RAC$\mathbb{SAM}$
Rev. R. Acad. Cien. Serie A. Mat.
VOL. **97** (2), 2003, pp. 273–286
Análisis Matemático / Mathematical Analysis

Recent developments in hypercyclicity

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Dedicated to the memory of Professor Klaus Floret

Abstract. In these notes we report on recent progress in the theory of hypercyclic and chaotic operators. Our discussion will be guided by the following fundamental problems: How do we recognize hypercyclic operators? How many vectors are hypercyclic? How many operators are hypercyclic? How big can non-dense orbits be?

Avances recientes en hiperciclicidad

Resumen. En estas notas informamos acerca de progresos recientes en la teoría de operadores hipercíclicos y caóticos. Nuestro estudio ha sido guiado por los siguientes problemas: ¿Cómo reconocemos los operadores hipercíclicos? ¿Cuántos vectores son hipercíclicos? ¿Cuántos operadores son hipercíclicos? ¿Qué tamano tienen las órbitas no densas?

1. Prologue: The basic concepts

Hypercyclicity is the study of linear operators that possess a dense orbit. Although the first examples of hypercyclic operators date back to the first half of the last century, a systematic study of this concept has only been undertaken since the mid-eighties. Seminal papers like the unpublished but widely disseminated thesis of Kitai [43], a highly original and broad investigation by Godefroy and Shapiro [32] and deep operator-theoretic contributions by Herrero [40], [41] were instrumental in creating a flourishing new area of analysis.

The survey [36] of 1999 tried to give a complete synopsis of hypercyclicity and the related area of universality. The intervening years have seen remarkable major advances. In particular, G. Costakis, A. Peris and S. Grivaux have solved two of the five[1] problems mentioned in [36]. Additionally, many other noteworthy results have been obtained and a number of foundational issues have been clarified. In this note we want to present some of these new developments.

For an updated bibliography on hypercyclicity and related areas such as universal functions, chaotic operators, transitive operators, supercyclic operators or hypercyclic semigroups the interested reader is referred to [37]. A very readable and detailed introduction to hypercyclicity from the point of view of linear dynamics is provided by unpublished notes of J. H. Shapiro [56].

The most general setting for hypercyclicity is that of a (real or complex) topological vector space, which will always be assumed to be Hausdorff. Depending on the result one wants to obtain additional structure

Presentado por José Bonet.
Recibido: 1 de Abril de 2003. *Aceptado*: 15 de junio de 2003.
Palabras clave / Keywords: Hypercyclic operators, chaotic operators, Hypercyclicity Criterion, hypercyclic subspaces, dense orbits.
Mathematics Subject Classifications: Primary 47A16

[1]The survey states six problems, but Problems 1 and 3 are equivalent.

like local convexity, metrizability, completeness or normability is needed. Since the Baire Category Theorem is vital in the more fundamental results on hypercyclicity, the underlying space is often assumed to be an F*-space*, that is, a complete metrizable topological vector space. In general, however, the reader will lose very little on assuming that we are working in Banach spaces. Throughout, by an *operator* we mean a continuous linear mapping.

Definition 1 *Let* X *be a topological vector space. Then an operator* $T : X \to X$ *is called* hypercyclic *if there is a vector* $x \in X$ *whose* orbit *under* T,

$$\text{orb}(T, x) := \{x, Tx, T^2x, \dots\},$$

is dense in X. *Every such vector* x *is called* hypercyclic *for* T.

Since the definition requires a countable dense set in X hypercyclicity can only occur in separable spaces. Another, less obvious, restriction is that the space X has to be infinite-dimensional as there are no hypercyclic operators in $\mathbb{R}^N$ or $\mathbb{C}^N$, cf. [36, Proposition 11].

Hypercyclicity is closely related to the well-known concept of transitivity from topological dynamics.

Definition 2 *Let* X *be a topological vector space. Then an operator* $T : X \to X$ *is called* (topologically) transitive *if for each pair* U *and* V *of non-empty open subsets of* X *there is some* $n \in \mathbb{N}$ *with*

$$T^n(U) \cap V \neq \emptyset.$$

It is easy to see that every hypercyclic operator is transitive, but the converse need not be true, see [17] or [7, Example] for simple examples. In many spaces, however, the two concepts coincide, cf. [56, 1.10] or [36, Theorem 3].

Theorem 1 (Birkhoff Transitivity Theorem) *Let* X *be a separable* F*-space. Then an operator* $T : X \to X$ *is hypercyclic if and only if it is transitive.* ■

The result follows from an application of the Baire Category Theorem. The breakdown of the theorem for general spaces has recently led to an increased interest in transitive operators in their own right. We will return to this in Section 4.

