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A Continuous Surjection from the Unit Interval onto the Unit Square

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ABSTRACT. We show that there exists a continuous surjection $\varphi: I \longrightarrow I^2$ which admits an *averaging operator in the sense of Pełczyński and which has the additional property that the* map $\varphi^\circ: f \longrightarrow f \circ \varphi$ is an isomorphism from $L_p(I^2)$ onto a subspace of $L_p(I)$, where $1 \leq p < \infty$.

1. INTRODUCTION

In [T] the author proved that for a wide class of pairs of compact metric spaces (K,K_1) there exists a continuous surjection $\psi: K \longrightarrow K_1$ admitting an averaging operator in the sense of Pełczyński, [P]. The results of [T] contain the important special case that there exists a continuous surjection $\varphi: I \longrightarrow I^2$, where $I = [0,1] \subset \mathbb{R}$, having a regular averaging operator (for the terminology, see below). The aim of this paper is to show that the definition of φ can be modified such that $\varphi^\circ: f \longrightarrow f \circ \varphi$ in addition becomes an isomorphism from $L_p(I^2)$ onto a subspace of $L_p(I)$, where $1 \leq p < \infty$ (Corollary 7 and Theorem 8). So, we get an operator $\varphi^\circ: C(I^2) \longrightarrow C(I)$ which has good properties *simultaneously* with respect to the sup- and L_p-norms. This result, while being of interest in itself, is connected with the study of some Fréchet function spaces, see Section 4.

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We introduce the notations and definitions used in this paper. If K is a compact metric space, we denote by $C(K)$ the Banach space of continuous, real or complex valued mappings, endowed with the sup-norm. If K_1 and K_2 are compact metric spaces and $\varphi: K_1 \longrightarrow K_2$ is a continuous surjection, we denote by φ° the linear isometry from $C(K_2)$ into $C(K_1)$ given by $\varphi^\circ f = f \circ \varphi$. If $\varphi^\circ(C(K_2))$ is 1-complemented in $C(K_1)$, i.e., if there exists a contractive projection from $C(K_1)$ onto $\varphi^\circ(C(K_2))$, we say that φ admits a regular averaging operator. For more details we recommend the reference [LT], Sections II.4.h.i; see also [P].

Let $\Delta \subset I$ be the "middle thirds"-Cantor set; see for example [R], p. 179. Using the homeomorphism

$$(\varepsilon_m)_{m=1}^{\infty} \to \sum_{m=1}^{\infty} 2\varepsilon_m 3^{-m},$$

where $\varepsilon_m = 0$ or 1 for all $m \in \mathbb{N}$, we identify the topological product

$$\prod_{m=1}^{\infty} \{0,1\} \tag{1}$$

with Δ. By $\psi: \Delta \longrightarrow [0,1]$ we denote the continuous surjection

$$\psi((\varepsilon_m)_{m=1}^{\infty}) = \sum_{m=1}^{\infty} \varepsilon_m 2^{-m}. \tag{2}$$

Each dyadic point of the form

$$\sum_{m=0}^{n} \varepsilon_m 2^{-m} \in I,$$

where $\varepsilon_n = 1$, $n \geq 1$, has two inverse images, $(\varepsilon_1,...,\varepsilon_n,0,0,0,...)$ and $(\varepsilon_1,...,\varepsilon_{n-1},0,1,1,1,...)$. The other points of I have only one inverse image. We define the discontinuous right inverse $\varrho: I \longrightarrow \Delta$ of ψ by

$$\varrho(x) = \min\{y \in \Delta \mid \psi(y) = x\}, \tag{3}$$

where "min" is taken with respect to the usual order of $I \supset \Delta$. The mapping ϱ° is an isometry from $C(\Delta)$ onto

$$D(I), \tag{4}$$

which is the subspace of $l_\infty(I)$ (the Banach space of bounded scalar valued functions on I endowed with the sup-norm) spanned by continuous functions and the characteristic functions of intervals with dyadic endpoints. It is easy to check that such characteristic functions are contained in $\varrho^\circ(C(\Delta))$, and that the other details of this statement also hold.

The elements of Δ^4 are considered as 4 x ∞-matrices consisting of numbers 0 or 1 (see (1.2)). We denote 4 x 1-matrices, i.e., the columns of elements of Δ^4, by $(\varepsilon_1,\varepsilon_2,\varepsilon_3,\varepsilon_4)^T$. By $\bar{0}$ (resp. $\bar{1}$) we denote a matrix which consists of numbers 0 (resp. 1) only; the dimension of such a matrix will be clear from context. If $A = (\varepsilon_{ij})$ is a matrix with $\varepsilon_{ij} = 0$ or 1 for all i and j, we denote by A^- the matrix (ε^-_{ij}), where

$$\varepsilon^-_{ij} = \begin{cases} 0, & \text{if } \varepsilon_{ij} = 1 \\ 1, & \text{if } \varepsilon_{ij} = 0. \end{cases} \tag{5}$$

The space of $4 \times m$-matrices, consisting of numbers 0 and 1, is denoted by Δ^4_m.

We denote by m_n the n-dimensional Lebesgue measure. We define the σ-algebra $\mathcal{M}$ of subsets of Δ by

$$\mathcal{M} = \{\psi^{-1}(\mathcal{A}) \mid \mathcal{A} \subset I \text{ is Lebesgue measurable}\},$$

and we define the measure μ_1 on $(\Delta,\mathcal{M})$ by $\mu_1(\mathcal{A}) = m_1(\psi(\mathcal{A}))$, where $\mathcal{A} \in \mathcal{M}$. Note that μ_1 is additive and even σ-additive in spite of the fact that ψ is not an injection: if $\mathcal{A} \subset \Delta$ and $\mathcal{B} \subset \Delta$ are disjoint, then $\psi(\mathcal{A}) \cap \psi(\mathcal{B})$ is contained in the subset of the dyadic points of I; this set has Lebesgue measure 0. We denote by μ_n the n-fold product of the measure μ_1.

