rho}{\partial t} - \operatorname{div}(D\nabla\frac{\partial\rho}{\partial t}), q), \quad \forall q \in H^1_{per}(\Omega)^n. \tag{6}$$

Throughout this paper, the same letter C shall denote positive constants that may change from line to line.

2 Preliminary results

In this section, we assume that A and C are not necessarily symmetric and positive definite and that the mobility κ is such that

$$\kappa \in \mathcal{C}(\mathbb{R}), \quad 0 < \kappa_0 \leq \kappa(s) \leq \kappa_1, \ \ \forall s \in \mathbb{R}. \tag{7}$$

We further assume that the potential f satisfies the following conditions:

$$\begin{cases} f \in \mathcal{C}^1(\mathbb{R}), \\ f(s) \geq -c_1, \ \ c_1 > 0, \quad \forall s \in \mathbb{R}, \\ |f'(s)| \leq c_2|s|^q + c_3, \quad c_2,\, c_3 > 0, \ \ \forall s \in \mathbb{R}, \end{cases} \tag{8}$$

where $q \geq 1$ if $n = 2$ and $q \in [1,6]$ if $n = 3$.
Now, we consider the following weak formulation:
Find $(\rho, w) : [0,T] \to H^1_{per}(\Omega) \times L^2(\Omega)$, such that $\rho(0) = \rho_0$, and for a.e. $t \in [0,T]$, $\forall T > 0$,

$$\frac{d}{dt}(L_1\rho, q) = -(\kappa(\rho)\nabla w, \nabla q), \quad \forall q \in H^1_{per}(\Omega), \tag{9}$$

$$(L_2 w, q) = \alpha(\nabla\rho, \nabla q) + (f'(\rho), q) + \beta(\frac{\partial\rho}{\partial t}, q) + (c.\nabla\frac{\partial\rho}{\partial t}, q) - (\mathrm{div}(D\nabla\frac{\partial\rho}{\partial t}), q), \quad \forall q \in H^1_{per}(\Omega). \tag{10}$$

We note that $a \cdot \nabla$ is antisymmetric on $H^1_{per}(\Omega)$, that is, $(a \cdot \nabla p, q) = -(p, a \cdot \nabla q), \forall p, q \in H^1_{per}(\Omega)$. We take $q = 1$ in (9) and observe that the average of ρ is conserved:

$$m(\rho(t)) = m(\rho_0), \quad \forall t \geq 0. \tag{11}$$

We now take $q = 1$ in (10) and obtain

$$m(w) = m(f'(\rho)). \tag{12}$$

Under assumptions (7) and (8) we prove the following result.

Theorem 1 *We assume that $\rho_0 \in H^1_{per}(\Omega)$, $a + b = 0$ and that $A = C$. Then, there exists a pair of functions (ρ, w) solution of* (9)–(10) *such that $\rho \in \mathcal{C}([0,T]; H^1_{per}(\Omega))$, $w \in L^2(0,T; H^1_{per}(\Omega))$ and $\frac{\partial\rho}{\partial t} \in L^2(0,T; H^1_{per}(\Omega))$.*

PROOF. The existence follows from standard arguments, using Galerkin approximations and then passing to the limit (see for instance [9] and [11]). In order to derive a priori estimates, we formally take $q = w$ in (9) and $q = \frac{\partial\rho}{\partial t}$ in (10). We obtain

$$\int_\Omega L_1 \frac{\partial\rho}{\partial t}\, w\, dx + \int_\Omega \kappa(\rho)|\nabla w|^2\, dx = 0; \tag{13}$$

and

$$\frac{d}{dt}\int_\Omega (\frac{\alpha}{2}|\nabla\rho|^2 + f(\rho))\, dx + \beta\int_\Omega |\frac{\partial\rho}{\partial t}|^2\, dx + \int_\Omega |D^{\frac{1}{2}}\nabla\frac{\partial\rho}{\partial t}|^2\, dx - \int_\Omega \frac{\partial\rho}{\partial t}\, L_2 w\, dx = 0, \tag{14}$$

