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Regularity, barrelledness and dual strong unions in locally convex spaces

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Abstract. Let $F_{\underline{n}(k',0)} = \cup\{F_{\underline{m}} : m \in \mathbb{N}\}$ be a dual strong union of E. In Section 3 we study the regularity properties of bounded sets of $F_{\underline{n}(k',0)}$. In Section 4 we consider the duality $\langle E, F_{\underline{n}(k',0)}\rangle$ and prove that $F_{\underline{n}(k',0)}$ with its Mackey, resp. Arens, resp. Schwartz, topology is an inductive limit of a Mackey, resp. Arens, resp. Schwartz, countable spectrum. In Section 5 we impose weak barrelledness conditions on $F_{\underline{n}(k',0)}$ and investigate their connection to the associated barrelled topology of E.

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Resumen. Sea $F_{\underline{n}(k',0)} = \cup\{F_{\underline{m}} : m \in \mathbb{N}\}$ una unión de duales fuertes de E. En la sección 3 estudiamos las propiedades de regularidad de los conjuntos acotados de $F_{\underline{n}(k',0)}$. En la sección 4 consideramos la dualidad $\langle E, F_{\underline{n}(k',0)}\rangle$ y demostramos que $F_{\underline{n}(k',0)}$ con su topología Mackey, Arens, o Schwartz, respectivamente es un límite inductivo de espectro numerable de Mackey, Arens, Schwartz, respectivamente. En la sección 5 imponemos condiciones de tonelación débiles en $F_{\underline{n}(k',0)}$ e investigamos su conexión con la topología tonelada asociada de E.

1 Introduction

Let E be a nonbarrelled locally convex space. A generalized dual strong sequence of E is an increasing sequence of bidual enlargements of E', directed upward the dual of the barrelled topology associated to E (Definition 1). The concept of a generalized dual strong sequence was introduced in [34] and [35]. It includes two basic elements: dual strong sequences and their unions. In this article we investigate the duality connection between a nonbarrelled space and its dual strong union. We focus our attention to general nonbarrelled spaces, continuing the comprehensive overall approach of [34] and [35].

Albeit being generic in the sense that it applies to any nonbarrelled space, the concept of a generalized dual strong sequence seems to be of great promise in the application-oriented research. We prove that a dual strong union is a resource of naturally arising "nice" properties, such as compactness, localization, regularity (Propositions 1, 2, 3, 5, Theorems 1, 2, 3). The rich structure of a dual strong union allows the generalized limits/mixed topologies approach, as well as the use of homological methods (see [2, 11, 12, 14, 23, 24, 39] for bibliography and insights). Representations of a dual strong union and its bounded disks used in this article resemble the generalized limits approach (Definitions 2 and 3).

The duality connection between a nonbarrelled space E and its dual strong union is perceptive to barrelledness, that is to say, a very weak barrelledness condition imposed on E by a dual strong union evokes

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the associated barrelled topology of E (Theorem 6). It is known that weak barrelledness topologies are difficult to reveal, unless they are Mackey or have some other distinguished features (see the enlightening introduction and references of [20]). More than thirty years ago Buchwalter and Schmets investigated in [4] the weak barrelled topologies on a DF-space of continuous functions, raising the question of discerning two weak barrelledness conditions that seemed ungraspably subtle. Recently Kąkol, Saxon and Todd answered their question by constructing a fine-grained example of two different non-Mackey weak barrelled topologies on the space of continuous functions, compatible with the same duality ([20, Example 5.4]).

Observing the sensitivity of the dual strong union to weak barelledness, we evoke the wb-topology, introduced by Roelcke in [25], identifying it as the strong topology, imposed on E by its dual strong union (Definition 4, Theorem 4). Then we actualize the notion of countable barrelledness, categorizing different weak barrellednes properties on E and investigating the conditions under which the wb-topology becomes weakly barrelled (Definition 5, Theorem 5). As a consequence of Theorem 5, we obtain that if E is l^{∞}-barrelled ([22, 8.2.13]), then E equipped with its weakest wb-topology $\sigma_{wb}(E, E')$ is $\aleph_0$-barrelled ([22, 8.2.1]). We suggest that the wb-topology is a fairly reasonable tool for identifying different weak barrelled conditions within the same duality (Proposition 7).

Despite their elusiveness, weak barrelled topologies associated to the spaces of continuous functions were defined and studied by Noureddine and Schmets ([21]). A thorough account of the research of the barrelled topologies associated to the spaces of continuous functions, along with important barrelledness criteria, can be found in [30] and [31]. A succinct guide on weak barrelledness is given in [10, p.38]. Some results on associated barrelled topologies for general spaces are obtained in [7, 32] and [33]. A description of barrelledness conditions is provided in [15, t.I, p.114] and [22, 4.9, 8.9]. The research on very strong barrelledness, although located outside the scope of this article, is presented in [10] and [22, Chapter 9]. An exceptionally concise and unique picture of regularity, localization and barrelledness is exhibited in Chapter One of [38].

Notice that for some nonbarrelled spaces the generalized dual strong sequence is trivial (i.e. the associated barrelled topology is reached after a finite number of bidual enlargements, as in $(E, \mu(E, E' + H))$ of [32], Corollary 4). Other nonbarrelled spaces appear to be quite suitable for starting the search after a significant dual strong union (see 6.3 or 6.5 of [10] for inspiring examples). We believe that the “size” of the gap needed between E and its associated barrelled space in order to construct a meaningful generalized dual strong sequence, is an important research topic per se.

2 Notations and setting

All topologies considered in this article are locally convex and Hausdorff. Let (E, τ) be a vector space over the field K of the real or complex numbers and E', resp. $E^{\star}$, its topological, resp. algebraic, dual. We denote by $\beta(E, E')$, $\mu(E, E')$, $\sigma(E, E')$ the strong, Mackey, weak topologies on E, respectively, and by $\beta^{\star}(E, E')$ the topology on E of uniform convergence on all strongly bounded subsets of E'.

A *disk* is an absolutely convex set. Given a bounded disk B, we denote by $Sp\, B$ the linear hull of B. By E_B we denote the $Sp\, B$ with the norm defined by the gauge of B. A bounded disk B is *barrelled*, resp. *Banach*, if E_B is *barrelled*, resp. a *Banach* space. A closed absorbing [bornivorous] disk is a *[bornivorous]* barrel. We say that E is $< dual >$ *locally [quasi-] barrelled*, (respectively $< dual >$ *locally [quasi-] complete*), if for any [strongly] bounded set A of $E <$ of $E' >$ there exists a closed [strongly] bounded disk B in $E <$ in $E' >$, such that $A \subseteq B$ and E_B is barrelled, (respectively a Banach space).

A locally convex topology on E is called an *[infra-]Schwartz topology* if for every closed equicontinuous disk K in E' there is another closed equicontinuous disk M such that K is [weakly] compact in the Banach space E_M, (for a Schwartz space see [15, t.I, p. 119] or [38, p. 205]; for an infra-Schwartz space see [5, III.3.9], or [17, p.91]). We say that (E, τ_{fc}) $[(E, \tau_{wfc})]$ is the *associated [infra-] Schwartz space* of (E, τ) if τ_{fc} $[\tau_{wfc}]$ is the finest [infra-] Schwartz topology coarser than τ, see [17], or [16, p. 62]; fc stands for *fast compact*, [22, 6.1.20]. The *Arens topology* $\kappa(E', E)$ on E' is the topology, formed by the polars of the compact disks of (E, τ), [1]. Notice that if $(E, \mu(E, E'))$ is a Fréchet space then the Arens topology

$\kappa(E', E)$ and the associated Schwartz topology $\mu_{fc}(E', E)$ coincide on E', [5, III.1.10].

