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On Nuclear Maps Between Spaces of Ultradifferentiable Jets of Roumieu Type

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To the memory of our friend Klaus Floret

Abstract. If K is a non void compact subset of $\mathbb{R}^r$, we give a condition under which the canonical injection from $\mathcal{E}_{\{M\},b}(K)$ into $\mathcal{E}_{\{M\},d}(K)$ is nuclear. We then consider the mixed case and obtain the existence of a nuclear extension map from $\mathcal{E}_{\{M_1\}}(F)_A$ into $\mathcal{E}_{\{M_2\}}(\mathbb{R}^r)_D$ where F is a proper closed subset of $\mathbb{R}^r$ and A and D suitable Banach disks. We finally apply this last result to the Borel case, i.e. when $F = \{0\}$.

Sobre aplicaciones nucleares entre espacios de jets ultradiferenciables de tipo Roumieu

Resumen. Si K es un compacto no vacío en $\mathbb{R}^r$, damos una condición suficiente para que la inyección canónica de $\mathcal{E}_{\{M\},b}(K)$ en $\mathcal{E}_{\{M\},d}(K)$ sea nuclear. Consideramos el caso mixto y obtenemos la existencia de un operador de extension nuclear de $\mathcal{E}_{\{M_1\}}(F)_A$ en $\mathcal{E}_{\{M_2\}}(\mathbb{R}^r)_D$ donde F es un subconjunto cerrado propio de $\mathbb{R}^r$ y A y D son discos de Banach adecuados. Finalmente aplicamos este último resultado al caso Borel, es decir cuando $F = \{0\}$.

1. Introduction

Let us first recall the following facts about the quasi-LB-spaces studied in [7].

We endow the set $\mathbb{N}^{\mathbb{N}}$ of the sequences of positive integers with the order $\leq$ defined by

$$(a_n)_{n\in\mathbb{N}} \leq (b_n)_{n\in\mathbb{N}} \iff a_n \leq b_n, \forall n \in \mathbb{N}.$$

A *quasi*-LB-*representation* in a locally convex space E is a set $\left\{ A_{\boldsymbol{a}} : \boldsymbol{a} \in \mathbb{N}^{\mathbb{N}} \right\}$ of Banach disks in E submitted to the following two requirements :
(a) $\cup \left\{ A_{\boldsymbol{a}} : \boldsymbol{a} \in \mathbb{N}^{\mathbb{N}} \right\} = E$;
(b) $(\boldsymbol{a}, \boldsymbol{b} \in \mathbb{N}^{\mathbb{N}}; \boldsymbol{a} \leq \boldsymbol{b}) \Rightarrow A_{\boldsymbol{a}} \subset A_{\boldsymbol{b}}$.
A *quasi*-LB-*space* is a locally convex space having a quasi-LB-representation.

Let us the remark that the Proposition 12 of [7] leads to the following property that will be used later on. *Let* $\left\{ A_{\boldsymbol{a}} : \boldsymbol{a} \in \mathbb{N}^{\mathbb{N}} \right\}$ *be a quasi*-LB-*representation in the locally convex space* E *and let* T *be a linear map with closed graph from* E *onto a Banach space* F. *Then for every compact subset* K *of* F, *there are* $\boldsymbol{a} \in \mathbb{N}^{\mathbb{N}}$ *and a compact subset* H *of* $E_{A_{\boldsymbol{a}}}$ *such that* $H \subset A_{\boldsymbol{a}}$ *and* $TH = K$.

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Throughout the paper, unless otherwise stated, r designates a positive integer and $\boldsymbol{M} = (M_n)_{n\in\mathbb{N}_0}$ designates a sequence of positive numbers which is
(a) *normalized*, i.e. $M_0 = 1$ and $M_n \geq 1$ for every $n \in \mathbb{N}$;
(b) *logarithmically convex*, i.e. $M_n^2 \leq M_{n-1}M_{n+1}$ for every $n \in \mathbb{N}$.

We now introduce the main locally convex spaces we shall deal with.

The space $\mathcal{E}^m(K)$.

For every non void compact subset K of $\mathbb{R}^r$ and integer $m \in \mathbb{N}_0$, $\mathcal{E}^m(K)$ is the Banach space introduced by Whitney. For the sake of completeness we recall that its elements are the Whitney m-jets $\boldsymbol{\varphi}$ on K, i.e. $\boldsymbol{\varphi}$ is a family $(\varphi_\alpha)_{\alpha\in\mathbb{N}_0^r,|\alpha|\leq m}$ of continuous functions on K such that

$$\sup_{|\alpha|\leq m} \sup_{\substack{x,y\in K\\ 0<|x-y|\leq t}} \frac{|(R^m\varphi_\alpha)(x,y)|}{|y-x|^{m-|\alpha|}} \to 0 \text{ if } t \to 0^+$$

where

$$(R^m\varphi_\alpha)(x,y) := \varphi_\alpha(y) - \sum_{|\beta|\leq m-|\alpha|} \varphi_{\alpha+\beta}(x)\frac{(y-x)^\beta}{\beta!}.$$

Its norm $\|\cdot\|_{\mathcal{E}^m(K)}$ is defined by

$$\|\boldsymbol{\varphi}\|_{\mathcal{E}^m(K)} := |\boldsymbol{\varphi}|_{\mathcal{E}^m(K)} + \sup_{|\alpha|\leq m} \sup_{\substack{x,y\in K\\ x\neq y}} \frac{|(R^m\varphi_\alpha)(x,y)|}{|y-x|^{m-|\alpha|}}$$

where

$$|\boldsymbol{\varphi}|_{\mathcal{E}^m(K)} := \sup_{|\alpha|\leq m} \|\varphi_\alpha\|_K .$$

The space $\mathcal{E}_{\{\boldsymbol{M}\}}(\mathbb{R}^r)$.

