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Hypersequents and Fuzzy Logic

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Abstract. Fuzzy logics based on t-norms and their residua have been investigated extensively from a semantic perspective but a unifying proof theory for these logics has, until recently, been lacking. In this paper we survey results of the authors and others which show that a suitable proof-theoretic framework for fuzzy logics is provided by *hypersequents*, a natural generalization of Gentzen-style sequents. In particular we present hypersequent calculi for the logic of left-continuous t-norms $\mathbf{MTL}$ and related logics, and for logics based on the three fundamental continuous t-norms, Gödel logic $\mathbf{G}$, Łukasiewicz logic **Ł**, and Product logic $\mathbf{\Pi}$.

Hypersecuentes y lógica borrosa

Resumen. Aunque se han investigado de forma extensiva las lógicas borrosas basadas en t-normas y sus residuos desde una perspectiva semántica, hasta ahora se carecía de una teoría unificadora de demostración para estas lógicas. En este trabajo se estudian los resultados de los autores y de otros investigadores que muestran que los hipersecuentes, una generalización natural de los secuentes al estilo de Gentzen, proporcionan un marco teórico adecuado para su demostración. En particular, se presentan los cálculos de los hipersecuentes para la lógica de t-normas continuas por la izquierda $\mathbf{MTL}$ y otras lógicas relacionadas, así como para las lógicas que se basan en las tres t-normas continuas fundamentales: la lógica de Gödel $\mathbf{G}$, la lógica de Łukasiewicz **Ł**y la lógica de Producto $\mathbf{\Pi}$.

1. Introduction

Fuzzy logics are many-valued logics that form a suitable basis for logical systems reasoning under uncertainty or vagueness. In recent years they have been identified in particular with logics where truth values are taken from the real unit interval $[0, 1]$, and conjunction and implication connectives are interpreted by t-norms[1] and their residua. Within this framework logics based both on fundamental t-norms, e.g. Gödel logic $\mathbf{G}$, Łukasiewicz logic **Ł** and Product logic $\mathbf{\Pi}$, and also basic classes of t-norms such as Monoidal t-norm logic $\mathbf{MTL}$ and Basic logic $\mathbf{BL}$, the logics of left-continuous and continuous t-norms respectively, are considered. Algebraic and axiomatic aspects of these logics have received a great deal of attention, e.g. in the monographs [22, 13, 20].

In this paper we present a *proof-theoretic* perspective on fuzzy logics. Analytic proof calculi for logics are not only an important theoretical tool, useful for understanding relationships between logics and proving metalogical properties like decidability, complexity, admissibility of rules and interpolation, but also the

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[1]T-norms are are widely used to combine uncertain or vague information in applications for approximate reasoning, knowledge representation and decision making. A detailed overview of results and applications is given in the monograph [25].

key to potential applications. Proof search algorithms can be used as the basis for "inference engines" in Artificial Intelligence for (fuzzy) knowledge representation, and reasoning in contexts of uncertainty and vagueness e.g. for tasks such as query-answering, consistency checking, abduction or revision. For such applications *analytic* proof methods are crucial, being not only good candidates for automated proof-search, but also, since they proceed by a stepwise decomposition of formulae, facilitating an *understanding* of proofs and allowing the extraction of explanatory information.

Analytic proof methods should be developed within a suitable *framework*, ideally one easy to understand and flexible enough to handle a wide range and diversity of logics. The best candidates for such a framework are *cut-free sequent calculi*, which deal with structures, called *sequents*, of the form:

$$A_1, \ldots, A_n \Rightarrow B_1, \ldots, B_m$$

usually understood intuitively as "A_1 and ... and A_n implies B_1 or ... or B_m", where the A_i and B_j are formulae of the logic in question. Sequent calculi have been provided for a wide range of logics, including classical, intuitionistic, modal and substructural logics, but have proved harder to come by for fuzzy logics, the main problem being that sequents do not cope well with *linearity* i.e. the fact that for truth values $x, y \in [0, 1]$, either $x \leq y$ or $y \leq x$. A solution to this problem was provided by Avron [2] and Pottinger [30] who independently introduced a generalization of sequents called *hypersequents*, which are just a multiset (or set or sequence) of ordinary sequents, written:

$$\Gamma_1 \Rightarrow \Delta_1 | \ldots | \Gamma_n \Rightarrow \Delta_n$$

where the $|$ symbol is interpreted as a meta-level "or", and intuitively a hypersequent is read as "one of the $\Gamma_i \Rightarrow \Delta_i$ holds".[2] In addition to rules operating on individual sequents, it is then possible to define rules which allows components to *interact*, e.g. the following "communication rule", where G represents an arbitary hypersequent:

$$\frac{G|\Gamma_1, \Pi_1 \Rightarrow \Delta_1, \Sigma_1 \quad G|\Gamma_2, \Pi_2 \Rightarrow \Delta_2, \Sigma_2}{G|\Gamma_1, \Gamma_2 \Rightarrow \Delta_1, \Delta_2 | \Pi_1, \Pi_2 \Rightarrow \Sigma_1, \Sigma_2}$$

In this paper we argue that the communication rule is a key ingredient in characterizing fuzzy logics proof-theoretically as *substructural logics* i.e. logics lacking certain structural rules (see e.g. [31] for a substantial treatment). A variant of this rule was first used in the context of fuzzy logic by Avron in [3] to provide a hypersequent calculus for Gödel logic $\mathbf{G}$, essentially by adding the communication rule to a hypersequent version of Gentzen's calculus for intuitionistic logic. Calculi were subsequently developed in the same framework for $\mathbf{MTL}$ by Baaz et al. [6] following work in e.g. [10] on the related **C** logics of Urquhart, and for $\mathbf{IMTL}$ and related t-norm based logics by Ciabattoni et al. in [9]. The problem of finding calculi for Łukasiewicz and Product logics was solved by the current authors in [29] and [28] (see also [26]) by adopting non-standard interpretations of hypersequents.

In this survey we present an overview of hypersequent calculi for all the major propositional [3] t-norm based fuzzy logics except $\mathbf{BL}$. We begin in Section 2 by introducing these logics algebraically and axiomatically, collecting completeness results from the literature, then in Section 3 we describe hypersequent calculi with a "standard" interpretation for fuzzy logics including $\mathbf{MTL}$, $\mathbf{IMTL}$ and $\mathbf{G}$. Finally, in Sections 4 and 5 we show that by defining alternative interpretations for hypersequents, calculi may also be obtained for **Ł**, $\mathbf{\Pi}$ and related logics.