There are several concepts of chaos for not necessarily linear mappings in the literature. Since many of them require, among others, the existence of a dense orbit it seems natural to introduce a notion of chaos into the theory of hypercyclicity. The first to consider this problem were Godefroy and Shapiro [32] who took up a definition of chaos due to Devaney [28, p. 50]. In F-spaces, the following has by now been generally accepted as the right notion of chaos in hypercyclicity; recall that a point $x \in X$ is called *periodic* for a mapping $T : X \to X$ if $T^N x = x$ for some $N \in \mathbb{N}$.

Definition 3 *Let* X *be an* F*-space. Then an operator* $T : X \to X$ *is called* chaotic *if*

(i) *it has a dense orbit, that is, it is hypercyclic, and*

(ii) *it has a dense set of periodic points.*

One may increase the symmetry between the two defining conditions by noting that condition (i) is equivalent to the existence of a dense set of hypercyclic vectors. This follows from the fact that if x has a dense orbit then so does every vector $T^n x$ because $\text{orb}(T, x) \setminus \text{orb}(T, T^n x)$ is finite, and the vectors $T^n x, n \in \mathbb{N}$, form a dense set.

As a consequence we see that in every neighbourhood of any point in X there are hypercyclic points and periodic points, which corresponds to the intuitive idea of chaos as the precarious alliance of regularity and irregularity.

Devaney's definition of chaos had in fact required a third condition, the so-called *sensitive dependence on initial conditions.* From the above discussion it is not surprising that, even for non-linear mappings, this condition follows from (i) and (ii), as was shown by Banks et al. [5]. Earlier, Godefroy and Shapiro [32, Proposition 6.1] had shown that for linear mappings hypercyclicity alone implies sensitive dependence on initial conditions.

Beyond F-spaces the appropriate definition of a chaotic operator is still under discussion. It seems that condition (i) should then be replaced by the requirement that T is transitive, which may then however lead to chaotic operators that are no longer hypercyclic, cf. Bonet [17]. Definitions of chaos for unbounded operators were recently proposed by Bès, Chan and Seubert [14] and by deLaubenfels, Emamirad and the author [27].

For further discussions of linear chaos we refer to Shapiro [56] and Feldman [30].

2. How do we recognize hypercyclic operators?

Throughout this section X will always be a separable F-space.

For certain special operators the Birkhoff Transitivity Theorem leads to a simple verification of their hypercyclicity, see the discussion at the end of Section 2 of [10]. In the vast majority of cases, however, a different and a priori only sufficient condition has been applied successfully in the literature to obtain hypercyclicity.

Theorem 2 (Hypercyclicity Criterion) *Let $T : X \to X$ be an operator. If there are dense subsets Y and Z of X and an increasing sequence (n_k) of positive integers such that*

(i) *for each $y \in Y$, $T^{n_k} y \to 0$,*

(ii) *for each $z \in Z$ there is a sequence (x_k) in X with*

$$x_k \to 0 \text{ and } T^{n_k} x_k \to z,$$

then T is hypercyclic.[2] ■

We have given here a reformulation of the usual statement of the criterion, cf. [15, Definition 1.2] or [36, Theorem 4]. In the present form it brings out more clearly that hypercyclicity is caused by intertwining collapse (in (i)) with blow-up (in (ii)).

The Hypercyclicity Criterion has evolved through a series of papers, starting with Kitai's thesis [43] and the independent investigation of Gethner and Shapiro [31] in the 1980's; an early form had already been given by Joó [42] in 1978.

In recent years, several variants of the criterion have been considered, see [36, Remark 3], [49, Theorem 1.1] and [29, Theorem 3.2]. All of them, however, were shown to be equivalent to the Hypercyclicity Criterion as stated above, see Peris [49, Theorem 2.3], Feldman [29, comment after Theorem 3.2] and Bermúdez, Bonilla and Peris [6, Section 2]. For additional interesting discussions we refer to Grivaux [35].

The effectiveness of the Hypercyclicity Criterion is such that, to date, every known hypercyclic operator in fact satisfies the Criterion. This has led to the following, cf. [36, Problem 1].

Great open problem in hypercyclicity *Does every hypercyclic operator satisfy the Hypercyclicity Criterion?*

The problem has so far evaded all attempts at being resolved. This has motivated the search for equivalent but less technical forms of the Hypercyclicity Criterion. The following result was obtained independently by Bernal and the author, Bès, Saldivia, and León, see [10, Remark 3.5] and [44].

[2] Adding as third condition that there is a dense subset W of X such that

(iii) *for each $w \in W$ there is some $n \in \mathbb{N}$ with $T^n w = w$*

we obtain a criterion for chaos.