noting that $(c \cdot \nabla\frac{\partial\rho}{\partial t}, \frac{\partial\rho}{\partial t}) = 0$. Assuming that $a + b = 0$ and $A = C$, we have $(L_1\frac{\partial\rho}{\partial t}, w) = (\frac{\partial\rho}{\partial t}, L_2 w)$, therefore

$$\begin{aligned}\int_\Omega (\frac{\alpha}{2}|\nabla\rho|^2 + f(\rho))\, dx + \beta\int_{\Omega_T} |\frac{\partial\rho}{\partial t}|^2\, dxdt + \int_{\Omega_T} |D^{\frac{1}{2}}\nabla\frac{\partial\rho}{\partial t}|^2\, dxdt \\ + \int_{\Omega_T} \kappa(\rho)|\nabla w|^2\, dxdt = \int_\Omega (\frac{\alpha}{2}|\nabla\rho_0|^2 + f(\rho_0))\, dx.\end{aligned} \tag{15}$$

Since $\rho_0 \in H^1_{per}(\Omega)$, we obtain

$$\int_\Omega (\frac{\alpha}{2}|\nabla\rho|^2 + f(\rho))\, dx + \beta\int_{\Omega_T} |\frac{\partial\rho}{\partial t}|^2\, dxdt + \int_{\Omega_T} |D^{\frac{1}{2}}\nabla\frac{\partial\rho}{\partial t}|^2\, dxdt + \int_{\Omega_T} \kappa(\rho)|\nabla w|^2\, dxdt \leq C; \tag{16}$$

hence

$$\|\rho\|_{L^\infty(0,T.H^1_{per}(\Omega))} \leq C, \quad \|\frac{\partial\rho}{\partial t}\|_{L^2(0,T;H^1_{per}(\Omega))} \leq C, \quad \|w\|_{L^2(0,T;H^1_{per}(\Omega))} \leq C; \tag{17}$$

using Sobolev embedding theorems and Poincaré's inequality. Finally, the fact that $\rho \in \mathcal{C}([0,T]; H^1_{per}(\Omega))$ follows from standard compactness results (see [9] and [11]). ■

Remark 1 *Theorem 1 is still true if, instead of $a + b = 0$ and $A = C$, we consider the following assumption: there exists a positive constant c_0 such that $\beta x^2 + \kappa(s)|y|^2 + Dz.z + (A - C)y.z + (a + b).yx \geq c_0(x^2 + |y|^2 + |z|^2)$, $\forall s, x \in \mathbb{R}, \forall y, z \in \mathbb{R}^n$.* □

3 A regularized problem

We denote by ψ and ϕ the functions

$$\psi(s) = \frac{\theta}{2}\left[(1+s)\ln\left(\frac{1+s}{2}\right) + (1-s)\ln\left(\frac{1-s}{2}\right)\right], \tag{18}$$

and $\phi(s) = \psi'(s)$, for $s \in]-1,1[$. We then have $f(s) = \frac{1}{2}(1-s^2) + \psi(s)$ and $f'(s) = -s + \phi(s)$.

The major difficulty in the study of problem (1)–(4) is that κ degenerates and $\phi(s)$ is singular at $s = \pm 1$ and, therefore, has no meaning if $\rho = \pm 1$ in an open set of non-zero measure. To overcome this difficulty, we consider a regularized problem as in [1] and [6]. The mobility κ is replaced by the non-degenerate function κ_ϵ defined by:

$$\kappa_\epsilon(s) = \begin{cases} \kappa(-1+\epsilon) & \text{if } s \le -1+\epsilon, \\ \kappa(s) & \text{if } |s| \le 1-\epsilon, \\ \kappa(1-\epsilon) & \text{if } s \ge 1-\epsilon; \end{cases} \tag{19}$$