[2] Avron [3] has also suggested interpreting a hypersequent as a *multiprocess* where hypersequent rules may be viewed e.g. as creating new processes, removing old ones, or exchanging information between processes. In the case of $\mathbf{G}$ this has led to an intepretation by Fermüller and Ciabattoni in terms of *dialogue games* [19].

[3] First-order calculi can be defined for several of these logics (for the case of Gödel logic see for example [8]) but this remains an area requiring further investigation.

2. T-Norm Based Fuzzy Logics

In this section we cover the main essentials[4] of the t-norm based approach, defining fuzzy logics in three ways: as logics based on (classes of) t-norms and their residua, as logics of residuated lattices, and as axiomatic systems. *Formulae* for all these logics are built inductively in the usual way from a set of propositional variables VAR with typical members p, q, r etc., binary connectives $\wedge$, $\vee$, $\odot$ and $\to$, and a constant $\perp$; we also define $\neg A =_{def} A \to \perp$, $\top =_{def} \neg\perp$ and $A \oplus B =_{def} \neg(\neg A \odot \neg B)$.

In the t-norm based approach we make two fundamental assumptions or (following Hájek [22]) "design choices": we take our set of truth values to be the real unit interval $[0, 1]$, and we consider *truth-functional* interpretations of connectives, that is, where the truth value of a compound formula is a function of the truth values of its subformulae. By then imposing some further intuitive restrictions to interpret *conjunction* i.e. commutativity, associativity and monotonicity, we obtain the following class of functions:

Definition 1 *A* t-norm *is a function* $* : [0, 1]^2 \to [0, 1]$ *such that for all* $x, y, z \in [0, 1]$*:*

1. $x * y = y * x$ *(Commutativity)*
2. $(x * y) * z = x * (y * z)$ *(Associativity)*
3. $x \leq y$ *implies* $x * z \leq y * z$ *(Monotonicity)*
4. $1 * x = x$ *(Identity)*

A natural (but not the only) way of obtaining a truth function $\Rightarrow$ for implication given a *left-continuous* t-norm $*$ is "residuation", which at an intuitive level equates to insisting that $x * (x \Rightarrow y)$ be no more true than y, and that subject to this restriction $x \Rightarrow y$ should be maximal.

Definition 2 *The* residuum *of a t-norm* $*$ *is an operation* $x \Rightarrow_* y =_{def} max\{z \mid x * z \leq y\}$.

Proposition 1 ([20]) *The residuum of a t-norm* $*$ *exists iff* $*$ *is* left-continuous.

The following are important examples of *continuous* t-norms and their residua:

	T-norm	Residuum
Łukasiewicz	$x *_Ł y = max(0, x + y - 1)$	$x \Rightarrow_Ł y = min(1, 1 - x + y)$
Gödel	$x *_G y = min(x, y)$	$x \Rightarrow_G y = \begin{cases} 1 & \text{if } x \leq y \\ y & \text{otherwise} \end{cases}$
Product	$x *_\Pi y = x.y$	$x \Rightarrow_\Pi y = \begin{cases} 1 & \text{if } x \leq y \\ y/x & \text{otherwise} \end{cases}$

In fact any continuous t-norm is locally isomorphic to one of these three (see e.g. [22] for details).

Residuation also provides suitable truth functions for *negation* as we can define $\neg x =_{def} x \Rightarrow_* 0$, giving Łukasiewicz negation $\neg x = 1 - x$ and Gödel (Product) negation $\neg 0 = 1$, $\neg x = 0$ for $x > 0$. Other truth functions considered important in fuzzy logic are *weak conjunction* $x \wedge y = min(x, y)$ and *weak disjunction* $x \vee y = max(x, y)$, which for a continuous t-norm $*$ with residuum $\Rightarrow_*$ may equivalently be defined as $x \wedge y =_{def} x * (x \Rightarrow_* y)$ and $x \vee y =_{def} ((x \Rightarrow_* y) \Rightarrow_* y) \wedge ((y \Rightarrow_* x) \Rightarrow_* x)$.

We now place such interpretations in a more general algebraic setting.

Definition 3 *An* ML-algebra *is a* bounded integral commutative residuated lattice *i.e. an algebra* $\langle L, \wedge, \vee, \odot, \to, \perp, \top\rangle$ *with universe* L*, binary operations* $\wedge$, $\vee$, $\odot$ *and* $\to$*, and constants* $\perp$ *and* $\top$*, such that:*

1. $\langle L, \wedge, \vee\rangle$ *is a bounded lattice with order* $\leq$*, top element* $\top$ *and bottom element* $\perp$.
2. $\langle L, \odot, \top\rangle$ *is a commutative semigroup with unit element* $\top$.
3. $\odot$ *and* $\to$ *form an adjoint pair i.e.* $z \leq x \to y$ *iff* $x \odot z \leq y$ *for all* $x, y, z \in L$.

We also define: $\neg x =_{def} x \to \perp$ *and* $x \oplus y =_{def} \neg(\neg x \odot \neg y)$.

[4] For greater detail we refer the reader to [22, 20].

To define validity in ML-algebras we exploit our use of the same symbol for an algebraic operation and the corresponding logical connective.

Definition 4 *A* valuation *for an ML-algebra $\mathcal{A}$ is a function $v : VAR \to L$ extended to formulae by:*

$$v(\#(A_1, \ldots, A_m)) = \#(v(A_1), \ldots, v(A_m)) \text{ where } \# \in \{\wedge, \vee, \odot, \to, \bot, \top\} \text{ and } m \text{ is the arity of } \#.$$

A formula A is valid *in $\mathcal{A}$ iff $v(A) = \top$ for all valuations v for $\mathcal{A}$.*

Refinements of ML-algebras suitable for fuzzy logics are defined as follows:

Definition 5 *An ML-algebra $\langle L, \wedge, \vee, \odot, \to, \bot, \top \rangle$ is:*

- dualizing *iff $\neg\neg x = x$ for all $x \in L$.*
- idempotent *iff $x \odot x = x$ for all $x \in L$.*
- prelinear *iff $\top = (x \to y) \vee (y \to x)$ for all $x, y \in L$.*
- divisible *iff $x \wedge y = x \odot (x \to y)$ for all $x, y \in L$.*
- weakly contracting *iff $x \wedge \neg x = \bot$ for all $x \in L$.*
- weakly cancellative *iff $\neg\neg x \leq (x \to (x \odot y)) \to y$ for all $x, y \in L$.*

Name	*Class of Residuated Lattices*
AMALL-algebra	*Dualizing ML-algebra*
MTL-algebra	*Prelinear ML-algebra*
IMTL-algebra	*Dualizing MTL-algebra*
SMTL-algebra	*Weakly contracting MTL-algebra*
BL-algebra	*Divisible MTL-algebra*
Ł-algebra	*Dualizing BL-algebra*
G-algebra	*Idempotent BL-algebra*
Π-algebra	*Weakly contracting weakly cancellative BL-algebra*

We write $\models_L A$ iff A is valid in all L-algebras.