Theorem 3 *An operator $T : X \to X$ satisfies the Hypercyclicity Criterion if and only if for each pair U and V of non-empty open subsets of X and each neighbourhood W of zero in X there is some $n \in \mathbb{N}$ with*

$$T^n(U) \cap W \neq \emptyset \text{ and } T^n(W) \cap V \neq \emptyset. \quad \blacksquare$$

This condition, which is in the spirit of the definition of transitivity, was introduced by Godefroy and Shapiro, who had already shown that it is implied by the Hypercyclicity Criterion, see [32, Corollaries 1.3 and 1.4].

The Hypercyclicity Criterion has recently helped to solve an open problem in a surprising way. By a well-known theorem of Ansari [2], [3, Note 3] every power T^N, $N \in \mathbb{N}$, of a hypercyclic operator T is itself hypercyclic, that is, there is some $x \in X$ such that the set $\{T^{kN}x : k = 0, 1, 2, \dots\}$ is dense in X, see also Section 5. One might ask if the sequences $(kN)_k$ can be replaced here by arbitrary increasing sequences (n_k) for which $\sup_k(n_{k+1} - n_k) < \infty$. Peris and Saldivia [51] have shown that this is indeed so (only) for operators T that satisfy the Hypercyclicity Criterion.

Theorem 4 *Let $T : X \to X$ be an operator. Then the following assertions are equivalent:*

(i) *For any increasing sequence (n_k) of positive integers with $\sup_k(n_{k+1} - n_k) < \infty$ there is some $x \in X$ such that the set $\{T^{n_k}x : k = 0, 1, 2, \dots\}$ is dense in X.*

(ii) *T satisfies the Hypercyclicity Criterion.*

In Banach spaces this result was obtained independently by Grivaux [35], whose approach also led to a new proof of Theorem 3. We give here a proof of Theorem 4, a variant of that by Peris and Saldivia [51], that is in turn based on Theorem 3.

PROOF. (ii) $\Longrightarrow$ (i). Let T satisfy the Hypercyclicity Criterion with sequence (n_k) and dense subsets Y and Z. Let (m_k) be an increasing sequence of positive integers with $\sup_k(m_{k+1} - m_k) < \infty$. Then it is easy to see that there is some $N \geq 0$ and subsequences (k_ν) and (k'_ν) such that $n_{k_\nu} = m_{k'_\nu} - N$ for all ν. Hence we have for all $y \in Y$

$$T^{m_{k'_\nu}}y = T^N T^{n_{k_\nu}}y \to 0,$$

and for each $z \in Z$ there is a sequence (x_k) in X with $x_k \to 0$ and

$$T^{m_{k'_\nu}}x_{k_\nu} = T^N T^{n_{k_\nu}}x_{k_\nu} \to T^N z.$$

Hence T satisfies the Hypercyclicity Criterion with sequence $(m_{k'_\nu})$ and dense subsets Y and $T^N(Z)$; note that T necessarily has dense range. This implies (i), see, for example, [36, Theorem 2].

(i) $\Longrightarrow$ (ii). We shall show that assertion (i) implies the condition stated in Theorem 3. Let U and V be non-empty open subsets of X and W a neighbourhood of zero in X. We choose non-empty open sets $U_1 \subset U$, $V_1 \subset V$ and $W_1 \subset W$ such that $U_1 - U_1 \subset W$ and $V_1 - W_1 \subset V$. Since T is hypercyclic there is some $m \in \mathbb{N}$ such that $T^m(W_1) \cap V_1 \neq \emptyset$. Hence there is a non-empty open set $W_2 \subset W_1$ with $T^m(W_2) \subset V_1$.

Now, by (i) there is some $l \in \mathbb{N}$ such that $T^{l+i}(U_1) \cap W_2 \neq \emptyset$ for $i = 0, \dots, m$ (otherwise we could find an increasing sequence (n_k) with $n_{k+1} - n_k \leq m + 1$ and $T^{n_k}(U_1) \cap W_2 = \emptyset$ for all k). In particular we have

$$T^l(U_1) \cap W_2 \neq \emptyset \quad \text{and} \quad T^{l+m}(U_1) \cap W_2 \neq \emptyset.$$

Writing $n = l + m$ we obtain

$$T^n(U) \cap W \neq \emptyset$$

and

$$\begin{aligned} T^n(W) \cap V &\supset T^{l+m}(U_1 - U_1) \cap (V_1 - W_2) = (T^m T^l(U_1) - T^{l+m}(U_1)) \cap (V_1 - W_2) \\ &\supset (T^m T^l(U_1) \cap V_1) - (T^{l+m}(U_1) \cap W_2) \neq \emptyset. \quad \blacksquare \end{aligned}$$

Another useful method for deriving the hypercyclicity of an operator consists in relating it to a known hypercyclic operator by means of a commutative diagram. This approach is well known in topological dynamics but its full potential in hypercyclicity was first realized by Martínez and Peris [47, Lemma 2.1].