and the logarithmic free energy $f(\rho)$ is replaced by the twice continuously differentiable function $f_\epsilon(s) = \frac{1}{2}(1-s^2) + \psi_\epsilon(s)$, where $\epsilon \in]0,1[$, and

$$\psi_\epsilon(s) = \begin{cases} \frac{\theta}{2}(1-s)\ln[\frac{1-s}{2}] + \frac{\theta}{4\epsilon}(1+s)^2 + \frac{\theta}{2}(1+s)\ln[\frac{\epsilon}{2}] - \frac{\theta\epsilon}{4} & \text{if } s \le -1+\epsilon, \\ \psi(s) & \text{if } |s| \le 1-\epsilon, \\ \frac{\theta}{2}(1+s)\ln[\frac{1+s}{2}] + \frac{\theta}{4\epsilon}(1-s)^2 + \frac{\theta}{2}(1-s)\ln[\frac{\epsilon}{2}] - \frac{\theta\epsilon}{4} & \text{if } s \ge 1-\epsilon. \end{cases} \tag{20}$$

The monotone function $\phi_\epsilon = \psi'_\epsilon$ has the following properties (see [1]):

- for all $r,\ s$, $\qquad f'_\epsilon(s)(r-s) \le f_\epsilon(r) - f_\epsilon(s) + \frac{1}{2}(r-s)^2;$ (21)
- $$\forall \epsilon \le \frac{1}{2}, \begin{cases} \theta(r-s)^2 \le (\phi_\epsilon(r) - \phi_\epsilon(s))(r-s), & \forall r,\ s, \\ \frac{\epsilon}{\theta}(\phi_\epsilon(r) - \phi_\epsilon(s))^2 \le (\phi_\epsilon(r) - \phi_\epsilon(s))(r-s), & \forall r,\ s; \end{cases} \tag{22}$$
- for ϵ sufficiently small, e.g. if $\epsilon \le \epsilon_0 = \dfrac{\theta}{8}$, then

$$f_\epsilon(s) \ge \frac{\theta}{8\epsilon}([s-1]_+^2 + [-1-s]_+^2) - 1 \ge -1 \quad \forall s, \tag{23}$$

where $[\cdot]_+ = \max\{\cdot, 0\}$.

We now study the corresponding regularized problem of (5)–(6):
Find $(\rho_\epsilon, J_\epsilon) : [0,T] \to H^1_{per}(\Omega) \times L^2(\Omega)^n$, such that $\rho_\epsilon(0) = \rho_0$, and for a.e. $t \in [0,T]$, $\forall T > 0$, $J_\epsilon = -\kappa_\epsilon(\rho_\epsilon)\nabla w_\epsilon$ and

$$\frac{d}{dt}(L_1\rho_\epsilon, q) = (J_\epsilon, \nabla q), \quad \forall q \in H^1_{per}(\Omega), \tag{24}$$

$$(J_\epsilon, q) = (\kappa_\epsilon(\rho_\epsilon)\nabla L_2^{-1}(-\alpha\Delta\rho_\epsilon + f'_\epsilon(\rho_\epsilon) + \beta\frac{\partial\rho_\epsilon}{\partial t} + c\cdot\nabla\frac{\partial\rho_\epsilon}{\partial t} - \operatorname{div}(D\nabla\frac{\partial\rho_\epsilon}{\partial t}), q), \quad \forall q \in H^1_{per}(\Omega)^n. \tag{25}$$

We prove the following result.