In fact we have already encountered some notable members of these classes.

Proposition 2 ([16]) *Let $*$ be a* left-continuous *t-norm with residuum $\Rightarrow_*$, then:*

$$\mathcal{A} = \langle [0, 1], min, max, *, \Rightarrow_*, 0, 1 \rangle$$

is an MTL-algebra, called a standard MTL-algebra*; if $\mathcal{A}$ is an IMTL-algebra (SMTL-algebra) then $\mathcal{A}$ is called a* standard IMTL-algebra (SMTL-algebra).

Proposition 3 ([22]) *Let $*$ be a* continuous *t-norm with residuum $\Rightarrow_*$, then:*

$$\mathcal{A} = \langle [0, 1], min, max, *, \Rightarrow_*, 0, 1 \rangle$$

is a BL-algebra, called a standard BL-algebra*; if $*$ is the Łukasiewicz, Gödel or Product t-norm then $\mathcal{A}$ is an Ł-algebra, G-algebra or Π-algebra respectively, called the* standard Ł-algebra, G-algebra *or* Π-algebra.

Remarkably, these standard algebras turn out to be *characteristic* for their respective classes.

Theorem 1 ([24, 15, 14, 22]) *For $\mathbf{L} \in \{\mathbf{MTL}, \mathbf{IMTL}, \mathbf{SMTL}, \mathbf{BL}, \mathbf{Ł}, \mathbf{G}, \mathbf{\Pi}\}$ a formula A is valid in all L-algebras iff A is valid in all standard L-algebras.*

Remark 1 *Of course there can also be said to exist standard ML-algebras and AMALL-algebras (in fact just the standard MTL-algebras and IMTL-algebras respectively); the point being that these algebras are not* characteristic *for ML-algebras and AMALL-algebras.*

We now define axiomatizations for t-norm based fuzzy logics as extensions of Höhle's *Monoidal logic* $\mathbf{ML}$ [23] and the *Affine multiplicative additive fragment of linear logic* $\mathbf{AMALL}$ (see e.g. [31] for details), the key axiom for fuzziness being the prelinearity axiom (PRL):

Definition 6 $\mathbf{HML}$ *consists of the following axioms and rules:*

$(A1)$	$\bot \to A$	$(A8)$	$((C \to A) \wedge (C \to B)) \to (C \to (A \wedge B))$
$(A2)$	$A \to (B \to A)$	$(A9)$	$A \to (A \vee B)$
$(A3)$	$(A \to (B \to C)) \to (B \to (A \to C))$	$(A10)$	$B \to (A \vee B)$
$(A4)$	$(A \to B) \to ((C \to A) \to (C \to B))$	$(A11)$	$(A \to B) \to ((C \to B) \to ((A \vee C) \to B))$
$(A5)$	$(A \wedge B) \to A$	$(A12)$	$A \to (B \to (A \odot B))$
$(A6)$	$(A \wedge B) \to B$	$(A13)$	$(A \to (B \to C)) \to ((A \odot B) \to C)$
$(A7)$	$A \to (B \to (A \wedge B))$		

$$(mp) \quad \frac{A \to B \quad A}{B}$$

Axiomatizations for fuzzy logics are defined as follows:

(INV)	$\neg\neg A \to A$	(PRL)	$(A \to B) \vee (B \to A)$
(G)	$A \to (A \odot A)$	(DIV)	$(A \odot (A \to B)) \to (B \odot (B \to A))$
(S)	$\neg(A \wedge \neg A)$	(Π)	$\neg\neg A \to ((A \to (A \odot B)) \to B)$

$\mathbf{HAMALL}$	*is*	$\mathbf{HML}$ *plus* (INV)	$\mathbf{HBL}$	*is*	$\mathbf{HMTL}$ *plus* (DIV)
$\mathbf{HMTL}$	*is*	$\mathbf{HML}$ *plus* (PRL)	$\mathbf{HG}$	*is*	$\mathbf{HBL}$ *plus* (G)
$\mathbf{HIMTL}$	*is*	$\mathbf{HMTL}$ *plus* (INV)	**HŁ**	*is*	$\mathbf{HBL}$ *plus* (INV)
$\mathbf{HSMTL}$	*is*	$\mathbf{HMTL}$ *plus* (S)	$\mathbf{H\Pi}$	*is*	$\mathbf{HBL}$ *plus* (S) *and* (Π)

Remark 2 *Many of these axiomatizations can be simplified, in some cases dramatically so; e.g. for* $\mathbf{BL}$ *we can remove (A5)-(A11) and use a language without* $\wedge$ *and* $\vee$*, defining* $A \wedge B =_{def} A \odot (A \to B)$ *and* $A \vee B =_{def} ((A \to B) \to B) \wedge ((B \to A) \to A)$*.*

It is straightforward to show that these calculi match the appropriate algebras, and therefore also standard algebras (where defined).