We define the homeomorphism η: $\Delta \longrightarrow \Delta^4$,

$$\eta:(\varepsilon_m)_{m=1}^{\infty}\to((\varepsilon_{4m-3})_{m=1}^{\infty},(\varepsilon_{4m-2})_{m=1}^{\infty},(\varepsilon_{4m-1})_{m=1}^{\infty},(\varepsilon_{4m})_{m=1}^{\infty})^T. \tag{6}$$

By

$$\sigma: I \times I \to I \tag{7}$$

we mean the continuous surjection which assigns to $(x,y) \in I^2$ the unique number $t \in I$ such that (x,y) belongs to the line segment joining $(0,t)$ with $(1,t^2)$. This map is used in the proof of the so called Milutin's lemma, see [LT], II.4.21.

2. CONSTRUCTION OF THE MAP φ

We first define the continuous surjection γ: $\Delta^4 \to \Delta^4$ as follows.

Let $A \in \Delta^4$; we write $A = (A_1,A_2,A_3,...)$ where each $A_m = (B_m,C_m)$ is a 4×2-matrix consisting of $(2m-1)$:th and $2m$:th columns B_m and C_m of A. We first define for all $m \in \mathbb{N}$ the 4×1-matrices D_m inductively as follows. Let $D_1 = C_1$. Let $m \in \mathbb{N}$, $m > 1$, and assume that D_k is defined for $k < m$. We first define $\Gamma_A^m:\{0,1\} \to \Delta_1^4$ by

$$\Gamma_A^m(0) = \begin{cases} D_{m-1}, & \text{if } A_{m-1} = \bar{0} \\ D_{m-1}^-, & \text{if } A_{m-1} \neq \bar{0}, \end{cases}$$

$$\Gamma_A^m(1) = \begin{cases} D_{m-1}, & \text{if } A_{m-1} = \bar{1} \\ D_{m-1}^-, & \text{if } A_{m-1} \neq \bar{1} \end{cases}$$

To define D_m we distinguish between several cases.

1º. If $B_m \neq C_m$ and $B_m \neq C_m^-$, we set $D_m = C_m$.

2º. Assume that $A_m = \bar{0}$ or $\bar{1}$. If $A_m = A_{m-1}$, we set $D_m = D_{m-1}$, and if $A_m \neq A_{m-1}$, we set $D_m = D_{m-1}^-$. Remark. If $A_m = \bar{0}$, we have $D_m = \Gamma_A^m(0)$, and if $A_m = \bar{1}$, we have $D_m = \Gamma_A^m(1)$.

3º. a) Assume that $B_m = C_m$, $A_m \neq \bar{0}$ and $A_m \neq \bar{1}$. If $\Gamma_A^m(0) = C_m$, we set $D_m = (0,0,0,0)^T$. If $\Gamma_A^m(0) \neq C_m$, we set $D_m = C_m$.

b) Assume $B_m = C_m^-$, $C_m \neq (0,0,0,0)^T$ and $C_m \neq (1,1,1,1)^T$. If $\Gamma_A^m(1) = C_m$, we set $D_m = (1,1,1,1)^T$ and $\Gamma_A^m(1) \neq C_m$, we define $D_m = C_m$.

4º. Consider the case $B_m = C_m^-$, and $C_m = (0,0,0,0)^T$ or $C_m = (1,1,1,1)^T$.

a) If $\Gamma_A^m(0) = (1,1,1,1)^T$ and $\Gamma_A^m(1) \neq (0,0,0,0)^T$, we set $D_m = (0,0,0,0)^T$.
b) If $\Gamma_A^m(1) = (0,0,0,0)^T$ and $\Gamma_A^m(0) \neq (1,1,1,1)^T$, we define $D_m = (1,1,1,1)^T$.
c) In the other cases we set $D_m = C_m$.

We define the element $D \in \Delta^4$ by $D = (D_1,D_2,D_3,\ldots)$. We define also for all m the mapping $\gamma_m\colon \Delta_{2m}^4 \longrightarrow \Delta_m^4$ by

$$\gamma_m((A_1,\ldots,A_m)) = (D_1,\ldots,D_m).$$

and we set

$$\gamma(A) = D.$$

In the following we denote $\bar{\psi}:=(\psi,\psi,\psi,\psi)\colon \Delta^4 \longrightarrow I^4$ and $\bar{\sigma}:=(\sigma,\sigma)\colon I^4 \longrightarrow I^2$.

Lemma 1. *The map $\gamma\colon \Delta^4 \longrightarrow \Delta^4$ is a continuous surjection which admits a continuous right inverse and for which the map*

$$\varphi=\bar{\sigma}\circ\bar{\psi}\circ\gamma\circ\eta\circ\varrho : I \to I^2 \tag{8}$$

is a continuous surjection.

Proof. The continuity of γ, the existence of a continuous right inverse of γ and the surjectivity of φ can be proved exactly as in Lemma 3.2 of [T]. Also the idea for the proof of the continuity of φ is the same as in [T], but because of the details it is necessary to give the proof here. In view of Lemma 3.1 of [T] it is enough to show that

$$J := \overline{\psi} \circ \gamma \circ \eta : \Delta \to I^4 \tag{9}$$

maps the elements, $a, \tilde{a} \in \Delta$ of the form

$$\begin{aligned} a &= (b,1,0,0,0,\ldots) \\ \tilde{a} &= (b,0,1,1,1,\ldots); \end{aligned} \tag{10}$$

where b is a finite sequence consisting of numbers 0 or 1, to the same element of I^4.

We denote $\eta(a) = A = (A_1, A_2, \ldots) \in \Delta^4$ and $\eta(\tilde{a}) = (\tilde{A}_1, \tilde{A}_2, \ldots) \in \Delta^4$, where A_m (respectively, $\tilde{A}_m$) consists of the $2m$-1:th and $2m$:th columns B_m and C_m of A (resp. $\tilde{B}_m, \tilde{C}_m, \tilde{A}$). In view of (10) and the definition of η, (6), there exists a unique number m such that $A_k = \tilde{A}_k$ for $k < m$ (if $m > 1$), $A_m \neq \tilde{A}_m$ and $A_k = \bar{0}$, $\tilde{A}_k = \bar{1}$ for $k > m$.

We now consider $\gamma(A) = D = (D_1, D_2, \ldots)$ and $\gamma(\tilde{A}) = \tilde{D} = (\tilde{D}_1, \tilde{D}_2, \ldots)$.

