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Structure theory of power series spaces of infinite type

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Dedicated to the memory of Klaus Floret

Abstract. The paper gives a complete characterization of the subspaces, quotients and complemented subspaces of a stable power series space of infinite type without the assumption of nuclearity, so extending previous work of M. J. Wagner and the author to the nonnuclear case. Various sufficient conditions for the existence of bases in complemented subspaces of infinite type power series spaces are also extended to the nonnuclear case.

Teoría estructural de espacios de series de potencias de tipo infinito

Resumen. Este artículo da una caracterización completa de los subespacios, cocientes y subespacios complementados de los espacios de series de potencias estables de tipo infinito sin el supuesto de nuclearidad, extendiendo trabajo previo de M. J. Wagner y el autor al caso no nuclear. Se extienden también varias condiciones suficientes para la existencia de bases en subespacios complementados de espacios de potencias de tipo infinito al caso no nuclear.

1. Introduction

In the present paper we extend the structure theory of nuclear stable power series spaces of infinite type as developed by M. J. Wagner and the author in [22],[29], [30] to the nonnuclear case, i.e. we characterize the subspaces, quotient spaces and complemented subspaces of the Fréchet space

$$\Lambda_\infty(\alpha) = \{x \in \mathbb{K}^{\mathbb{N}_0} \,:\, |x|_t^2 = \sum_{n=0}^{\infty} |x_n|^2 e^{2t\alpha_n} < +\infty \text{ for all } t \in \mathbb{R}\}.$$

Here $\alpha = (\alpha_0, \alpha_1, \ldots)$ is an increasing sequence of nonnegative numbers tending to infinity and satisfying

$$\limsup_n \frac{\alpha_{2n}}{\alpha_n} < +\infty. \tag{1}$$

Condition (1) is equivalent to $\Lambda_\infty(\alpha) \times \Lambda_\infty(\alpha) \cong \Lambda_\infty(\alpha)$ and called stability.

In the above mentioned work of Wagner and the author it was additionally assumed that

$$\limsup_n \frac{\log n}{\alpha_n} < +\infty.$$

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This condition is equivalent to the nuclearity of $\Lambda_\infty(\alpha)$. In the present paper we present the theory without this assumption.

While the characterizing conditions remain the same and also the proof of the necessity of these conditions, the proof for the sufficiency is essentially different and provides also a new access for the nuclear case. The difference is the following: in the nuclear case the proof was based of the T. and Y. Komura imbedding theorem, resp. its analogue for α-nuclear spaces due to Ramanujan and Terzioğlu [17], and on a splitting theorem for exact sequences of Fréchet spaces. The splitting theorem is also valid in the nonnuclear case. However we do not have an equivalent to the above mentioned imbedding theorems. Therefore the imbedding and quotient maps will be given by a direct construction.

In a last section we consider results of Vogt [23], Aytuna, Krone and Terzioğlu [1], [2], Wagner [32], Kondakov [9] and Dubinsky and Vogt [6], [7] on the structure of complemented subspaces of $\Lambda_\infty(\alpha)$ and give proofs in the new framework, without assuming nuclearity. Also here in some parts new methods had to be developed.

The paper is partly based on lectures which the author has given in Wuppertal in 1987/88 but never has published. The author thanks M. J. Wagner for useful conversations.

Throughout the paper we will study Fréchet-Hilbert spaces, i.e. Fréchet spaces admitting a fundamental system of seminorms $\|\;\|_0 \le \|\;\|_1 \le \dots$ given by semiscalar products, $\|x\|_k^2 = \langle x, x\rangle_k$. If not otherwise stated a fundamental system of seminorms in a Fréchet-Hilbert space will always be assumed to be of this form. So the local Banach spaces E_k are Hilbert spaces.

We will use common notation for locally convex spaces and Fréchet spaces in particular also for their local Banach spaces and the linking maps between them. The scalar field is always $\mathbb{K}$, where $\mathbb{K}$ is either $\mathbb{R}$ or $\mathbb{C}$. For all this we refer to [12].

2. Power series spaces of infinite type and related invariants

In the present paper we will study only power series spaces of infinite type as defined above. We will not assume nuclearity. The space $\Lambda_\infty(\alpha)$ and likewise the sequence α will be called stable if α satisfies condition (1). For more information on power series spaces we refer to [12, Section 29] and to [4]. For examples see also [18, Chapter 8].

Throughout the paper we use for the local Banach space of $\Lambda_\infty(\alpha)$ with respect to $|\;|_t$ the notation

$$\Lambda_t^\alpha := \{x = (x_0, x_1, \dots) \,:\, |x|_t^2 := \sum_{j=0}^\infty |x_j|^2 e^{2t\alpha_j} < +\infty\}.$$

Of course, these are Hilbert spaces isomorphic to ℓ_2, in particular $\Lambda_0^\alpha = \ell_2$.

An analogous extension of the structure theory to the nonnuclear case for finite type power series spaces has been given in [27]. From this paper we quote the following result [27, Theorem 3.2] which is based on [30, Satz 2.4].

Proposition 1 *If α is stable then there exists an exact sequence*

$$0 \longrightarrow \Lambda_\infty(\alpha) \longrightarrow \Lambda_\infty(\alpha) \longrightarrow \Lambda_\infty(\alpha)^{\mathbb{N}} \longrightarrow 0. \quad \blacksquare$$

To describe the characteristic properties of power series spaces and their subspaces, quotient spaces and complemented subspaces we need two types of invariants. The first one describes the asymptotic behavior of the relative semiaxes of the ellipsoids which form the neighborhoods of zero.

Let X be a linear space and $V \subset U$ absolutely convex subsets. We set for linear subspaces $F, G \subset X$:

$$\delta(V, U; F) = \inf\{\delta > 0 \,:\, V \subset \delta U + F\}$$
$$\gamma(V, U; G) = \inf\{\gamma > 0 \,:\, V \cap G \subset \gamma U\}$$

and with this we notation

$$\delta_n(V,U) = \inf\{\delta(V,U;F) \,:\, \dim F \leq n\}$$
$$\gamma_n(V,U) = \inf\{\gamma(V,U;G) \,:\, \operatorname{codim} G \leq n\}.$$

$\delta_n(V,U)$ is called the n-th Kolmogoroff diameter, $\gamma_n(V,U)$ the n-th Gelfand diameter. For the behavior of the diameters see [19] or [4, Section I, 6].

We call U a Hilbert disc if $U = \{x \,:\, \|x\| \leq 1\}$ and $||x||$ is given by a semiscalar product, i.e. $\|x\|^2 = \langle x, x\rangle$. The following is well known (see [19, II,3.(3)]).

Lemma 1 *If V an U are Hilbert discs then $\delta_n(V,U) = \gamma_n(V,U)$ for all n and they coincide with the singular numbers of the canonical map $E_V \hookrightarrow E_U$.* ■

Here E_U and E_V denote the respective local Hilbert spaces and for the last assertion we assumed that $E_V \hookrightarrow E_U$ is compact, i.e. that V is precompact with respect to U. The following lemma is immediately clear.

Lemma 2 *If $V \subset U$ are absolutely convex subsets of the linear space X, $X_0 \subset X$ a linear subspace and $q\colon X \longrightarrow X/X_0$ the quotient map, then $\delta_n(qV, qU) \leq \delta_n(V,U)$ and $\gamma_n(V \cap X_0, U \cap X_0) \leq \gamma_n(V,U)$.* ■

We will use the following property of the diameters (cf. [20]).

Lemma 3 *Let $\| \,\|_0 \leq \| \,\|_1 \leq \| \,\|_2$ be seminorms on the linear space X and $\| \,\|_1^2 \leq \| \,\|_0 \| \,\|_2$, then $\gamma_n(U_2, U_0) \leq \gamma_n^2(U_1, U_0)$ for all $n \in \mathbb{N}_0$.*

PROOF. Let $G \subset X$ be a linear subspace with $\operatorname{codim} G \leq n$ and $U_2 \cap G \subset \gamma U_0$. From the assumption we derive that $(\gamma U_0) \cap U_2 \subset \sqrt{\gamma} U_1$. Inserting the first inclusion into the second we see that $U_2 \cap G \subset \sqrt{\gamma} U_1$. Therefore $\gamma(U_2, U_1; G) \leq \sqrt{\gamma}$ and, in consequence, $\gamma(U_2, U_1; G)^2 \leq \gamma(U_2, U_0; G)$. Since obviously $\gamma(U_2, U_0; G)^2 \leq \gamma(U_2, U_1; G)^2 \gamma(U_1, U_0; G)^2$ we have $\gamma(U_2, U_0; G) \leq \gamma(U_1, U_0; G)^2$. Taking infima over G proves the result. ■

The following definition goes back to Ramanujan-Terzioğlu [17] (there and in [30] under the name of $\Lambda_{\mathbb{N}}(\alpha)$-nuclearity). We use the Kolmogoroff diameters for the definition as it is done in [30]. The difference is that we do not assume nuclearity, i. e. we do not assume $\limsup_n \frac{\log n}{\alpha_n} < \infty$.

Definition 1 *A Fréchet-Hilbert space E is called α-nuclear if for every absolutely convex neighborhood U of zero and every $t > 0$ there is another such neighborhood $V \subset U$, so that*

$$\lim_n e^{t\alpha_n} \delta_n(V,U) = 0.$$

Remark 1 In view of Lemma 1 we could have used also the condition

$$\lim_n e^{t\alpha_n} \gamma_n(V,U) = 0. \quad ■$$

Clearly α-nuclearity is invariant under topological linear isomorphisms. As an immediate consequence of Lemma 2 we obtain.

Proposition 2 *α-nuclearity is inherited by subspaces and quotient spaces.* ■

Lemma 4 *$\Lambda_\infty(\alpha)$ is α-nuclear.*

PROOF. If $U_t = \{x \in \Lambda_\infty(\alpha) : |x|_t \leq 1\}$ and $s < t$ then it is immediate that

$$U_t \subset \operatorname{span}\{e_0, \dots, e_{n-1}\} + e^{(s-t)\alpha_n} U_s$$

where $e_0, e_1, \dots$ are the canonical basis vectors. Therefore $\delta_n(U_t, U_s) \leq e^{(s-t)\alpha_n}$. ■

Remark 2 If $\operatorname{codim} G \leq n$ and $U_t \cap G \subset \gamma U_s$ then there is $x \in \operatorname{span}\{e_0, \dots, e_n\} \cap G$, $\|x\|_t = 1$ and we have

$$\gamma^2 \geq |x|_s^2 = \sum_{j=0}^{n} |x_j|^2 e^{2s\alpha_j} \geq e^{2(s-t)\alpha_n}$$

hence $e^{(s-t)\alpha_n} \leq \gamma_n(U_t, U_s)$ which means that we have even

$$\gamma_n(U_t, U_s) = \delta_n(U_t, U_s) = e^{(s-t)\alpha_n}. \quad ■$$

From Lemma 4 and Proposition 2 we obtain:

Proposition 3 *If E is isomorphic to a subspace or a quotient space of $\Lambda_\infty(\alpha)$ then E is α-nuclear.* ■

Just for the sake of completeness we notice that every closed subspace or quotient space of a Fréchet-Hilbert space is again a Fréchet-Hilbert space.

The second type of invariants are those who describe the interpolational properties of the seminorm system defining E. They have been considered, under different names, by Dragilev, Zaharjuta, Dubinsky, Robinson, Wagner and the author. The importance of the properties (DN) and (Ω) is based on the structure theory of power series spaces as developed in [22],[29],[30] and, in particular, on the (DN)-(Ω)-Splitting Theorem (see Theorem 2 below). For the properties of (DN) and (Ω) see [12, Sections 29, 30, 31].