Theorem 2 ([23, 31, 16, 22]) *For* $\mathbf{L} \in \{\mathbf{ML}, \mathbf{AMALL}, \mathbf{MTL}, \mathbf{IMTL}, \mathbf{SMTL}, \mathbf{BL}$, **Ł**, $\mathbf{G}, \mathbf{\Pi}\}$ *a formula* A *is derivable in* $\mathbf{HL}$ *iff* A *is valid in all L-algebras.*

Corollary 1 *For* $\mathbf{L} \in \{\mathbf{MTL}, \mathbf{IMTL}, \mathbf{SMTL}, \mathbf{BL}$, **Ł**, $\mathbf{G}, \mathbf{\Pi}\}$ *a formula* A *is derivable in* $\mathbf{HL}$ *iff* A *is valid in all standard L-algebras.*

The weakest t-norm based fuzzy logic is *Monoidal t-norm logic* $\mathbf{MTL}$ introduced by Esteva and Godo in [16] and confirmed to be the logic of left-continuous t-norms in [24]. Also defined in [16] are $\mathbf{IMTL}$ and $\mathbf{SMTL}$, proved in [15] to be the logics of left-continuous t-norms with an involutive negation (i.e. where $\neg\neg x = x$) and weak contraction (i.e. where $x \wedge \neg x = \bot$) respectively. $\mathbf{BL}$ is Hájek's *Basic fuzzy logic*, proved in [14] to be the logic of continuous t-norms, which has as extensions the famous many-valued logics Łukasiewicz logic **Ł** and Gödel logic $\mathbf{G}$, plus the more recently introduced Product logic $\mathbf{\Pi}$, all these logics being studied extensively in the monograph [22].

We end this section with a diagrammatic representation of the relationships between these and some other well-known logics.[5]

[5] Note that a more complete diagram of the hierarchy of t-norm based logics is presented in [17].

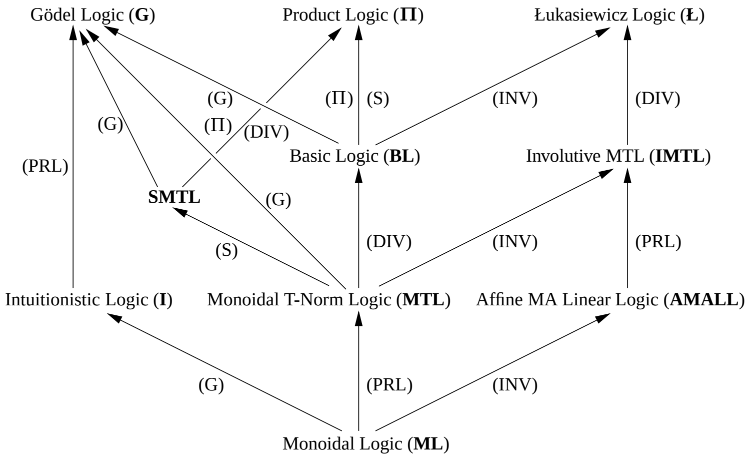

Figure 1. Relationships between fuzzy logics

3. Hypersequent Calculi

Hypersequents were introduced independently by Avron in [2] and Pottinger in [30] as a natural and easily understood generalization of Gentzen sequents, defined as follows:

Definition 7 *A* hypersequent *is a multiset of the form:*

$$\Gamma_1 \Rightarrow \Delta_1 | \ldots | \Gamma_n \Rightarrow \Delta_n$$

where for $i = 1, \ldots, n$, $\Gamma_i \Rightarrow \Delta_i$ *is an ordered pair of multisets of formulae (i.e. a sequent) called a* component *of the hypersequent. If* Δ_i *contains at most one formula for* $i = 1, \ldots, n$ *then the hypersequent is said to be* single-conclusion*, otherwise* multiple-conclusion*.*

The symbol $|$ may be read as a kind of *meta-level disjunction*, inspiring the following interpretation:

Definition 8 *The* standard interpretation *of a hypersequent* $G = \Gamma_1 \Rightarrow \Delta_1 | \ldots | \Gamma_n \Rightarrow \Delta_n$ *is the formula:*

$$\phi^G =_{def} (\odot\Gamma_1 \to \oplus\Delta_1) \vee \ldots \vee (\odot\Gamma_n \to \oplus\Delta_n)$$

where $*\{A_1, \ldots, A_m\} = A_1 * \ldots * A_m$ *for* $* \in \{\odot, \oplus\}$, $\odot\emptyset = \top$, $\oplus\emptyset = \bot$, *and* $\models G$ *iff* $\models \phi^G$.

Like sequent calculi, hypersequent calculi consist of axioms, logical rules and structural rules, the latter being divided however into two categories. *Internal* rules deal with formulae within components as in sequent calculi, and may include a distinguished "cut" rule corresponding to the transitivity of deduction. *External* rules manipulate whole components; for example the external weakening and contraction rules (EW) and (EC) add and remove components as follows:

$$(EW) \quad \frac{G}{G|\Gamma \Rightarrow \Delta} \qquad (EC) \quad \frac{G|\Gamma \Rightarrow \Delta|\Gamma \Rightarrow \Delta}{G|\Gamma \Rightarrow \Delta}$$

We begin here by introducing a hypersequent calculus for $\mathbf{ML}$, using G, G_1, G_2 etc. as metavariables to denote (possibly empty) hypersequents called *side-hypersequents*, and C, C_1, C_2 etc. to denote multisets with at most one element. Note also that since we have defined hypersequents as multisets of components, and components as pairs of multisets of formulae, multiplicity but not order are important in these rules.

Definition 9 $\mathbf{GML}$ *has the following axioms and rules:*

Axioms

$$(ID)\quad A \Rightarrow A \qquad\qquad (\bot)\quad \bot \Rightarrow \qquad\qquad (\top)\quad \Rightarrow \top$$

Structural Rules

$$(EW) \text{ and } (EC)$$

$$(WL)\quad \frac{G|\Gamma \Rightarrow C}{G|\Gamma, A \Rightarrow C} \qquad\qquad (WR)\quad \frac{G|\Gamma \Rightarrow}{G|\Gamma \Rightarrow C}$$

Logical Rules

$$(\rightarrow, l)\quad \frac{G|\Gamma_1 \Rightarrow A \quad G|\Gamma_2, B \Rightarrow C}{G|\Gamma_1, \Gamma_2, A \rightarrow B \Rightarrow C} \qquad (\rightarrow, r)\quad \frac{G|\Gamma, A \Rightarrow B}{G|\Gamma \Rightarrow A \rightarrow B}$$

$$(\odot, l)\quad \frac{G|\Gamma, A, B \Rightarrow C}{G|\Gamma, A \odot B \Rightarrow C} \qquad (\odot, r)\quad \frac{G|\Gamma_1 \Rightarrow A \quad G|\Gamma_2 \Rightarrow B}{G|\Gamma_1, \Gamma_2 \Rightarrow A \odot B}$$

$$(\wedge_i, l)_{i=1,2}\quad \frac{G|\Gamma, A_i \Rightarrow C}{G|\Gamma, A_1 \wedge A_2 \Rightarrow C} \qquad (\wedge, r)\quad \frac{G|\Gamma \Rightarrow A \quad G|\Gamma \Rightarrow B}{G|\Gamma \Rightarrow A \wedge B}$$

$$(\vee, l)\quad \frac{G|\Gamma, A \Rightarrow C \quad G|\Gamma, B \Rightarrow C}{G|\Gamma, A \vee B \Rightarrow C} \qquad (\vee_i, r)_{i=1,2}\quad \frac{G|\Gamma \Rightarrow A_i}{G|\Gamma \Rightarrow A_1 \vee A_2}$$

$$(\neg, l)\quad \frac{G|\Gamma \Rightarrow A}{G|\Gamma, \neg A \Rightarrow} \qquad (\neg, r)\quad \frac{G|\Gamma, A \Rightarrow}{G|\Gamma \Rightarrow \neg A}$$