Let us briefly recall its definition as introduced in [3]. Its elements are the C^∞-functions f on $\mathbb{R}^r$ such that, for every compact subset K of $\mathbb{R}^r$, there are $A > 0$ and $h > 0$ such that

$$|\mathrm{D}^\alpha f(x)| \leq Ah^{|\alpha|}M_{|\alpha|}, \quad \forall x \in K, \forall \alpha \in \mathbb{N}_0^r.$$

Its topology is defined as follows. For every ball $b(m) := \{\, x \in \mathbb{R}^r : |x| \leq m\}$ and $h > 0$, one introduces the Banach space $\mathcal{E}_{\{\boldsymbol{M}\},m,h}(\mathbb{R}^r)$: its elements are the restrictions to $b(m)$ of the C^∞-functions f on $\mathbb{R}^r$ such that

$$\pi_{\{\boldsymbol{M}\},m,h}(f) := \sup_{\alpha\in\mathbb{N}_0^r} \frac{\|\mathrm{D}^\alpha f\|_{b(m)}}{h^{|\alpha|}M_{|\alpha|}} < \infty$$

and endows it with the norm $\pi_{\{\boldsymbol{M}\},m,h}$. Finally one sets

$$\mathcal{E}_{\{\boldsymbol{M}\}}(R^r) = \operatorname*{proj}_{m\in\mathbb{N}} \operatorname*{ind}_{k\in\mathbb{N}} \mathcal{E}_{\{\boldsymbol{M}\},m,k}(R^r);$$

so $\mathcal{E}_{\{\boldsymbol{M}\}}(\mathbb{R}^r)$ is a locally convex space. More precisely as a Hausdorff projective limit of a sequence of (LB)-spaces it is a quasi-LB-space (cf. [7]).

The space $\mathcal{E}_{\{\boldsymbol{M}\}}(F)$.

In this notation, F designates a proper closed subset of $\mathbb{R}^r$. A *jet* $\boldsymbol{\varphi}$ on F is a family $\boldsymbol{\varphi} = (\varphi_\alpha)_{\alpha\in\mathbb{N}_0^r}$ of continuous function on F. The elements of $\mathcal{E}_{\{\boldsymbol{M}\}}(F)$ are the Whitney jets of class $\{\boldsymbol{M}\}$ on F, i.e. the jets $\boldsymbol{\varphi}$ on F such that for every compact subset K of F, there is $h > 0$ such that

$$|\boldsymbol{\varphi}|_{K,h} := \sup_{\alpha\in\mathbb{N}_0^r} \frac{\|\varphi_\alpha\|_K}{h^{|\alpha|}\boldsymbol{M}_{|\alpha|}} < \infty$$

and

$$\|\boldsymbol{\varphi}\|_{K,h} := \sup_{m\in\mathbb{N}_0} \sup_{\substack{\alpha\in\mathbb{N}_0^r\\|\alpha|\le m}} \sup_{\substack{x,y\in K\\x\neq y}} \frac{|(R^m\varphi_\alpha)(x,y)|}{h^{m+1}M_{m+1}} \frac{(m-|\alpha|+1)!}{|y-x|^{m-|\alpha|+1}} < \infty.$$

Clearly $\mathcal{E}_{\{\boldsymbol{M}\}}(F)$ is a vector space. To define its topology, one proceeds as follows.

If $F = K$ is compact, then for every positive integer s, $\mathcal{E}_{\{\boldsymbol{M}\},s}(K)$ denotes the vector subspace of $\mathcal{E}_{\{\boldsymbol{M}\}}(K)$ the elements $\boldsymbol{\varphi}$ of which verify $|\boldsymbol{\varphi}|_{K,s} + \|\boldsymbol{\varphi}\|_{K,s} < \infty$, endowed with the norm $|\cdot|_{K,s} + \|\cdot\|_{K,s}$; it is a Banach space. Then we set

$$\mathcal{E}_{\{\boldsymbol{M}\}}(K) = \operatorname*{ind}_{s\in\mathbb{N}} \mathcal{E}_{\{\boldsymbol{M}\},s}(K),$$

a Hausdorff (LB)-space indeed hence a quasi-LB-space.

If F is not compact, we consider a sequence $(H_s)_{s\in\mathbb{N}}$ of compact subsets of $\mathbb{R}^r$ such that $H_s = H_s^{\circ,-} \subset H_{s+1}^\circ$, $K_s = H_s \cap F \neq \emptyset$ and $\mathbb{R}^r = \cup_{s=1}^\infty H_s$ and set

$$\mathcal{E}_{\{\boldsymbol{M}\}}(F) = \operatorname*{proj}_{s\in\mathbb{N}} \mathcal{E}_{\{\boldsymbol{M}\}}(K_s).$$

It is a Hausdorff (LF)-space hence a quasi-LB-space.

Results.

We establish that the canonical map from $\mathcal{E}^{r+1}(K)$ into $\mathcal{E}^0(K)$ is nuclear; this extends a result of Komatsu (cf. [3]). We then obtain a result establishing the nuclearity of the canonical injection from $\mathcal{E}_{\{\boldsymbol{M}\},b}(K)$ into $\mathcal{E}_{\{\boldsymbol{M}\},d}(K)$ for some $d > b$. Finally we consider the mixed problem. In this case two sequences $\boldsymbol{M}_1$ and $\boldsymbol{M}_2$ are used; they are submitted to a condition of the type

$$L \subset \left\{ (\mathrm{D}^\alpha f|_F)_{\alpha\in\mathbb{N}_0^r} : f \in \mathcal{E}_{\{\boldsymbol{M}_2\}}(\mathbb{R}^r) \right\}$$

where L is a vector subspace of $\mathcal{E}_{\{\boldsymbol{M}_1\}}(F)$. We obtain a result providing the existence of a nuclear extension map from a subspace of L into $\mathcal{E}_{\{\boldsymbol{M}_2\}}(\mathbb{R}^r)$. We finally examine the application of this last result to the Borel case, i.e. when F reduces to $\{0\}$.