Let E be a Fréchet space, $\| \ \|_0 \leq \| \ \|_1 \leq \dots$ a fundamental system of seminorms and $\| \ \|_0^* \geq \| \ \|_1^* \geq \dots$ the dual extended real valued norms in E', where for any seminorm $\| \ \|$ we are using the notation

$$\|y\|^* = \sup_{\|x\| \leq 1} |y(x)|$$

for $y \in E'$.

Definition 2 *E has property* (DN) *if there is p so that for any k there is K and $C > 0$ with*

$$\| \ \|_k^2 \leq C \| \ \|_p \| \ \|_K.$$

$\| \ \|_p$ is called a dominating norm.

An equivalent formulation is given in the following lemma for the proof of which we refer to [12].

Lemma 5 *E has property* (DN) *if and only if there is p, so that for every k and $0 < \tau < 1$ there exists K and $C > 0$ with*

$$\| \ \|_k \leq C \| \ \|_p^{1-\tau} \| \ \|_K^{\tau}. \quad ■$$

Definition 3 *E has property (Ω) if for every p there is q such that for any k there is $0 < \vartheta < 1$ and $C > 0$ with*

$$\| \ \|_q^* \leq C \| \ \|_p^{*\,1-\vartheta} \| \ \|_k^{*\,\vartheta}.$$

The most prominent example of a space with properties (DN) and (Ω) is the following.

Lemma 6 *$\Lambda_\infty(\alpha)$ has properties* (DN) *and* (Ω).

PROOF. By Hölder's inequality we obtain easily for $t_0 < t_1 < t_2$

$$|x|_{t_1} \leq |x|_{t_0}^{\frac{t_2-t_1}{t_2-t_0}} |x|_{t_2}^{\frac{t_1-t_0}{t_2-t_0}}. \tag{2}$$

Taking into account that

$$|y|_t^{*2} = \sum_j |y_j|^2 e^{-2t\alpha_j}$$

we obtain in the same way

$$|y|_{t_1}^* \leq |y|_{t_0}^{*\frac{t_2-t_1}{t_2-t_0}} |y|_{t_2}^{*\frac{t_1-t_0}{t_2-t_0}}. \quad \blacksquare \tag{3}$$

Directly from the definition we see:

Lemma 7 *(1) Property* (DN)*is inherited by subspaces.*

(2) Property (Ω) *is inherited by quotient spaces.* ■

From Lemmas 6, 7 and Proposition 3 we conclude:

Proposition 4 *(1) If E is isomorphic to a subspace of $\Lambda_\infty(\alpha)$ then it is α-nuclear and has property* (DN).

(2) If E is isomorphic to a quotient space of $\Lambda_\infty(\alpha)$ then it is α-nuclear and has property (Ω). ■

Of course, the arguments of the proof of Lemma 6 hold also if instead of $\Lambda_\infty(\alpha)$ we take a space of the following form

$$\Lambda_\infty(\alpha, I) = \{x \in \mathbb{K}^I \ : \ |x|_t^2 = \sum_{i\in I} |x_i|^2 e^{2t\alpha_i} < +\infty \text{ for all } t \in \mathbb{R}\}.$$

Here $\alpha = (\alpha_i)_{i\in I}$ is a family of nonnegative numbers and I is any index set. In particular we have formulas (2) and (3). For any index set J we set $\Sigma_\infty(J) = \Lambda_\infty(\alpha, I)$ where $I = \mathbb{N} \times J$ and $\alpha_{n,j} = n$. Moreover we set $\Sigma_\infty = \Sigma_\infty(\mathbb{N})$. Then $\Sigma_\infty(J)$ and Σ_∞ have properties (DN)and (Ω).

At this point we want to extend some results of [25] to the case of Fréchet-Hilbert spaces.

Theorem 1 *Let E be a Fréchet-Hilbert space and J a dense subset of E, then*

(1) E has property (DN) *if and only if E is isomorphic to a subspace of $\Sigma_\infty(J)$.*

(2) E has property (Ω) if and only if E is isomorphic to a quotient space of $\Sigma_\infty(J)$. ■

In particular we have

Corollary 1 *Let E be a separable Fréchet-Hilbert space then*

(1) E has property (DN) *if and only if E is isomorphic to a subspace of Σ_∞.*

(2) E has property (Ω) if and only if E is isomorphic to a quotient space of Σ_∞. ■

We can read Corollary 1 also as a characterization of the subspaces and quotient spaces of Σ_∞, which is isomorphic to $\mathscr{D}_{L_2}$ (see [21] or [24, Theorem 3.2]).

From Theorem 1 we get immediately

Corollary 2 *If E is a Fréchet-Hilbert space and has property* (DN), *then there is a one-parameter family $(|\ |_t)_{t\in\mathbb{R}}$ of Hilbert norms on E which generates the topology and so that $t \mapsto \log|x|_t$ is convex and increasing for any $x \in E$.* ■

Corollary 3 *If E is a Fréchet-Hilbert space and has property (Ω) then there is a one-parameter family of extended real valued Hilbert seminorms $(|\ |_t^*)_{t\in\mathbb{R}}$ so that the sets $B_t = \{y \in E' : |y|_t^* \leq 1\}$ are a fundamental system of bounded sets in E' and $t \mapsto \log|x|_t^*$ is convex and decreasing for each $y \in E'$.* ■

The proof of Theorem 1 is exactly the same as the proof of Theorems 2.2 and 3.2 in [25]. One has to replace the spaces $\ell^\infty(I)$ and $\ell^1(I)$ by $\ell^2(I)$ and apply the following theorem instead of [25, Lemma 1.3.]. For a proof of this theorem see e.g. [12, Section 30].

Theorem 2 *Let $0 \longrightarrow F \longrightarrow G \longrightarrow E \longrightarrow 0$ be an exact sequence of Fréchet-Hilbert spaces and assume that F has property (Ω) and E has property* (DN), *then the sequence splits.* ■

An essential ingredient in the proof of Theorem 2 is the following lemma (see [12, Lemma 30.7]) which we will use in the sequel.

Lemma 8 *Let E and F be Fréchet-Hilbert spaces. Assume that E has property* (DN) *with dominating norm $\|\ \|_0$ and that F has property (Ω). Let $\|\ \|_0$ be a continuous seminorm on F and $\|\ \|_1$ be chosen for $\|\ \|_0$ according to (Ω). Then for every $\varphi \in L(E, F_1)$ and $\varepsilon > 0$ there exist $\psi \in L(E, F_0)$, $\chi \in L(E, F)$ so that $\sup_{\|x\|_0\leq 1} \|\psi x\|_0 \leq \varepsilon$ and $\imath_1^o \circ \varphi = \psi + \imath^o \circ \chi$.* ■

Here F_0 and F_1 are the local Hilbert spaces generated by $\|\ \|_0$ and $\|\ \|_1$, respectively.

We will need the following interpolation result for Hilbert scales. For any two index sets I, J and families $(\alpha_i)_{i\in I}, (\beta_j)_{j\in J}$ of nonnegative real numbers we put

$$G_t = \left\{ x = (x_i)_{i\in I} :\ |x|_t^2 = \sum_{i\in I} e^{2\alpha_i t}|x_i|^2 < +\infty \right\}$$

$$H_t = \left\{ x = (x_j)_{j\in J} :\ |x|_t^2 = \sum_{j\in J} e^{2\beta_j t}|x_j|^2 < +\infty \right\}.$$

These are Hilbert spaces, equipped with their natural scalar products. In the following lemma $\|\ \|_t$ denotes the norm of an operator $G_t \to H_t$. The following result is well known, see [11, Theorem 1.11].

Lemma 9 *Let $T \in L(G_0, H_0)$ and $TG_1 \subset H_1$. Then $TG_t \subset H_t$ and $\|T\|_t \leq \|T\|_0^{1-t}\|T\|_1^t$ for all $t \in [0,1]$.* ■

We will make immediate use of Lemma 9 to prove the following useful fact.

Lemma 10 *If $\|\ \|$ is a Hilbert norm on $\Lambda_\infty(\alpha)$ and $|\ |_0 \leq \|\ \| \leq C|\ |_\tau$, $C \geq 1$. Then there is an automorphism U of $\Lambda_\infty(\alpha)$ so that $|Ux|_0 = \|x\|$ and*

$$|x|_t \leq |Ux|_t \leq C|x|_{t+\tau}$$

for all $x \in \Lambda_\infty(\alpha)$, $t \geq 0$.

PROOF. We denote by H_0 the Hilbert space generated by $\| \ \|$ and by $(\ , \ )$ its scalar product. For every $K > \tau$ we consider the canonical map $\imath_K^o \colon \Lambda_K^\alpha \hookrightarrow H_0$. It is compact, let s_n be the singular numbers. We set

$$\beta_n = -\frac{1}{K} \log s_n.$$

Then the Schmidt representation takes the form

$$\imath_K^o x = \sum_{n=0}^{\infty} e^{-K\beta_n} \langle x, e_n \rangle_K f_n, \tag{4}$$

where $(e_n)_n, (f_n)_n$ are orthonormal bases in Λ_K^α and H_0, respectively. If we set $U_t = \{x : |x|_t \leq 1\}$, $V = \{x : \|x\| \leq 1\}$ then $\frac{1}{C} U_\tau \subset V \subset U_0$ leads to

$$\delta_n(U_K, U_0) \leq \delta_n(U_K, V) \leq C \delta_n(U_K, U_\tau)$$

i.e. (see Remark 2)

$$e^{-K\alpha_n} \leq e^{-K\beta_n} \leq C e^{-(K-\tau)\alpha_n}. \tag{5}$$

We put

$$u_K x = ((x, f_n))_{n \in \mathbb{N}_0}.$$

We have

$$|x|_0 \leq |u_K x|_0 = \|x\| \leq C |x|_\tau$$

and, by use of (5) for the first inequality and (4) for the equation in the middle row,

$$\begin{aligned}
\frac{1}{C} |u_K x|_{K-\tau} &= \frac{1}{C} \left(\sum_{n=0}^{\infty} e^{2(K-\tau)\alpha_n} |(x, f_n)|^2 \right)^{\frac{1}{2}} \\
&\leq \left(\sum_{n=0}^{\infty} e^{2K\beta_n} |(x, f_n)|^2 \right)^{\frac{1}{2}} = |x|_K \\
&\leq \left(\sum_{n=0}^{\infty} e^{2K\alpha_n} |(x, f_n)|^2 \right)^{\frac{1}{2}} = |u_K x|_K.
\end{aligned}$$

By use of Lemma 9, applied to u_K and its inverse, we obtain

$$|x|_t \leq |u_K x|_t \leq C |x|_{t+\tau}$$

for all $0 \leq t \leq K - \tau$.

For every $k \in \mathbb{N}$ the set $\{u_K : K \geq k + \tau\}$ is an equicontinuous subset of $L(\Lambda_{k+\tau}^\alpha, \Lambda_k^\alpha)$. Since $\Lambda_t^\alpha \longrightarrow \Lambda_s^\alpha$ is compact for $t > s$ the set is relatively compact in $L(\Lambda_{k+\tau+1}^\alpha, \Lambda_{k-1}^\alpha)$ for every $k \in \mathbb{N}$. Therefore we may, by use of a diagonal procedure, find a subsequence u_{K_n}, so that u_{K_n} converges in $L(\Lambda_{k+\tau+1}^\alpha, \Lambda_{k-1}^\alpha)$ for every $k \in \mathbb{N}$. Since the same applies to $(u_K^{-1})_K$ we may choose the subsequence so that also $(u_{K_n}^{-1})_n$ converges in $L(\Lambda_{k+1}^\alpha, \Lambda_{k-1}^\alpha)$ for all $k \in \mathbb{N}$, and we set for $x \in \Lambda_\infty(\alpha)$:

$$Ux = \lim_{n \to \infty} u_{K_n} x, \quad Vx = \lim_{n \to \infty} u_{K_n}^{-1} x.$$

and certainly $U, V \in L(\Lambda_\infty(\alpha))$. Of course, we first take the limits in the local Banach spaces separately and then see that those results define elements $Ux \in \Lambda_\infty(\alpha)$, $Vx \in \Lambda_\infty(\alpha)$, respectively.