An increasing sequence of disks $\{A_n,\, n \in \mathbb{N}\}$ in a locally convex space E is called *an absorbent sequence* if its union is absorbing, ([6]). It is *bounded-absorbent* if any bounded set of E is absorbed by some A_n, (ibid.). Two well-known classics on absorbent sequences are [36] of Valdivia and the mentioned previously [6] of De Wilde and Houet. Given an absorbent sequence $\{A_m,\, m \in \mathbb{N}\}$ in (E, τ), we denote by $\tau(A)$, [resp. $\mu(A)$,$\sigma(A)$], the finest locally convex topology on E, agreeing with τ, [resp. $\mu(E, E')$, $\sigma(E, E')$], on each A_m, ([14, 23, 25, 27, 38, 40]). If necessary, the sequence $\{A_m,\, m \in \mathbb{N}\}$ may be replaced by $\{2^m A_m,\, m \in \mathbb{N}\}$. This replacement has no impact on $\tau(A)$, however it allows different representations of the base of 0-neighborhoods of $\tau(A)$, [25, Lemma 1]. The reader is referred to [22, 8.1, 8.5, 8.9] for a thorough review on the topology $\tau(A)$, including references and credits.

An *inductive spectrum* is a family $\{(E_i,\, \eta_i) : \, i \in \mathcal{I}\}$ of ordered by inclusion vector subspaces E_i of E, equipped with a topology η_i and with continuous inclusion maps $(E_k,\, \eta_k) \to (E_j,\, \eta_j)$, $k \le j$. For $E = \cup\{E_i : \, i \in \mathcal{I}\}$, the inductive limit $(E,\, \eta) = \operatorname{ind\,lim}\{(E_i,\, \eta_i) : \, i \in \mathcal{I}\}$ is the finest locally convex topology on E relative to which the maps $(E_i,\, \eta_i) \to E$ are continuous. A comprehensive information on inductive limits can be found in [2, 11, 15, 22, 38, 39]. The definitions below are confined to our needs. We say that $(E,\, \eta) = \operatorname{ind\,lim}\{(E_n, \eta_n) : \; n \in \mathbb{N}\}$ is *strict* if η_j induces η_i on E_i, $i \le j$. A strict countable inductive limit is *hyperstrict*, if E_n is closed in E_{n+1} for every $n \in \mathbb{N}$, [15, t. 1, p. 125]. A bounded set B of $(E, \eta) = \operatorname{ind\,lim}\{\{(E_n,\, \eta_n) : \, n \in \mathcal{I}\}\}$ is *regular* if there exists $n = n(B) \in \mathbb{N}$ such that B is contained and bounded in $(E_n,\, \eta_n)$ ([38, p. 163]). It is α-regular if there exists $n = n(B) \in \mathbb{N}$ such that B is contained in $(E_n,\, \eta_n)$, and β-regular if whenever it is contained in some $(E_n,\, \eta_n)$, it is bounded in $(E_k,\, \eta_k)$ for some $k \ge n$, [22, 8.5.11]. Finally, it is *quasi-regular* if there exists $n = n(B) \in \mathbb{N}$ and a bounded set M in $(E_n,\, \eta_n)$ such that B is contained in the η-closure of M, [8]. An inductive limit $(E,\, \eta) = \operatorname{ind\,lim}(E_n,\, \eta_n)$ is *regular* (resp. *α-regular*, *β-regular*, *quasi-regular*) if any bounded set of (E, η) is regular (resp. α-regular, β-regular, quasi-regular). An inductive limit $(E,\, \eta) = \operatorname{ind\,lim}(E_n,\, \eta_n)$ is *(weakly) compactly regular* if any compact subset of (E, η) is contained and (weakly) compact in $(E_n,\, \eta_n)$, [22, 8.5.32]. Generally, $(E, \eta) = \operatorname{ind\,lim}(E_n,\, \eta_n)$ is *$(\mathcal{F}_1, \mathcal{F}_2)$-regular* if any subset of (E, η) of a class $\mathcal{F}_1(E)$ is contained in some $(E_n,\, \eta_n)$ and belongs to a class $\mathcal{F}_2(E_n)$, [39, p. 111]. Notice that a hyperstrict inductive limit is regular, a strict inductive limit is not necessarily regular, and a regular strict inductive limit is not necessarily hyperstrict, [22, 8.4.16, 8.5.14].

We say that $(E, \eta) = \operatorname{ind\,lim}(E_n,\, \eta_n)$ has the Retakh's property (M), [resp. (M_0)], if there is an increasing sequence $\{U_n : \, U_n \subseteq U_{n+1},\, U_n \subseteq (E_n,\, \eta_n),\, n \in \mathbb{N}\}$ of absolutely convex 0-neighbourhoods, satisfying the following: for every $n \in \mathbb{N}$ there exists a positive integer $k \ge n$ such that for any $m \ge k$ the topologies, [resp. the weak topologies], induced by $(E_m,\, \eta_m)'$ and $(E_k,\, \eta_k)'$, coincide on U_n ([38, p. 158, p. 164], [22, 8.9.16]). An LF-space is [weakly] acyclic if and only if it satisfies the property (M), [(M_0)]. We refer to [3, 8, 13, 39] for an up-to-date bibliography on [weak] acyclicity of countable inductive spectra.

3 Representation of a dual strong union and its bounded sets

We use slightly changed notations of [35]. Let $\mathbb{N} = 1, 2, 3, \ldots$, $\mathbb{N}_0 = \{0\} \cup \mathbb{N}$, $N_0^1 = \mathbb{N}_0$. For $k \ge 2$, denote $\underline{n} = n_1 n_2 \ldots n_k \in N_0^k$. Define recursively an order relation $\le^*$ on N_0^k in the following way:

1). If $k = 1$, then $\underline{m} \le^* \underline{n}$ is the usual order relation $m_1 \le n_1$, for each $m_1 \in \mathbb{N}_0$, $n_1 \in \mathbb{N}_0$.

2). If $k \ge 2$, then $\underline{m} \le^* \underline{n}$ if and only if $((m_1 < n_1) \vee ((m_1 = n_1) \wedge (m_2 \ldots m_k \le^* n_2 \ldots n_k)))$ for every $\underline{m} \in N_0^k$ and $\underline{n} \in N_0^k$.

We use $\underline{n} <^* \underline{m}$ for $((\underline{n} \le^* \underline{m}) \wedge (\underline{n} \ne \underline{m}))$. We denote by $\underline{0}$ the least element of $(N_0^k, \le^*)$. For a given $\underline{n} \in (N_0^k, \le^*)$, we denote by $\underline{n+1}$ the least of the elements $\{\underline{m} \in (N_0^k, \le^*) : \underline{n} <^* \underline{m}\}$.

For a fixed $k \ge 2$, we denote by $\underline{n}(k', 0)$ an element $\underline{n} \in (N_0^k, \le^*)$ such that $n_{k'} \ne 0$ for some k', satisfying: $1 \le k' \le k - 1$, and $n_p = 0$ for each p, satisfying: $k' + 1 \le p \le k$. For example, if $k = 4$ and $k' = 2$ then $\underline{n}(2, 0)$ is a quadruple $n = n_1 n_2 n_3 n_4 \in (N_0^4, \le^*)$ such that $n_2 \ne 0$ and $n_3 = n_4 = 0$.