Theorem 5 *Let X and Y be F-spaces and $S : Y \to Y$, $T : X \to X$ operators. Suppose that there is a continuous mapping*[3] *$\phi : Y \to X$ of dense range such that the following diagram commutes:*

$$\begin{array}{ccc} Y & \xrightarrow{S} & Y \\ {\scriptstyle\phi}\big\downarrow & & \big\downarrow{\scriptstyle\phi} \\ X & \xrightarrow{T} & X, \end{array}$$

that is, we have $T \circ \phi = \phi \circ S$. If S is hypercyclic, satisfies the Hypercyclicity Criterion or is chaotic, respectively, then T has the same property. If $y \in Y$ is hypercyclic or periodic for S then $\phi(y)$ is hypercyclic or periodic for T, respectively. ■

We note that, with a proof as in [47], the result remains true for any topological vector spaces X and Y, independently of whether in the definition of chaos we use the concept of hypercyclicity or that of transitivity.

In the language of topological dynamics, T is called a *quasi-factor of* S and S a *quasi-extension of* T, see [56, Definition 1.12]. In the case when Y is a dense subspace of X, where Y carries a possibly stronger topology than the one inherited from X, and $\phi : Y \to X$ is the canonical injection we get Shapiro's *Hypercyclicity Comparison Principle*: $T : X \to X$ is hypercyclic if $T|_Y : Y \to Y$ is well-defined, continuous and hypercyclic. We refer to [55, p. 111] and [36, Proposition 9].

3. How many vectors are hypercyclic?

The simple answer to the question is: Many – if there are any. Of course, there are operators with no hypercyclic vectors like the identity operator. But as soon as an operator is hypercyclic the set of hypercyclic vectors becomes huge, in several respects. First one may note that for every hypercyclic operator $T : X \to X$ on any topological vector space X

> *the set of hypercyclic vectors of T is dense*,

because, as we had seen in the introductory section, with any vector x also every $T^n x,\ n \in \mathbb{N}$, has a dense orbit. Next, if X is an F-space then it is not difficult to see that

> *the set of hypercyclic vectors of T is a dense G_δ-set, hence has a complement of first Baire category*,

which has as an immediate consequence that

> *every vector in X is the sum of two hypercyclic vectors for T*,

see [36, Proposition 8(a)].

This final observation, however, seems to impose, in general, a restriction on the size of the set of all hypercyclic vectors: If, after adding the non-hypercyclic zero vector, this set becomes a linear subspace then every non-zero vector in X is hypercyclic, which is very rarely the case. Defining a *hypercyclic subspace* as one in which every non-zero vector is hypercyclic we must therefore conclude that the set of all hypercyclic

[3] In the case of the Hypercyclicity Criterion, the proof given in [47] requires that $\phi(0) = 0$. In fact, this can be assumed without loss of generality. For, if $\phi(0) = x_0$ is arbitrary then the commutativity of the diagram implies that x_0 is a fixed point of T. When we now define $\psi(y) = \phi(y) - x_0$ for $y \in Y$ then the assumptions of the theorem also hold for ψ, and we have $\psi(0) = 0$. I am grateful to Alfredo Peris for this clarification.

vectors is in most cases no hypercyclic subspace. This suggested the problem if, nonetheless, the set of hypercyclic vectors contains hypercyclic subspaces of high dimensions. Quite surprisingly, Herrero [40] and Bourdon [21] showed independently that, indeed, every hypercyclic operator on complex Hilbert space possesses a dense, and thus infinite-dimensional hypercyclic subspace. While Bourdon's technique immediately extended to complex locally convex spaces, Bès [11] showed that the same result also holds in the real setting. The final step was recently taken by Wengenroth [57] who was able to extend the result to arbitrary topological vector spaces.

In fact, Bourdon's technique can be used to obtain common dense hypercyclic subspaces for countably many commuting operators on F-spaces, as was noted by Grivaux [34, Section 4]. Using, in addition, Wengenroth's ideas one may obtain the following result.

Proposition 1 *Let X be a topological vector space and $\mathcal{M}$ a non-empty set of commuting operators $T : X \to X$. If the operators $T \in \mathcal{M}$ share a hypercyclic vector then they share a dense hypercyclic subspace.*

PROOF. We fix a common hypercyclic vector $x \in X$ and an operator $S \in \mathcal{M}$. Then

$$L := \text{span}\{x, Sx, S^2x, \dots\}$$

is a dense subspace of X. It suffices to show that each non-zero vector in L is hypercyclic for each $T \in \mathcal{M}$. We first note that

$$L = \{p(S)x : p \text{ a polynomial}\}.$$

Now, by the assumption of commutativity, we have

$$T^n(p(S)x) = p(S)(T^n x),\ n \in \mathbb{N}_0,$$

and $\{T^n x : n \in \mathbb{N}_0\}$ is dense in X. Hence $p(S)x$ is hypercyclic for T if $p(S)$ has dense range.