It is clear from the definition of γ that $D_k = \tilde{D}_k$ for $k < m$. Moreover, the matrix A_m contains an element equal to 1 and, similarly, $\tilde{A}_m$ contains an element 0. Hence, we have $A_m \neq A_{m+1}$ and $\tilde{A}_m \neq \tilde{A}_{m+1}$. By 2º we get $D_{m+1} = D_m^-$ and $\tilde{D}_{m+1} = \tilde{D}_m^-$, and, moreover, $D_k = D_{m+1}$ and $\tilde{D}_k = \tilde{D}_{m+1}$ for $k > m+1$. So, we have

$$\gamma \circ \eta(a) = D = (D_1, D_2, \ldots, D_{m-1}, D_m, D_m^-, D_m^-, D_m^-, \ldots),$$

$$\gamma \circ \eta(\tilde{a}) = \tilde{D} = (D_1, D_2, \ldots, D_{m-1}, \tilde{D}_m, \tilde{D}_m^-, \tilde{D}_m^-, \tilde{D}_m^-, \ldots).$$

Let us consider the i:th ($1 \leq i \leq 4$) rows $(d_1^{(i)}, d_2^{(i)}, \ldots) \in \Delta$ and $(\tilde{d}_1^{(i)}, \tilde{d}_2^{(i)}, \ldots) \in \Delta$ of D and $\tilde{D}$, respectively. We have $d_j^{(i)} = \tilde{d}_j^{(i)}$ for $1 \leq j < m$. Moreover, $d_k^{(i)} = d_m^{(i)-}$ and $\tilde{d}_k^{(i)} = \tilde{d}_m^{(i)-}$ for all $k > m$. This shows that

$$\psi((d_1^{(i)}, d_2^{(i)}, \ldots)) = \psi((\tilde{d}_1^{(i)}, \tilde{d}_2^{(i)}, \ldots))$$

and, hence, $J(a) = \overline{\psi}(D) = \overline{\psi}(\tilde{D}) = J(\tilde{a})$.

Theorem 2. *The map* φ: $I \longrightarrow I^2$ *(see (8)) has a regular averaging operator.*

The proof is the same as that of Theorem 3.3 of [T].

3. MAIN RESULT

We now show that φ also has the additional property that φ° defines an isomorphism from $L_p(I^2)$ into $L_p(I)$ for $1 \leq p < \infty$.

Lemma 3. *Let* $m > 1$ *and* $A \in \Delta^4_{2m-2}$ *and* $D \in \Delta^4_1$. *There exist exactly 16 different matrices* $A_m \in \Delta^4_2$ *such that*

$$\gamma_m((A,A_m))=(\gamma_{m-1}(A),D). \tag{11}$$

Proof. The proof of this lemma consists of a straightforward but elaborate verification of the different cases in the definition of γ_m. The numbers 1º-4º refer there.

i) We first assume that $D \neq (0,0,0,0)^T$ and $D \neq (1,1,1,1)^T$. There exist 14 vectors $B \in \Delta^4_1$ such that $B \neq D$ and $B \neq D^{\cdot}$. By 1º we see that $A_m = (B,D)$ satisfies (11) for all such B.

Next we check if the cases $A_m = \bar{0}, \bar{1}$, (D,D) or $(D^{\cdot},D)$ could satisfy (11). First, if $\Gamma^m_A(0) = D$, then, by 2º and 3º a), $A_m = \bar{0}$ satisfies (11) and $A_m = (D,D)$ does not (use the remark in 2º). If $\Gamma^m_A(0) \neq D$, then $A_m = (D,D)$ satisfies (11) and $A_m = \bar{0}$ does not. Hence, in every case exactly one of the matrices $\bar{0}$ and (D,D) satisfies (11). In the same way we see that exactly one the matrices $\bar{1}$ and $(D^{\cdot},D)$ satisfies (11).

Since $D \neq (0,0,0,0)^T,(1,1,1,1)^T$ we see that 4º cannot produce other matrices satisfying (11). Finally, by 1º-4º, a matrix A_m of the form $A_m = (B,C)$, where $C \neq D$ and, moreover, either B or C is different from $(0,0,0,0)^T$ and $(1,1,1,1)^T$, cannot satisfy (11).

Summing up, we see that (11) holds for exactly 16 different matrices $A \in \Delta_2^4$.

ii) We assume $D = (0,0,0,0)^T$. Again there exist 14 vectors $B \in \Delta_1^4$, $B \neq D$ and $B \neq D^\cdot$. By 1º, (11) holds for $A_m = (B,D)$.

It follows immediately from 1º-4º that a matrix $A_m = (B_m,C_m)$, where $C_m \neq D$, can satisfy (11) only if $C_m = D^\cdot$ and $B_m = D$ or $D^\cdot$, or if $B_m = C_m = \Gamma_A^m(0)$ (see 3º a)). Hence, we need only to consider such cases and the cases $A_m = \bar{0}$ and $A_m = (D^\cdot,D)$. We should find exactly two matrices of these types satisfying (11).

a) We assume $\Gamma_A^m(0) = (0,0,0,0)^T$. Then $A_m = \bar{0}$ satisfies (11). Moreover, if $\Gamma_A^m(1) = (0,0,0,0)^T$, then also $A_m = \bar{1}$ works, by 2º, and, by 4ºb), the cases $A_m = (D^\cdot,D)$, $A_m = (D,D^\cdot)$ do not work. If $\Gamma_A^m(1) \neq (0,0,0,0)^T$, then $A_m = \bar{1}$ (see 2º) and $A_m = (D,D^\cdot)$ (see 4º c)) do not work but $A_m = (D^\cdot,D)$ does, by 4º c). So we get altogether two positive cases.

b) We assume $\Gamma_A^m(0) = (1,1,1,1)^T$ so that $A_m = \bar{0}$ does not work. If $\Gamma_A^m(1) = (0,0,0,0)^T$, then by 2º, $A_m = \bar{1}$ satisfies (11). Moreover, by 4º c), $(D^\cdot,D)$ satisfies (11) and $(D,D^\cdot)$ does not. If $\Gamma_A^m(1) \neq (0,0,0,0)^T$, then $A_m = \bar{1}$ does not satisfy (11), but by 4º a), $(D^\cdot,D)$ and $(D,D^\cdot)$ do.