Definition 1 *For a locally convex space (E, τ) and a fixed integer $k \in \mathbb{N}$, define a mapping of $(N_0^k, \leq^*)$ into the set of subspaces of E^* in the following way:*

1. $F_{\underline{0}} = E'$,
2. $F_{\underline{n+1}} = (E, \beta(E, F_{\underline{n}}))'$
3. $F_{\underline{n}(k',0)} = \cup\{F_{\underline{m}} : \underline{m} <^* \underline{n}(k', 0)\}$

For a given $k \geq 2$, the well ordered chain $\{F_{\underline{n}} : \underline{n} \in (N_0^k, \leq^*)\}$ is called the generalized dual strong sequence of the dual pair $\langle E, E' \rangle$. For k=1, or for a fixed $k \geq 2$ and $n_1 n_2 ... n_{k-1} \in N_0^{k-1}$, the sequence $\{F_{\underline{n}} : n_k \in N_0\}$ is called a dual strong sequence of $\langle E, E' \rangle$. A subspace $F_{\underline{n}(k',0)}$is called a dual strong union. If $k' = k - 1$ then $F_{\underline{n}(k-1,0)}$ is called an initial dual strong union.

Notice that $(E, \beta(E, F_{\underline{n}}))$ is barrelled $\Leftrightarrow$ $(E, (\mu(E, F_{\underline{n+1}})))$ is barrelled $\Leftrightarrow$ $F_{\underline{m}} = F_{\underline{n+1}}$ for every $\underline{m} \in N_0^k$ such that $\underline{n} <^* \underline{m}$, [34, Remark 3.4]. In the sequel we shall use the abbreviation d.s.u. for a dual strong union.

The following representation of a d.s.u. by an increasing sequence of its subspaces is pivotal to our investigation.

Definition 2 *For $k \geq 2$ and $1 \leq k' \leq k-1$, $F_{\underline{n}(k',0)} = \cup\{F_{\underline{m}} : m_{k'+1} \in \mathbb{N}\}$, where $m_{k'} = n_{k'} - 1$, and $m_i = n_i$ for any i, satisfying: $((1 \leq i \leq k) \wedge (i \neq k') \wedge (i \neq k'+1))$. The sequence $\{F_{\underline{m}} : m_{k'+1} \in \mathbb{N}\}$ is called the representation of the d.s.u. $F_{\underline{n}(k',0)}$.*

We shall use the notation $F_{\underline{n}(k',0)} = \cup\{F_{\underline{m}} : m \in \mathbb{N}\}$ for the representation of $F_{\underline{n}(k',0)}$, suggesting that $m_{k'+1} = m$. Notice that an initial d.s.u. is represented by a dual strong sequence, and a non-initial d.s.u. is represented by a sequence of dual strong unions.

Definitions 1 and 2 imply that $(F_{\underline{n}(k',0)}, \sigma(F_{\underline{n}(k',0)}, E)) = \operatorname{ind}\lim\{F_{\underline{m}}\, \sigma(F_{\underline{m}}, E) : m \in \mathbb{N}\}$. Obviously this inductive limit is strict, not hyperstrict, β-regular, and satisfies the Retakh's property (M), as well as (M_0). Some of these observations are summarized in the following proposition.

Proposition 1 *Let E be a locally convex space, $F_{\underline{n}(k',0)}$ a d.s.u. and $F_{\underline{n}(k',0)} = \cup\{F_{\underline{m}} : m \in \mathbb{N}\}$ its representation. Let $(F_{\underline{n}(k',0)}, \eta) = \operatorname{ind}\lim\{(F_{\underline{m}}, \eta_m) : m \in \mathbb{N}\}$ such that $(F_{\underline{n}(k',0)}, \eta)' = (F_{\underline{m}, \eta_m})' = E$ for each $m \in \mathbb{N}$. The following statements are true:*

(a) $(F_{\underline{n}(k',0)}, \eta)$ *is β-regular.*

(b) *any α-regular set of $(F_{\underline{n}(k',0)}, \eta)$ is regular and weakly relatively compact.*

(c) $(F_{\underline{n}(k',0)}, \eta)$ *satisfies the Retakh's property (M_0).*

Continuing [39, p. 111], we say that $(E, \eta) = \operatorname{ind}\lim(E_n, \eta_n)$ is $\beta - (\mathcal{F}_1, \mathcal{F}_2)$ if whenever a subset of (E, η) of a class $\mathcal{F}_1(E)$ is contained in some (E_n, η_n), it belongs to a class $\mathcal{F}_2(E_k)$ for some $k \geq n$. The part a of the Proposition 1 may be reformulated in the following way: $(F_{\underline{n}(k',0)}, \eta)$ is β-(bounded, weakly relatively compact)-regular.

The next theorem reveals the nature of closed bounded disks of a dual strong union. It was proved in [34] for $k = 2$. The same arguments are valid for $k \geq 2$.

Theorem 1 *Let E be a locally convex space, $F_{\underline{n}(k',0)}$ a d.s.u., and $F_{\underline{n}(k',0)} = \cup\{F_{\underline{m}} : m \in \mathbb{N}\}$ its representation. Any closed bounded disk of $(F_{\underline{n}(k',0)}, \sigma(F_{\underline{n}(k',0)}, E))$ is either compact and regular or a countable union of an increasing sequence of compact disks $\{A_m : m \in \mathbb{N}\}$ such that $A_m \in F_m$ for each $m \in \mathbb{N}$.*

PROOF. Without loss of generality we prove the theorem for $k = 3$ and $k' = 1$. For a fixed $n \in N_0$, let B be a closed bounded disk of $(F_{n+1,0,0}, \sigma(F_{n+1,0,0}, E))$. Let $B_m = B \cap F_{n,m,0}$. Then $B = \cup\{B_m : m \in N_0\}$. Denote by A_{m+1} the closure of B_m in $(F_{n,m+1,0}, \sigma(F_{n,m,1}, E))$-compact disk of $F_{n,m,1}$. Therefore A_{m+1} is $\sigma(F_{n,m+1,0}, E)$-compact. Since $B = \cup\{A_m : m \in \mathbb{N}\}$, where A_m is a compact disk of $(F_{n,m,0}, \sigma(F_{n,m,0}, E))$ and $A_m \subseteq A_{m+1}$ for each $m \in \mathbb{N}$. ■

The next definition is derived from Theorem 1.

Definition 3 *Let $F_{\underline{n}(k',0)}$ be a d.s.u., $F_{\underline{n}(k',0)} = \cup\{F_{\underline{m}} : m \in \mathbb{N}\}$ its representation, and B a closed bounded disk of $(F_{\underline{n}(k',0)}, \sigma(F_{\underline{n}(k',0)}, E))$. We say that $B = \cup\{A_m : m \in \mathbb{N}\}$ is the wc-representation of B, if $\{A_m : m \in \mathbb{N}\}$ is the increasing sequence of weakly compact regular disks of Theorem 1.*

By saying that a bounded set B of a d.s.u. $F_{\underline{n}(k',0)}$ is regular, we intend to the regularity of B in $(F_{\underline{n}(k',0)}, \sigma(F_{\underline{n}(k',0)}, E)) = \operatorname{ind\,lim}\{F_{\underline{m}}, \sigma(F_{\underline{m}}, E) : m \in \mathbb{N}\}$.