It can easily be seen that $UV = VU = \mathrm{id}$, hence U is an automorphism. We have

$$|Ux|_0 = \lim_n |U_{k_n} x|_0 = \|x\|$$

and we have for any $t > 0$

$$|x|_t \leq \lim_n |U_{k_n} x|_t = |Ux|_t \leq C|x|_{t+\tau}.$$

This proves the result. ■

An endomorphism U of $\Lambda_\infty(\alpha)$ for which there are constants $C > 0$ and $\tau \geq 0$ so that

$$|Ux|_t \leq C\,|x|_{t+\tau}, \quad x \in \Lambda_\infty(\alpha)$$

for all $t > 0$ we call uniformly tame. A description of uniformly tame endomorphisms is contained in the following lemmas.

For any endomorphism $U \in L(\Lambda_\infty(\alpha))$ there is a matrix $(u_{k,j})_{k,j\in\mathbb{N}_0}$ so that

$$Ux = \left(\sum_{j=0}^{\infty} u_{k,j} x_j \right)_{k\in\mathbb{N}_0}, \quad x \in \Lambda_\infty(\alpha).$$

Of course, we have $u_{k,j} = \langle Ue_j, e_k \rangle$ if e_k and e_j denote the canonical basis vectors and $\langle\ ,\ \rangle$ the ℓ_2-scalar product.

Lemma 11 *Let α be strictly increasing and $U \in L(\Lambda_\infty(\alpha))$ be uniformly tame, then its matrix is upper triangular.*

PROOF. From the continuity estimates we get

$$|u_{k,j}|e^{t\alpha_k} \leq |Ue_j|_t \leq C\,|e_j|_{t+\tau} = C\,e^{(t+\tau)\alpha_j}.$$

Therefore

$$|u_{k,j}| \leq C\,e^{t(\alpha_j - \alpha_k) + \tau\alpha_j}$$

for all $t > 0$ which implies the result. ■

If α is not necessarily strictly increasing we have to replace upper triangular by blockwise upper triangular, where the blocks are given by the sets of indices on which α is constant.

Lemma 12 *Let α be strictly increasing and $S \in L(\ell_2)$ have an upper triangular matrix, then $S \in L(\Lambda_\infty(\alpha))$ and S is uniformly tame with $\tau = 0$.*

PROOF. We may assume that $\|S\|_{L(\ell_2)} = 1$. Then in particular $|s_{j,j}| \leq 1$ for all j. Let $(d_0, d_1, \dots)$ with $\sup_j |d_j| \leq 4$ be a sequence so that $|s_{j,j} + d_j| = 4$ for all j.

We set $Dx = (d_0 x_0, d_1 x_1, \dots)$ and $H = S + D$. Let $H_0 x = (h_{o,o} x_0, h_{1,1} x_1, \dots)$ be the diagonal part of H and $H_+ = H - H_0$ then $H_+ = S_+$ where S_0 and S_+ are defined in an analogous way. Therefore $\|H_+\| = \|S_+\| \leq \|S\| + \|S_0\| \leq 2$, all norms taken in $L(\ell_2)$.

Because of $H = H_0(I + H_0^{-1} H_+)$ and $\|H_0^{-1} H_+\| \leq \frac{1}{2}$ the operator H is invertible in $L(\ell_2)$ and $\|H^{-1}\| \leq 1$. So we have

$$|x|_0 \leq |Hx|_0 \leq 5|x|_0.$$

We apply Lemma 10 to $\|x\| = |Hx|_0$ and get a uniformly tame automorphism U of $\Lambda_\infty(\alpha)$ with

$$|x|_t \leq |Ux|_t \leq 5|x|_t, \quad x \in \Lambda_\infty(\alpha)$$

for all $t > 0$, so that $|Ux|_0 = |Hx|_0$ for all x. By Lemma 11 the matrix of U is upper triangular.

Therefore $V = U \circ H^{-1}$ is unitary in ℓ_2 and has an upper triangular matrix. This implies that V is diagonal and $|v_{j,j}| = 1$ for all j and therefore $|Hx|_t = |V^{-1}Ux|_t \leq 5|x|_t$ for all $x \in \Lambda_\infty(\alpha)$ and $t > 0$. Finally we obtain

$$|Sx|_t \leq |Hx|_t + |Dx|_t \leq 9|x|_t, \quad x \in \Lambda_\infty(\alpha)$$

for all $t > 0$. This proves the result. ■

Since we can modify any α to a strictly increasing sequence without changing $\Lambda_\infty(\alpha)$ we obtain from Lemma 12 as an immediate consequence:

Lemma 13 *Let $S \in L(\ell_2)$ have an upper triangular matrix, then $S \in L(\Lambda_\infty(\alpha))$.* ∎

3. Subspaces of power series spaces of infinite type

We characterize in this section the subspaces of a stable power series space $\Lambda_\infty(\alpha)$. We assume that that the Fréchet-Hilbert space E has property (DN) and is α-nuclear. By Corollary 1 the space E is isomorphic to a subspace of Σ_∞. We may assume that it is a subspace of Σ_∞.

By $|\ |_t$ we denote the canonical norms on Σ_∞ and also their restrictions to E.

We assume chosen a sequence $0 = t_0 < t_1 < \dots$ with $\lim_k t_k = \infty$. We set $\|\ \|_k = |\ |_{t_k}$ on E and $U_k = \{x \in E \,:\, \|x\|_k \leq 1\}$.

By definition the $\|\ \|_0 \leq \|\ \|_1 \leq \dots$ are Hilbertian norms on E so that with suitable $0 < \tau_k < 1$

$$\|\ \|_k \leq \|\ \|_{k-1}^{1-\tau_k} \|\ \|_{k+1}^{\tau_k}. \tag{6}$$

By the choice of our seminorms the space E is countably normed, i.e. the canonical map $\imath_k^l : E_k \longrightarrow E_l$ is injective for all $k \geq l$.

The fundamental lemma in this section is:

Lemma 14 *Assume that $\delta_n(U_k, U_0) \leq e^{-k\alpha_n}$ for all $n \geq n_0$. Then there is a sequence β with $\beta_n \geq a_n$ for all $n \geq n_0$ and an injective map $\varphi_k \in L(E, \Lambda_\infty(\beta))$, so that for all $x \in E$*

1. $|\varphi_k(x)|_0 \leq 2\|x\|_0$,
2. $|\varphi_k(x)|_k \geq \frac{1}{2}\,\|x\|_k$.

Moreover φ_k extends to an isomorphism $\hat{\varphi}_k : E_0 \longrightarrow \ell_2 = \Lambda_0^\beta$.

PROOF. We set $s = 2t_k$, $E_{(s)}$ the local Banach space of $|\ |_s$ and $\imath_l^{(s)} : E_l \longrightarrow E_{(s)}$, $\imath_{(s)}^l : E_{(s)} \longrightarrow E_l$ the respective linking maps. By assumption the map $\imath_k^0$, hence also $\imath_{(s)}^0 : E_{(s)} \longrightarrow E_0$ is compact. Let s_n be the singular numbers of $\imath_{(s)}^0$. We set

$$\beta_n = -\frac{1}{2k}\log s_n.$$

Then the Schmidt representation takes the form

$$\imath_{(s)}^0 x = \sum_{n=0}^{\infty} e^{-2k\beta_n} \langle x, e_n\rangle_s f_n. \tag{7}$$

Here $(e_n)_n$ is an orthonormal basis of $E_{(s)}$ and $(f_n)_n$ an orthonormal basis of E_0.

The choice of s and (2) applied to Σ_∞ implies $\|\ \|_k^2 \leq \|\ \|_0|\ \ |_s$. If $\delta_n(U_k, U_0) \leq e^{-k\alpha_n}$ we obtain, by use of Lemmas 1 and 3, with $U_{(s)} = \{x \in E \,:\, |x|_s \leq 1\}$

$$e^{-2k\beta_n} = \delta_n(U_{(s)}, U_0) \leq \delta_n^2(U_k, U_0) \leq e^{-2k\alpha_n}.$$

Therefore $\alpha_n \leq \beta_n$ for all $n \geq n_0$.

We set $\varphi x = (\langle x, f_n\rangle_0)_{n\in\mathbb{N}_0}$ and obtain a unitary map $\varphi\colon E_0 \longrightarrow \ell_2 = \Lambda_0^\beta$. From (7) we see that

$$\varphi \circ \imath_{(s)}^0(x) = (e^{-2k\beta_n}\langle x, e_n\rangle_s)_n$$

which means that $\varphi \circ \imath_{(s)}^0$ is a unitary map $E_{(s)} \longrightarrow \Lambda_{2k}^\beta$. We denote by u its inverse. Then u is unitary $\Lambda_{2k}^\beta \longrightarrow E_{(s)}$ and we have

$$\begin{aligned} |u\,\xi|_s &= |\xi|_{2k} \\ \|\imath_{(s)}^0 u\,\xi\|_0 &= |\xi|_0 \end{aligned}$$

for all $\xi \in \Lambda_{2k}^{\beta}$. The latter because obviously

$$\imath_{(s)}^{0} \circ u = \varphi^{-1}|_{\Lambda_{2k}^{\beta}}. \tag{8}$$

By Lemma 9, applied for maps between the Hilbert spaces of Λ_t^{β} and Σ_t we obtain

$$\|\imath_{(s)}^{k} u\,\xi\|_k \le |\xi|_k. \tag{9}$$

Notice that, by assumption, $E \subset \Sigma_\infty$ and therefore, by natural identification, $E_k \subset \Sigma_{t_k}$, $E_{(s)} \subset \Sigma_s$. By this identification we have $\|\imath_{(s)}^{0} u\,\xi\|_0 = |u\xi|_0$ and $\|\imath_{(s)}^{k} u\,\xi\|_k = |u\xi|_{t_k}$.

We apply Lemma 8 (with $\varepsilon = \frac{1}{2}$) to $\varphi \circ \imath_{(s)}^{0} \in L(E_{(s)}, \Lambda_{2k}^{\beta})$. We obtain maps $\varphi_k \in L(E, \Lambda_\infty(\beta)), \psi \in L(E_0, \Lambda_k^{\beta})$ so that $\sup_{\|x\|_0 \le 1} |\psi x|_k \le \frac{1}{2}$ and

$$\varphi \circ \imath^0 = \varphi_k + \psi \circ \imath^0.$$

We have to verify the desired properties for φ_k.

For $x \in E$ we have

$$\begin{aligned} |\varphi_k x|_0 \le& |\varphi(\imath^0 x)|_0 + |\psi(\imath^0 x)|_k \\ \le& 2\|x\|_0. \end{aligned}$$

This proves part 1. Moreover φ_k extends to $\hat{\varphi}_k \in L(E_0, \Lambda_0^{\beta})$ and we have $\hat{\varphi}_k = \varphi - \psi = \varphi(I - \varphi^{-1}\psi)$. Since $\|\varphi^{-1}\psi\|_{L(E_0)} \le \frac{1}{2}$ the map $\hat{\varphi}_k$ is invertible.