We now assume that X is a vector space over $\mathbb{C}$. Since then $p(S)$, which we may assume to be non-zero, factorizes into linear factors it suffices to show that

$$S - \lambda I$$

has dense range for each $\lambda \in \mathbb{C}$, i.e., that the subspace

$$K := \overline{\{(S - \lambda I)(y) : y \in X\}}$$

of X coincides with X.

If this is not the case the quotient X/K is a topological vector space of dimension at least one. Let $q : X \to X/K$ denote the corresponding quotient map. Then

$$q((S - \lambda I)y) = 0 \text{ for all } y \in X,$$

hence

$$q(Sy) = \lambda q(y).$$

By induction we obtain in particular

$$q(S^n x) = \lambda^n q(x) \text{ for } n \in \mathbb{N}_0.$$

Since $\{S^n x : n \in \mathbb{N}_0\}$ is dense in X and q is continuous and surjective we conclude that

$$\{\lambda^n q(x) : n \in \mathbb{N}_0\}$$

is a dense subset of X/K. This is clearly a contradiction because this set is, for any λ, a nowhere dense subset of a one-dimensional subspace of X/K. This completes the proof in the complex case.

The case of real scalars can be treated similarly, or one applies a complexification technique, see [57]. We omit the details. ■

The proof also shows that, for any given operator $T \in \mathcal{M}$, one may find a common hypercyclic subspace that is invariant under T.

The proposition reduces the problem of existence of common dense hypercyclic subspaces for commuting operators to the problem of the existence of just one common hypercyclic vector. The latter problem has recently attracted some attention. By a classical theorem of Rolewicz [52], the weighted backward shifts

$$\lambda B : l^2 \to l^2,\ (x_1, x_2, x_3, \dots) \mapsto (\lambda x_2, \lambda x_3, \lambda x_4, \dots)$$

are hypercyclic for every scalar λ with $|\lambda| > 1$, where l^2 is the Hilbert space of square-summable sequences. Salas [54] has posed the problem of studying the set of common hypercyclic vectors of the operators $\lambda B, |\lambda| > 1$; in particular, it was not even clear if this set is non-empty. The latter question was answered in the affirmative independently by Abakumov and Gordon [1] and Peris [48]; Abakumov and Gordon [1] and Costakis and Sambarino [26] have even shown that the set of common hypercyclic vectors is a dense G_δ-set. Costakis and Sambarino [26] have obtained further (uncountable) families of operators that share hypercyclic vectors. For the commuting families among them Proposition 1 implies then also the existence of common dense hypercyclic subspaces. In particular we have:

Corollary 1 *The operators $\lambda B : l^2 \to l^2, |\lambda| > 1$, share a common dense invariant hypercyclic subspace.* ■

In the case of *countable* families of hypercyclic operators on an F-space the existence of a common hypercyclic vector is an immediate consequence of the Baire Category Theorem. Thus we have the following result that is due to Grivaux [34, Section 4].

Corollary 2 *Countably many commuting hypercyclic operators $T_k, k \in \mathbb{N}$, on an F-space share a common dense hypercyclic subspace.* ■

In particular, on the space $H(\mathbb{C})$ of entire functions the operators D of differentiation and T of translation,

$$(Df)(z) = f'(z),\ (Tf)(z) = f(z+1),$$

commute and are hypercyclic by classical theorems of MacLane [46] and Birkhoff [16]. Thus they have a common dense hypercyclic subspace. This answers positively Problem 1 of Aron, García and Maestre [4].

When we consider non-commuting operators new methods are called for to produce common dense hypercyclic subspaces. Using techniques that helped her to solve a problem of Halperin, Kitai and Rosenthal, see Section 4, Grivaux [34] was able to obtain the following important and surprisingly general result.

Theorem 6 *Countably many hypercyclic operators $T_k, k \in \mathbb{N}$, on a Banach space always share a common dense hypercyclic subspace.* ■

A related result in arbitrary F-spaces with additional assumptions on the operators T_k is due to Bernal und Calderón [9, Theorem 3.1].

4. How many operators are hypercyclic?

Experience has shown that hypercyclicity of an operator is not as rare a phenomenon as one might at first think. Already in 1969, Rolewicz [52] had asked if indeed every, by necessity separable and infinite-dimensional, Banach space supports a hypercyclic operator. This was answered positively by Ansari [3] and Bernal [8], while Bonet and Peris [20] extended the result to Fréchet spaces.

Theorem 7 *Every separable infinite-dimensional Fréchet space admits a hypercyclic operator.* ■

The corresponding problem for chaotic operators has, perhaps surprisingly, a different answer. While Rolewicz' weighted backward shifts are simple examples of chaotic operators on Hilbert space, Bonet, Martínez and Peris [19] have used the work of Gowers and Maurey to produce a counterexample in the setting of Banach spaces.