c) Assume $\Gamma_A^m(0) \neq D$ and $\Gamma_A^m(0) \neq D^\cdot$. The case $A_m = \bar{0}$ does not work. By 3º a), $A_m = (\Gamma_A^m(0),\Gamma_A^m(0))$ satisfies (11). If $\Gamma_A^m(1) = (0,0,0,0)^T$, then $A_m = \bar{1}$ works, and by 4º b), $A_m = (D,D^\cdot)$ and $A_m = (D^\cdot,D)$ do not. If $\Gamma_A^m(1) \neq (0,0,0,0)^T$, then $A_m = \bar{1}$ does not work, and by 4º c), $A_m = (D^\cdot,D)$ satisfies (11) and $A_m = (D,D^\cdot)$ does not.

iii) The case $D = (1,1,1,1)^T$ is analogous to ii). But since the point in this kind of proofs is a careful verification of all the details, we want to give the proof also in this case.

By 1º, there exist 14 vectors $B \in \Delta_1^4$, $B \neq D$, $B \neq D^\cdot$ such that (11) holds for $A_m = (B,D)$. From now on we need only to consider the cases $A_m = \bar{1},\bar{0},(D^\cdot,D),(D,D^\cdot)$ and $(\Gamma_A^m(1)^\cdot,\Gamma_A^m(1))$.

a) We assume $\Gamma_A^m(1) = D$. Now $A_m = \bar{1}$ works. If $\Gamma_A^m(0) = D$, then also $\bar{0}$ works but $(D^{\cdot},D)$ and $(D,D^{\cdot})$ do not. If $\Gamma_A^m(0) \neq D$, then $\bar{0}$ and $(D,D^{\cdot})$ do not work but $(D^{\cdot},D)$ does.

b) Assume $\Gamma_A^m(1) = D^{\cdot}$. If $\Gamma_A^m(0) = D$, then $\bar{0}$ and $(D^{\cdot},D)$ satisfy (11) but $\bar{1}$ and $(D,D^{\cdot})$ do not. If $\Gamma_A^m(0) \neq D$, then the cases $A_m = \bar{0}$ and $A_m = \bar{1}$ are negative and the cases $(D^{\cdot},D)$ and $(D,D^{\cdot})$ are positive.

c) Assume $\Gamma_A^m(1) \neq D,D^{\cdot}$. By 3º b), $A_m = (\Gamma_A^m(1)^{\cdot},\Gamma_A^m(1))$ satisfies (11). If $\Gamma_A^m(0) = D$, then also $\bar{0}$ works but $\bar{1}$, $(D^{\cdot},D)$ and $(D,D^{\cdot})$ do not. If $\Gamma_A^m(0) \neq D$, then $\bar{0},\bar{1}$ and $(D,D^{\cdot})$ do not work but $(D^{\cdot},D)$ does.

We have now gone through all the cases.

Corollary 4. *Given $m \in N$ and $D \in \Delta_m^4$ there exist exactly 2^{4m} different matrices $A \in \Delta_{2m}^4$ such that $\gamma_m(A) = D$.*

Proof. Let $A = (A_1,...,A_m) \in \Delta_{2m}^4$, where $A_m \in \Delta_2^4$, and let $A' = (A_1,...,A_{m-1})$. Since the matrix formed by the first m-1 columns of $\gamma_m(A)$ is equal to $\gamma_{m-1}(A')$, we can prove Corollary 4 using Lemma 3 and induction with respect to the number of the columns of D. Note that by definition, the 16 matrices (B,C), where $B \in \Delta_1^4$, are the preimages of C with respect to γ_1.

Lemma 5. *Let $K(i) \in N$ for all $i = 1,2,3,4$ and let $a_m^{(i)} \in \{0,1\}$ for all i and for all $m \leq K(i)$. Let us denote by $A \subset \Delta^4$ the set*

$$A=\{(x_m^{(i)})_{\substack{m\in\mathbf{N}\\1\leq i\leq 4}}\in\Delta^4 \,|\, x_m^{(i)}=a_m^{(i)} \text{ for } m\leq K(i)\}. \tag{12}$$

We have

$$\mu_4(\gamma^{-1}(A))=\prod_{i=1}^{4} 2^{-K(i)}. \tag{13}$$

Proof. Let $K = \max \{K(i) \mid 1 \leq i \leq 4\}$. Let us introduce the set

$$\mathcal{A}_K = \{ (x_m^{(i)})_{\substack{1\leq m\leq K \\ 1\leq i\leq 4}} \in \Delta_K^4 \mid x_m^{(i)} = a_m^{(i)} \textit{ for } m \leq K(i) \}.$$

It is a direct consequence of the definition of γ and γ_K that $A \in \gamma^1(\mathcal{A})$ if and only if $A = (A_1, B)$, where $A_1 \in \Delta_{2K}^4$, $B \in \Delta^4$ and A_1 satisfies

$$\gamma_K(A_1) \in \mathcal{A}_K. \tag{14}$$

For a fixed $A_1 \in \gamma_K^{-1}(\mathcal{A}_K) \subset \Delta_{2K}^4$

$$\mu_4(\{(A_1,B) \in \Delta^4 \mid B \in \Delta^4\}) = 2^{-8K}. \tag{15}$$

We thus need only to calculate $\#(\gamma_K^{-1}(\mathcal{A}_K))$. (We denote by $\#(C)$ the cardinality of the set C). But by Corollary 4,

$$\#(\gamma_K^{-1}(\mathcal{A}_K)) = 2^{4K} \#(\mathcal{A}_K). \tag{16}$$

On the other hand it is elementary to see that

$$\#(\mathcal{A}_K) = \prod_{i=1}^{4} 2^{K-K(1)}.$$

Since the sets $\{(A,B) \mid B \in \Delta^4\}$ and $\{(A_1,B) \mid B \in \Delta^4\}$ are disjoint for $A \neq A_1$, we get by (15) and (16)

$$\mu_4(\gamma^1(\mathcal{A})) = 2^{-8K} \#(\gamma_K^{-1}(\mathcal{A}_K)) = 2^{-4K} \prod_{i=1}^{4} 2^{K-K(i)} = \prod_{i=1}^{4} 2^{-K(i)}.$$

Proposition 6. *There exist positive constants C_1 and C_2 such that*

$$C_1 m_1(\varphi^{-1}(\mathcal{A})) \leq m_2(\mathcal{A}) \leq C_2 m_1(\varphi^{-1}(\mathcal{A}))$$

holds for all rectangles $\mathcal{A} \subset I^2$.