The map $\imath^0$ is injective, hence also $\varphi_k = \hat{\varphi}_k \circ \imath^0$.

We may write $\hat{\varphi}_k^{-1}$ by its Neumann series

$$\begin{aligned} \hat{\varphi}_k^{-1} =& \varphi^{-1} + \varphi^{-1}\psi\varphi^{-1} \sum_{n=0}^{\infty} (\psi\varphi^{-1})^n \\ =& \varphi^{-1} + \varphi^{-1} \circ v \end{aligned}$$

where

$$v = \psi \sum_{n=0}^{\infty} (\varphi^{-1}\psi)^n \varphi^{-1}.$$

Hence $v \in L(\Lambda_0^{\beta}, \Lambda_k^{\beta})$ and $\|v\|_{L(\Lambda_0^{\beta}, \Lambda_k^{\beta})} \le 1$.

For $x \in E$ and $\xi = \varphi_k x$ we obtain

$$\imath^0 x = \varphi^{-1}(\xi) + \varphi^{-1}(v\xi) = \varphi^{-1}(\xi + v\xi). \tag{10}$$

Because of (9) the map $\imath_{(s)}^{k} \circ u$ extends to $u_k \in L(\Lambda_k^{\beta}, E_k)$. Due to (8), for $\eta \in \Lambda_k^{\beta}$ we have $\varphi^{-1}\eta = \imath_k^0 u_k \eta$.

With $\eta = \xi + v\xi$ we obtain from (10), using that $\imath_k^0$ is injective, $\imath^k x = u_k(\xi + v\xi)$.

Therefore we have

$$\|x\|_k \le |\xi + v\xi|_k \le 2|\xi|_k.$$

This completes the proof. ■

Lemma 15 *If* $\limsup_n \frac{\alpha_{2n+1}}{\alpha_n} < d < +\infty$ *and* $\beta_n \ge \alpha_n$ *for all* $n \ge n_0$, *then there exists a strictly monotonous sequence* $(n(\nu))_{\nu \in \mathbb{N}_0}$ *and* $b \ge 0$ *so that* $\beta_\nu \le \alpha_{n(\nu)} \le b + d\beta_\nu$ *for all* $\nu \in \mathbb{N}_0$.

PROOF. We set $m(\nu) = \sup\{n : \alpha_n \leq \beta_\nu\}$ and $n(\nu) = m(\nu) + \nu + 1$. Then by assumption we have $m(\nu) \geq \nu$ for $\nu \geq n_0$ and $n(\nu)$ is strictly increasing. Moreover $\beta_\nu < \alpha_{n(\nu)}$ for all ν. We choose $n_1 \geq n_0$ so that $\alpha_{2n+1} \leq d\alpha_n$ for $n \geq n_1$. For $\nu \geq n_1$ we have

$$\alpha_{n(\nu)} \leq \alpha_{2m(\nu)+1} \leq d\alpha_{m(\nu)} \leq d\beta_\nu.$$

Setting $b = \sup\{\alpha_{n(\nu)} : \nu < n_1\}$ we obtain the result. ■

Lemma 16 *Under the assumptions of Lemma 14 and the assumption that* $\limsup_n \frac{\alpha_{2n}}{\alpha_n} < +\infty$ *we obtain: for every* k *there is an injective map* $\varphi_k \in L(E, \Lambda_\infty(\alpha))$ *so that for all* $x \in E$

1. $|\varphi_k(x)|_0 \leq 2\|x\|_0$
2. $|\varphi_k x|_k \geq \frac{1}{2}\|x\|_k$.

PROOF. We choose β and φ_k as in Lemma 14. We use Lemma 15 to find a sequence $n(\nu)$ with $\beta_\nu \leq \alpha_{n(\nu)} \leq b + d\beta_\nu$ and define for $\xi = (\xi_0, \xi_1, \ldots)$ a sequence $\varphi\,\xi$ by

$$(\varphi\,\xi)_n = \begin{cases} \xi_\nu & \text{if } n = n(\nu) \\ 0 & \text{otherwise.} \end{cases}$$

Then φ gives an isomorphic imbedding $\Lambda_\infty(\beta) \hookrightarrow \Lambda_\infty(\alpha)$ and an isometric imbedding $\ell_2 \hookrightarrow \ell_2$. If we replace φ_k by $\phi_k := \varphi \circ \varphi_k$, then $\phi_k \in L(E, \Lambda_\infty(\alpha))$ is injective, we have

$$|\phi_k(x)|_0 \leq |\varphi_k(x)|_0 \leq 2\|x\|_0$$

and with $\xi = \varphi_k x$

$$\begin{aligned} |\phi_k x|_k =& |\varphi(\varphi_k x)|_k = \left(\sum_\nu |\xi_\nu|^2 e^{2k\alpha_{n(\nu)}}\right)^{\frac{1}{2}} \\ \geq& \left(\sum_\nu |\xi_\nu|^2 e^{2k\beta_\nu}\right)^{\frac{1}{2}} = |\varphi_k x|_k \geq \frac{1}{2}\|x\|_k. \quad ■ \end{aligned}$$

We are now in the position to prove the main theorem of this section. For the nuclear case see [30, Satz 3.2].

Theorem 3 *Let* α *be stable. A Fréchet space* E *is isomorphic to a subspace of* $\Lambda_\infty(\alpha)$ *if and only if* E *is an* α*-nuclear Fréchet-Hilbert space with property* (DN).

PROOF. If E is isomorphic to a subspace of $\Lambda_\infty(\alpha)$ then it is Fréchet-Hilbert and, by Proposition 4, it is α-nuclear and has property (DN).

We have to show the converse implication. We set $n(i,k) = 2^k + i2^{k+1} - 1$ and get a bijective map $\mathbb{N}_0 \times \mathbb{N}_0 \longrightarrow \mathbb{N}_0$. We set $\alpha_{i,k} := \alpha_{n(i,k)}$ and obtain an isomorphism $\Lambda_\infty(\alpha) \cong \Lambda$ where

$$\Lambda := \{x = (x_{i,k})_{i,k\in\mathbb{N}_0} : \|x\|_t^2 = \sum_{i,k} |x_{i,k}|^2 e^{2t\alpha_{i,k}} < +\infty \text{ for all } t \in \mathbb{R}\}.$$

The isomorphism is given by $(x_n)_{n\in\mathbb{N}_0} \mapsto (x_{n(i,k)})_{i,k\in\mathbb{N}_0}$.

In E we choose a fundamental system of Hilbertian norms as in Lemma 14. For every $k \in \mathbb{N}_0$ we choose $\varphi_k \in L(E, \Lambda_\infty(\alpha))$ according to Lemma 16 and set

$$\varphi_k(x) = (\varphi_{i,k}(x))_{i\in\mathbb{N}_0}.$$

We define for $x \in E$

$$\phi(x) := (2^{-k} e^{-k\alpha_{i,k}} \varphi_{i,k}(x))_{i,k\in\mathbb{N}_0}.$$

We have to show that $\phi(x) \in \Lambda$ and $\phi \in L(E, \Lambda)$.

From the stability of α we get $b_k \geq 0$, $d_k \in \mathbb{N}$ so that

$$\alpha_i \leq \alpha_{i,k} \leq b_k + d_k \alpha_i.$$

We may assume that $b_k \leq b_{k+1}$ and $d_k \leq d_{k+1}$ for all k. Now, we fix m and choose M, C such that

$$|\varphi_k x|_{md_m} \leq C \|x\|_M$$

for $k = 0, \ldots, m-1$. We get

$$\begin{aligned}
\|\phi x\|_m^2 &\leq \sum_{k=0}^{m-1} 4^{-k} \left(\sum_{i=0}^{\infty} |\varphi_{i,k}(x)|^2 e^{2(m-k)\alpha_{i,k}} \right) + \sum_{k=m}^{\infty} 4^{-k} \left(\sum_{i=0}^{\infty} |\varphi_{i,k}(x)|^2 \right) \\
&\leq \sum_{k=0}^{m-1} 4^{-k} e^{2mb_m} |\varphi_k(x)|_{md_m}^2 + \sum_{k=m}^{\infty} 4^{-k} |\varphi_k(x)|_0^2 \\
&\leq \frac{4}{3}(e^{2mb_m} C^2 + 4) \|x\|_M^2.
\end{aligned}$$

On the other hand we have for $m \in \mathbb{N}$

$$\|\phi x\|_{2m} \geq 2^{-m} \left(\sum_{i=0}^{\infty} |\varphi_{i,m}(x)|^2 e^{2m\alpha_{i,m}} \right)^{\frac{1}{2}} \geq 2^{-m} |\varphi_m(x)|_m \geq 2^{-m-1} \|x\|_m.$$

Therefore ϕ is an isomorphic imbedding $E \hookrightarrow \Lambda \cong \Lambda_\infty(\alpha)$. ■

4. Quotient spaces of power series spaces of infinite type

In this section we characterize the quotient spaces of a stable power series space $\Lambda_\infty(\alpha)$. We assume that that the Fréchet-Hilbert space E has property (Ω) and is α-nuclear. By Corollary 1 the space E is isomorphic to a quotient of Σ_∞. Let $q : \Sigma_\infty \longrightarrow E$ be the quotient map.

By $|\ |_t$ we denote the quotient seminorms under q of the canonical norms $|\ |_t$ on Σ_∞.

We assume chosen a sequence $0 = t_0 < t_1 < \ldots$ with $\lim_k t_k = \infty$. We set $\|\ \|_k = |\ |_{t_k}$ and $U_k = \{x : \|x\|_k \leq 1\}$.

By definition the $\|\ \|_0 \leq \|\ \|_1 \leq \ldots$ are Hilbertian seminorms on E so that with suitable $0 < \vartheta_k < 1$ and $C_k > 0$

$$\|\ \|_k^* \leq C_k \|\ \|_{k-1}^{*\,1-\vartheta_k} \|\ \|_{k+1}^{*\,\vartheta_k}. \tag{11}$$

The fundamental lemma in this section is:

Lemma 17 *Assume that $\delta_n(U_{k+1}, U_k) \leq e^{-2p\alpha_n}$ for all $n \geq n_0$. Then there is a sequence β with $\beta_n \geq \alpha_n$ for all $n \geq n_0$ and a map $\varphi_k \in L(\Lambda_\infty(\beta), E)$ so that for all $\xi \in \Lambda_\infty(\beta)$*

1. $\|\varphi_k \xi\|_k \leq 2|\xi|_0$
2. $\|\varphi\, \xi\|_{k+1} \geq \frac{1}{2}|\xi|_p$.

Moreover φ_k extends to an isomorphism $\hat{\varphi}_k \colon \ell_2 = \Lambda_0^\beta \longrightarrow E_k$.

PROOF. By assumption the map $\imath^k_{k+1}$ is compact. We set

$$\beta_n = -\frac{1}{2p}\log \delta_n(U_{k+1}, U_k).$$

Then the Schmidt representation takes the form

$$\imath^k_{k+1} x = \sum_{n=0}^{\infty} e^{-2p\beta_n} \langle x, e_n\rangle_{k+1} f_n$$

where $(e_n)_{n\in\mathbb{N}_0}$ is an orthonormal sequence in E_{k+1}, $(f_n)_{n\in\mathbb{N}_0}$ an orthonormal basis of E_k.