Theorem 8 *Every separable infinite-dimensional Hilbert space admits a chaotic operator, while there is a separable reflexive infinite-dimensional Banach space that does not support any chaotic operator.* ■

Going beyond Fréchet spaces one encounters additional obstacles: On the space $\varphi = \bigoplus_{n=1}^{\infty} \mathbb{C}$ of finite sequences with its natural inductive limit topology there are even no hypercyclic operators, cf. [20], [36].

In contrast, Bonet, Frerick, Peris and Wengenroth [18] have shown that the space φ supports transitive operators. Their work was motivated by Bermúdez and Kalton [7] who have shown that, while every (not necessarily separable) infinite-dimensional Hilbert space supports a transitive operator, there are no transitive operators on l^∞ or, more generally, on any non-reflexive quotient of a von Neumann algebra.

The presumably difficult problem of characterizing those spaces that carry transitive, hypercyclic, or chaotic operators has not been solved yet, see also [7, p. 1454]. In particular, the existence of hypercyclic operators in any separable (non-locally convex) F-space is still open, cf. [36, Problem 5].

Suppose now that a space admits hypercyclic operators. Motivated by the fact that the existence of a single hypercyclic *vector* implies the existence of a dense set of such vectors one might ask if the analogous statement is true for operators. A simple argument shows that this is not the case, at least if understood in its most natural sense. To see this, consider an operator $T : X \to X$ on a Banach space X with $\|T\| \leq 1$. Then we have for every $x \in X$

$$\|T^n x\| \leq \|T\|^n \|x\| \leq \|x\|, \quad n \in \mathbb{N}_0,$$

so that T only has bounded orbits and hence cannot be hypercyclic. Thus, when we endow the space $L(X)$ of all operators $T : X \to X$ with the operator norm topology then there are no hypercyclic operators in the closed unit ball $\{T \in L(X) : \|T\| \leq 1\}$. As a consequence the hypercyclic operators do not form a dense set in $L(X)$.

Now, on $L(X)$ there is another natural, and weaker topology, the strong operator topology (SOT) in which convergence is defined as pointwise convergence at every $x \in X$. Thus the question comes to life again: Is the set of hypercyclic operators SOT-dense in $L(X)$, where X is, say, a separable infinite-dimensional Banach space? In [23], Chan succeeded in giving a positive answer in Hilbert space, which was generalized by Bès and Chan [12] to Fréchet spaces. In a subsequent paper Bès and Chan [13] note that these results are in fact a consequence of the following very general statement on SOT-density that is due to Hadwin, Nordgren, Radjavi and Rosenthal [38].

Proposition 2 *Let X be a separable Fréchet space and $T : X \to X$ an operator. If*

(LI) *for every $n \in \mathbb{N}$ there are $x_1, \ldots, x_n \in X$ such that*

$$x_1, \ldots, x_n, Tx_1, \ldots, Tx_n$$

is linearly independent,

then the set

$$\{S^{-1}TS \ : \ S : X \to X \text{ an isomorphism}\}$$

is SOT-dense in $L(X)$. ■

This result allowed Bès and Chan [13], among other things, to give a new proof of their earlier results and to extend them to chaotic operators.

Theorem 9 *Let X be separable infinite-dimensional Fréchet space.*

(a) *The set of hypercyclic operators on X is SOT-dense in $L(X)$.*

(b) *The set of chaotic operators on X is either empty or SOT-dense in $L(X)$.*

PROOF. It follows from Theorem 5 that the sets described in (a) and (b) are invariant under the mapping $T \mapsto S^{-1}TS$, where $S : X \to X$ is an isomorphism. Now let T be an operator in one of these sets. Then T possesses a hypercyclic vector x. This implies that for each $n \in \mathbb{N}$

$$x, T^2x, T^4x, \ldots, T^{2n-2}x, Tx, T^3x, T^5x, \ldots, T^{2n-1}x$$

is linearly independent because otherwise some $T^m x$ would be a linear combination of $x, Tx, \ldots, T^{m-1}x$, which would imply that the orbit of x under T lies in some finite-dimensional subspace of X; this is clearly impossible. Hence T satisfies condition (LI) so that by the previous proposition the sets in (a) and (b) are either empty or SOT-dense in $L(X)$. To complete the proof we remark that by Theorem 7 every separable infinite-dimensional Fréchet space admits a hypercyclic operator. ■

In spite of what was said before some density results are possible even under the stronger operator norm topology in $L(X)$ when X is a complex separable infinite-dimensional Hilbert space. By Chan [23] the linear span of the set of hypercyclic operators is norm dense in $L(X)$ which was improved independently by Bès and Chan [13] and León [45] who showed that the set of sums of two hypercyclic operators is norm dense in $L(X)$. This led these authors to ask if every operator on X is in fact the sum of two hypercyclic, or even of two chaotic operators. Grivaux [33] could recently answer these questions positively. Her deep result is the more remarkable in that it breaks down for general Banach spaces, as she has shown by using the work of Gowers and Maurey.