Cut Rule

$$(CUT)\quad \frac{G|\Gamma_1, A \Rightarrow C \quad G|\Gamma_2 \Rightarrow A}{G|\Gamma_1, \Gamma_2 \Rightarrow C}$$

In this calculus the use of hypersequents is in fact unnecessary; (EW) and (EC) only apply to one component at a time and hence do not increase the expressive power of hypersequent calculi over sequent calculi. To prove the key prelinearity axiom (PRL) however, we require a rule permitting *interactions* between components; the most generally useful being the following (single-conclusion) "communication" rule:

$$(COM_I)\quad \frac{G|\Gamma_1, \Pi_1 \Rightarrow A \quad G|\Gamma_2, \Pi_2 \Rightarrow B}{G|\Gamma_1, \Gamma_2 \Rightarrow A|\Pi_1, \Pi_2 \Rightarrow B}$$

A hypersequent calculus using (COM_I) has been defined for $\mathbf{MTL}$ by Baaz et al. in [6] (see also [10] for connections with calculi for Urquhart's **C** logics).

Definition 10 $\mathbf{GMTL}$ *consists of the same rules and axioms as* $\mathbf{GML}$ *together with* (COM_I).

Example 1 (COM_I) *allows us to prove* (PRL) *as follows:*

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{A \Rightarrow A \quad B \Rightarrow B}{A \Rightarrow B|B \Rightarrow A}\,(COM_I)}{A \Rightarrow B| \Rightarrow B \rightarrow A}\,(\rightarrow, r)}{\Rightarrow A \rightarrow B| \Rightarrow B \rightarrow A}\,(\rightarrow, r)}{\Rightarrow A \rightarrow B| \Rightarrow (A \rightarrow B) \vee (B \rightarrow A)}\,(\vee, r)}{\Rightarrow (A \rightarrow B) \vee (B \rightarrow A)| \Rightarrow (A \rightarrow B) \vee (B \rightarrow A)}\,(\vee, r)}{\Rightarrow (A \rightarrow B) \vee (B \rightarrow A)}\,(EC)$$

A calculus for $\mathbf{AMALL}$ is obtained as a multiple-conclusion version of $\mathbf{GML}$, noting once again that the extra expressive power of hypersequents is unnecessary for this logic.

Definition 11 $\mathbf{GAMALL}$ *has the following axioms and rules:*

Axioms

$$(ID)\quad A \Rightarrow A \qquad\qquad (\bot)\quad \bot \Rightarrow \qquad\qquad (\top)\quad \Rightarrow \top$$

Structural Rules

$$(EW) \text{ and } (EC)$$

$$(WL)\ \frac{G|\Gamma \Rightarrow \Delta}{G|\Gamma, A \Rightarrow \Delta} \qquad\qquad (WR)\ \frac{G|\Gamma \Rightarrow \Delta}{G|\Gamma \Rightarrow A, \Delta}$$

Logical Rules

$$(\rightarrow, l)\ \frac{G|\Gamma_1 \Rightarrow A, \Delta_1 \quad G|\Gamma_2, B \Rightarrow \Delta_2}{G|\Gamma_1, \Gamma_2, A \rightarrow B \Rightarrow \Delta_1, \Delta_2} \qquad (\rightarrow, r)\ \frac{G|\Gamma, A \Rightarrow B, \Delta}{G|\Gamma \Rightarrow A \rightarrow B, \Delta}$$

$$(\odot, l)\ \frac{G|\Gamma, A, B \Rightarrow \Delta}{G|\Gamma, A \odot B \Rightarrow \Delta} \qquad (\odot, r)\ \frac{G|\Gamma_1 \Rightarrow A, \Delta_1 \quad G|\Gamma_2 \Rightarrow B, \Delta_2}{G|\Gamma_1, \Gamma_2 \Rightarrow A \odot B, \Delta_1, \Delta_2}$$

$$(\oplus, l)\ \frac{G|\Gamma_1, A \Rightarrow \Delta_1 \quad G|\Gamma_2, B \Rightarrow \Delta_2}{G|\Gamma_1, \Gamma_2, A \oplus B \Rightarrow \Delta_1, \Delta_2} \qquad (\oplus, r)\ \frac{G|\Gamma \Rightarrow A, B, \Delta}{G|\Gamma \Rightarrow A \oplus B, \Delta}$$

$$(\wedge_i, l)_{i=1,2}\ \frac{G|\Gamma, A_i \Rightarrow \Delta}{G|\Gamma, A_1 \wedge A_2 \Rightarrow \Delta} \qquad (\wedge, r)\ \frac{G|\Gamma \Rightarrow A, \Delta \quad G|\Gamma \Rightarrow B, \Delta}{G|\Gamma \Rightarrow A \wedge B, \Delta}$$

$$(\vee, l)\ \frac{G|\Gamma, A \Rightarrow \Delta \quad G|\Gamma, B \Rightarrow \Delta}{G|\Gamma, A \vee B \Rightarrow \Delta} \qquad (\vee_i, r)_{i=1,2}\ \frac{G|\Gamma \Rightarrow A_i, \Delta}{G|\Gamma \Rightarrow A_1 \vee A_2, \Delta}$$

$$(\neg, l)\ \frac{G|\Gamma \Rightarrow A, \Delta}{G|\Gamma, \neg A \Rightarrow \Delta} \qquad (\neg, r)\ \frac{G|\Gamma, A \Rightarrow \Delta}{G|\Gamma \Rightarrow \neg A, \Delta}$$

Cut Rule

$$(CUT)\ \frac{G|\Gamma_1, A \Rightarrow \Delta_1 \quad G|\Gamma_2 \Rightarrow A, \Delta_2}{G|\Gamma_1, \Gamma_2 \Rightarrow \Delta_1, \Delta_2}$$

A hypersequent calculus for $\mathbf{IMTL}$ has been obtained by Ciabattoni et al. [9] by adding a multiple-conclusion communication rule to $\mathbf{GAMALL}$.