As in the proof of Lemma 14, we get $\alpha_n \leq \beta_n$ for all $n \geq n_0$. We set

$$\varphi\xi = \sum_{n=0}^{\infty} \xi_n e^{2p\beta_n} e_n$$

and obtain an isometry φ from Λ^{β}_{2p} into E_{k+1}. Because of

$$\imath^k_{k+1}\varphi\xi = \sum_{n=0}^{\infty} \xi_n f_n$$

the map $\imath^k_{k+1} \circ \varphi$ extends to a unitary map $\chi\colon \ell_2 = \Lambda^{\beta}_0 \longrightarrow E_k$.

We put $s = \frac{1}{2}(t_{k+1} - t_k)$. We denote by $E_{(s)}$ the local Hilbert space of $|\ |_s$ and by $\imath^{(s)}$, $\imath^{(s)}_{k+1}$, $\imath^k_{(s)}$ the respective canonical maps. We apply Lemma 8 to the map $\varphi \in L(\Lambda_\infty(\beta), E_{k+1})$ defined by φ and the seminorms $|\ |_s \leq \|\ \|_{k+1}$ on E. We obtain $\varphi_k \in L(\Lambda_\infty(\beta), E)$, $\psi \in L(\Lambda_\infty(\beta), E_{(s)})$ so that $\sup_{|\xi|_0\leq 1} |\psi\xi|_s \leq \frac{1}{2}$ and

$$\imath^{(s)}_{k+1} \circ \varphi = \imath^{(s)} \circ \varphi_k + \psi.$$

We have to verify the desired properties for φ_k. We have

$$\imath^k \circ \varphi_k = \imath^k_{k+1} \circ \varphi - \imath^k_{(s)} \circ \psi.$$

Therefore $\|\varphi_k\xi\|_k \leq 2|\xi|_0$ for $\xi \in E$. Moreover $\imath^k \circ \varphi_k$ extends to a map $\hat{\varphi}_k \in L(\ell_2, E_k)$ which can be written as

$$\hat{\varphi}_k = \chi \circ (\mathrm{id} - \chi^{-1} \circ \imath^k_{(s)} \circ \hat{\psi}).$$

Here $\hat{\psi}$ denotes, as always, the continuous extension. Since χ is invertible and $\|\chi^{-1} \circ \imath^k_{(s)} \circ \hat{\psi}\|_{L(\ell_2)} \leq \frac{1}{2}$ the map $\hat{\varphi}_k$ is invertible.

We have for $x \in E_k$

$$\chi^{-1}(x) = (\langle x, f_n\rangle_k)_{n\in\mathbb{N}_0}.$$

Therefore for $x \in E_{k+1}$

$$\chi^{-1}(\imath^k_{k+1} x) = (e^{-2p\beta_n}\langle x, e_n\rangle_{k+1})_{n\in\mathbb{N}_0}.$$

For $x \in E$ we define $h(x) := \chi^{-1}(\imath^k x)$ and obtain

$$\begin{aligned} |h(x)|_0 &\leq \|x\|_k \\ |h(x)|_{2p} &\leq \|x\|_{k+1}. \end{aligned}$$

Therefore by Lemma 9, applied to Σ_∞ and $\Lambda_\infty(\beta)$, we get

$$|h(x)|_p \leq |x|_s.$$

This implies that $\chi^{-1} \circ i^k_{(s)} \circ \psi$ extends to a map $T \in L(\Lambda^\beta_0, \Lambda^\beta_p)$ and we have $\|T\|_{L(\Lambda^\beta_0, \Lambda^\beta_p)} \leq \frac{1}{2}$. Therefore $\mathrm{id} - T$ is invertible in $L(\Lambda^\beta_p)$, with $\|(\mathrm{id} - T)^{-1}\|_{L(\Lambda^\beta_p)} \leq 2$.

For $\xi \in \Lambda_\infty(\beta)$ we set $x = (\mathrm{id} - T)\xi$ and have $\varphi_k(\xi) = \chi(x)$ hence $x = h(\varphi_k(\xi))$ which implies

$$|\xi|_p \leq 2|x|_p \leq 2\|\varphi_k(\xi)\|_{k+1}.$$

This completes the proof. ■

We assume now that α is stable and may choose $d \in \mathbb{N}$, so that

$$\limsup_n \frac{\alpha_{2n+1}}{\alpha_n} < d.$$

Lemma 18 *Under the assumptions of Lemma 17 and the assumption that $\delta_n(U_{k+1}, U_k) \leq e^{-2dp\alpha_n}$ for $n \geq n_0$ and $\limsup_n \frac{\alpha_{2n+1}}{\alpha_n} < d$ we obtain: there is a map $\varphi_k \in L(\Lambda_\infty(\alpha), E)$ and $c_k > 0$, so that*

1. $\|\varphi_k \xi\|_k \leq 2|\xi|_0$ *for all* $\xi \in \Lambda_\infty(\alpha)$

2. $|\varphi'_k y|^*_p \geq c_k \|y\|^*_{k+1}$ *for all* $y \in E'$.

Moreover φ_k extends to a surjection $\hat{\varphi}_k \colon \Lambda^\alpha_0 \longrightarrow E_k$.

Proof. From Lemma 17, applied to $(d\alpha_n)_n$, we get for given k a sequence β with $\beta_n \geq d\alpha_n$ for $n \geq n_0$ and a map $\varphi_k \in L(\Lambda_\infty(\beta), E)$ with the properties 1. and 2. in Lemma 17.

From Lemma 15 we get a increasing sequence $(n(\nu))_{\nu \in \mathbb{N}_0}$ and $b > 0$ so that

$$\frac{1}{d}\beta_\nu \leq \alpha_{n(\nu)} \leq b + \beta_\nu.$$

We set, as in the proof of Lemma 16

$$(\varphi\xi)_n = \begin{cases} \xi_\nu & \text{if } n = n(\nu) \\ 0 & \text{otherwise.} \end{cases}$$

Then φ gives an isomorphic imbedding $\Lambda_\infty(\beta) \hookrightarrow \Lambda_\infty(\alpha)$ and an isometric imbedding $\ell_2 \hookrightarrow \ell_2$. We have $|\varphi x|_t \leq e^{tb}|x|_t$ for all $t \in \mathbb{R}$.

We define $L \in L(\Lambda_\infty(\alpha), \Lambda_\infty(\beta))$ by

$$(L\xi)_{\nu \in \mathbb{N}_0} = (\xi_{n(\nu)})_{\nu \in \mathbb{N}_0}.$$

Then $L \circ \varphi = \mathrm{id}$ and $|L\xi|_t \leq |x|_{dt}$ for all $t \geq 0$. We put $\phi_k = \varphi_k \circ L$. Then $\phi_k \in L(\Lambda_\infty(\alpha), E)$.

Moreover 1. is satisfied since $|L\xi|_0 \leq |\xi|_0$ and, because L is surjective, also $\hat{\phi}_k$ is surjective. As for 2. we notice that for $y \in E'$

$$\begin{aligned} \frac{1}{2}\|y\|^*_{k+1} \leq& |\varphi'_k y|^*_p = |(\varphi' \circ L' \circ \varphi'_k)y|^*_p \\ \leq& e^{pb}|(L' \circ \varphi'_k)y|^*_p = e^{pb}|\phi'_k y|^*_p. \quad ■ \end{aligned}$$

We are now ready to prove the main theorem of this section. For the nuclear case see [30, Satz 3.4].

Theorem 4 *Let α be stable. A Fréchet space E is isomorphic to a quotient space of $\Lambda_\infty(\alpha)$ if and only if E is an α-nuclear Fréchet-Hilbert space with property (Ω).*

PROOF. If E is isomorphic to a quotient space of $\Lambda_\infty(\alpha)$ then it is Fréchet-Hilbert and, by Proposition 4, it is α-nuclear and has property (Ω).

We have to show the converse implication. As in the proof of Theorem 3 we set $n(i,k)=2^k+i2^{k+1}-1$ and $\alpha_{i,k}:=\alpha_{n(i,k)}$ and use the isomorphic representation $\Lambda_\infty(\alpha)\cong\Lambda$ where

$$\Lambda := \{x=(x_{i,k})_{i,k\in\mathbb{N}_0} \,:\, \|x\|_t^2=\sum_{i,k}|x_{i,k}|^2 e^{2t\alpha_{i,k}}<+\infty \text{ fore all } t\in\mathbb{R}\}.$$

For every $k\in\mathbb{N}_0$ we choose $d_k\in\mathbb{N}$, $b_k\geq 0$ so that $\alpha_{i,k}\leq b_k+d_k\alpha_i$ for all i and set $p_k=kd_k$. We choose by induction a sequence $t_0=0<t_1<t_2<\dots$ so that the seminorms $\|\;\|_k=|\;|_{t_k}$ satisfy the assumptions of Lemma 18 with $p=p_k$. For each $k\in\mathbb{N}_0$ let ψ_k be the map from Lemma 18.

For $x=(x_{i,k})_{i,k\in\mathbb{N}_0}$ and $k\in\mathbb{N}_0$ set $x_k=(x_{i,k})_{i\in\mathbb{N}_0}$. Since $i\leq n(i,k)$, hence $\alpha_i\leq\alpha_{n(i,k)}=\alpha_{i,k}$, we have $x_k\in\Lambda_\infty(\alpha)$ and

$$\sum_{k=0}^\infty |x_k|_t^2\leq\sum_{k=0}^\infty\sum_{i=0}^\infty|x_{i,k}|^2e^{2t\alpha_{i,k}}=\|x\|_t^2$$

for all $t\geq 0$.

We put

$$\Psi(x):=\sum_{k=1}^\infty 2^{-\frac{k}{2}}\psi_k(x_k)$$

for $x=(x_k)_k=(x_{i,k})_{i,k}\in\Lambda$. We show that the series converges and defines $\Psi\in L(\Lambda,E)$.

This follows from

$$\begin{aligned}\|\Psi x\|_m^2\leq&\left(\sum_{k=1}^{m-1}2^{-\frac{k}{2}}\|\psi_k x_k\|_m+\sum_{k=m}^\infty 2^{-\frac{k}{2}}\|\psi_k x_k\|_k\right)^2\\ \leq& C^2\sum_{k=1}^{m-1}|x_k|_M^2+4\sum_{k=m}^\infty|x_k|_0^2\\ \leq&(C^2+4)\sum_{k=0}^\infty|x_k|_M^2\\ \leq&(C^2+4)\|x\|_M^2\end{aligned}$$

with suitable C and M depending from m.

Since $\psi_k\in L(\Lambda_\infty(\alpha),E)$ and $\hat{\psi}_k$ is surjective from ℓ_2 onto E_k, we see that $R(\Psi)$ is dense in E. Let m be given. We obtain for $y\in E'$:

$$\begin{aligned}\|\Psi'y\|_m^*\geq&2^{-\frac{m}{2}}\sup\{|(\psi_m'(y))(\xi)|\,:\,\sum_i|\xi_i|^2e^{2m\alpha_{i,m}}\leq 1\}\\ \geq&2^{-\frac{m}{2}}\sup\{|(\psi_m'(y))(\xi)|\,:\,|\xi|_{md_m}\leq e^{-mb_m}\}\\ \geq&2^{-\frac{m}{2}}\cdot e^{-mb_m}|\psi_m'(y)|_{md_m}^*\\ \geq&2^{-\frac{m}{2}}e^{-mb_m}c_{md_m}\|y\|_{m+1}^*.\end{aligned}$$

The surjectivity criterion [12, 26.1] then shows that Ψ is surjective. ■

5. Complemented subspaces of power series spaces of infinite type

Our next task is to characterize the complemented subspaces of $\Lambda_\infty(\alpha)$ by invariants and study their structure.