Lemma 1 *We assume that $\rho_0 \in H^1_{per}(\Omega)$ with $\|\rho_0\|_{L^\infty(\Omega)} \le 1$, $a = b = 0$ and that $A = C$. Then, for all $\epsilon \le \epsilon_0$, there exists a pair of functions $(\rho_\epsilon, J_\epsilon)$ solution of* (24)–(25) *such that*

$$\|\rho_\epsilon\|_{L^\infty(0,T;H^1_{per}(\Omega))} \le C, \ \|J_\epsilon\|_{L^2(\Omega_T)} \le C, \ \|\frac{\partial\rho_\epsilon}{\partial t}\|_{L^2(0,T;H^1_{per}(\Omega))} \le C, \tag{26}$$

and

$$\|[\rho_\epsilon - 1]_+\|_{L^\infty(0,T;L^2(\Omega))} + \|[-\rho_\epsilon - 1]_+\|_{L^\infty(0,T;L^2(\Omega))} \le C\epsilon^{\frac{1}{2}}, \tag{27}$$

where C is independent of ϵ.

PROOF. Since κ_ϵ and f_ϵ satisfy assumptions (7) and (8) respectively, we deduce from Theorem 1 that, for all $\epsilon > 0$, there exists a solution $(\rho_\epsilon, w_\epsilon)$ for the regularized counterpart of (9)–(10) such that $\rho_\epsilon \in \mathcal{C}([0,T]; H^1_{per}(\Omega))$, $w_\epsilon \in L^2(0,T; H^1_{per}(\Omega))$ and $\frac{\partial \rho_\epsilon}{\partial t} \in L^2(0,T; H^1_{per}(\Omega))$. As in the proof of Theorem 1, the solution $(\rho_\epsilon, w_\epsilon)$ further satisfies the following estimate

$$\begin{aligned} \int_\Omega (\frac{\alpha}{2}|\nabla \rho_\epsilon|^2 + f_\epsilon(\rho_\epsilon))\,dx + \beta \int_{\Omega_T} |\frac{\partial \rho_\epsilon}{\partial t}|^2\,dxdt + \int_{\Omega_T} |D^{\frac{1}{2}} \nabla \frac{\partial \rho_\epsilon}{\partial t}|^2\,dxdt \\ + \int_{\Omega_T} \kappa_\epsilon(\rho_\epsilon)|\nabla w_\epsilon|^2\,dxdt \le \int_\Omega (\frac{\alpha}{2}|\nabla \rho_0|^2 + f(\rho_0))\,dx \le C, \end{aligned} \tag{28}$$

hence, $\operatorname{ess\,sup}_{t\in[0,T]} \|\nabla \rho_\epsilon\| \le C$, $\|[\kappa_\epsilon(\rho_\epsilon)]^{\frac{1}{2}} \nabla w_\epsilon\|_{L^2(\Omega_T)} \le C$, $\|\frac{\partial \rho_\epsilon}{\partial t}\|_{L^2(0,T;H^1_{per}(\Omega))} \le C$, and

$$\operatorname*{ess\,sup}_{t\in[0,T]} \int_\Omega ([\rho_\epsilon - 1]_+^2 + [-1-\rho_\epsilon]_+^2)\,dx \le C\epsilon, \tag{29}$$

which follows from (23); noting that $f_\epsilon(\rho_0) \le f(\rho_0)$ for ϵ sufficiently small. Since $m(\rho_\epsilon) = m(\rho_0)$, Poincaré's inequality yields $\operatorname{ess\,sup}_{0\le t\le T} \|\rho_\epsilon\|_{H^1_{per}(\Omega)} \le C$, where the generic constant C does not depend on ϵ, $\epsilon \le \epsilon_0$, for a sufficiently small ϵ_0. ■

Theorem 2 *Let the assumptions of Lemma* 1 *hold. Then, there exists a solution* (ρ, J) *of* (5)–(6) *such that* $\rho \in \mathcal{C}([0,T]; H^1_{per}(\Omega))$, $J \in L^2(\Omega_T)$, $|\rho| \le 1$ *a.e. in* Ω_T *and* $\frac{\partial \rho}{\partial t} \in L^2(0,T; H^1_{per}(\Omega))$.