Let us mention that the problem of the existence of extension maps in the mixed setting has also been examined in [1], [2], [4] and [5].

2. A nuclearity result about the $\mathcal{E}_{\{\boldsymbol{M}\},s}(K)$ spaces

For the sake of completeness, let us mention with proof the following Lemma that was obtained by Komatsu in [3], under the assumption that K is *regular*, i.e. K has a finite number of connected components and there is a constant $C > 0$ such that any two points x, y of any connected component H of K are the endpoints of a rectifiable curve contained in H and of length $\le C\,|x-y|$. Of course if K is convex, it is regular. Let us recall the following property that will be used later on: *if the compact subset K is regular, then, for every $m \in \mathbb{N}_0$, the norms $|\cdot|_{\mathcal{E}^m(K)}$ and $\|\cdot\|_{\mathcal{E}^m(K)}$ are equivalent on $\mathcal{E}^m(K)$* (cf. [6] page 76, for instance).

Lemma 1 *For every non void compact subset K of $\mathbb{R}^r$, the continuous linear map*

$$J : \mathcal{E}^{r+1}(K) \to \mathcal{E}^0(K); \quad \boldsymbol{\varphi} = (\varphi_\alpha)_{|\alpha|\le r+1} \mapsto \varphi_0$$

is nuclear.

PROOF. Let h be a positive integer such that K is contained in the interior of $[-h,h]^r$. We are going to use the following Banach space $\mathrm{C}_H^{r+1}(\pi H)$: its elements are the C^{r+1}-functions on πH with support contained in H and its norm is $\|\cdot\| := \sup_{|\alpha|\le r+1} \|\mathrm{D}^\alpha\cdot\|_{\pi H}$.

The Whitney extension theorem provides the existence of a continuous linear extension map E from $\mathcal{E}^{r+1}(K)$ into $\mathrm{C}_H^{r+1}(\pi H)$, i.e. such that $(\mathrm{D}^\alpha E\boldsymbol{\varphi})(x) = \varphi_\alpha(x)$ for every $\boldsymbol{\varphi} \in \mathcal{E}^{r+1}(K)$, $x \in K$ and $\alpha \in \mathbb{N}_0^r$

such that $|\alpha| \leq r+1$. Let us denote by $\|E\|$ the norm of this map E. For every $k \in \mathbb{Z}^r$, we then designate by v_k the continuous linear functional

$$v_k \colon \mathrm{C}_H^{r+1}(\pi H) \to \mathbb{C}; \quad g \mapsto \int_{\pi H} g(y) \mathrm{e}^{-iyk/h}\, dy.$$

It is well known that if $\|v_k\|$ is the norm of v_k, there is a constant $L > 0$ such that $\|v_k\| \leq L(1+|k|)^{-1-r}$ for every $k \in \mathbb{Z}^r$. Finally we set $u_k := v_k \circ E$ for every $k \in \mathbb{Z}^r$ as well as $\psi_k(x) := (2\pi h)^{-r} \mathrm{e}^{ixk/h}$ for every $k \in \mathbb{Z}^r$ and $x \in K$.

Of course for every $k \in \mathbb{Z}$, ψ_k belongs to $\mathcal{E}^0(K)$ and u_k to the dual of $\mathcal{E}^{r+1}(K)$. If we designate by $|u_k|$ the norm of u_k, we successively get

$$\begin{aligned}\sum_{k\in\mathbb{Z}^r} |u_k|\, \|\psi_k\|_{\mathcal{E}^0(K)} &\leq (2\pi h)^{-r} \sum_{k\in\mathbb{Z}^r} \|v_k\|\, \|E\| \\ &\leq \frac{L\,\|E\|}{(2\pi h)^r} \sum_{k\in\mathbb{Z}^r} \frac{1}{(1+|k|)^{r+1}} < \infty.\end{aligned}$$

Hence the conclusion since for every $\boldsymbol{\varphi} \in \mathcal{E}^{r+1}(K)$, we have

$$\begin{aligned}(J\boldsymbol{\varphi})(x) &= (2\pi h)^{-r} \sum_{k\in\mathbb{Z}^r} \mathrm{e}^{ixk/h} \int_{\pi H} (E\boldsymbol{\varphi})(y) \mathrm{e}^{-iyk/h}\, dy \\ &= \sum_{k\in\mathbb{Z}^r} \langle \boldsymbol{\varphi}, u_k\rangle\, \psi_k(x), \quad \forall x \in K. \quad \blacksquare\end{aligned}$$

Theorem 1 *Let K be a non empty convex and compact subset of $\mathbb{R}^r$ and let the sequence $\boldsymbol{M}$ verify the following condition: there are positive constants P and Q such that $M_{n+1} \leq PQ^n M_n$ for every $n \in \mathbb{N}_0$.*

Then for every $b \in \mathbb{N}$, there is an integer $d > b$ such that the continuous linear injection from $\mathcal{E}_{\{\boldsymbol{M}\},b}(K)$ into $\mathcal{E}_{\{\boldsymbol{M}\},d}(K)$ is nuclear.