First we prove a partial replacement for the Ramanujan and Terzioğlu imbedding theorem from [17].

Lemma 19 *If Q is a quotient of $\Lambda_\infty(\alpha)$, equipped with the quotient seminorms $\| \ \|_t$ of the norms $| \ |_t$, $t \in \mathbb{R}$, then for any k there is $S_k \in L(Q, \Lambda_\infty(\alpha))$, so that $\sup_{\|x\|_k \leq 1} |S_k x|_0 < +\infty$ and S_k induces an isometry $\hat{S}_k \colon Q_k \hookrightarrow \Lambda_0^0 = \ell_2$.*

PROOF. We may assume $k = 0$ and $\dim Q_0 = \infty$. Let $\varphi \colon \Lambda_\infty(\alpha) \longrightarrow Q$ be the quotient map. For $K > 0$ the map $\imath_K^0 \colon Q_K \longrightarrow Q_0$ is compact. Let s_n be its singular numbers. We set

$$\beta_n = -\frac{1}{K} \log s_n.$$

Then its Schmidt representation takes the form

$$\imath_K^0 x = \sum_n e^{-K\beta_n} \langle x, e_n \rangle_K f_n,$$

where $(e_n)_{n\in\mathbb{N}_0}$ is an orthonormal system in Q_K, $(f_n)_{n\in\mathbb{N}_0}$ an orthonormal basis of Q_0.

By Remark 2 and Lemma 2 we get

$$e^{-K\beta_n} \leq e^{-K\alpha_n}$$

i.e. $\beta_n \geq \alpha_n$ for all n.

We set for $x \in \Lambda_\infty(\alpha)$

$$T_K x := (\langle x, f_n \rangle)_n.$$

Then we have

$$|T_K x|_0^\beta = \|x\|_0, \qquad |T_K x|_K^\beta = \|x\|_K,$$

hence

$$|T_K x|_0^\alpha = \|x\|_0, \qquad |T_K x|_K^\alpha \leq \|x\|_K.$$

and therefore by Lemma 9, applied to $T_K \circ q$,

$$|T_k x|_k^\alpha \leq \|x\|_k \text{ for all } 0 \leq k \leq K.$$

Here $| \ |_t^\beta$ and $| \ |_t^\alpha$ denote the norms in $\Lambda_\infty(\beta)$ and $\Lambda_\infty(\alpha)$, respectively.

We argue like in the proof of Lemma 10 to find a subsequence $(T_{K_n})_{n\in\mathbb{N}}$ and an operator $T \in L(Q, \Lambda_\infty(\alpha))$ so that $T_{K_n} x \longrightarrow Tx$ for all $x \in Q$.

For every $x \in Q$ we have

$$|Tx|_0 = \lim_n |T_{K_n} x|_0 = \|x\|_0.$$

Setting $S_0 := T$ we obtain the result. ■

As an immediate consequence we obtain:

Lemma 20 *If Q is a quotient of $\Lambda_\infty(\alpha)$ then there is an imbedding $S \colon Q \hookrightarrow \Lambda_\infty(\alpha)^{\mathbb{N}}$.*

PROOF. We set $Sx = (S_k x)_{k\in\mathbb{N}}$, S_k as in Lemma 19. ■

We can now give the characterization of the complemented subspaces of $\Lambda_\infty(\alpha)$. For the nuclear case see [30, Satz 3.5].

Theorem 5 *Let α be stable. A Fréchet space E is isomorphic to a complemented subspace of $\Lambda_\infty(\alpha)$ if and only if E is an α-nuclear Fréchet-Hilbert space with properties* (DN) *and* (Ω).

PROOF. If E is isomorphic to a complemented subspace of $\Lambda_\infty(\alpha)$ then it is Fréchet-Hilbert and by Proposition 4, it is α-nuclear and has properties (DN) and (Ω). We have to prove the converse implication. Since E is an α-nuclear Fréchet-Hilbert space with property (DN) we know from Theorem 3 that E is isomorphic to a subspace of $\Lambda_\infty(\alpha)$, this means that there is an isomorphic imbedding $j\colon E \hookrightarrow \Lambda_\infty(\alpha)$.

We set $Q := \Lambda_\infty(\alpha)/jE$. By Lemma 20 there is an imbedding $S\colon Q \hookrightarrow \Lambda_\infty(\alpha)^{\mathbb{N}}$.

We consider the exact sequence

$$0 \longrightarrow \Lambda_\infty(\alpha) \longrightarrow \Lambda_\infty(\alpha) \xrightarrow{\varphi} \Lambda_\infty(\alpha)^{\mathbb{N}} \longrightarrow 0$$

from Proposition 1 and set $\tilde{Q} = \varphi^{-1}(SQ)$. We can set up the following commutative diagram with exact rows and columns:

$$\begin{tikzcd}
 & & 0 & 0 & \\
0 \arrow[r] & E \arrow[r, "j"] & \Lambda_\infty(\alpha) \arrow[r, "q"] \arrow[u] & Q \arrow[r] \arrow[u] & 0 \\
0 \arrow[r] & E \arrow[r] \arrow[u, "="'] & H \arrow[r, "q_0"] \arrow[u, "\varphi_0"'] & \tilde{Q} \arrow[r] \arrow[u, "\varphi"'] & 0 \\
 & & \Lambda_\infty(\alpha) \arrow[r, "="] \arrow[u] & \Lambda_\infty(\alpha) \arrow[u] & \\
 & & 0 \arrow[u] & 0 \arrow[u] &
\end{tikzcd}$$

Here $H = \{(x, y) \in \Lambda_\infty(\alpha) \times \tilde{Q} \,:\, qx = \varphi y\}$ is a Fréchet-Hilbert space, $\varphi_0(x, y) = x$, $q_0(x, y) = y$.

Since by assumption E has property (Ω) and $\tilde{Q}$ as a subspace of $\Lambda_\infty(\alpha)$ has property (DN) the second row splits by Theorem 2.

Since $\Lambda_\infty(\alpha)$ has properties (DN) and (Ω) the first column splits by the same theorem. Therefore we have

$$E \oplus \tilde{Q} \cong H \cong \Lambda_\infty(\alpha) \oplus \Lambda_\infty(\alpha) \cong \Lambda_\infty(\alpha).$$

The last isomorphism exists since α is stable. ■

6. About the structure of complemented subspaces of power series spaces of infinite type

After having characterized the complemented subspaces of power series spaces by invariants we will now closer study their structure. In [14, Theorem14] (see also [13]) Mityagin proved that every complemented subspace of a finite type power series space has a basis and is again isomorphic to a finite type power series space. He posed the problem whether the same is true for infinite type power series spaces, see [14, Problem 15]. In [15] he showed that that every complemented subspace of an infinite type power series space with an unconditional basis is isomorphic to an infinite type power series space. So the problem remains whether every complemented subspace of an infinite type power series space has an unconditional basis. This problem is, even in the nuclear case, still unsolved. A solution for the nuclear case has been announced without proof in Kondakov [10]. A proof proposed by the same author in [8] is apparently mistaken. Nevertheless there are many partial solutions. We will describe some of them.

A principal tool for that is the following result, in the nuclear case it was first proved in [23].

Lemma 21 *If α is stable and E is a complemented subspace of $\Lambda_\infty(\alpha)$, then $E \oplus \Lambda_\infty(\alpha) \cong \Lambda_\infty(\alpha)$.*

PROOF. Let F be a complement of E in $\Lambda_\infty(\alpha)$, i.e. $E \oplus F \cong \Lambda_\infty(\alpha)$, then

$$\Lambda_\infty(\alpha)^{\mathbb{N}} \cong E^{\mathbb{N}} \oplus F^{\mathbb{N}} \cong E \oplus E^{\mathbb{N}} \oplus F^{\mathbb{N}} \cong E \oplus \Lambda_\infty(\alpha)^{\mathbb{N}}.$$

We use again the exact sequence from Proposition 1. We add the exact sequence

$$0 \longrightarrow 0 \longrightarrow E \xrightarrow{\mathrm{id}} E \longrightarrow 0$$

and obtain by use of the previous isomorphism

$$0 \longrightarrow \Lambda_\infty(\alpha) \longrightarrow \Lambda_\infty(\alpha) \oplus E \longrightarrow \Lambda_\infty(\alpha)^{\mathbb{N}} \longrightarrow 0.$$

Like in the proof of Theorem 5 we obtain the following diagram

$$\begin{tikzcd}
 & & 0 & 0 & \\
0 \arrow[r] & \Lambda_\infty(\alpha) \arrow[r] & \Lambda_\infty(\alpha) \arrow[r] \arrow[u] & \Lambda_\infty(\alpha)^{\mathbb{N}} \arrow[r] \arrow[u] & 0 \\
0 \arrow[r] & \Lambda_\infty(\alpha) \arrow[r] & H \arrow[r] \arrow[u] & \Lambda_\infty(\alpha) \oplus E \arrow[r] \arrow[u] & 0 \\
 & & \Lambda_\infty(\alpha) \arrow[u] & \Lambda_\infty(\alpha) \arrow[u] & \\
 & & 0 \arrow[u] & 0 \arrow[u] &
\end{tikzcd}$$

Here $H \subset \Lambda_\infty(\alpha) \oplus \Lambda_\infty(\alpha) \oplus E$ is a Fréchet-Hilbert space. By use of Theorem 2 the first column and the second row split and we get

$$\Lambda_\infty(\alpha) \oplus E \cong \Lambda_\infty(\alpha) \oplus \Lambda_\infty(\alpha) \oplus E \cong H \cong \Lambda_\infty(\alpha) \oplus \Lambda_\infty(\alpha) \cong \Lambda_\infty(\alpha).$$

Here we used several times the stability of α. ■

In the nuclear case the following theorem was first proved in [23].

Theorem 6 *Let α be stable. If E is isomorphic to a complemented subspace of $\Lambda_\infty(\alpha)$ and $\Lambda_\infty(\alpha)$ isomorphic to a complemented subspace of E, then $E \cong \Lambda_\infty(\alpha)$.*

PROOF. Let $E \oplus F \cong \Lambda_\infty(\alpha)$ and $G \oplus \Lambda_\infty(\alpha) \cong E$, then $G \oplus \Lambda_\infty(\alpha) \oplus F \cong \Lambda_\infty(\alpha)$. Therefore G is isomorphic to a complemented subspace of $\Lambda_\infty(\alpha)$. By Lemma 21 we have

$$E \cong G \oplus \Lambda_\infty(\alpha) \cong \Lambda_\infty(\alpha). \quad ■$$

We will now present sufficient conditions so that a Fréchet-Hilbert space with properties (DN) and (Ω) has a basis. To formulate them we need one more concept which is due Aytuna-Krone-Terzioğlu [2]. Let E be a Fréchet-Hilbert-Schwartz space with properties (DN) and (Ω). Let $\| \ \|_p$ be a dominating norm and choose, for given p, a q according to (Ω). We set $U_k = \{x : \|x\|_k \leq 1\}$ for all k.

Definition 4 *The sequence*

$$\alpha_n := -\log \delta_n(U_q, U_p)$$

is called an associated exponent sequence of E and $\Lambda_\infty(\alpha)$ the associated power series space.