Proof. 1º. We first find positive constants d_1 and d_2 such that

$$d_1\mu_4(\gamma^{-1}(\mathcal{A}))\leq\mu_4(\mathcal{A})\leq d_2\mu_4(\gamma^{-1}(\mathcal{A})) \tag{17}$$

holds for all rectangles $\mathcal{A}\subset\Delta^4$,

$$\mathcal{A}=\prod_{i=1}^{4}[a^{(i)},\tilde{a}^{(i)}] \tag{18}$$

where $a^{(i)},\tilde{a}^{(i)}\in\Delta$ for all i. (Intervals $[a^{(i)},\tilde{a}^{(i)}]$ in Δ are defined with respect to the natural order of Δ.)

Let us denote for $x=(x_m)_{m=1}^{\infty}$, $y=(y_m)_{m=1}^{\infty}\in\Delta$, $x\neq y$,

$$n(x,y)=min\{m\,|\,x_m\neq y_m\} \tag{19}$$

and let $m(a,i)$ (resp. $m(\tilde{a},i)$) stand for the smallest number m such that $m > n(a^{(i)},\tilde{a}^{(i)})$ and $a_m^{(i)}=0$ (resp. $\tilde{a}_m^{(i)}=1$). Then (*) $x\in[a^{(i)},\tilde{a}^{(i)}]$ if either $x_m = a_m^{(i)}$ for $m<m(a,i)$ and $x_{m(a,i)}=1$, or $x_m=\tilde{a}_m^{(i)}$ for $m<m(\tilde{a},i)$ and $x_{m(\tilde{a},i)} = 0$. Moreover, (**) if $x\in[a^{(i)},\tilde{a}^{(i)}]$ then either $x_m=a_m^{(i)}$ for $m<m(a,i)$, or $x_m=\tilde{a}_m^{(i)}$ for $m<m(\tilde{a},i)$. Denoting $M(i):=\min\{m(a,i),m(\tilde{a},i)\}$ for all $1\leq i\leq 4$ we thus get

$$2^{-M(i)}\leq\mu_1([a,\tilde{a}])\leq 2^{-M(i)+2}$$

so that,

$$\prod_{i=1}^{4}2^{-M(i)}\leq\mu_4(\mathcal{A})\leq 2^8\prod_{i=1}^{4}2^{-M(i)}. \tag{20}$$

Let us denote, for all i, $b_m^{(i)}=a_m^{(i)}$ and $b_{M(i)}=1$, if $M(i)=m(a,i)$, or $b_m^{(i)}=\tilde{a}_m^{(i)}$ and $b_{M(i)}=0$, if $M(i)=m(\tilde{a},i)$. From (*) we see that $\mathcal{A}$ contains the set

$$\mathcal{B}=\{(x_m^{(i)})\in\Delta^4\,|\,x_m^{(i)}=b_m^{(i)}\ for\ m\leq M(i)\}. \tag{21}$$

By Lemma 5,

$$\mu_4(\gamma^{-1}(\mathcal{A}))\geq\mu_4(\gamma^{-1}(\mathcal{B}))=\prod_{i=1}^{4} 2^{-M(i)}. \tag{22}$$

Moreover, by (**) $\mathcal{A}$ is contained in the union of the 16 sets

$$\mathcal{B}(\varepsilon_1,\varepsilon_2,\varepsilon_3,\varepsilon_4)=\{(x_m^{(i)})\in\Delta^4 \mid x_m^{(i)}=c_m^{(i)} \textit{ for } m<M(i)\} \tag{23}$$

where $\varepsilon_i \in \{0,1\}$ for $i = 1,...,4$ and $c_m^{(i)} = a_m^{(i)}$ for $m < M(i)$, if $\varepsilon_i = 0$, or $c_m^{(i)} = \tilde{a}_m^{(i)}$ for $m < M(i)$, if $\varepsilon_i = 1$. Hence, by Lemma 5

$$\mu_4(\gamma^{-1}(\mathcal{A}))\leq\mu_4(\bigcup_{\varepsilon_i\in\{0,1\}} \gamma^{-1}(\mathcal{B}(\varepsilon_1,\varepsilon_2,\varepsilon_3,\varepsilon_4)))\leq 2^8\prod_{i=1}^{4} 2^{-M(i)}. \tag{24}$$

Combining (20), (22) and (24) we see that (17) holds with $d_1 = 2^{-8}$, $d_2 = 2^8$.

2º. We find constants $c_1,c_2 > 0$ such that

$$c_1\mu_1(\eta^{-1}(\mathcal{A}))\leq\mu_4(\mathcal{A})\leq c_2\mu_1(\eta^{-1}(\mathcal{A})) \tag{25}$$

holds for all 4-rectangles $\mathcal{A}\subset\Delta^4$. We define the elements $a^{(i)}$ and $\tilde{a}^{(i)}$ and the numbers $m(a,i)$, $m(\tilde{a},i)$, $b_m^{(i)}$ and $M(i)$ as in 1º. Let $M = \max\{M(i) \mid i = 1,...,4\}$. Note that by (20) we again have

$$\prod_{i=1}^{4} 2^{-M(i)}\leq\mu_4(\mathcal{A})\leq 2^8\prod_{i=1}^{4} 2^{-M(i)}. \tag{26}$$

Let us define the set $\mathcal{B}\subset\mathcal{A}$ as in 1º, (21). From the definition of η, (6), we see that

$$\eta^{-1}(\mathcal{B})=\{(x_m)_{m=1}^{\infty}\in\Delta_{4M} \mid x_{4(m-1)+i}=b_m^{(i)} \text{ for all } 1\leq i\leq 4 \text{ and } m\leq M(i)\}. \tag{27}$$