Proposition 2 *Let E be a locally convex space, $F_{\underline{n}(k',0)}$ a d.s.u. and $F_{\underline{n}(k',0)} = \cup\{F_{\underline{m}} : m \in \mathbb{N}\}$ its representation. Any closed barrelled disk of $F_{\underline{n}(k',0)}$ is weakly compact and regular.*

PROOF. If B is a closed barrelled disk of $(F_{\underline{n}(k',0)}, \sigma(F_{\underline{n}(k',0)}, E))$, then E_B is Baire-like, [28]. By Theorem 1, B admits a wc-representation $B = \cup\{A_m : m \in \mathbb{N}\}$. Therefore $E_B = \cup\{2^m A_m : m \in \mathbb{N}\}$, hence $B \subseteq \lambda A_m$ for some $m \in \mathbb{N}$ and positive λ ([6, Corollary 1]; see also [36, Theorem 6]). We conclude that B is weakly compact and regular. ■

Proposition 3 *Let E be a locally convex space, $F_{\underline{n}(k',0)}$ a d.s.u., and $F_{\underline{n}(k',0)} = \cup\{F_{\underline{m}} : m \in \mathbb{N}\}$ its representation. Let $(F_{\underline{n}(k',0)}, \eta) = \operatorname{ind\,lim}\{(F_{\underline{m}}, \eta_m : m \in \mathbb{N})\}$ such that $(F_{\underline{n}(k',0)}, \eta)' = (F_{\underline{m}}, \eta_m)' = E$ for each $m \in \mathbb{N}$. The following statements are equivalent.*

(a) $(F_{\underline{n}(k',0)}, \eta)$ *is quasi-regular.*

(b) $(F_{\underline{n}(k',0)}, \eta)$ *is α-regular.*

(c) $(F_{\underline{n}(k',0)}, \eta)$ *is regular.*

(d) $(F_{\underline{n}(k',0)}, \eta)$ *is weakly compactly regular.*

(e) $(E, \mu(E, F_{\underline{n}(k',0)}))$ *is dual locally barrelled.*

(f) $(E, \mu(E, F_{\underline{n}(k',0)}))$ *is dual locally complete.*

(g) $(E, \mu(E, F_{\underline{n}(k',0)}))$ *is barrelled.*

(h) $(E, \mu(E, F_{\underline{n}(k',0)}))$ *is the associated barrelled topology of E.*

PROOF. (a) ⇒ (b) ⇒ (c) ⇒ (a) follow from Proposition 1. (c) ⇒ (d): since $(F_{\underline{n}(k',0)}, \eta)$ is regular, any weakly compact set K of $F_{\underline{n}(k',0)}$ is contained in some step. Since $(F_{\underline{n}(k',0)}, \eta)$ and its steps have the same dual, K is weakly compact in the step. (d) ⇒ (c): if $(F_{\underline{n}(k',0)}, \eta)$ is not regular then by Proposition 1, it is not α-regular. Therefore there is a bounded sequence $\{u_n\}$ in $(F_{\underline{n}(k',0)}, \eta)$ such that $\{u_n\} \not\subset F_{\underline{m}}$ for any $m \in \mathbb{N}$. Since $2^{-n} u_n$ converges to zero in $(F_{\underline{n}(k',0)}, \eta)$, $\{2^{-n} u_n\}$ is compact, hence $(F_{\underline{n}(k',0)}, \eta)$ is not compactly regular. (c) ⇒ (g): if $(F_{\underline{n}(k',0)}, \eta)$ is regular then by Proposition 1, any bounded disk of $(F_{\underline{n}(k',0)}, \eta)$ is weakly relatively compact, hence $(E, \mu(E, F_{\underline{n}(k',0)}))$ is barrelled. (g) ⇒ (f) ⇒ (e) are obvious. (e) ⇒ (c) follows from Proposition 2. (g) ⇔ (h) follows from [35, Proposition 3.1]. ■

4 Topologies of the dual pair $\langle E,\, F_{\underline{n}(k',0)}\rangle$

Let $\{\eta_i :\ i \in I\}$ be a set of locally convex topologies on E, equipped with the usual (partial) order relation $\eta_i \leq \eta_j$. The topology $\sup\{\eta_i :\ i \in I\}$, (resp. $\inf\{\eta_i :\ i \in I\}$), is the weakest, (resp. finest), locally convex topology on E, finer, (resp. weaker), than each $\eta_i,\ i \in I$. Notice that the topology $\inf\{\eta_i,\ i \in I\}$ is not necessarily Hausdorff.

Theorem 2 *Let E be a locally convex space, $F_{\underline{n}(k',0)}$ a d.s.u. , and $F_{\underline{n}(k',0)} = \cup\{F_{\underline{m}} :\ m \in \mathbb{N}\}$ its representation. For each $m \in \mathbb{N}$, let η_m be a topology on E, such that $\mu(E, F_{\underline{m}}) \leq \eta_m \leq \beta(E, F_{\underline{m}})$. Then $\mu(E, F_{\underline{n}(k',0)}) = \sup\{\eta_m,\ m \in \mathbb{N}\}$.*

PROOF. First, we shall prove the theorem for $\eta_m = \mu(E, F_{\underline{m}})$. Clearly $\mu(E, F_{\underline{m}}) \leq \mu(E, F_{\underline{n}(k',0)})$ for each $m \in \mathbb{N}$, therefore $\sup\{\mu(E, F_{\underline{m}}) :\ m \in \mathbb{N}\} \leq \mu(E, F_{\underline{n}(k',0)})$. Let ζ be a locally convex topology on E such that $\zeta \geq \mu(E, F_{\underline{m}})$ for each $m \in \mathbb{N}$. Then $(E,\zeta)' \supseteq F_{\underline{m}}$ for each $m \in \mathbb{N}$, therefore $(E,\zeta)' \supseteq F_{\underline{n}(k',0)}$. By Proposition 2, any $\sigma(F_{\underline{n}(k',0)}, E)$-compact disk is contained in some $F_{\underline{m}}$, and therefore it is $\mu(E, F_{\underline{m}})$-equicontinuous for some $m \in \mathbb{N}$. Hence any $\sigma(F_{\underline{n}(k',0)}, E)$-compact disk is ζ-equicontinuous, therefore $\zeta \geq \mu(E, F_{\underline{n}(k',0)})$. Thus $\mu(E, F_{\underline{n}(k',0)}) = \sup\{\mu(E, F_{\underline{m}}) :\ m \in \mathbb{N}\}$. Since $\mu(E, F_{\underline{m}}) \leq \eta_m \leq \beta(E, F_{\underline{m}}) \leq \mu(E, F_{\underline{m+1}})$ for any $m \in \mathbb{N}$, we conclude that $\mu(E, F_{\underline{n}(k',0)}) = \sup\{\eta_m :\ m \in \mathbb{N}\}$. ■

Proposition 4 *Let $\{F_{\underline{n}} :\ \underline{n} \in \mathbb{N}\}$ be the generalized dual strong sequence of $\langle E,\, E'\rangle$, $F_{\underline{n}(k',0)}$ a d.s.u., and $F_{\underline{n}(k',0)} = \cup\{F_{\underline{m}} :\ m \in \mathbb{N}\}$ its representation. Then $\mu(E, F_{\underline{n}(k',0)}) = \sup\{\mu(E, F_{\underline{m}}) :\ \underline{m} <^* \underline{n}(k',0)\} = \sup\{\beta^*(E, F_{\underline{m}}) :\ \underline{m} <^* \underline{n}(k',0)\} = \sup\{\beta(E, F_{\underline{m}}) :\ \underline{m} <^* \underline{n}(k',0)\}$.*