From [2] we know the following:

Lemma 22 $\Lambda_\infty(\alpha)$ *depends only on* E*, not on the choice of* U_p, U_q*.* ■

By use of Lemma 3 this implies that, in particular, E is α-nuclear. As a consequence of Lemma 22 we obtain:

Corollary 4 *If* E *is a Fréchet-Hilbert-Schwartz space with properties* (DN) *and* (Ω) *and* $\Lambda_\infty(\alpha)$ *its associated power series space and if* $E \cong F$ *then* F *has the same properties, in particular* $\Lambda_\infty(\alpha)$ *is its associated power series space.* ■

This yields

Proposition 5 *If*

$$\lambda(A) := \{x = (x_n)_{n\in\mathbb{N}_0} \,:\, \|x\|_k^2 := \sum_{j=0}^{\infty} |x_j|^2 a_{j,k} < +\infty \text{ for all } k\}$$

has properties (DN) *and* (Ω)*, and* $\Lambda_\infty(\alpha)$ *is its associated power series space then* $\lambda(A) \cong \Lambda_\infty(\alpha)$*.*

PROOF. By [29, Satz 2.7] there is a power series space $\Lambda_\infty(\beta)$ so that $\lambda(A) \cong \Lambda_\infty(\beta)$. Since then clearly $\Lambda_\infty(\beta)$ is the associated power series space of $\lambda(A)$, we have by Corollary 4 $\Lambda_\infty(\beta) = \Lambda_\infty(\alpha)$. ■

This means that, if a Fréchet-Hilbert-Schwartz space with properties (DN) and (Ω) is isomorphic to a Köthe space $\lambda(A)$ defined as above, then it is isomorphic to its associated power series space. Since, by Theorem 5 with suitable α, it is always isomorphic to a complemented subspace of some $\Lambda_\infty(\alpha)$ the result of Mityagin [15] shows that this is always the case if it has an unconditional basis.

We will use a simplified version of the proofs of Lemma 14 and Lemma 17 to show the following.

Lemma 23 *Let* E *be a Fréchet-Hilbert-Schwartz space with properties* (DN) *and* (Ω)*. Let* $\|\ \|_0$ *be a Hilbertian dominating norm,* $\|\ \|_1$ *chosen for* $\|\ \|_0$ *according to* (Ω) *and*

$$\alpha_n = -\log \delta_n(U_1, U_0)$$

where $U_j = \{x \in E \,:\, \|x\|_j \le 1\}$*.*

Then there exist maps $\psi \in L(\Lambda_\infty(\alpha), E)$*,* $\varphi \in L(E, \Lambda_\infty(\alpha))$ *so that* ψ *extends to an isomorphism* $\psi_0\colon \ell_2 \longrightarrow E_0$*,* φ *extends to an isomorphism* $\varphi_0\colon E_0 \longrightarrow \ell_2$ *and we have*

$$\sup_{|\xi|_0 \le 1} |\xi - \varphi_0 \circ \psi_0(\xi)|_0 < \frac{1}{2}.$$

PROOF. By assumption $\imath_1^0$ is compact and its Schmidt representation takes the form

$$\imath_1^0 x = \sum_{n=0}^{\infty} e^{-\alpha_n} \langle x, e_n\rangle_1 f_n$$

where $(e_n)_n$ is an orthonormal sequence in E_1 and $(f_n)_n$ an orthonormal basis of E_0.

We set $\tilde\varphi(x) = (\langle x, f_n\rangle_0)_{n\in\mathbb{N}_0}$ and obtain a unitary map $\tilde\varphi\colon E_0 \longrightarrow \ell_0 = \Lambda_0^\alpha$ for which we have

$$\tilde\varphi \circ \imath_1^0(x) = (e^{-\alpha_n}\langle x, e_n\rangle_1)_{n\in\mathbb{N}_0}$$

which means that $\tilde\varphi\circ\imath_1^0$ defines an isometry $\tilde\varphi_1$ from $F := \overline{\operatorname{span}\{e_0, e_1, \ldots\}}$ onto Λ_1^α with $F^\perp = \ker \tilde\varphi\circ\imath_1^0 = \ker \imath_1^0$. We set $\tilde\psi = \tilde\varphi^{-1}$. Then $\tilde\psi$ is a unitary map $\ell_2 = \Lambda_0^\alpha \longrightarrow E_0$ so that there is a map $\tilde\psi_1\colon \Lambda_1^\alpha \longrightarrow E_1$

with $\imath_1^0 \circ \tilde{\psi}_1 = \tilde{\psi}$. We apply Lemma 8 to $\tilde{\varphi}_1$ and $\tilde{\psi}_1$ and obtain maps $\varphi \in L(E, \Lambda_\infty(\alpha))$, $\psi \in L(\Lambda_\infty(\alpha), E)$ so that with $0 < \varepsilon < 1$ to be determined later

$$\begin{aligned} \tilde{\varphi} \circ \imath^0 =& \varphi + \chi \circ \imath^0 \quad \text{on } E \\ \tilde{\psi} =& \imath^0 \circ \psi + \eta \quad \text{on } \Lambda_\infty(\alpha) \end{aligned}$$

where $\chi \in L(E_0, \Lambda_0^\alpha)$ with $\sup_{\|x\|_0 \le 1} |\chi(x)|_0 < \varepsilon$ and $\eta \in L(\Lambda_0^\alpha, E_0)$ with $\sup_{|\xi|_0 \le 1} \|\eta(\xi)\|_0 < \varepsilon$.

Therefore φ extends to $\varphi_0 = \tilde{\varphi} - \chi \in L(E_0, \ell_2)$ and ψ to $\psi_0 = \tilde{\psi} - \eta \in L(\ell_2, E_0)$. Since $\|\chi\| < 1$ and $\|\eta\| < 1$ φ_0 and ψ_0 are invertible.

Moreover

$$\varphi_0 \circ \psi_0 = \mathrm{id} - \chi \circ \tilde{\psi} - \tilde{\varphi} \circ \eta + \chi \circ \eta$$

and therefore

$$\sup_{|\xi|_0 \le 1} |\xi - \varphi_0 \circ \psi_0(\xi)|_0 < 2\varepsilon + \varepsilon^2 < \frac{1}{2}$$

for small $\varepsilon > 0$. ■

Corollary 5 *Under the assumptions of Lemma 23 there exist maps $\psi \in L(\Lambda_\infty(\alpha), E)$, $\varphi \in L(E, \Lambda_\infty(\alpha))$ so that ψ extends to a unitary map $\psi_0 \colon \ell_2 \longrightarrow E_0$ and φ extends to a unitary map $\varphi_0 \colon E_0 \longrightarrow \ell_2$.*

PROOF. We choose ψ and φ according to Lemma 23. Then we apply Lemma 10 to the norm $\|x\| = \|\psi x\|_0$ and obtain an automorphism U of $\Lambda_\infty(\alpha)$ so that $|Ux|_0 = \|\psi x\|_0$. We do the same with the norm $\|x\| = \|\varphi^{-1} x\|_0$ and obtain an automorphism V of $\Lambda_\infty(\alpha)$ so that $|Vx|_0 = \|\varphi^{-1} x\|_0$. Finally we replace ψ by $\psi \circ U^{-1}$ and φ by $V \circ \varphi$. ■

An important step now is contained in the following lemma. The method for the construction of S is due to Aytuna, Krone and Terzioğlu [1], the difference here is again, that we don't need nuclearity.

Lemma 24 *Let α be stable, $T \in L(\Lambda_\infty(\alpha))$ so that T induces a unitary map in $L(\ell_2)$. Then there is $S \in L(\Lambda_\infty(\alpha))$, so that $P = T \circ S$ is a projection in $\Lambda_\infty(\alpha)$, orthogonal in ℓ_2, and $R(P) \cong \Lambda_\infty(\alpha)$.*

PROOF. Let $e_j = (0, \ldots, 0, 1, 0, \ldots) \in \Lambda_\infty(\alpha)$ and $f_j = Te_j$. We choose inductively vectors $g_n \in \Lambda_\infty(\alpha)$ with following properties:

(1) $g_n \in \operatorname{span}\{f_0, \ldots, f_{2n}\}$

(2) $g_n \perp g_0, \ldots, g_{n-1}$ in ℓ_2

(3) $g_n \perp e_0, \ldots, e_{n-1}$ in ℓ_2

(4) $|g_n|_0 = 1$.

This is possible since $\dim \operatorname{span}\{f_0, \ldots, f_{2n}\} = 2n + 1$. Due to (1) we have

$$g_n := \sum_{k=0}^{2n} \mu_{k,n} f_k = T(\sum_{k=0}^{2n} \mu_{k,n} e_k).$$

We set

$$h_n = \sum_{k=0}^{2n} \mu_{k,n} e_k$$

and obtain an orthonormal system $(h_n)_{n \in \mathbb{N}_0}$. We set $\mu_{k,n} = 0$ for $k > 2n$.

We define

$$Sx := \sum_{n=0}^{\infty} \langle x, g_n \rangle h_n.$$

This means $S = T^{-1} \circ P$ where P is the orthogonal projection onto $\overline{\mathrm{span}\{g_0, g_1, \ldots\}}$. We have to show that S defines a map in $L(\Lambda_\infty(\alpha))$.

We do that in two steps. First we define a map $\varphi \in L(\ell_2)$ by

$$\varphi(x) = \sum_{n=0}^{\infty} \langle x, g_n \rangle e_n.$$

For the matrix elements $\varphi_{k,j} = \langle \varphi e_j, e_k \rangle = \langle e_j, g_k \rangle$ we have $\varphi_{k,j} = 0$ for $k > j$. Therefore, by Lemma 13, $\varphi \in L(\Lambda_\infty(\alpha))$.

Next we define a map $\psi \in L(\ell_2)$ by

$$\psi(x) = \sum_{n=0}^{\infty} \langle x, e_n \rangle h_n.$$

For the matrix elements $\psi_{k,j} = \langle \psi e_j, e_k \rangle = \langle h_j, e_k \rangle$ we obtain that $\psi_{k,j} = 0$ for $k > 2j$.

We define $\tilde{\psi} \in L(\ell_2)$ by $\tilde{\psi} = \psi \circ A$ and $Ax = (x_{2n})_{n \in \mathbb{N}_0}$ for $x = (x_n)_{n \in \mathbb{N}_0}$. Then we have

$$\langle \tilde{\psi} e_j, e_k \rangle = \begin{cases} \langle \psi e_\nu, e_k \rangle & : \quad j = 2\nu \\ 0 & : \quad j = 2\nu + 1. \end{cases}$$

This means that $\tilde{\psi}_{k,j} = \langle \tilde{\psi} e_j, e_k \rangle = 0$ for $k > j$. By Lemma 13 we obtain that $\tilde{\psi} \in L(\Lambda_\infty(\alpha))$.

Now we set

$$(Bx)_j = \begin{cases} x_\nu & : \quad j = 2\nu \\ 0 & : \quad j = 2\nu + 1. \end{cases}$$

for $x = (x_n)_{n \in \mathbb{N}_0}$. Due to the stability of α we have $B \in L(\Lambda_\infty(\alpha))$ and therefore $\psi = \tilde{\psi} \circ B \in L(\Lambda_\infty(\alpha))$.

Since obviously $S = \psi \circ \varphi$ we have shown that $S \in L(\Lambda_\infty(\alpha))$. It remains to show that $R(P) \cong \Lambda_\infty(\alpha)$.

The map $T \circ \psi \in L(\Lambda_\infty(\alpha), R(P))$ is injective and, because of $(T \circ \psi) \circ \varphi = T \circ S = P$, also surjective. Therefore it is an isomorphism. ■

From Lemmas 23, 24 and Theorems 5, 6 we derive the following theorem which was shown in the nuclear case by Aytuna, Krone and Terzioğlu in [1].