Let us denote

$$\mathcal{C}=\{(x_m)_{m=1}^{4M}\in\Delta_{4M}\mid x_{4(m-1)+i}=b_m^{(i)} \text{ for all } 1\leq i\leq 4 \text{ and } m\leq M(i)\}. \tag{28}$$

Now $x=(x_m)\in\eta^{-1}(\mathcal{B})$ if and only if $x=(y,z)$, where $y\in\mathcal{C}$ and $z\in\Delta$. For a fixed $y\in\mathcal{C}$ we have

$$\mu_I(\{(y,z)\mid z\in\Delta\})=2^{-4M}. \tag{29}$$

We calculate $\#(\mathcal{C})$. In view of (28), the elements of $\mathcal{C}$ are vectors with $4M$ components out of which $\Sigma_{i=1}^4 M(i)$ are fixed and thus $4M-\Sigma_{i=1}^4 M(i)$ may be chosen arbitrarily from the set $\{0,1\}$. So,

$$\#(\mathcal{C})=2^{4M-\Sigma_{i=1}^4 M(i)}. \tag{30}$$

Combining this with (29) we get

$$\mu_1(\eta^{-1}(\mathcal{A}))\geq\mu_1(\eta^{-1}(\mathcal{B}))=\prod_{i=1}^{4}2^{-M(i)}. \tag{31}$$

To get an upper estimate for $\mu(\eta^{-1}(\mathcal{A}))$ we define the 16 sets $\mathcal{B}(\varepsilon_1,\varepsilon_2,\varepsilon_3,\varepsilon_4)$, where $\varepsilon_i\in\{0,1\}$, as in 1º, (23). Since these sets are of the same form as $\mathcal{B}$ above, we get by (31)

$$\mu_1(\eta^{-1}(\mathcal{B}(\varepsilon_1,\varepsilon_2,\varepsilon_3,\varepsilon_4)))=\prod_{i=1}^{4}2^{-M(i)+1}. \tag{32}$$

Since the union of all the sets $\mathcal{B}(\varepsilon_1,\varepsilon_2,\varepsilon_3,\varepsilon_4)$ contains $\mathcal{A}$, we get from (32)

$$\mu_1(\eta^{-1}(\mathcal{A}))\leq 2^8\prod_{i=1}^{4}2^{-M(i)}. \tag{33}$$

Combining (26), (31) and (33) yields (25) with $c_1=2^{-8}$, $c_2=2^8$.

3º. We consider the map $\bar{\sigma}$. If $[a,b]\subset I$, the definition of σ implies

$$m_2(\sigma^{-1}([a,b]))=((b+b^2)-(a+a^2))/2=(b-a)(1+b+a)/2.$$

Hence, $m_1([a,b])/2 \le m_2(\sigma^{-1}([a,b]) \le 2m_1([a,b])$, and so

$$m_4(\bar{\sigma}^{-1}(\mathcal{A}))/4 \le m_2(\mathcal{A}) \le 4m_4(\bar{\sigma}^{-1}(\mathcal{A})) \tag{34}$$

for all rectangles $\mathcal{A} \subset I^2$.

4º. Our statement now follows by combining (17), (25) and (34); the maps ϱ and $\bar{\psi}$ are measure preserving. Note that if $\mathcal{A} \subset I^2$ is a rectangle, then $\sigma^{-1}(\mathcal{A})$ is not a 4-rectangle but there is no difficulty to approximate it as well as we wish by finite unions of 4-rectangles. Moreover, if $\mathcal{A} \subset \Delta^4$ is a rectangle, then $\gamma^{-1}(\mathcal{A})$ need not be. However, the lower and upper estimates for $\mu_4(\gamma^{-1}(\mathcal{A}))$ are done using 4-rectangles in Δ^4, see (21) and (23), respectively. Hence, we need also the result of 2º only for rectangles.

Corollary 7. *There exist positive constants C_1 and C_2 such that for all $f \in C(I^2)$*

$$C_1\int_I |f\circ\varphi(x)|\,dx \le \int_{I^2} |f(x)|\,dx \le C_2\int_I |f\circ\varphi(x)|\,dx. \tag{35}$$

Proof. If $(A_i)_{i=1}^n$ is a sequence of disjoint rectangles in I^2, we have for all sequences $(a_i)_{i=1}^n$ of scalars

$$\int_{I^2} \Big|\sum_{i=1}^n a_i\chi_i(x)\Big|\,dx=\sum_{i=1}^n |a_i|m_2(A_i),$$

$$\int_I \Big|\sum_{i=1}^n a_i\chi_i\circ\varphi(x)\Big|\,dx=\sum_{i=1}^n |a_i|m_1(\varphi^{-1}(A_i)),$$

where χ_i is the characteristic function of A_i. So, for simple functions of this form (35) follows from Proposition 6, and for continuous functions we get the statement by approximation.

We immediately get the following

Theorem 8. *The operator φ° can be extended to an isomorphism from $L_p(I^2)$ into $L_p(I)$, where $1 \leq p < \infty$.*

4. ON THE SPACES $C(\Omega) \cap L_p(\Omega)$

If $\Omega \subset \mathbb{R}^n$, $n \geq 1$, is an open set, we denote by $C(\Omega) \cap L_p(\Omega)$, $1 \leq p < \infty$, the Fréchet space of continuous, L_p-integrable functions from Ω into $\mathbb{K}$. The topology of this space is determined by the seminorms

$$p_0(f)=(\int_\Omega |f|^p)^{1/p},$$

$$p_k(f)=\sup_{x\in\Omega_k}|f(x)|, \quad k\in\mathbb{N} \tag{36}$$

where $(\Omega_k)_{k=1}^\infty$ is an increasing sequence of compact subsets of Ω, whose union is Ω. For more details on these spaces we refer to [BT].