Definition 12 $\mathbf{GIMTL}$ *has the same rules and axioms as* $\mathbf{GAMALL}$ *and also:*

$$(COM_C)\ \frac{G|\Gamma_1, \Pi_1 \Rightarrow \Delta_1, \Sigma_1 \quad G|\Gamma_2, \Pi_2 \Rightarrow \Delta_2, \Sigma_2}{G|\Gamma_1, \Gamma_2 \Rightarrow \Delta_1, \Delta_2|\Pi_1, \Pi_2 \Rightarrow \Sigma_1, \Sigma_2}$$

An elegant hypersequent calculus for $\mathbf{G}$ (the first for a fuzzy logic) was defined by Avron in [3].

Definition 13 $\mathbf{GG}$ *has the same rules and axioms as* $\mathbf{GMTL}$ *and also:*

$$(CL)\ \frac{G|\Gamma, A, A \Rightarrow C}{G|\Gamma, A \Rightarrow C}$$

Remark 3 $\mathbf{GG}$ *can be viewed as a calculus for intuitionistic logic extended by the communication rule.*

Hypersequent calculi have also been provided by Ciabattoni and Ferrari [11] for *finite-valued* Gödel logics $\mathbf{G_n}$ i.e. logics with the same connectives as $\mathbf{G}$ but truth value set $[0, \frac{1}{n-1}, \ldots, \frac{n-2}{n-1}, 1]$, by adding the following rule to $\mathbf{GG}$:

$$(G_n)\ \frac{G|\Gamma_1, \Gamma_2 \Rightarrow A_1 \quad G|\Gamma_2, \Gamma_3 \Rightarrow A_2 \ \ldots \ G|\Gamma_{n-1}, \Gamma_n \Rightarrow A_{n-1}}{G|\Gamma_1 \Rightarrow A_1|\ldots|\Gamma_{n-1} \Rightarrow A_{n-1}|\Gamma_n \Rightarrow}$$

An in-depth survey of hypersequent calculi for Gödel logics including also first-order and propositional quantifier versions, is provided in [5].

The flexibility of the hypersequent formulation means that calculi can be defined for various other fuzzy logics defined in the literature, for example:[6]

[6] Hypersequent calculi for logics obtained by adding bounded contraction to $\mathbf{MTL}$ and $\mathbf{IMTL}$ are also defined in [9].

Definition 14 $\mathbf{GSMTL}$ *has the same rules and axioms as* $\mathbf{GMTL}$ *and also:*

$$(Q) \quad \frac{G|\Gamma_1, \Gamma_2 \Rightarrow}{G|\Gamma_1 \Rightarrow |\Gamma_2 \Rightarrow}$$

We now collect soundness and completeness results for these hypersequent calculi.

Theorem 3 *For* $\mathbf{L} \in \{\mathbf{ML}, \mathbf{AMALL}, \mathbf{MTL}, \mathbf{IMTL}, \mathbf{SMTL}, \mathbf{G}\}$*,* G *is derivable in* $\mathbf{GL}$ *iff* $\models_L G$*.*

The key result here is *cut-elimination*, proved for $\mathbf{ML}$ and $\mathbf{AMALL}$ in e.g. [31], for $\mathbf{MTL}$ by Baaz et al. [6], $\mathbf{IMTL}$ by Ciabattoni et al. [9], $\mathbf{SMTL}$ by Ciabattoni (unpublished proof) and $\mathbf{G}$ by Avron [3].

Theorem 4 ([31, 6, 9, 3]) *For* $\mathbf{L} \in \{\mathbf{ML}, \mathbf{AMALL}, \mathbf{MTL}, \mathbf{IMTL}, \mathbf{SMTL}, \mathbf{G}\}$*,* (CUT) *can be eliminated from* $\mathbf{GL}$*.*

An important by-product of cut-elimination is that all these calculi (without (CUT)) enjoy the *subformula property*, i.e. all formulae occurring in a cut-free proof are subformulae of the hypersequent to be proved, and are therefore *analytic*.

4. Łukasiewicz Logic

In this section we define a very natural hypersequent calculus for **Ł**, the catch being that to do so we have to abandon the standard interpretation of hypersequents. We begin by considering an alternative standard algebra for **Ł**, obtained by knocking down the set of truth values from $[0, 1]$ to $[-1, 0]$.

Proposition 4 ([29]) *Let* $[-1, 0]_{Ł} =_{def} \langle[-1, 0], min, max, \odot, \to, -1, 0\rangle$ *where* $x \odot y =_{def} max(-1, x + y)$ *and* $x \to y =_{def} min(0, y - x)$*.* A *is valid in* $[-1, 0]_{Ł}$ *iff* A *is valid in* $[0, 1]_{Ł}$*.*

We use this algebra to give a non-standard reading of hypersequents as follows:

Definition 15 $\models^*_{Ł} \Gamma_1 \Rightarrow \Delta_1 | \ldots | \Gamma_n \Rightarrow \Delta_n$ *iff for all* $[-1, 0]_{Ł}$ *valuations* v*:*

$$\sum v(\Gamma_i) \leq \sum v(\Delta_i) \text{ for some } i,\ 1 \leq i \leq n, \text{ where } v(\Gamma) = \{v(A) : A \in \Gamma\}.$$

For *formulae* this interpretation gives us the usual notion of validity for **Ł**, i.e. we have that a formula A is valid in $[0, 1]_{Ł}$ iff $\models^*_{Ł} \Rightarrow A$. Alternatively, we get the same reading by using the standard interpretation of hypersequents for Meyer and Slaney's *Abelian logic* $\mathbf{A}$, a logic with a characteristic model in the reals, and embedding **Ł** into $\mathbf{A}$ (see [29] for details).

We now present a *cut-free* hypersequent calculus for **Ł** based on this interpretation, taking $\to$ and $\bot$ as primitive connectives and defining $\neg A =_{def} A \to \bot$, $A \odot B =_{def} \neg(A \to \neg B)$, $A \wedge B =_{def} A \odot (A \to B)$ and $A \vee B =_{def} (A \to B) \to B$.