Theorem 7 *If E is a Fréchet-Hilbert-Schwartz space with properties* (DN) *and* (Ω) *and its associated power series space $\Lambda_\infty(\alpha)$ is stable, then $E \cong \Lambda_\infty(\alpha)$.*

PROOF. From Corollary 5 we get $\varphi \in L(E, \Lambda_\infty(\alpha))$, $\psi \in L(\Lambda_\infty(\alpha), E)$ so that $T := \varphi \circ \psi$ extends to a unitary map in $L(\ell_2)$. Then by Lemma 24 we get $S \in L(\Lambda_\infty(\alpha))$, so that $P = T \circ S$ is a projection in $\Lambda_\infty(\alpha)$ with $R(P) \cong \Lambda_\infty(\alpha)$.

We set $\pi := \psi \circ S \circ P \circ \varphi \in L(E)$ and obtain a projection. $P \circ \varphi \in L(R(\pi), R(P))$ is an isomorphism, since $\psi \circ S|_{R(P)}$ is its inverse. As the assumptions imply that E is α-nuclear (see the remark after Lemma 22), Theorems 5 and 6 yield the result. ■

Stability of the associated power series space is one condition which implies the existence of a basis in E, and by far the most important for analysis since in applications in analysis the associated power series usually can be calculated and is stable. Another condition is the following, see [6], [7].

Definition 5 *$\Lambda_\infty(\alpha)$ is called tame if, up to equivalence, α has the following form: there are strictly increasing sequences $n(k)$ in $\mathbb{N}_0$ with $n(0) = 0$ and $\beta_k > 0$ so that*

(1) $\alpha_n = \beta_k$ *for* $n(k) \le n < n(k+1)$

(2) $\lim_k \frac{\beta_{k+1}}{\beta_k} = +\infty$.

These spaces have the following properties (see [6, Proposition 1], [7, Theorem 1.3]).

Theorem 8 *The following are equivalent*

(1) $\Lambda_\infty(\alpha)$ *is tame.*

(2) There exists d *so that for every* $A \in L(\Lambda_\infty(\alpha))$ *there is* b *with*

$$|Ax|_k \le C_k |x|_{dk+b}$$

for all k *with suitable* C_k.

(3) There are countably many functions $\sigma_m(\cdot)$, $m \in \mathbb{N}$ *so that for every* $A \in L(\Lambda_\infty(\alpha))$ *there is* m *with*

$$|Ax|_k \le C_k |x|_{\sigma_m(k)}$$

for all k *with suitable* C_k.

(4) The set of finite limit points of $\{\frac{\alpha_\mu}{\alpha_\nu} \,:\, \mu, \nu \in \mathbb{N}_0\}$ *is bounded.* ■

If α has, without equivalence, the form given in the definition, then α is called blockwise unstable. Then we have in a natural way

$$\Lambda_\infty(\alpha) \cong \{x = (x_0, x_1, \ldots) \in \prod_k \ell_2(m(k)) \,:\, |x|_t^2 = \sum_k |x_k|^2 e^{2t\beta_k} < +\infty \text{ for all } t \in \mathbb{R}\} \tag{12}$$

where $m(k) = n(k+1) - n(k)$ and $\ell_2(m(k))$ is the $m(k)$-dimensional Hilbert space.

Every $A \in L(\Lambda_\infty(\alpha))$ corresponds to a matrix $(A_{j,\nu})_{j,\nu \in \mathbb{N}_0}$ where $A_{j,\nu} \in L(\ell_2(m(\nu)), \ell_2(m(j)))$. We put

$$A^0_{j,\nu} := \delta_{j,\nu} A_{j,\nu}, \quad A^1_{j,\nu} = A_{j,\nu} - A^0_{j,\nu}.$$

The following result, which is a generalization of a result in [3], is contained in [7, Lemma 2.1].

Lemma 25 *If* α *is blockwise unstable,* $A \in L(\Lambda_\infty(\alpha))$ *and* $|Ax|_0 \le C|x|_0$, *then for each* $\varepsilon > 0$ *the set* $A^1 U_\varepsilon$ *is relatively compact in* $\Lambda_\infty(\alpha)$, *where*

$$U_\varepsilon = \{x \in \Lambda_\infty(\alpha) \,:\, |x|_\varepsilon \le 1\}.$$

PROOF. We need to prove only that $A^1 U_\varepsilon$ is bounded in $L(\Lambda_\infty(\alpha))$. Fix $t > 0$.

(1) For $\nu > j \ge j_0$ we obtain from the continuity of A in ℓ_2

$$\left| \sum_{\nu=j+1}^{\infty} A_{j,\nu} x_\nu \right| \le C \left(\sum_{\nu=j+1}^{\infty} |x_\nu|^2 \right)^{\frac{1}{2}}.$$

Therefore

$$\left| \sum_{\nu=j+1}^{\infty} A_{j,\nu} x_\nu \right| e^{t\beta_j} \le C e^{-\beta_j} \left(\sum_{\nu=j+1}^{\infty} |x_\nu|^2 e^{2\varepsilon\beta_\nu} \right)^{\frac{1}{2}}$$

if j_0 is so large that $(t+1)\beta_j \le \varepsilon \beta_{j+1}$.

(2) For $\nu < j$, $j \geq j_0$ we find $C_{t+2}, \sigma(t+2)$ so that

$$|Ax|_{t+2} \leq C_{t+2}|x|_{\sigma(t+2)}$$

and therefore

$$\|A_{j,\nu}\|e^{(t+2)\beta_j} \leq C_{t+2}e^{\sigma(t+2)\beta_\nu}.$$

Here $\|A_{j,\nu}\|$ denotes the norm in $L(\ell_2(m(\nu)), \ell_2(m(j)))$. From this we get

$$\begin{aligned}\|A_{j,\nu}\|e^{t\beta_j} \leq & C_{t+2}e^{\sigma(t+2)\beta_\nu - 2\beta_j}\\ \leq & C_{t+2}e^{-\beta_j}\end{aligned}$$

if j_0 is so large that $\sigma(t+2)\beta_{j-1} \leq \beta_j$ for $j \geq j_0$.

(3) From all this we get

$$\begin{aligned}|A^1x|_t \leq & e^{t\beta_{j_0}} \left(\sum_{j=1}^{j_0} \left| \sum_{\nu=0}^{\infty} A_{j,\nu}x_\nu \right|^2 \right)^{\frac{1}{2}} + \left(\sum_{j=j_0}^{\infty} \left| \sum_{\nu=j+1}^{\infty} A_{j,\nu}x_\nu \right|^2 e^{2t\beta_j} \right)^{\frac{1}{2}}\\ & + \left(\sum_{j=j_0}^{\infty} \left| \sum_{\nu=0}^{j-1} A_{j,\nu}x_\nu \right|^2 e^{2t\beta_j} \right)^{\frac{1}{2}}\\ \leq & e^{t\beta_{j_0}}C|x|_0 + C\left(\sum_j e^{-2\beta_j} \right)^{\frac{1}{2}} |x|_\varepsilon + C_{t+2}\left(\sum_j j^2 e^{-2\beta_j} \right)^{\frac{1}{2}} |x|_0.\end{aligned}$$

The last estimate holds since

$$\begin{aligned}\left| \sum_{\nu=0}^{j-1} A_{j,\nu}x_\nu \right| e^{t\beta_j} \leq & \left(\sum_{\nu=0}^{j-1} \|A_{j,\nu}\|^2 e^{2t\beta_j} \right)^{\frac{1}{2}} |x|_0\\ \leq & C_{t+2}j^{\frac{1}{2}}e^{-\beta_j}|x|_0. \quad \blacksquare\end{aligned}$$

Lemma 26 *Let $\Lambda_\infty(\alpha)$ be tame, $A \in L(\Lambda_\infty(\alpha))$ and $\sup_{|x|_0 \leq 1} |x - Ax|_0 < 1$. Then A is invertible.*

PROOF. We may assume that α is blockwise unstable and that, in the notation of Definition 5, all $\beta_j \in \mathbb{N}$. We calculate $x - A^0x$ as follows: we put $B(t)x = (e^{it\beta_j}x)_j$ for $x = (x_j)_j \in \Lambda_\infty(\alpha)$ in the representation (12). Then $B(t) \in L(\Lambda_\infty(\alpha))$ and $B(t)$ is unitary on ℓ_2. We obtain

$$x - A^0x = \frac{1}{2\pi}\int_0^{2\pi} B(t)(I - A)B(-t)x\,dt.$$

From this we derive easily

$$\sup_{|x|_0 \leq 1} |x - A^0x|_0 < 1.$$

Therefore A^0 is invertible in $\ell_2 = \Lambda_0^\alpha$. ${A^0}^{-1}$ being a blockwise diagonal map defines a map in $L(\Lambda_\infty(\alpha))$. Hence we obtain

$${A^0}^{-1} \circ A = I + {A^0}^{-1} \circ A^1$$

where ${A^0}^{-1} \circ A^1$ is compact in $L(\Lambda_\infty(\alpha))$. This implies that ${A^0}^{-1} \circ A$ is a Fredholm map with index $= 0$ in $L(\Lambda_\infty(\alpha))$.

The assumption implies that A is invertible in $\ell_2 = \Lambda_0^\alpha$. Therefore $\ker {A^0}^{-1} \circ A = \{0\}$. This proves the result. $\blacksquare$

We obtain the following theorem, for the nuclear case see Wagner [32, Theorem 5] and Kondakov [9].

Theorem 9 *If E is a Fréchet-Hilbert-Schwartz space with properties* (DN) *and* (Ω) *and its associated power series space $\Lambda_\infty(\alpha)$ is tame, then $E \cong \Lambda_\infty(\alpha)$.*

PROOF. We apply Lemma 26 to $A = \varphi \circ \psi$ where $\varphi \in L(E, \Lambda_\infty(\alpha))$ and $\psi \in L(\Lambda_\infty(\alpha), E)$ are the maps of Lemma 23. Notice that α in Lemma 23 is the associated exponent sequence.

Lemma 26 yields that A is invertible. We set $\chi := A^{-1} \circ \varphi \in L(E, \Lambda_\infty(\alpha))$. Then $\chi \circ \psi = \mathrm{id}$. From this we conclude that $P := \psi \circ \chi$ is a projection in E. If $P(x) = 0$ then $|x|_0 = 0$ hence $x = 0$. So $\ker P = 0$ and $P = \mathrm{id}$, i.e. $\psi \circ \chi = \mathrm{id}$. So χ is an isomorphism. ∎

The preceding theorem is a generalization of the following theorem shown in [6, Theorem], [7, Theorem 2.4].

Theorem 10 *If $\Lambda_\infty(\alpha)$ is tame then every complemented subspace of $\Lambda_\infty(\alpha)$ has a basis.*

PROOF. We have to show that Theorem 9 implies Theorem 10. Let E be complemented in $\Lambda_\infty(\alpha)$ and F a complement. Let $\Lambda_\infty(\beta)$ and $\Lambda_\infty(\gamma)$ be the associated power series spaces of E and F, respectively. Then clearly $\Lambda_\infty(\alpha) = \Lambda_\infty(\beta) \oplus \Lambda_\infty(\gamma)$. where $\oplus$ means that to get $\Lambda_\infty(\alpha)$ we have to take an increasing common rearrangement of α and β. Therefore there is a subsequence $\tilde{\alpha} = (\alpha_{j_\nu})_{\nu \in \mathbb{N}_0}$ so that $\Lambda_\infty(\tilde{\alpha}) = \Lambda_\infty(\beta)$. From there it is easily seen that $\Lambda_\infty(\beta)$ is tame, hence $E \cong \Lambda_\infty(\beta)$ by Theorem 9. ∎

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