Theorem 10 (a) *Every operator on a complex separable infinite-dimensional Hilbert space is the sum of two chaotic, and hence of two hypercyclic operators.*

(b) *There is a separable infinite-dimensional Banach space on which not every operator is the sum of two hypercyclic operators.* ■

The question of abundance of hypercyclic operators can also be interpreted in a different direction, where we will now consider Banach spaces. Since hypercyclicity is defined by the existence of a dense orbit one may ask if such orbits may be prescribed. Now, every dense orbit is linearly independent as we have seen in the proof of Theorem 9; so the question becomes if for every dense linearly independent sequence in a Banach space X there exists a, necessarily hypercyclic, operator T on X and an $x \in X$ such that the orbit orb(T, x) contains the given sequence. This problem was raised in 1985 by Halperin, Kitai and Rosenthal [39], see also [36, Problem 6], who had given a positive answer in Hilbert space. Grivaux [34] was recently able to solve the problem for general Banach spaces. Her proof is based on the following result that should also be of independent interest.

Proposition 3 *Let (x_n) and (y_n) be two dense linearly independent sequences in a Banach space X. Then there exists an isomorphism S on X such that*

$$\{Sx_n : n \in \mathbb{N}\} = \{y_n : n \in \mathbb{N}\}. \quad ■$$

This result enabled Grivaux to solve the problem of Halperin, Kitai and Rosenthal by distorting an arbitrary hypercyclic operator.

Theorem 11 *Let X be a Banach space and (y_n) a dense linearly independent sequence in X. Then there exists a, necessarily hypercyclic, operator T on X and an $x \in X$ such that*

$$\text{orb}(T, x) = \{y_n : n \in \mathbb{N}\}.$$

PROOF. By the Theorem of Ansari and Bernal, see Theorem 7, there exists a hypercyclic operator $\widetilde{T}$ in X; note that X is necessarily separable and infinite-dimensional. If $\widetilde{x}$ is a hypercyclic vector for $\widetilde{T}$ then its orbit $\{\widetilde{T}^n\widetilde{x} : n \in \mathbb{N}_0\}$ is a dense linearly independent sequence. By the preceding proposition there exists an isomorphism S on X such that

$$\{Sy_n : n \in \mathbb{N}\} = \{\widetilde{T}^n\widetilde{x} : n \in \mathbb{N}_0\}.$$

We now consider the operator $T = S^{-1}\widetilde{T}S$ on X and the vector $x = S^{-1}\widetilde{x}$. Then we have for $n \in \mathbb{N}_0$

$$T^n x = S^{-1}\widetilde{T}^n Sx = S^{-1}\widetilde{T}^n\widetilde{x},$$

so that

$$\text{orb}(T, x) = \{T^n x : n \in \mathbb{N}_0\} = \{y_n : n \in \mathbb{N}\}$$

which had to be shown. ■

Grivaux also showed that the operator T can always be chosen in such a way that one may take $x = y_1$.

In an interesting addition to Grivaux's work, Bonet, Frerick, Peris and Wengenroth [18] have shown that neither Proposition 3 nor Theorem 11 extends to arbitrary Fréchet spaces. In fact, one may even find a counterexample to Theorem 11 in the space $\omega = \mathbb{C}^{\mathbb{N}}$ of all complex sequences.

In Section 3 we have seen that every hypercyclic operator on a Banach space has a dense hypercyclic subspace. As a consequence of Theorem 11 Grivaux proved that, conversely, every dense subspace of countable infinite dimension in a Banach space X is a hypercyclic invariant subspace of some operator on X. This, in turn, implies the following, see Grivaux [34].

Theorem 12 *Let X be a normed space of countable infinite dimension. Then there exists an operator T on X so that each non-zero vector in X is hypercyclic for T. In other words, T has no non-trivial invariant closed subset.* ■

This result cannot be extended to separable normed spaces. In fact, Bonet, Frerick, Peris and Wengenroth [18] have shown, using a theorem of Valdivia, that every separable infinite-dimensional Fréchet space contains a hyperplane that supports no transitive, hence no hypercyclic, operator.

5. How big can non-dense orbits be?

Suppose that a given operator is *not* hypercyclic. The question one might then ask is how big its orbits can be without ever becoming dense. Possibly the first one to look at a problem of this type was Herrero [41] who, instead of demanding the density of one orbit, allowed to take the union of finitely many orbits,

$$\bigcup_{i=1}^{N} \text{orb}(T, x_i) = \{T^n x_i : n \in \mathbb{N}_0, i = 1, \ldots, N\},$$

for obtaining a dense set. Such an operator T is called *multi-hypercyclic*. In fact, Herrero conjectured that, in Hilbert space, this situation can only occur if T is already hypercyclic, more precisely, if one of the x_i alone has a dense orbit, see also [36, Problem 4].