PROOF. Let the map $J \colon \mathcal{E}^{r+1}(K) \to \mathcal{E}^0(K)$ as well as the $u_k \in \mathcal{E}^{r+1}(K)'$ and $\psi_k \in \mathcal{E}^0(K)$ be defined as in the Proposition 1 and its proof. We then order the family $(u_k, \psi_k)_{k\in\mathbb{Z}^r}$ as a sequence $(w_j, \phi_j)_{j\in\mathbb{N}}$; this leads to

$$\sum_{j=1}^{\infty} |w_j| < \infty \;\text{ and }\; J\boldsymbol{\varphi} = \varphi_0 = \sum_{j=1}^{\infty} \langle \boldsymbol{\varphi}, w_j\rangle\, \phi_j, \quad \forall \boldsymbol{\varphi} \in \mathcal{E}^{r+1}(K).$$

To every $\boldsymbol{\varphi} \in \mathcal{E}_{\{\boldsymbol{M}\},b}(K)$ and $\alpha \in \mathbb{N}_0^r$, let us associate the $(r+1)$-jet

$$\boldsymbol{\varphi}^{(\alpha)} := (\varphi_{\alpha+\beta})_{\beta\in\mathbb{N}_0^r, |\beta|\leq r+1}.$$

Obviously we have $\boldsymbol{\varphi}^{(\alpha)} \in \mathcal{E}^{r+1}(K)$ hence

$$J\boldsymbol{\varphi}^{(\alpha)} = \varphi_\alpha = \sum_{j=1}^{\infty} \left\langle \boldsymbol{\varphi}^{(\alpha)}, w_j \right\rangle \phi_j. \tag{1}$$

Now we choose an integer l such that

$$l > b \text{ and } \frac{bQ^{r+1}}{l} < \frac{1}{(1+r)^{2r}}.$$

Then for every $j \in \mathbb{N}$ and $\alpha \in \mathbb{N}_0^r$, we designate by $u_{\alpha,j}$ the continuous linear functional defined on $\mathcal{E}_{\{\boldsymbol{M}\},b}(K)$ by

$$\langle \boldsymbol{\varphi}, u_{\alpha,j}\rangle := \frac{3\left\langle \boldsymbol{\varphi}^{(\alpha)}, w_j\right\rangle}{l^{|\alpha|} M_{|\alpha|}}, \quad \forall \boldsymbol{\varphi} \in \mathcal{E}_{\{\boldsymbol{M}\},b}(K),$$

and we denote its norm by $|||u_{\alpha,j}|||$. From the inequality (1), we get

$$\|\varphi_\alpha\|_K \leq \frac{1}{3} l^{|\alpha|} M_{|\alpha|} \sum_{j=1}^{\infty} |\langle \boldsymbol{\varphi}, u_{\alpha,j}\rangle|$$

hence

$$|\boldsymbol{\varphi}|_{K,l} = \sup_{\alpha\in\mathbb{N}_0^r} \frac{\|\varphi_\alpha\|_K}{l^{|\alpha|} M_{|\alpha|}} \leq \frac{1}{3} \sum_{\alpha\in\mathbb{N}_0^r} \sum_{j=1}^{\infty} |\langle \boldsymbol{\varphi}, u_{\alpha,j}\rangle| \,. \tag{2}$$

As K is convex, it is regular. Therefore the norms $|\cdot|_{\mathcal{E}^{r+1}(K)}$ and $\|\cdot\|_{\mathcal{E}^{r+1}(K)}$ are equivalent on $\mathcal{E}^{r+1}(K)$: there is $A > 0$ such that

$$\|\cdot\|_{\mathcal{E}^{r+1}(K)} \leq A \, |\cdot|_{\mathcal{E}^{r+1}(K)} \text{ on } \mathcal{E}^{r+1}(K).$$

This successively leads to

$$\left|\left\langle \boldsymbol{\varphi}^{(\alpha)}, w_j \right\rangle\right| \leq |w_j| \left\|\boldsymbol{\varphi}^{(\alpha)}\right\|_{\mathcal{E}^{r+1}(K)} \leq A \, |w_j| \left|\boldsymbol{\varphi}^{(\alpha)}\right|_{\mathcal{E}^{r+1}(K)}$$

with

$$\begin{aligned}\left|\boldsymbol{\varphi}^{(\alpha)}\right|_{\mathcal{E}^{r+1}(K)} &= \sup_{|\beta|\leq r+1} \|\varphi_{\alpha+\beta}\|_K \leq b^{|\alpha|+r+1} M_{|\alpha|+r+1} \sup_{|\beta|\leq r+1} \frac{\|\varphi_{\alpha+\beta}\|_K}{b^{|\alpha+\beta|} M_{|\alpha+\beta|}} \\ &\leq b^{|\alpha|+r+1} M_{|\alpha|+r+1} \, |\boldsymbol{\varphi}|_{K,b}\end{aligned}$$

hence

$$\left|\left\langle \boldsymbol{\varphi}^{(\alpha)}, w_j \right\rangle\right| \leq A \, |w_j| \, b^{|\alpha|+r+1} M_{|\alpha|+r+1} (|\boldsymbol{\varphi}|_{K,b} + \|\boldsymbol{\varphi}\|_{K,b}).$$

As the inequalities $M_{|\alpha|+1} \leq PQ^{|\alpha|} M_{|\alpha|}$, $M_{|\alpha|+2} \leq PQ^{|\alpha|+1} M_{|\alpha|+1}, \ldots$ lead to

$$M_{|\alpha|+r+1} \leq P^{r+1} Q^{|\alpha|(r+1)} Q^{r(r+1)/2} M_{|\alpha|},$$

we finally obtain

$$|\langle \boldsymbol{\varphi}, u_{\alpha,j}\rangle| \leq 3A \, |w_j| \left(\frac{bQ^{r+1}}{l}\right)^{|\alpha|} (bPQ^{r/2})^{r+1} (|\boldsymbol{\varphi}|_{K,b} + \|\boldsymbol{\varphi}\|_{K,b}). \tag{3}$$