PROOF. Follows from Definition 1 and Theorem 2. ■

Proposition 5 *Let $\{F_{\underline{n}} :\ \underline{n} \in N_0^k\}$ be the generalized dual strong sequence, $F_{\underline{n}(k',0)}$ a d.s.u. and $F_{\underline{n}(k',0)} = \cup\{F_{\underline{m}} :\ m \in \mathbb{N}\}$ its representation. Let K be a disk of E. The following statements are equivalent.*

(a) *K is compact in $(E, \mu(E, F_{\underline{n}(k',0)}))$*

(b) *K is compact in $(E, \mu(E, F_{\underline{n}}))$, for each $\underline{n} \in N_0^k$, satisfying: $\underline{n} < \underline{n}(k',0)$.*

(c) *K is compact in $(E, \mu(E, F_{\underline{n}}))$, for each $m \in \mathbb{N}$ of the representation $F_{\underline{n}(k',0)} = \cup\{F_{\underline{m}} :\ m \in \mathbb{N}\}$.*

PROOF. (a) ⇒ (b) ⇒ (c): obviously, if K is $\mu(E, F_{\underline{n}(k',0)})$-compact then it is $\mu(E, F_{\underline{n}})$-compact for each $\underline{n} \in N_0^k$, satisfying $\underline{n} < \underline{n}(k',0)$. (c) ⇒ (a): by Theorem 2, $(E, \mu(E, F_{\underline{n}(k',0)}))$ is isomorphic to a closed subspace (diagonal) of the topological product $\prod\{(E, \mu(E, F_{\underline{m}})) :\ m \in \mathbb{N}\}$. Therefore by Tychonoff theorem, A is $\mu(E, F_{\underline{n}(k',0)})$-compact whenever it is $\mu(E, F_{\underline{m}})$-compact for each $m \in \mathbb{N}$. ■

The next theorem states that $\mu(F_{\underline{n}(k',0)}, E)$, the Arens topology of $(F_{\underline{n}(k',0)}, \mu(F_{\underline{n}(k',0)}, E))$ and the associated (infra-) Schwartz topology of $(F_{\underline{n}(k',0)}, \mu(F_{\underline{n}(k',0)}, E))$ satisfy the requirements of Propositions 1 and 3.

Theorem 3 *Let E be a locally convex space, $F_{\underline{n}(k',0)}$ a d.s.u., and $F_{\underline{n}(k',0)} = \cup\{F_{\underline{m}} :\ m \in \mathbb{N}\}$ its representation.. The following statements are true.*

(a) $(F_{\underline{n}(k',0)}, \mu(F_{\underline{n}(k',0)}, E)) = \operatorname{ind\,lim}\{F_{\underline{m}}, \mu(F_{\underline{m}}, E) :\ m \in \mathbb{N}\}$

(b) $(F_{\underline{n}(k',0)}, \kappa(F_{\underline{n}(k',0)}, E)) = \operatorname{ind\,lim}\{F_{\underline{m}}, \kappa(F_{\underline{m}}, E) :\ m \in \mathbb{N}\}$

(c) $(F_{\underline{n}(k',0)}, \mu_{fc}(F_{\underline{n}(k',0)}, E)) = \operatorname{ind\,lim}\{F_{\underline{m}}, \mu_{fc}(F_{\underline{m}}, E) :\ m \in \mathbb{N}\}$

(d) $(F_{\underline{n}(k',0)}, \mu_{wfc}(F_{\underline{n}(k',0)}, E)) = \operatorname{ind}\lim\{F_{\underline{m}}, \mu_{wfc}(F_{\underline{m}}, E) : \ m \in \mathbb{N}\}$

PROOF. We identify E with a subspace of the algebraic dual of $F_{\underline{n}(k',0)}$.

- (a) Let $(F_{\underline{n}(k',0)}, \eta) = \operatorname{ind}\lim(F_{\underline{m}}, \mu(F_{\underline{m}}, E))$. Clearly the canonical inclusions $(F_{\underline{m}}, \mu(F_{\underline{m}}, E)) \to (F_{\underline{m+1}}, \mu(F_{\underline{m+1}}, E)) \to (F_{\underline{n}(k',0)}, \mu(F_{\underline{n}(k',0)}, E))$ are continuous, therefore $\eta \geq \mu(F_{\underline{n}(k',0)}, E)$. Hence by proving that $(F_{\underline{n}(k',0)}, \eta)' = E$, we conclude that $\eta = \mu(F_{\underline{n}(k',0)}, E)$. If $g \in (F_{\underline{n}(k',0)}, \eta)'$, then its restriction on $F_{\underline{m}}$ is $\mu(F_{\underline{m}}, E)$-continuous for every $m \in \mathbb{N}$. Therefore for every $m \in \mathbb{N}$ there exists $f_m \in E$ such that g and f_m coincide on $F_{\underline{m}}$ and $\{f_m : \ m \in \mathbb{N}\}$ converges pointwise to g. Since for each $m \in \mathbb{N}$, $f_{m+1} - f_m = 0$ on $F_{\underline{m}}$ and $F_{\underline{m}}$ separates the points of E, there exists $f \in E$ such that $f_m = f$, for each $m \in \mathbb{N}$. Hence, $g = f \in E$ and we conclude that $(F_{\underline{n}(k',0)}, \eta)' = E$. Therefore, $\eta = \mu(F_{\underline{n}(k',0)}, E)$.
- (b) Let $(F_{\underline{n}(k',0)}, \nu) = \operatorname{ind}\lim(F_{\underline{m}}, \kappa(F_{\underline{m}}, E))$. Clearly, $\nu \geq \kappa(F_{\underline{n}(k',0)}, E)$. Since $\kappa(F_{\underline{m}}, E) \leq \mu(F_{\underline{m}}, E)$, it follows from (a) that $\nu \leq \mu(F_{\underline{n}(k',0)}, E)$. Hence, $(F_{\underline{n}(k',0)}, \nu)' = E$. Let U be a closed absolutely convex 0-neighbourhood in $(F_{\underline{n}(k',0)}, \nu)$. Then the polar U^o of U in E is $\sigma(E, F_{\underline{n}(k',0)})$-compact, and we conclude that U^o is $\sigma(E, F_{\underline{m}})$-compact for each $m \in \mathbb{N}$. On the other hand, for every $m \in \mathbb{N}$, there exists a closed 0-neighbourhood V_m in $(F_{\underline{m}}, \kappa(F_{\underline{m}}, E))$ such that $V_m \subseteq U \cap F_{\underline{m}}$. The polar $(V_m)^o$ of V_m in E is $\mu(E, F_{\underline{m}})$-compact. By bipolar theorem $(U \cap F_{\underline{m}})^o$ equals to the closure of U^o in $(E, \sigma(E, F_{\underline{m}}))$ for each $n \in \mathbb{N}$. hence, by the Proposition 5, U^o is $\mu(E, F_{\underline{n}(k',0)})$-compact, therefore U is a 0-nbgh in $(F_{\underline{n}(k',0)}, \kappa(F_{\underline{n}(k',0)}, E))$. Hence, $\nu \leq \kappa(F_{\underline{n}(k',0)}, E)$, and we conclude that $\nu = \kappa(F_{\underline{n}(k',0)}, E)$.
- (c) and (d) follow from the Proposition 5.2 of [35]. ■

Notice that by Proposition 5.2 and Observations 5.1, 5.2 of [35], each of the inductive limits in c and d is strict. Clearly no one of c or d is hyperstrict.