A particular case of this conjecture was considered by Ansari. Suppose that T is a hypercyclic operator on a locally convex space X, and let $N \in \mathbb{N}, N \geq 2$. Then the operator T^N is multi-hypercyclic. To see this one has to note that for all $x \in X$

$$\text{orb}(T, x) = \bigcup_{i=1}^{N} \text{orb}(T^N, x_i)$$

with $x_i = T^{i-1}x$; and this set will be dense if x is hypercyclic for T. Now, Ansari [2], [3, Note 3] has shown that in this case Herrero's conjecture is true: T^N is hypercyclic and $x_1 = x$ is hypercyclic for T^N whenever x is hypercyclic for T. Thus T and T^N have the same hypercyclic vectors.

The full conjecture of Herrero was settled in the affirmative independently by Costakis [25] and Peris [50] for general locally convex spaces. Finally, Wengenroth [57] showed how to extend the result to arbitrary topological vector spaces.

Theorem 13 *Every multi-hypercyclic operator T on a topological vector space is hypercyclic. More precisely, if*

$$\bigcup_{i=1}^{N} \mathrm{orb}(T, x_i)$$

is dense in X then $\mathrm{orb}(T, x_i)$ *is dense for some $i \in \{1, \dots, N\}$.* ■

The argument given above then implies the following extension of Ansari's result.

Theorem 14 *If an operator T on a topological vector space is hypercyclic then so is T^N for every $N \geq 2$. In fact, T and T^N have the same hypercyclic vectors.* ■

The beginning of the proof of Theorem 13 might be like this. Suppose that there are $x_1, \dots, x_N \in X$ such that

$$X = \overline{\bigcup_{i=1}^{N} \mathrm{orb}(T, x_i)},$$

which equals

$$\bigcup_{i=1}^{N} \overline{\mathrm{orb}(T, x_i)}.$$

We may assume that N is chosen minimal. If $N = 1$ we are done. Otherwise we have that

$$X \setminus \bigcup_{i=1}^{N-1} \overline{\mathrm{orb}(T, x_i)}$$

is non-empty. On the other hand, this set is open and, by necessity, contained in $\overline{\mathrm{orb}(T, x_N)}$. This implies that the closure of $\mathrm{orb}(T, x_N)$ has an interior point, that is, $\mathrm{orb}(T, x_N)$ is somewhere dense. From here the proof proceeds, but Peris [50] wondered if, for locally convex spaces, this statement in itself suffices to make $\mathrm{orb}(T, x_N)$ dense and hence T hypercyclic. This question was answered positively by Bourdon and Feldman [22], the final extension to topological vector spaces is once more due to Wengenroth [57].

Theorem 15 *If an operator T on a topological vector space has a somewhere dense orbit then this orbit is dense and T is hypercyclic.* ■

The result can also be phrased like this: any orbit of any operator on a topological vector space is either nowhere dense or dense – a strikingly simple and general statement.

For a more detailed exposition of the work of Costakis, Peris, Bourdon and Feldman we refer to Shapiro [56, Section 8].

The question posed in the heading of this section has also been studied in other directions. Answering a question of Feldman [29], Chan and Sanders [24] have given an example of an operator in Hilbert space that has a weakly dense but not norm-dense orbit; in fact their operator has no dense orbits, that is, it is not hypercyclic. On the other hand, Feldman [29] has considered orbits in Banach spaces X that come within bounded distance of every point, that is, orbits that meet every ball of radius R in X, where $R > 0$ is a suitable preassigned number. He has shown that, even in Hilbert space, an orbit may come within bounded distance of every point without being dense, but he has also shown that every operator in a Banach space that possesses such an orbit must be hypercyclic.

6. Epilogue: Further work

In this note we have discussed some of the recent progress in hypercyclicity. The selection of material was motivated by personal preferences and is by no means exhaustive. Work that we have not touched upon here includes investigations on

– closed hypercyclic subspaces,

– supercyclic operators,

– hypercyclic semigroups of operators;

in addition, many papers construct and investigate

– specific hypercyclic operators and universal families.

The interested reader is referred to the updated bibliography [37].

Acknowledgement. These notes constitute an expanded version of some material presented in the Seminario de Análisis Funcional of the Departamento de Análisis Matemático, Universidad de Sevilla, in October 2002.

I wish to express my deep gratitude to the whole department and in particular to Luis Bernal and Carmen Calderón for the invitation to Sevilla and for their hospitality during the visit. I am also grateful to Alfredo Peris for a rectifying remark on a preliminary version of this paper, and to the referee for several helpful suggestions and comments.

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