So if we set $B := 3A(bPQ^{r/2})^{r+1}$, we get

$$|||u_{\alpha,j}||| \leq B \, |w_j| \, (1+r)^{-2r|\alpha|}, \quad \forall \alpha \in \mathbb{Z}^r, \forall j \in \mathbb{N}.$$

For every $s \in \mathbb{N}$, this leads to

$$\sum_{|\alpha|=s} |||u_{\alpha,j}||| \leq B \, |w_j| \, s^r (1+r)^{-2rs} \leq B \, |w_j| \, (2^s/2^{2s})^r = 2^{-s} B \, |w_j|$$

hence

$$\sum_{j=1}^{\infty} \sum_{\alpha\in\mathbb{N}_0^r} |||u_{\alpha,j}||| \leq \sum_{j=1}^{\infty} \sum_{s=0}^{\infty} \sum_{|\alpha|=s} |||u_{\alpha,j}||| \leq 2B \sum_{j=1}^{\infty} |w_j| < \infty. \tag{4}$$

Given $\boldsymbol{\varphi} \in \mathcal{E}_{\{\boldsymbol{M}\},b}(K)$ real and $m \in \mathbb{N}_0$, the finite jet $(\varphi_\alpha)_{|\alpha|\leq m+1}$ belongs of course to $\mathcal{E}^{m+1}(K)$ and the extension theorem of Whitney provides a real function $f \in \mathrm{C}^{m+1}(\mathbb{R}^r)$ such that $\mathrm{D}^\alpha f(x) = \varphi_\alpha(x)$ for every $x \in K$ and $\alpha \in \mathbb{N}_0^r$ such that $|\alpha| \leq m+1$. If we fix $\alpha \in \mathbb{N}_0^r$ such that $|\alpha| \leq m$ as well as two points x and y of $\mathbb{R}^r$, the limited Taylor formula provides the existence of some $\theta \in]0,1[$ such that $\mathrm{D}^\alpha f(y)$ is equal to

$$\sum_{|\beta|\le m-|\alpha|} \mathrm{D}^{\alpha+\beta} f(x) \frac{(y-x)^\beta}{\beta!} + \sum_{|\beta|=m+1-|\alpha|} \mathrm{D}^{\alpha+\beta} f(x+\theta(y-x)) \frac{(y-x)^\beta}{\beta!}.$$

If x and y belong to K, we have $x+\theta(y-x) \in K$ since K is convex and this formula applies as well if we replace $\mathrm{D}^{\alpha+\beta} f$ by $\varphi_{\alpha+\beta}$. If $\boldsymbol{\varphi}$ is not real, we may split it into its real and imaginary parts and therefore get

$$|(R^m\varphi_\alpha)(x,y)| \le 2 \sum_{|\beta|=m+1-|\alpha|} \|\varphi_{\alpha+\beta}\|_K \frac{|y-x|^{m+1-|\alpha|}}{\beta!}$$

for every $\boldsymbol{\varphi} \in \mathcal{E}_{\{\boldsymbol{M}\},b}(K)$, $m \in \mathbb{N}_0$ and $\alpha \in \mathbb{N}_0^r$ such that $|\alpha| \le m$ hence successively

$$\begin{aligned}
&|(R^m\varphi_\alpha)(x,y)| \\
&\le 2\frac{l^{m+1}M_{m+1}}{(m+1-|\alpha|)!} \sum_{|\beta|=m+1-|\alpha|} \frac{\|\varphi_{\alpha+\beta}\|_K}{l^{|\alpha+\beta|}M_{|\alpha+\beta|}} \frac{|y-x|^{m+1-|\alpha|}}{\beta!}(m+1-|\alpha|)! \\
&\le 2\frac{l^{m+1}M_{m+1}}{(m+1-|\alpha|)!} |\boldsymbol{\varphi}|_{K,l} |y-x|^{m+1-|\alpha|} \sum_{|\beta|=m+1-|\alpha|} \frac{(m+1-|\alpha|)!}{\beta!} \\
&\le 2\frac{l^{m+1}M_{m+1}}{(m+1-|\alpha|)!} |\boldsymbol{\varphi}|_{K,l} |y-x|^{m+1-|\alpha|} r^{m+1-|\alpha|} \\
&\le 2\frac{(rl)^{m+1}M_{m+1}}{(m+1-|\alpha|)!} |\boldsymbol{\varphi}|_{K,l} |y-x|^{m+1-|\alpha|}
\end{aligned}$$

and finally

$$\frac{|(R^m\varphi_\alpha)(x,y)|}{(rl)^{m+1}M_{m+1}} \frac{(m+1-|\alpha|)!}{|y-x|^{m+1-|\alpha|}} \le 2\,|\boldsymbol{\varphi}|_{K,l}\,.$$

Obviously we have $|\boldsymbol{\varphi}|_{K,rl} \le |\boldsymbol{\varphi}|_{K,l}$ for every $\boldsymbol{\varphi} \in \mathcal{E}_{\{\boldsymbol{M}\},b}(K)$. Therefore if J_1 is the canonical injection from $\mathcal{E}_{\{\boldsymbol{M}\},b}(K)$ into $\mathcal{E}_{\{\boldsymbol{M}\},rl}(K)$, we get

$$|J_1\boldsymbol{\varphi}|_{K,rl} + \|J_1\boldsymbol{\varphi}\|_{K,rl} = |\boldsymbol{\varphi}|_{K,rl} + \|\boldsymbol{\varphi}\|_{K,rl} \le 3\,|\boldsymbol{\varphi}|_{K,l}\,.$$

Applying the inequality (2) leads then to

$$|J_1\boldsymbol{\varphi}|_{K,rl} + \|J_1\boldsymbol{\varphi}\|_{K,rl} \le \sum_{\alpha\in\mathbb{N}_0^r} \sum_{j=1}^{\infty} |\langle \boldsymbol{\varphi}, u_{\alpha,j}\rangle|\,, \quad \forall \boldsymbol{\varphi} \in \mathcal{E}_{\{\boldsymbol{M}\},b}(K).$$

This last relation combined with the inequality (4) imply that the linear map J_1 is quasi-nuclear.