5 Weak barrelledness and the wb-topology on E

A barrel U of (E, τ) is called a $\aleph_0$-*barrel* if there is a sequence $\{U_n : \ n \in \mathbb{N}\}$ of closed absolutely convex 0-neighbourhoods of (E, τ) such that $U = \cap\{U_n : \ n \in \mathbb{N}\}$. Notice that finite intersections of [bornivorous] $\aleph_0$-barrels are [bornivorous] $\aleph_0$-barrels.

Our next definition specifies a topology introduced by Roelcke in [25] in connection with the finest locally convex topology $\tau(A)$ agreeing with τ on an absorbent sequence $\{A_m : \ m \in \mathbb{N}\}$ of (E, τ).

Definition 4 *Let (E, τ) be a locally convex space. Denote by $\tau_{wb}[\tau^*_{wb}]$ a topology on E, such that a [bornivorous] barrel V of (E, τ) is a 0-neighbourhood of $\tau_{wb}[\tau^*_{wb}]$ if and only if there exists a [bornivorous] $\aleph_0$-barrel U of (E, τ) such that $U \subseteq V$. We say that $\tau_{wb}[\tau^*_{wb}]$ is the [bornivorous] wb-topology of (E, τ), (wb stands for weak barrelledness).*

Roelcke observes that $\tau \leq \tau(A) \leq \tau_{wb} \leq \beta(E, E')$ and if $\{\bar{A}^{\tau}_m : \ m \in \mathbb{N}\}$ is bounded-absorbent, then $\tau \leq \tau(A) \leq \tau^*_{wb} \leq \inf(\tau_{wb}, \beta^*(E, E'))$, [25, Theorems 3 and 5]. We denote by $\mu_{wb}(E, E')$, $[\mu^*_{wb}(E, ('E))]$, respectively $\sigma_{wb}(E, E'), [\sigma^*_{wb}(E, E')]$, the [bornivorous] wb-topology for the Mackey, respectively weak, topology of (E, τ). Our next proposition regards the weakest quasi-wb-topology for $\mu(E, E')$ and the associated Schwartz topology $\mu_{fc}(E, E')$. We believe the result is essentially known, although we did not find a direct reference.

Proposition 6 *Let E be a locally convex space. Then $\sigma(E, E') \leq \mu_{fc}(E, E') \leq \sigma^*_{wb}(E, E')$.*

PROOF. A closed disk K of $(E', \sigma(E, E'))$ is $\mu_{fc}(E, E')$-equicontinuous if and only if K is compact in E_M for some Banach disk $M \subseteq E'$, [38, p. 205], hence if and only if it is contained in the closed absolutely convex hull of a sequence converging to 0 in E_M ([5, III.1.5, III.1.7]; see also [22, 6.1.21]). Therefore a 0-neighbourhood of $\mu_{fc}(E, E')$ is an $\aleph_0$-barrel of $(E, \mu(E, E'))$. By [22, 3.2.7], it is a bornivorous $\aleph_0$-barrel. ■

The next theorem affiliates the [bornivorous] wb-topology of E with a dual strong union.

Theorem 4 *Let $F_{\underline{n}(k',0)}$ be a dual strong union of the duality $\langle E,\ E'\rangle$. The following statements are true.*

(a) $\mu_{wb}(E, F_{\underline{n}(k',0)}) = \beta(E, F_{\underline{n}(k',0)})$.

(b) $\mu^*_{wb}(E, F_{\underline{n}(k',0)}) = \beta^*(E, F_{\underline{n}(k',0)})$

PROOF. Let B be a closed bounded disk of $(F_{\underline{n}(k',0)}, \sigma(F_{\underline{n}(k',0)}, E))$. Applying Theorem 1 and Definition 3, we have $B = \bigcup\{A_m :\ m \in \mathbb{N}\}$. Then by the bipolar theorem any barrel of $(E, \sigma(E, F_{\underline{n}(k',0)}))$ is an $\aleph_0$-barrel of $(E, \mu(E, F_{\underline{n}(k',0)}))$. ■

The next definition was introduced in [33, Definitions 1 and 2]. It conceptualized a class of spaces studied by Valdivia in [37]. The version below befits our needs.

Definition 5 *A locally convex space (E, τ) is called $\aleph_0 - \eta$-[quasi]-barrelled if η is compatible with the duality $\langle E,\ E'\rangle$ and any [bornivorous] $\aleph_0$-barrel of (E, η) is a 0-neighbourhood in (E, τ).*

We remind two well-known extremes of Definition 5. The $\aleph_0 - \tau$-[quasi-]-barrelled space (E, τ) is precisely the countably [quasi-] barrelled space of [18], or the d-[infra-] barrelled space of [30], or the $\aleph_0$-[quasi-]-barrelled space of [22, 8.2.1]. The $\aleph_0 - \sigma(E, E')$-[quasi-]-barrelled space is the σ-barrelled [σ-evaluable] space of [6], or the σ-[infra-] barrelled space of [30], or the l^∞-[quasi-] barrelled space of [22, 8.2.13]. Notice that if (E, τ) is $\aleph_0 - \sigma(E, E')$-[quasi-]-barrelled and E' admits an infinite-dimensional bounded Banach disk, then τ is strictly finer than $\sigma(E, E')$, [22, 6.1.21].

Theorem 5 *Let (E, τ) be an $\aleph_0 - \eta$-[quasi-] barrelled space. The following is true.*

(a) $\sigma(E, E') \leq \eta \leq \eta_{wb} \leq \tau \leq \mu(E, E')$, *[*$\sigma(E, E') \leq \eta \leq \eta^*_{wb} \leq \tau \leq \mu(E, E')$*]*.

(b) *(E, t) is $\aleph_0 - \eta$-[quasi-] barrelled for any locally convex topology t such that $\eta_{wb} \leq t \leq \mu(E, E')$, [$\eta^*_{wb} \leq t \leq \mu(E, E')$].*

(c) *(E, t) is $\aleph_0 - \eta_{wb}$-barrelled, [$\aleph_0 - \eta^*_{wb}$-quasi-barrelled], for any locally convex topology t such that $\eta_{wb} \leq t \leq \mu(E, E')$, [$\eta^*_{wb} \leq t \leq \mu(E, E')$].*

(d) *(E, t) is $\aleph_0 - \sigma(E, E')$-[quasi-] barrelled for any locally convex topology t such that $\sigma_{wb} \leq t \leq \mu(E, E')$, [$\sigma^*_{wb} \leq t \leq \mu(E, E')$].*

(e) *(E, t) is $\aleph_0 - \sigma_{wb}(E, E')$-barrelled, [$\aleph_0 - \sigma^*_{wb}(E, E')$-quasi-barrelled], for any locally convex topology t such that $\sigma_{wb} \leq t \leq \mu(E, E')$, [$\sigma^*_{wb} \leq t \leq \mu(E, E')$].*

PROOF. We start with the barrelled case.

(a) is obvious.

(b) since (E, τ) is $\aleph_0 - \eta$-barrelled, (E, η) and (E, η_{wb}) have the same $\aleph_0$-barrels. Therefore, (E, η_{wb}) is $\aleph_0 - \eta$-barelled. Obviously,(E, t) is $\aleph_0 - \eta$-barrelled for any topology t such that $\eta_{wb} \leq t \leq \mu(E, E')$.