The isomorphic classification of such spaces is an open problem. Probably the most interesting question in this area is, whether the spaces $C(\mathbb{R}) \cap L_p(\mathbb{R})$ and $C(\mathbb{R}^2) \cap L_p(\mathbb{R}^2)$ are isomorphic to each other. It is not difficult to see, using a natural imbedding, that $C(\mathbb{R}) \cap L_p(\mathbb{R})$ is isomorphic to a complemented subspace of $C(\mathbb{R}^2) \cap L_p(\mathbb{R}^2)$. (First, select a continuous cut-off function with compact support $\varphi \in C(\mathbb{R})$ such that $0 \leq \varphi \leq 1$ on $\mathbb{R}$ and $\varphi = 1$ for every $x \in [0,1]$. Put $E:=C(\mathbb{R}^2) \cap L_p(\mathbb{R}^2)$ and $F:= C(\mathbb{R}) \cap L_p(\mathbb{R})$ and define $T: F \rightarrow E$ by $Tf(x,y) = f(x)\varphi(y)$ for all $f\in F$, $x,y \in \mathbb{R}$, and $S: E \rightarrow F$ by $Sg(x):=\int_0^1 g(x,y)dy$ for all $g \in E$, $x \in \mathbb{R}$. It is a direct matter to check that T and S are continuous linear maps such that $S \circ T$ is the identity of F. So, $P:=T \circ S$ is a continuous projection on E whose image is isomorphic to F.) So, in view of the decomposition method of Pełczyński, the crucial problem is, whether $C(\mathbb{R}^2) \cap L_p(\mathbb{R}^2)$ is isomorphic to a complemented subspace of $C(\mathbb{R}) \cap L_p(\mathbb{R})$. Using the results in earlier sections we can prove that the space defined on $\mathbb{R}^2$ is isomorphic to a subspace of $C(\mathbb{R}) \cap L_p(\mathbb{R})$, but to prove the complementedness we would still need a "measure preserving" continuous surjection $\varphi: I \longrightarrow I^2$ and a projection P which is simultaneously bounded $C(I) \longrightarrow \varphi^\circ(C(I^2))$ and $L_p(I) \longrightarrow \varphi^\circ(L_p(I^2))$, and this result is not (yet?) available.

So, let us prove what we can do.

Lemma 9. *There exists an enumeration $(Q_n)_{n\in Z}$ of the family of closed squares $(Q_{n,m})_{n,m\in Z}$, where $Q_{n,m} = \{(x,y) \in \mathbb{R}^2 \mid n \le x \le n+1,\ m \le y \le m+1\}$, such that Q_n and Q_{n+1} have a common side for all $n \in Z$.*

There is no difficulty to make such an enumeration for example according to the following picture:

		-17	-16	15	16	17	18	
	-13	-14	-15	14	13	12	19	
	-12	-3	-2	1	2	11	20	
	-11	-4	-1	0	3	10	21	
	-10	-5	-6	5	4	9	22	
	-9	-8	-7	6	7	8	23	
				27	26	25	24	
				28	29			

Figure 1.

Proposition 10. *For all p, $1 \le p < \infty$, the space $C(\mathbb{R}^2) \cap L_p(\mathbb{R}^2)$ is isomorphic to a subspace of $C(\mathbb{R}) \cap L_p(\mathbb{R})$.*

Proof. Let φ be the map constructed in Section 2. It can be verified from the definition that $\varphi(0) = (0,0)$ and $\varphi(1) = (1,1)$. Let $\psi^{(1)}$ be a homeomorphism from I^2 onto itself such that $\psi^{(1)}(0,0) = (0,0)$ and $\psi^{(1)}(1,1) = (1,0)$. It is then clear that also the operator ψ°, where $\psi:=\psi^{(1)} \circ \varphi$, is an isometry from $C(I^2)$ onto a subspace of $C(I)$ and an isomorphism from $L_p(I^2)$ onto a subspace of $L_p(I)$. Let $(Q_n)_{n=-\infty}^{\infty}$ be the sequence of closed squares as in Lemma 9. We claim that it is possible to choose a sequence of continuous surjections $\varphi^{(n)}:[n,n+1] \longrightarrow Q_n$ such that

(i)) each $\varphi^{(n)}$ is of the form $\tau_n^{(2)} \circ r_n \circ \varphi \circ \tau_n$ or $\tau_n^{(2)} \circ r_n \circ \psi \circ \tau_n$, where τ_n is the translation from $[n,n+1]$ onto I, r_n is an isometry from I^2 onto itself, and $\tau_n^{(2)}$ is the translation from I^2 onto Q_n, and

(ii)) $\varphi^{(n-1)}(n) = \varphi^{(n)}(n)$ for all $n \in Z$.

Note that ii) means in particular that

$$\varphi^{(n-1)}(n) \in Q_n. \tag{37}$$

To prove this we first choose $\varphi^{(0)}$ of the form i) such that $\varphi^{(0)}(0) \in Q_{-1}$ and $\varphi^{(0)}(1) \in Q_1$. Assume that $n \geq 1$ and that $\varphi^{(k)}$ is constructed for $-n+1 \leq k \leq n-1$ such that i) holds for these $\varphi^{(k)}$ and such that $\varphi^{(k-1)}(k) = \varphi^{(k)}(k)$ for $-n+1 < k \leq n-1$ and $\varphi^{(n-1)}(n) \in Q_n$ and $\varphi^{(-n+1)}(-n+1) \in Q_{-n}$. By Lemma 9 the squares Q_n and Q_{n+1} have one common side S_n. So, it is possible to join one of the endpoints, say s_n, of S_n and $\varphi^{(n-1)}(n)$ by one side of Q_n or one diagonal of Q_n. We thus can find a map $\varphi^{(n)}$ which is of the form i) and satisfies $\varphi^{(n)}(n) = \varphi^{(n-1)}(n)$ and $\varphi^{(n)}(n+1) = s_n \in Q_{n+1}$. (If s_n and $\varphi^{(n-1)}(n)$ are the endpoints of a side of Q_n, we can take a map of the form $\tau_n^{(2)} \circ r_n \circ \psi \circ \tau_n$, and if s_n and $\varphi^{(n-1)}(n)$ are in the opposite corners of Q_n, i.e. they are the endpoints of a diagonal of Q_n, we can take a map of the form $\tau_n^{(2)} \circ r_n \circ \varphi \circ \tau_n$. In each case r_n is the combination of some rotation and reflection.) The map $\varphi^{(-n)}$ is defined analogously.