Definition 16 **GŁ** *has the following axioms and rules:*

Axioms

$$(ID) \quad A \Rightarrow A \qquad\qquad (\Lambda) \quad \Rightarrow \qquad\qquad (\bot) \quad \bot \Rightarrow A$$

Structural rules

$$(EW), (EC) \text{ and } (WL)$$

$$(S) \quad \frac{G|\Gamma_1, \Gamma_2 \Rightarrow \Delta_1, \Delta_2}{G|\Gamma_1 \Rightarrow \Delta_1 | \Gamma_2 \Rightarrow \Delta_2} \qquad (M) \quad \frac{G|\Gamma_1 \Rightarrow \Delta_1 \quad G|\Gamma_2 \Rightarrow \Delta_2}{G|\Gamma_1, \Gamma_2 \Rightarrow \Delta_1, \Delta_2}$$

Logical Rules

$$(\to, l) \quad \frac{G|\Gamma, B \Rightarrow A, \Delta}{G|\Gamma, A \to B \Rightarrow \Delta} \qquad (\to, r) \quad \frac{G|\Gamma \Rightarrow \Delta \quad G|\Gamma, A \Rightarrow B, \Delta}{G|\Gamma \Rightarrow A \to B, \Delta}$$

Although several of the axioms and structural rules of **GŁ** are familiar from previous calculi (note that the standard rules for $\wedge$ and $\vee$ are also derivable), there are certain non-standard aspects to this calculus. In particular weakening is only allowed on the left, and the axiom $(\bot)$ is only applicable when there is exactly one formula on the right. Also the logical rules $(\to, l)$ and $(\to, r)$ run contrary to expectation in that the former has one premise and the latter two, the exact opposite of the standard rules. However notice that of the two "new" structural rules, (S) is just a simplification of the communication rule (COM_C) using the axiom (Λ), while (M) is a "weaker" version of the weakening rules (WL) and (WR).

Example 2 *We illustrate* **GŁ** *with the following proof:*

$$
\dfrac{\Rightarrow \qquad \dfrac{\dfrac{\Rightarrow}{(A\to B)\to B\Rightarrow}\,(WL) \qquad \dfrac{\dfrac{\dfrac{\dfrac{B\Rightarrow B \quad A\Rightarrow A}{B,A\Rightarrow A,B}\,(M)}{B,B\to A\Rightarrow A}\,(\to,l) \qquad \dfrac{\dfrac{B\Rightarrow B \quad A\Rightarrow A}{B,A\Rightarrow A,B}\,(M)}{B,B\to A,A\Rightarrow A,B}\,(WL)}{B,B\to A\Rightarrow A,A\to B}\,(\to,r)}{(A\to B)\to B,B\to A\Rightarrow A}\,(\to,l)}{(A\to B)\to B\Rightarrow (B\to A)\to A}\,(\to,r)}{\Rightarrow ((A\to B)\to B)\to((B\to A)\to A)}\,(\to,r)
$$

Soundness and completeness results for **GŁ** are proved in [29] by relating **GŁ** to a hypersequent calculus for Abelian logic, and then proving the soundness and completeness for this latter calculus semantically.

Theorem 5 ([29]) *G is derivable in* **GŁ** *iff* $\models^*_{Ł} G$.

Alternatively, we can use the completeness of an axiomatization for **Ł** and prove cut-elimination for **GŁ** extended with one of the following (inter-derivable) rules:

$$
(CUT)\quad \dfrac{G|\Gamma_1, A\Rightarrow \Delta_1 \quad G|\Gamma_2\Rightarrow A,\Delta_2}{G|\Gamma_1,\Gamma_2\Rightarrow\Delta_1,\Delta_2} \qquad\qquad (GCUT)\quad \dfrac{G|\Gamma, A\Rightarrow A,\Delta}{G|\Gamma\Rightarrow\Delta}
$$

Theorem 6 ([12]) *Cut-elimination holds for* **GŁ** + (CUT) *and* **GŁ** + $(GCUT)$.

We end this section by remarking that a hypersequent calculus for the *bounded* Łukasiewicz logic $\mathbf{ŁB_n}$ which characterizes the intersection of k-valued Łukasiewicz logics for $k \leq n$, has been obtained in [12] by adding the following rule to **GŁ**:

$$
(nC)\quad \dfrac{G|\Gamma,\overbrace{\Pi,\ldots,\Pi}^{n-1},\bot\Rightarrow\overbrace{\Sigma,\ldots,\Sigma}^{n-1},\Delta}{G|\Pi\Rightarrow\Sigma|\Gamma\Rightarrow\Delta}
$$

We also conjecture that by adding further rules we can obtain calculi for *finite-valued* Łukasiewicz logics.

5. Product Logic

To obtain a hypersequent calculus for $\mathbf{\Pi}$ we again use a non-standard reading of hypersequents, although in this case (unlike for **Ł**) we able to give an interpretation as a formula of the logic.

Definition 17 *Given a hypersequent* $G = \Gamma_1 \Rightarrow \Delta_1|\ldots|\Gamma_n \Rightarrow \Delta_n$ *we define:*

$$
\phi^G =_{def} (\odot\Gamma_1 \to \odot\Delta_1)\vee\ldots\vee(\odot\Gamma_n\to\odot\Delta_n)
$$

where $\odot\{A_1,\ldots,A_m\} = A_1\odot\ldots\odot A_m$, $\odot\emptyset = \top$, *and we write* $\models_\Pi G$ *iff* $\models_\Pi \phi^G$.

A hypersequent calculus based on this interpretation has been defined in [28], taking $\to$, $\odot$, $\neg$ and $\bot$ as primitive, and defining $A\wedge B =_{def} A\odot(A\to B)$ and $A\vee B =_{def} ((A\to B)\to B)\wedge((B\to A)\to A)$.