In the same way we may obtain an integer $d > rl$ such that the canonical injection J_2 from $\mathcal{E}_{\{\boldsymbol{M}\},rl}(K)$ into $\mathcal{E}_{\{\boldsymbol{M}\},d}(K)$ is quasi-nuclear. Therefore we know that the canonical injection $J := J_2 \circ J_1$ from $\mathcal{E}_{\{\boldsymbol{M}\},b}(K)$ into $\mathcal{E}_{\{\boldsymbol{M}\},d}(K)$ is nuclear.

Hence the conclusion. ■

3. Mixed problem: general case

Theorem 2 *Let $\boldsymbol{M}_1 = (M_{1,n})_{n\in\mathbb{N}_0}$ and $\boldsymbol{M}_2 = (M_{2,n})_{n\in\mathbb{N}_0}$ be two sequences of positive numbers which are normalized and logarithmically convex. Let moreover A and B be Banach disks in $\mathcal{E}_{\{\boldsymbol{M}_1\}}(F)$ such that $A \subset B$ and the canonical injection from $\mathcal{E}_{\{\boldsymbol{M}_1\}}(F)_A$ into $\mathcal{E}_{\{\boldsymbol{M}_1\}}(F)_B$ is nuclear.*

If

$$\mathcal{E}_{\{\boldsymbol{M}_1\}}(F)_B \subset \left\{ (D^\alpha f|_F)_{\alpha\in\mathbb{N}_0^r} : f \in \mathcal{E}_{\{\boldsymbol{M}_2\}}(\mathbb{R}^r) \right\},$$

then there are an absolutely convex compact subset D of $\mathcal{E}_{\{\boldsymbol{M}_2\}}(\mathbb{R}^r)$ and a nuclear linear extension map from $\mathcal{E}_{\{\boldsymbol{M}_1\}}(F)_A$ into $\mathcal{E}_{\{\boldsymbol{M}_2\}}(\mathbb{R}^r)_D$.

PROOF. Let us designate by H the vector subspace of $\mathcal{E}_{\{\boldsymbol{M}_2\}}(\mathbb{R}^r)$ the elements of which verify

$$Sf := (\mathrm{D}^\alpha f|_F)_{\alpha\in\mathbb{N}_0^r} \in \mathcal{E}_{\{\boldsymbol{M}_1\}}(F)_B.$$

Of course the map $S\colon H \to \mathcal{E}_{\{\boldsymbol{M}_1\}}(F)_B$ so defined is linear and surjective.

As $S^{-1}\{0\}$ clearly is a closed vector subspace of $\mathcal{E}_{\{\boldsymbol{M}_2\}}(\mathbb{R}^r)$, we may consider $H/S^{-1}\{0\}$ as a vector subspace of $\mathcal{E}_{\{\boldsymbol{M}_2\}}(\mathbb{R}^r)/S^{-1}\{0\}$. If we consider the canonical quotient map

$$Q\colon \mathcal{E}_{\{\boldsymbol{M}_2\}}(\mathbb{R}^r) \to \mathcal{E}_{\{\boldsymbol{M}_2\}}(\mathbb{R}^r)/S^{-1}\{0\},$$

this allows to define the injective linear map

$$T\colon \mathcal{E}_{\{\boldsymbol{M}_1\}}(F)_B \to \mathcal{E}_{\{\boldsymbol{M}_2\}}(\mathbb{R}^r)/S^{-1}\{0\}$$

by $T(Sf) = Qf$ for every $f \in H$.

We now prove that this map T has a closed graph. Let $(\boldsymbol{\varphi}_j)_{j\in J}$ be a net in $\mathcal{E}_{\{\boldsymbol{M}_1\}}(F)_B$ converging to 0 and such that the net $(T\boldsymbol{\varphi}_j)_{j\in J}$ converges to u in $\mathcal{E}_{\{\boldsymbol{M}_2\}}(\mathbb{R}^r)/S^{-1}\{0\}$. Let f be an element of $\mathcal{E}_{\{\boldsymbol{M}_2\}}(\mathbb{R}^r)$ such that $Qf = u$. As u belongs to the closure of $H/S^{-1}\{0\}$ in $\mathcal{E}_{\{\boldsymbol{M}_2\}}(\mathbb{R}^r)/S^{-1}\{0\}$, f itself belongs to the closure of H in $\mathcal{E}_{\{\boldsymbol{M}_2\}}(\mathbb{R}^r)$. Now let $\{V_i : i \in I\}$ be a fundamental system of neighbourhoods of f in $\mathcal{E}_{\{\boldsymbol{M}_2\}}(\mathbb{R}^r)$ and let L be the subset of the elements (i,j) of $I \times J$ such that $T\boldsymbol{\varphi}_j \in QV_i$. We order L with $\leq$ defined by

$$(i_1, j_1) \leq (i_2, j_2) \Longleftrightarrow \big(V_{i_2} \subset V_{i_1} \text{ and } j_1 \leq j_2\big).$$

For every $(i,j) \in L$, we choose an element $f_{i,j} \in V_i$ such that $Qf_{i,j} = T\boldsymbol{\varphi}_j$. Of course the net $(f_{i,j})_{(i,j)\in(L,\leq)}$ converges to f; in particular, for every $\alpha \in \mathbb{N}_0^r$, the net $(\mathrm{D}^\alpha f_{i,j})_{(i,j)\in(L,\leq)}$ converges pointwise to $\mathrm{D}^\alpha f$ and as

$$Sf_{i,j} = (T^{-1}\circ Q)f_{i,j} = T^{-1}(Qf_{i,j}) = T^{-1}(T\boldsymbol{\varphi}_j) = \boldsymbol{\varphi}_j,$$

we get that the net $(\mathrm{D}^\alpha f_{i,j})_{(i,j)\in(L,\leq)}$ converges pointwise to 0 on F, i.e. $\mathrm{D}^\alpha f|_F = 0$ hence $f \in S^{-1}\{0\}$. This implies $u = 0$ and so T has a closed graph.