Defining

$$\phi(t)=\varphi^{(n)}(t) \text{ for } t\in[n,n+1],\ n\in Z, \tag{38}$$

we get a continuous surjection from $\mathbb{R}$ onto $\mathbb{R}^2$. We claim that $\phi^{\circ}:f\longrightarrow f\circ\phi$

is the desired isomorphism from $C(\mathbb{R}^2) \cap L_p(\mathbb{R}^2)$ onto a subspace of $C(\mathbb{R}) \cap L_p(\mathbb{R})$. Since ϕ is a continuous surjection and since $\phi^{-1}(K)$ is compact for all compact $K \subset \mathbb{R}^2$, $\phi°$ is an isomorphism from $C(\mathbb{R}^2)$ onto a subspace of $C(\mathbb{R})$. It is thus enough to prove the corresponding statement between L_p-spaces. For all $f \in C(\mathbb{R}^2) \cap L_p(\mathbb{R}^2)$

$$\int_{\mathbb{R}} |f\circ\phi|^p = \sum_{n\in Z} \int_n^{n+1} |f\circ\phi|^p = \sum_{n\in Z} \int_0^1 |f\circ\tau_n^{(2)}\circ r_n\circ\phi_n|^p \tag{39}$$

where, for all n, ϕ_n equals φ or ψ. According to the definition of ψ we can find positive constants c_1 and c_2 such that

$$c_1\int_{I^2} |f\circ\tau_n^{(2)}\circ r_n|^p \le \int_0^1 |f\circ\tau_n^{(2)}\circ r_n\circ\phi_n|^p \le c_2\int_{I^2} |f\circ\tau_n^{(2)}\circ r_n|^p \tag{40}$$

for all f and n. Since r_n is an isometry, we can further write

$$\sum_{n\in Z} \int_{I^2} |f\circ\tau_n^{(2)}\circ r_n|^p = \sum_{n\in Z} \int_{I^2} |f\circ\tau_n^{(2)}|^p = \sum_{n\in Z} \int_{Q_n} |f|^p = \int_{\mathbb{R}^2} |f|^p. \tag{41}$$

Combining (39), (40) and (41) we see that $\phi°$ is also an isomorphism from $L_p(\mathbb{R}^2)$ onto a subspace of $L_p(\mathbb{R})$.

NOTE ADDED IN PROOF

After the paper "A continuous surjection from the unit interval onto the unit square" and the reference [T] in it, "Averaging operators on spaces of continuous functions" were submitted, I realized that some of the results of [T] were already proved by B. Hoffmann in "An injective characterization of Peano spaces", Topol. and Appl. 11 (1980), 37-46. This is why [T] does not appear anywhere. In this note we give the missing details of the proofs of Lemma 1 and Theorem 2 of "A continuous surjection from the unit interval onto the unit square".

Proof of Lemma 1: The map γ: $\Delta^4 \longrightarrow \Delta^4$ is continuous, since the first m columns of $\gamma(A)$, $A \in \Delta^4$ depend only on the first $2m$ columns of A. We show that γ is a surjection having a continuous right inverse. Let $D=(D_1,D_2,...) \in \Delta^4$, where each $D_m \in \Delta_1^4$. We define the element $A \in \Delta^4$, using the same notation as in the definition of γ, as follows. Let $B_1 = C_1 = D_1$. For $m > 1$ we set $C_m = D_m$ and for B_m we choose a matrix which is not equal to anyone of the matrices D_m, D_m^-, $\bar{0}$ or $\bar{1}$. We set $A = (A_1,A_2,...)$, where $A_m = (B_m,C_m)$ for all m. It follows now directly from 1º in the definition of γ that $\gamma(A) = D$. Moreover, the element A depends continuously on D, since the first $2m$ columns of A depend only on the first m columns of D.

We denote by γ^{-1} the continuous right inverse of γ constructed above.

Finally, we show that φ is a surjection. Since $\bar{\sigma}$ and $\bar{\psi}$ are surjections, it is enough to prove that $\gamma \circ \eta \circ \varrho$ is surjective. Let $D \in \Delta^4$ be arbitrary and let $A = (A_1,A_2,...) = \gamma^{-1}(D)$. Each A_m, $m > 1$, contains both numbers 0 and 1. Hence, $\eta^{-1}(A)$ is of the form $(\varepsilon_m)_{m=1}^{\infty}$, where both numbers 0 and 1 occur as ε_m for arbitrarily large m. But for such sequences we have

$$\varrho(\sum_{m=1}^{\infty} \varepsilon_m 2^{-m})=(\varepsilon_m)_{m=1}^{\infty}$$

so that $\gamma \circ \eta \circ \varrho$ is surjective.

Proof of Theorem 2. The original proof of Milutin's lemma, which is also presented in [LT], Proposition 2.4.21, shows that the continuous map $\sigma \circ (\psi,\psi)$: $\Delta^2 \longrightarrow I$ admits a regular averaging operator. Hence, the same is also true for

$$\bar{\sigma} \circ \bar{\psi}: \Delta^4 \longrightarrow I^2.$$

Let P, $\|P\| = 1$, be a projection from $C(\Delta^4)$ onto $(\bar{\sigma} \circ \bar{\psi})^\circ (C(I^2))$.

Let γ: $\Delta^4 \longrightarrow \Delta^4$ be as above. The operator $f \mapsto \gamma^\circ P(\gamma^{-1})^\circ f$ is a contractive projection from $C(\Delta^4)$ onto $(\bar{\sigma} \circ \bar{\psi} \circ \gamma)^\circ (C(I^2))$. Hence, also $(\bar{\sigma} \circ \bar{\psi} \circ \gamma \circ \eta)^\circ (C(I^2)$ is a 1-complemented subspace of $C(\Delta)$.

Let $\varrho: I \longrightarrow \Delta$ be the discontinuous map defined in (3). By (4), ϱ° is an isometry from $C(\Delta)$ onto $D(I)$ so that there exists a contractive projection R from $D(I)$ onto $\varphi^\circ(C(I^2))$, where

$$\varphi := \tilde{\sigma} \circ \tilde{\psi} \circ \gamma \circ \eta \circ \varrho.$$

By Lemma 1, φ is continuous so that $\varphi^\circ(C(I^2))$ is a subspace of $C(I) \subset D(I)$. The restriction of R to $C(I)$ gives the desired projection.

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