Definition 18 $\mathbf{G\Pi}$ *consists of the following axioms and rules:*

Axioms

$$(ID)\quad A \Rightarrow A \qquad\qquad (\Lambda)\quad \Rightarrow \qquad\qquad (\bot)\quad \Gamma, \bot \Rightarrow \Delta$$

Structural rules

$$(EW), (EC), (S), (M) \text{ and } (WL)$$

Logical rules

$$(\to, l)\quad \frac{G|\Gamma, B \Rightarrow A, \Delta \quad G|\Gamma, \neg A \Rightarrow \Delta}{G|\Gamma, A \to B \Rightarrow \Delta} \qquad (\to, r)\quad \frac{G|\Gamma \Rightarrow \Delta \quad G|\Gamma, A \Rightarrow B, \Delta}{G|\Gamma \Rightarrow A \to B, \Delta}$$

$$(\odot, l)\quad \frac{G|\Gamma, A, B \Rightarrow \Delta}{G|\Gamma, A \odot B \Rightarrow \Delta} \qquad (\odot, r)\quad \frac{G|\Gamma \Rightarrow A, B, \Delta}{G|\Gamma \Rightarrow A \odot B, \Delta}$$

$$(\neg, l)\quad \frac{G|\Gamma \Rightarrow A}{G|\Gamma, \neg A \Rightarrow \Delta} \qquad (\neg, r)\quad \frac{G|\Gamma \Rightarrow \Delta \quad G|\Gamma, A \Rightarrow \bot}{G|\Gamma \Rightarrow \neg A, \Delta}$$

Note that this calculus has much in common with **GŁ**, i.e. the axioms (ID), (Λ), all of the structural rules, and the logical rule $(\to, r)$. Moreover, the extra premise in the $(\to, l)$ rule, and the axioms and rules for $\bot$ and $\neg$ may be viewed as dealing with the special case of multiplication by zero in $\mathbf{\Pi}$.

Example 3 *We illustrate* $\mathbf{G\Pi}$ *with a proof of the axiom* (Π)*:*

$$\dfrac{\Rightarrow \qquad \dfrac{\dfrac{\Rightarrow}{\neg\neg A \Rightarrow}\,(WL) \qquad \dfrac{\dfrac{\dfrac{\dfrac{A \Rightarrow A \quad B \Rightarrow B}{A, B \Rightarrow A, B}\,(M)}{A \odot B \Rightarrow A, B}\,(\odot, l)}{\neg\neg A, A \odot B \Rightarrow A, B}\,(WL) \qquad \dfrac{\neg A \Rightarrow \neg A}{\neg\neg A, \neg A \Rightarrow B}\,(\neg, l)}{\neg\neg A, A \to (A \odot B) \Rightarrow B}\,(\to, l)}{\neg\neg A \Rightarrow (A \to (A \odot B)) \to B}\,(\to, r)}{\Rightarrow \neg\neg A \to ((A \to (A \odot B)) \to B)}\,(\to, r)$$

The completeness of $\mathbf{G\Pi}$ is proved semantically in [28] by first proving the completeness of an extended calculus, then showing that the extra rules are admissible in $\mathbf{G\Pi}$.

Theorem 7 ([28]) G *is derivable in* $\mathbf{G\Pi}$ *iff* $\models_\Pi G$.

Finally in this section we remark that a calculus for *Cancellative hoop logic* $\mathbf{CHL}$, defined in [18] as a logic with product conjunction and implication defined on the half-open interval $(0, 1]$, is obtained in [28] by removing the axiom $(\bot)$ rule from **GŁ** and adding the rules for $\odot$ of $\mathbf{G\Pi}$.

6. Concluding Remarks

Our aim in this paper has been to show that hypersequents provide an appropriate level of generality for defining analytic calculi for fuzzy logics. To support this view we have surveyed calculi for logics based on several important classes of left-continuous t-norms, such as $\mathbf{MTL}$, $\mathbf{IMTL}$ and $\mathbf{SMTL}$, and logics based on the fundamental continuous t-norms, **Ł**, $\mathbf{G}$ and $\mathbf{\Pi}$, the logic $\mathbf{BL}$ being the only significant omission.

What these results make very clear is that fuzzy logics are also *substructural logics*, the key added ingredient proof-theoretically being variants of the so-called "communication rule". Bearing this in mind a natural next step is to investigate related substructural logics such as those obtained by removing structural rules like weakening (WL) and (WR). Preliminary results in this direction have been obtained in [26]. We may also consider *first-order* logics obtained by adding "standard" (e.g. those for intuitionistic or classical logics) quantifier rules to hypersequent calculi. This has been achieved for $\mathbf{G}$ with completeness results for the $[0, 1]$ interval in [8], and for $\mathbf{MTL}$ in [6], and should also be possible for $\mathbf{IMTL}$ and related logics.

However, since first-order $\mathbf{Ł}$ and $\mathbf{\Pi}$ based on the $[0,1]$ interval are not recursively enumerable (see e.g. [22] for details), systems obtained for these logics will necessarily correspond only to *fragments*, and require further investigation. Back at the propositional level it would clearly also be desirable to find calculi for other t-norm based logics defined in the literature such as the logics based on (weak) nilpotent minimum t-norms $\mathbf{WNM}$ and $\mathbf{NM}$ defined in [16], and in particular Basic logic $\mathbf{BL}$. In the case of the latter, the divisibility axiom (DIV) does not seem to be easy to capture proof-theoretically. Nevertheless it may be possible to obtain a calculus for this logic, either as for $\mathbf{Ł}$ by considering an alternative interpretation of hypersequents, or by considering structures more complicated than hypersequents.

We conclude by mentioning some related work in the literature. First note that *sequent calculi* have been provided for some fuzzy logics, including $\mathbf{G}$ [4], $\mathbf{Ł}$ [29] and $\mathbf{\Pi}$ [28], that while typically not as uniform or elegant as the corresponding hypersequent calculi, may be more suitable for proof search or for proving properties like interpolation. We also remark that other frameworks may be more suited to automated reasoning in fuzzy logics, for example Hähnle's *labelled tableaux* calculus for $\mathbf{Ł}$ [21], Baaz and Fermüller *sequent-of-relations* calculus for $\mathbf{G}$ [7], and the authors' *goal-directed methods* for $\mathbf{G}$, $\mathbf{Ł}$ and $\mathbf{\Pi}$ [27, 26]. However, the only other analytic and purely logical systems provided for fuzzy logics have been the *multiple-sequent* calculi of Aguzzoli and Gerla [1] which exploit the fact that a formula valid in $\mathbf{Ł}$, $\mathbf{G}$ or $\mathbf{\Pi}$ is valid also in an n-valued logic where n is a function of the number of occurrences of variables in the formula. Such calculi provide a valuable perspective on the connection between finite and infinite valued logics, but are not really suitable for proof search, and, being tailored to the semantics of the particular logic, do not cohere well with calculi for other families of logics.

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