(c) follows from (b).

(d) if $\{f_n :\ n \in \mathbb{N}\}$ is bounded in $(E', \sigma(E, E'))$ and K is an η-equicontinuous disk of E', then $\{(K \cup f_n) :\ n \in \mathbb{N}\}$ is a bounded sequence of η-equicontinuous sets of E'. Since (E, τ) is $\aleph_0 - \eta$-barrelled, $\cup\{(K \cup f_n) :\ n \in \mathbb{N}\}$ is η-equicontinuous. Hence $\{f_n :\ n \in \mathbb{N}\}$ is weakly relatively compact, therefore (E, τ) is $\aleph_0$-$\sigma(E, E')$-barrelled. Using (b) we conclude that if $\sigma_{wb} \leq t \leq \mu(E, E')$, then (E, t) is $\aleph_0 - \sigma(E, E')$-barrelled.

(e) if (E,τ) is $\aleph_0-\sigma(E,E')$-barrelled, then $(E,\sigma(E,E'))$ and $(E,\sigma_{wb}(E,E'))$ have the same $\aleph_0$-barrels. The conclusion follows then from (c).

The quasi-barrelled case follows immediately after noticing that if (E,τ) is $\aleph_0-\eta$-quasi-barrelled, then (E,η) and (E,η^*_{wb}) have the same bornivorous $\aleph_0$-barrels. ∎

In [20, Example 5.4] Kąkol, Saxon and Todd constructed a non-Mackey countably barrelled DF-space (E,τ) that admits another topology γ, such that $\tau<\gamma<\mu(E,E')$ and (E,γ) is not countably barrelled. Kąkol, Saxon and Todd noticed that (E,γ) is $\aleph_0-\sigma(E,E')$-barrelled. Definition 5 and Theorem 5 "upgrade" the barrelledness condition of (E,γ), claiming that it is $\aleph_0-\tau$-barreled, as well as $\aleph_0-\tau_{wb}$-barrelled.

Our next proposition links the wb-topology of Definition 4 and the weak barrelledness of Definition 5. It also unifies Corollaries 1, 2 of Theorem 3 and Corollaries 1, 2 of Theorem 5 in [25]. The parts a and b are well known Theorems 1 and 2 of [18], rephrased in the terms of Definitions 4 and 5.

Proposition 7 *Let (E,τ) be a locally convex space. Consider the following statements.*

(a) $\tau=\tau_{wb}$, *[*$\tau=\tau^*_{wb}$*].*

(b) (E,τ) *is* $\aleph_0-\tau$*-[quasi-] barrelled.*

(c) $\tau_{wb}[\tau^*_{wb}]$ *is compatible with the duality* $\langle E,\,E'\rangle$*.*

(d) $(E,\tau_{wb})[(E,\tau^*_{wb})]$ *is* $\aleph_0-\tau$*-[quasi-]-barrelled.*

Then: (a) ⇔ (b), (a) ⇒ (c), and (c) ⇔ (d).

PROOF. (a) ⇔ (b) are Theorems 1 and 2 of [18]. (a) ⇒ (c) is obvious. (c) ⇒ (d): since $\tau_{wb}[\tau^*_{wb}]$ is compatible with the duality $\langle E,\,E'\rangle$, any [strongly] bounded countable union of τ-equicontinuous sets of E' is weakly relatively compact. Therefore, $(E,\tau_{wb})[(E,\tau^*_{wb})]$ is $\aleph_0-\tau$-[quasi]-barrelled. (d) ⇒ (c) is embedded in Definition 5. ∎

Generally, for a locally convex space (E,τ) the topology $\tau_{wb}[\tau^*_{wb}]$ is not necessarily $\aleph_0-\sigma(E,E')$-[quasi]-barrelled. Moreover, it is not necessarily compatible with the duality $\langle E,\,E'\rangle$. Example 2.1 of [29] exhibits a dual locally complete Mackey space that is not $\aleph_0-\sigma(E,E')$-barrelled. Applying Proposition 7, it means that the wb-topology $\sigma_{wb}(E,E')$ of a dual locally complete space is not necessarily compatible with the duality $\langle E,\,E'\rangle$. However for the space $C_c(X)$ the following proposition is true.

Proposition 8 *Let $E=C_c(X)$ be the space of continuous real-valued functions on a completely regular topological space X equipped with the compact-open topology. The following statements are equivalent.*

(a) $E=C_c(X)$ *is dual locally complete.*

(b) *the wb-topology* $\sigma_{wb}(E,E')$ *for* $E=C_c(X)$ *is compatible with the duality* $\langle E,\,E'\rangle$*.*

(c) $E=C_c(X)$ *is* $\aleph_0-\sigma(E,E')$*-barrelled.*

(d) $E=C_c(X)$ *is* $\aleph_0-\sigma_{wb}(E,E')$*-barrelled.*

PROOF. (a) ⇔ (c) is Theorem 4.1 of [4]. (b) ⇔ (c) follows from Proposition 7. (c) ⇔ (d) follows from Theorem 5. ∎

Our next theorem reveals the sensitivity of the dual pair $\langle E,\,F_{\underline{n}(k',0)}\rangle$ to weakest barrelledness condition.

Theorem 6 *Let $F_{\underline{n}(k',0)}$ be a dual strong union of the duality $\langle E,\,E'\rangle$. The following statements are equivalent.*

(a) *any [strongly] bounded set of* $(F_{\underline{n}(k',0)},\mu(F_{\underline{n}(k',0)},E))$ *is* α*-regular.*

(b) *any [strongly] bounded set of* $(F_{\underline{n}(k',0)}, \mu(F_{\underline{n}(k',0)}, E))$ *is quasi-regular.*

(c) $(E, \mu(F_{\underline{n}(k',0)}, E))$ *is dual locally [quasi-] barrelled.*

(d) $(E, \mu(F_{\underline{n}(k',0)}, E))$ *is dual locally [quasi-] complete.*

(e) *the topology* $\sigma_{wb}(E, F_{\underline{n}(k',0)})[\sigma^*_{wb}(E, F_{\underline{n}(k',0)})]$ *is compatible with the duality* $\langle E,\ F_{\underline{n}(k',0)}\rangle$.

(f) $\mu_{wb}(E, F_{\underline{n}(k',0)}) = \mu(E, F_{\underline{n}(k',0)}), [\mu^*_{wb}(E, F_{\underline{n}(k',0)}) = \mu(E, F_{\underline{n}(k',0)})]$

(g) $(E, \mu(E, F_{\underline{n}(k',0)}))$ *is* $\aleph_0 - \sigma(E, F_{\underline{n}(k',0)})$*-[quasi-] barrelled.*

(h) $(E, \mu(E, F_{\underline{n}(k',0)}))$ *is [quasi-] barrelled.*

PROOF. (a) $\Leftrightarrow$ (b) follows from Proposition 1.
(a) $\Rightarrow$ (h) follows from Proposition 1.b and Theorem 2.
(c) $\Rightarrow$ (h) follows from Proposition 2.
(f) $\Leftrightarrow$ (h) follows from Theorem 4.
(h) $\Rightarrow$ (g) is well known.
(g) $\Rightarrow$ (e) follows from Theorem 5. (a).
(e) $\Rightarrow$ (g) follows from Proposition 7 and Theorem 5 (d).
(g) $\Rightarrow$ (d) is well known.
(d) $\Rightarrow$ (c) is obvious.
(c) $\Rightarrow$ (a) follows from Proposition 2. ■

6 Final remarks

We conclude this article with remarks regarding bounded disks in dual strong unions and weak barrelledness conditions of Definitions 4, 5.