PROOF. It follows from Lemma 1 that there exists a subsequence (which we still denote by $(\rho_\epsilon, J_\epsilon)_{\epsilon>0}$) and a pair of functions (ρ, J) such that

$$\begin{aligned} &\rho_\epsilon,\ \nabla\rho_\epsilon \to \rho,\ \nabla\rho && \text{strongly in } L^2(\Omega_T) \text{ and } a.e. \text{ in } \Omega_T, \\ &\frac{\partial \rho_\epsilon}{\partial t} \rightharpoonup \frac{\partial \rho}{\partial t} && \text{weakly in } L^2(0,T;H^1_{per}(\Omega)), \\ &J_\epsilon \rightharpoonup J && \text{weakly in } L^2(\Omega_T). \end{aligned}$$

We then pass to the limit in the regularized problem (24)–(25) (see [6] for more details). ■

Remark 2 *The more interesting case is when* $a + b \neq 0$ *and* $A \neq C$. *But, we are unable to deal this situation. The technique used in the proof of Theorem 2 failed.* □

Remark 3 *Theorem 2 is still true even if* $D = 0$ *and* $\beta = 0$. *Indeed, we have* $\|q\|^2 + \|A^{\frac{1}{2}}\nabla q\|^2 = (L_1 q, q) \le c\|L_1 q\|_{-1}\|\nabla q\|$, $c > 0$, $\forall q \in \dot{H}^1_{per}(\Omega)$, *and therefore there exist two positive constants* c_1, c_2 *such that* $c_1\|\nabla \frac{\partial \rho_\epsilon}{\partial t}\|_{L^2(\Omega_T)} \le \|\frac{\partial L_1 \rho_\epsilon}{\partial t}\|_{L^2(0,T;(H^1_{per}(\Omega))')} \le c_2\|J_\epsilon\|_{L^2(\Omega_T)}$, $\forall \epsilon > 0$, *which leads to the result together with the other estimates.* □

References

[1] Barrett, J. W. and Blowey, J. F. (1996). Finite element approximation of the Cahn-Hilliard equation with concentration dependent mobility, *Math. Comp*. **68**, 487–517.

[2] Bonfoh, A. and Miranville, A. (2001). On Cahn-Hilliard-Gurtin equations, in Proceedings of the 3rd World Congress of Nonlinear Analysts, *Nonlin. Anal.*, **47** , 3455–3466.

[3] Cahn, J. W. (1961). On spinodal decomposition, *Acta Metall.*, **9**, 795–801.

[4] Cahn, J. W. and Hilliard, J.E. (1958). Free energy of a non-uniform system I. Interfacial free energy, *J. Chem. Phys.*, **2**, 258–267.

[5] Debussche, A. and Dettori, L. (1995). On the Cahn-Hilliard equation with a logarithmic free energy, *Nonlinear Anal.* **24**, 1491–1514.

[6] Elliott, C. M. and Garcke, H. (1996). On the Cahn-Hilliard equation with degenerate mobility, *SIAM J. Math. Anal.*, **27**, 404–423.

[7] Efendiev, M. and Miranville, A. New models of Cahn-Hilliard-Gurtin equations, *Cont. Mech. Thermodynamics* (to appear).

[8] Gurtin, M. (1996). Generalized Ginzburg-Landau and Cahn-Hilliard equations based on a microforce balance, *Phys. D*, **92**, 178–192.

[9] Lions, J. L. (1969). Quelques Méthodes de Résolutions des Problèmes aux Limites Non Linéaires, Dunod, Paris.

[10] Miranville, A. (2000). Some generalizations of the Cahn-Hilliard equation, *Asymptotic Anal.*, **22**, 235–259.

[11] Temam, R. (1997). it Infinite Dimensional Dynamical Systems in Mechanics and Physics, (2nd Edition) Springer-Verlag, Berlin, Heidelberg, New York.

A. Sanih Bonhof
University Kuala Lumpur,
Malaysia France Institute,
Sec 14, Jln Teras Jernang,
43650 Bdr Baru Bangi,
Selangor D.E., Malaysia
sanbonf@yahoo.fr