Since $\mathcal{E}_{\{\boldsymbol{M}_2\}}(\mathbb{R}^r)/S^{-1}\{0\}$ is a quasi-LB-space, the map T is continuous (cf. Corollary 1.5 of [7]). Therefore TA and TB are Banach disks in $H/S^{-1}\{0\}$. Let us respectively denote by E and G the Banach spaces generated by TA and TB: we have $E \subset G$ and by use of the hypothesis, the canonical injection $W\colon E \to G$ is nuclear. Let us denote by $\|\cdot\|$ the norm in E and its conjugate as well, and by $|\cdot|$ the norm in G. So we know there are sequences $(u'_n)_{n\in\mathbb{N}}$ in E' and $(v_n)_{n\in\mathbb{N}}$ in G such that

$$\begin{cases} \|u'_n\| = 1 \text{ for every } n \in \mathbb{N}, \\ \displaystyle\sum_{n=1}^{\infty} |v_n| < \infty, \\ Wu = \displaystyle\sum_{n=1}^{\infty} \langle u, u'_n\rangle\, v_n \text{ for every } u \in E. \end{cases}$$

For every $n \in \mathbb{N}$, if we set

$$\lambda_n := \Big(\sum_{j=n}^{\infty} |v_j|\Big)^{1/2} - \Big(\sum_{j=n+1}^{\infty} |v_j|\Big)^{1/2} \text{ and } \rho_n := (|v_n|/\lambda_n)^{1/2},$$

we get

$$\left|\frac{v_n}{\lambda_n \rho_n}\right| = \left(\frac{|v_n|}{\lambda_n}\right)^{1/2} = \left(\left(\sum_{j=n}^{\infty} |v_j|\right)^{1/2} + \left(\sum_{j=n+1}^{\infty} |v_j|\right)^{1/2}\right)^{1/2}$$

hence $\rho_n = |v_n/(\lambda_n \rho_n)| \to 0$ if $n \to \infty$. Therefore the closed absolutely convex hull P of the set $\{ v_n/(\lambda_n \rho_n) : n \in \mathbb{N}\}$ in G is compact. Moreover it is clear that the sequence $(v_n/\lambda_n = \rho_n v_n/(\lambda_n \rho_n))_{n\in\mathbb{N}}$ converges to 0 in G_P; therefore the closed absolutely convex hull M of $\{ v_n/\lambda_n : n \in \mathbb{N}\}$ is a compact subset of G_P.

Now let $\left\{ A_{\boldsymbol{a}} : \boldsymbol{a} \in \mathbb{N}^{\mathbb{N}} \right\}$ be a quasi-LB representation of $\mathcal{E}_{\{\boldsymbol{M}_2\}}(\mathbb{R}^r)$. To every $\boldsymbol{a} \in \mathbb{N}^{\mathbb{N}}$, we associate the set $B_{\boldsymbol{a}} := Q^{-1}(a_1 P) \cap A_{\boldsymbol{a}^0}$ where $\boldsymbol{a}^0$ denotes the sequence $(a_n^0 := a_{n+1})_{n\in\mathbb{N}}$. It is then clear that $\left\{ B_{\boldsymbol{a}} : \boldsymbol{a} \in \mathbb{N}^{\mathbb{N}} \right\}$ is a quasi-LB representation of the subspace $L = \cup_{\boldsymbol{a}\in\mathbb{N}^{\mathbb{N}}} B_{\boldsymbol{a}} = Q^{-1} G_P$ of $\mathcal{E}_{\{\boldsymbol{M}_2\}}(\mathbb{R}^r)$.

The map

$$R\colon L \to G_P; \quad f \mapsto Qf$$

is linear and surjective and has a closed graph. As M is a compact subset of G_P, the property of the quasi-LB spaces mentioned in the introduction provides $\boldsymbol{b} \in \mathbb{N}^{\mathbb{N}}$ and a compact subset D of $L_{B_{\boldsymbol{b}}}$ such that $RD = QD = M$. Let us denote by $|||\cdot|||$ the norm of L_D as well as the one of G_M. From $|||v_n/\lambda_n||| \le 1$, we deduce

$$\sum_{m=1}^{\infty} |||v_n||| \le \sum_{n=1}^{\infty} \lambda_n \le \sum_{n=1}^{\infty} |v_n|^{1/2} < \infty.$$

For every $n \in \mathbb{N}$, we then choose an element $g_n \in L_D$ such that $Rg_n = v_n$ and $|||g_n||| \le 2\,|||v_n|||$.