A locally convex space (E, t) *satisfies the (BBC) condition* if there exists a weaker Hausdorff locally convex topology τ on E, such that every bounded set of (E, t) is contained in a t-bounded τ-compact disk of (E, t), [3]. The reader is referred to [7], [19] or [22, Ch. 7] for a definition of a B-complete space, and to [31, IV.1.1] for a definition of a hemi-compact space and related properties of spaces of continuous functions defined on a hemi-compact space.

Remark 1 *If* B *is a closed bounded disk of a d.s.u.* $(F_{\underline{n}(k',0)}, \sigma(F_{\underline{n}(k',0)}, E))$*, then there exists a locally convex topology* $\sigma(B)$ *on* SpB *such, that:*

(a) B *is hemi-compact when equipped with the induced* $\sigma(B)$*-topology.*

(b) $(SpB, \sigma(B))$ *is* B*-complete.*

(c) $\sigma(B)$ *is the finest topology on* SpB*, having the same compact disks as the induced* $\sigma(F_{\underline{n}(k',0)}, E)$ *topology.*

(d) *any locally convex topology on* SpB*, finer that* $\sigma(B)$*, satisfies the (BBC) condition.*

PROOF. Let $F_{\underline{n}(k',0)} = \cup\{F_{\underline{m}} : m \in \mathbb{N}\}$ be the representation of $F_{\underline{n}(k',0)}$, and $B = \cup\{A_m : m \in \mathbb{N}\}$ the wc-representation of B. Denote by $\sigma(B)$ the finest locally convex topology on SpB, agreeing with $\sigma(F_{\underline{n}(k',0)}, E)$ on each A_m, [22, 8.1.16].
(a) follows from Definition 3 and Definition IV.1.1 of [31].
(b) By [25, Lemma 1], $\sigma(B)$ admits a base of 0-neighbourhoods closed in $(SpB, \sigma(F_{\underline{n}(k',0)}, E))$. Hence by the arguments of [19, Theorem 3.1], $(SpB, \sigma(B))$ is B-complete.
(c) Since each A_m is compact, it follows from the Banach-Dieudonne theorem that $\sigma(B)$ is the finest linear,

as well as the finest general topology, agreeing with $\sigma(F_{\underline{n}(k',0)}, E)$ on each A_m, [15, t. I, p. 230]; [22, 8.1.8.].
(d) If τ is a locally convex topology on SpB, finer that $\sigma(B)$, then any closed τ-bounded disk is $\sigma(B)$-bounded, hence contained in some $2^m A_m : m \in \mathbb{N}$. Therefore, (SpB, τ) satisfies the (BBC)-condition. ■

Remark 2 *Let $F_{\underline{n}(k',0)} = \cup\{F_{\underline{m}} : m \in \mathbb{N}\}$ be the representation of a d.s.u. $F_{\underline{n}(k',0)}$, and $B = \cup\{A_m : m \in \mathbb{N}\}$ the wc-representation of a bounded disk B of $(F_{\underline{n}(k',0)}, \sigma(F_{\underline{n}(k',0)}, E))$. Denote by E_m the Banach space with the unit ball A_m, $m \in \mathbb{N}$. Then:*

(a) $(SpB, t(B)) = \operatorname{ind}\lim\{E_m : m \in \mathbb{N}\}$ *is a regular (LB)-space, satisfying the conditions $(BBC)_n$ and $(CNC)_n$ of [3], thus allowing a construction of a natural predual, [3, Section 1(b), Theorem 1].*

(b) $(SpB, t(B))$ *)) admits an equivalent inductive sequence of normed spaces $\{Sp(B \cap F_{\underline{m}})\} : m \in \mathbb{N}$ with the unit ball $B \cap F_{\underline{m}}$ weakly closed in $F_{\underline{m}}$, for each $m \in \mathbb{N}$.*

PROOF. (a) is well known. (b) is embedded in the Theorem 1. ■

A wealth of important topologies with compactness, regularity, localization and other "nice" properties coexist on SpB. Results related to SpB may be found in [2, 3, 6, 11, 12, 13, 27, 36], (this reference list is far from being exhausting; in particular we would like to notice a remarkably simple proof of Grothendieck-Floret factorization theorem, given in Section 2 of [13], see also [22, 8.5.38]).

As mentioned in the introduction, the research on bounded sets of $(F_{\underline{n}(k',0)}, \sigma(F_{\underline{n}(k',0)}, E))$ seems to be fascinating. Take, for example, the set **K** of all compact disks of $(F_{\underline{n}(k',0)}, \sigma(F_{\underline{n}(k',0)}, E))$, and consider two topologies on $F_{\underline{n}(k',0)}$: the finest locally convex topology $\sigma(\mathbf{K})$ agreeing with $\sigma(F_{\underline{n}(k',0)}, E)$ on each $A \in \mathbf{K}$ and the inductive limit $(F_{\underline{n}(k',0)}, t(\mathbf{K})) = \operatorname{ind}\lim\{E_A : A \in \mathbf{K}\}$. The topologies $\sigma(\mathbf{K})$ and $t(\mathbf{K})$ may be regarded in the setting of mixed topologies ([14, 23, 40], see also Section 5 of [26]), or treated by homological methods ([24, 39]). Notice that by [5, III.2.4.] and Proposition 2, $t(\mathbf{K})$ is the associated ultrabornological topology for $(F_{\underline{n}(k',0)}, \sigma(F_{\underline{n}(k',0)}, E))$ and by [5, III.2.8], the dual of $(F_{\underline{n}(k',0)}, t(\mathbf{K}))$ is the completion of the Schwartz space $(E, (E, \mu_{fc}(E, F_{\underline{n}(k',0)})))$, associated to $(E, \mu(E, F_{\underline{n}(k',0)}))$. It should be noticed that **K** is not a base for a compact bornology on $(F_{\underline{n}(k',0)}, \sigma(F_{\underline{n}(k',0)}, E))$ unless $\mu(E, F_{\underline{n}(k',0)})$ is the associated barrelled topology for $(E, \mu(E, E'))$, (see [24] for a definition of bornology). Indeed, by Proposition 2, $(E, \mu(E, F_{\underline{n}(k',0)}))$ is barrelled if and only if it is dual locally complete, therefore by [9], if and only if the closed absolutely convex hull of any convergent to zero sequence of $(F_{\underline{n}(k',0)}, \sigma(F_{\underline{n}(k',0)}, E))$ is compact and hence absorbed by some $A \in \mathbf{K}$. We believe that the research of weakly compact subsets of a dual strong union is going to be fruitful and promising.

Regarding the topologies of Definitions 4 and 5, we do not know whether the σ_{wb}-and η_{wb}-barrelledness of an $\aleph_0 - \eta$-barrelled space (E, τ) affects the topologies between σ_{wb} and η_{wb}. Speaking informally, Theorem 5 presents the wb-topology as a "demarcation line" of barrelledness. It simply states that the weak barrelledness condition of an $\aleph_0 - \eta$-barrelled space (E, τ) starts from σ_{wb}, then jumps to η_{wb} and keeps going until $\mu(E, E')$. But what happens between σ_{wb} and η_{wb}?

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