Now for every $\boldsymbol{\varphi} \in \mathcal{E}_{\{\boldsymbol{M}_1\}}(F)_A$, we consider the series

$$\sigma\boldsymbol{\varphi} = \sum_{n=1}^{\infty} \langle T\boldsymbol{\varphi}, u_n' \rangle\, g_n.$$

We denote by $\tilde{T}\colon E' \to \mathcal{E}_{\{\boldsymbol{M}_1\}}(F)_A'$ the transposed of $T\colon \mathcal{E}_{\{\boldsymbol{M}_1\}}(F)_A \to E$ and set $w_n' = \tilde{T} u_n'$ for every $n \in \mathbb{N}$. If $|\cdot|$ denotes the norm of $\mathcal{E}_{\{\boldsymbol{M}_1\}}(F)_A$ and of its conjugate space, we have $|w_n'| = 1$ for every $n \in \mathbb{N}$ hence

$$\sum_{n=1}^{\infty} |w_n'|\ |||g_n||| = \sum_{n=1}^{\infty} |||g_n||| \le 2 \sum_{n=1}^{\infty} |||v_n||| < \infty.$$

Moreover we also have

$$\sigma\boldsymbol{\varphi} = \sum_{n=1}^{\infty} \langle T\boldsymbol{\varphi}, u_n' \rangle\, g_n = \sum_{n=1}^{\infty} \langle \boldsymbol{\varphi}, w_n' \rangle\, g_n.$$

Therefore σ is a linear and nuclear map from $\mathcal{E}_{\{\boldsymbol{M}_1\}}(F)_A$ into $\mathcal{E}_{\{\boldsymbol{M}_2\}}(\mathbb{R}^r)_D$. To conclude we then have just to compute successively

$$\begin{aligned}\left((\mathrm{D}^{\alpha}(\sigma\boldsymbol{\varphi}))|_F\right)_{\alpha\in\mathbb{N}_0^r} &= S\sigma\boldsymbol{\varphi} = T^{-1}Q\sigma\boldsymbol{\varphi} = T^{-1}Q\Big(\sum_{n=1}^{\infty} \langle T\boldsymbol{\varphi}, u_n' \rangle\, g_n\Big) \\ &= T^{-1} \sum_{n=1}^{\infty} \langle T\boldsymbol{\varphi}, u_n' \rangle\, v_n = T^{-1}WT\boldsymbol{\varphi} = T^{-1}T\boldsymbol{\varphi} = \boldsymbol{\varphi}\end{aligned}$$

which proves that σ is an extension map. ∎

4. Mixed problem: Borel setting

In the case $F = \{0\}$, the space $\mathcal{E}_{\{\boldsymbol{M}_1\}}(F)$ has to be replaced by the space $\Lambda_{\{\boldsymbol{M}_1\}}$ defined as follows. It is the vector space of the families $\boldsymbol{c} = (c_\alpha)_{\alpha\in\mathbb{N}_0^r}$ of complex numbers for which there is $h > 0$ such that

$$|\boldsymbol{c}|_h := \sup_{\alpha\in\mathbb{N}_0^r} \frac{|c_\alpha|}{h^{|\alpha|} M_{1,|\alpha|}} < \infty.$$

Then
a) $\Lambda_{\{\boldsymbol{M}_1\},h}$ denotes the Banach space of the elements $\boldsymbol{c}$ of $\Lambda_{\{\boldsymbol{M}_1\}}$ for which $|\boldsymbol{c}|_h < \infty$, endowed with the norm $|\cdot|_h$;
b) $\Lambda_{\{\boldsymbol{M}_1\}}$ is the inductive limit $\mathrm{ind}_{n\in\mathbb{N}} \Lambda_{\{\boldsymbol{M}_1\},n}$. It is a Hausdorff (LB)-space hence a quasi-LB space: in fact if we denote by B_n the closed unit ball of $\Lambda_{\{\boldsymbol{M}_1\},n}$, it is clear that $\left\{ A_{\boldsymbol{a}} = a_2 B_{a_1} : \boldsymbol{a} \in \mathbb{N}^{\mathbb{N}} \right\}$ is a quasi-LB representation of $\Lambda_{\{\boldsymbol{M}_1\}}$.

The proof of the following Lemma is standard since a multiplication operator is nuclear whenever its symbol is absolutely summable.

Lemma 2 *With the notations just introduced, for every $\boldsymbol{a}$, $\boldsymbol{b} \in \mathbb{N}^{\mathbb{N}}$ such that $a_1 < b_1$, the canonical injection from $(\Lambda_{\{\boldsymbol{M}_1\}})_{a_2 B_{a_1}}$ into $(\Lambda_{\{\boldsymbol{M}_1\}})_{b_2 B_{b_1}}$ is nuclear.*

In particular, the space $\Lambda_{\{\boldsymbol{M}_1\}}$ is complete, nuclear and conuclear. ■

As a direct consequence of the main theorem, we then get the following result.

Theorem 3 *If the inclusion*

$$\Lambda_{\{\boldsymbol{M}_1\},m+1} \subset \left\{ (f^{(\alpha)}(0))_{\alpha\in\mathbb{N}_0^r} : f \in \mathcal{E}_{\{\boldsymbol{M}_2\}}(\mathbb{R}^r) \right\}$$

holds then there are an an absolutely convex compact subset D of $\mathcal{E}_{\{\boldsymbol{M}_2\}}(\mathbb{R}^r)$ and a linear nuclear extension map from $\Lambda_{\{\boldsymbol{M}_1\},m}$ into $\mathcal{E}_{\{\boldsymbol{M}_2\}}(\mathbb{R}^r)_D$. ■

Finally we obtain the following Corollary as a direct consequence of Grothendieck's factorization theorem.

Corollary 1 *If the inclusion*

$$\Lambda_{\{\boldsymbol{M}_1\}} \subset \left\{ (f^{(\alpha)}(0))_{\alpha\in\mathbb{N}_0^r} : f \in \mathcal{E}_{\{\boldsymbol{M}_2\}}(\mathbb{R}^r) \right\}$$

holds then, for every Banach disk B of $\Lambda_{\{\boldsymbol{M}_1\}}$, there are an absolutely convex compact subset D of $\mathcal{E}_{\{\boldsymbol{M}_2\}}(\mathbb{R}^r)$ and a linear nuclear extension map from $\Lambda_{\{\boldsymbol{M}_1\},B}$ into $\mathcal{E}_{\{\boldsymbol{M}_2\}}(\mathbb{R}^r)_D$. ■

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