RAC$\mathbb{SAM}$
Rev. R. Acad. Cien. Serie A. Mat.
VOL. **97 (**2), 2003, pp. 257–261
Análisis Matemático / Mathematical Analysis
Comunicación Preliminar / Preliminary Communication

Small ball properties for Fréchet spaces

Leonhard Frerick and Alfredo Peris

Dedicated to the memory of Professor Klaus Floret

Abstract. We give characterizations of certain properties of continuous linear maps between Fréchet spaces, as well as topological properties on Fréchet spaces, in terms of generalizations of Behrends and Kadets small ball property.

Propiedades de bola pequeña para espacios de Fréchet

Resumen. Caracterizamos ciertas propiedades para aplicaciones lineales y continuas entre espacios de Fréchet, así como propiedades topológicas en espacios de Fréchet, en términos de propiedades de bola pequeña inspiradas en el concepto de la propiedad de bola pequeña introducido por Behrends y Kadets.

Let (M, d) be a metric space. Following Behrends and Kadets [1], (M, d) has the small ball property (sbp), by definition, if for all $\delta > 0$ there are $(\delta_n)_{n\in\mathbb{N}} \in]0, \delta[^{\mathbb{N}}$ decreasing and converging to zero, and $(x_n)_{n\in\mathbb{N}} \in M^{\mathbb{N}}$ such that the union of the balls $B(x_n, \delta_n) := \{y \in M : d(x_n, y) < \delta_n\}$ covers M.

If $D \subset M$, an equivalent formulation gives that $(D, d|_D)$ has the (sbp) if and only if, for *each* decreasing zero sequence $(\epsilon_n)_{n\in\mathbb{N}}$ of positive numbers, there is a sequence $(A_n)_{n\in\mathbb{N}}$ of finite subsets of M such that

$$D \subset \bigcup_{n\in\mathbb{N}} B(A_n, \epsilon_n),$$

where $B(A, \delta) := \{x \in M : d(x, A) < \delta\}$. This formulation motivates us to give the following general definition:

Definition 1 *Let (M, d) be a metric space and let $\mathscr{A}$ be a family of subsets of M (shortly, $\mathscr{A} \subset 2^M$) stable under finite unions. $D \subset M$ has the $\mathscr{A}$-*small ball property *($\mathscr{A}$-*sbp*) if, for each decreasing zero sequence $(\varepsilon_n)_{n\in\mathbb{N}}$ of positive numbers, there is $(A_n)_{n\in\mathbb{N}} \in \mathscr{A}^{\mathbb{N}}$ such that*

$$D \subset \bigcup_{n\in\mathbb{N}} B(A_n, \varepsilon_n).$$

Presentado por José Bonet.
Recibido: 16 de Abril de 2003. *Aceptado*: 8 de Octubre de 2003.
Palabras clave / Keywords: small ball properties for metric spaces, topological properties of Fréchet spaces, reflexivity, compact and weakly compact operators
Mathematics Subject Classifications: 46A04, 46A11, 46A25, 46A63, 47B07, 54E35.

From now on we assume that the family $\mathscr{A}$ is stable under finite unions. In our definition of the *$\mathscr{A}$-sbp*, it is enough to require that for every $\delta > 0$ there are $(\delta_n)_{n\in\mathbb{N}} \in]0,\delta[^{\mathbb{N}}$ decreasing and converging to zero, and $(A_n)_{n\in\mathbb{N}} \in \mathscr{A}^{\mathbb{N}}$ such that

$$D \subset \bigcup_{n\in\mathbb{N}} B(A_n, \delta_n).$$

There are many such families of subsets of Fréchet spaces which are natural and provide characterizations of several known properties. To show this we need a result which is essential for the rest of the note. It is an extended version of a result of Behrends and Kadets which allows us, in addition, to consider images of maps.

Proposition 1 *Let $T : (M_1, d_1) \to (M_2, d_2)$ be a uniformly continuous map between metric spaces where (M_1, d_1) is in addition complete and let $\mathscr{A} \subset 2^{M_2}$. If $T(M_1)$ has the $\mathscr{A}$-sbp then there exists an $r > 0$ such that for all $\varepsilon > 0$ there are $x \in M_1$ and $A \in \mathscr{A}$ with $T(B(x,r)) \subset B(A,\varepsilon)$.*

PROOF. Assume that the condition does not hold. Then there is an increasing function $\varepsilon : [0,\infty) \to [0,\infty)$ with $\lim_{r\to 0} \varepsilon(r) = 0$ such that

$$T(B(x,r)) \not\subset B(A, 2\varepsilon(2r)) \tag{1}$$

for all $r > 0, x \in M_1$ and $A \in \mathscr{A}$. The uniform continuity ensures the existence of an increasing function $\delta : [0,\infty) \to [0,\infty)$ with $\lim_{\varepsilon\to 0} \delta(\varepsilon) = 0$ such that

$$T(B(x, \delta(\varepsilon))) \subset B(T(x), \varepsilon)$$

for all $x \in M_1$ and $\varepsilon > 0$. We may assume that $\gamma(r) := \delta \circ \varepsilon(r) \leq \frac{r}{2}, r > 0$. Since $T(M_1)$ has the $\mathscr{A}$-sbp there is a sequence $(\varepsilon_n)_{n\in\mathbb{N}}$ with $\varepsilon(1) > \varepsilon_1 > \cdots > \varepsilon_n \to 0$ and $(A_n)_{n\in\mathbb{N}} \in \mathscr{A}^{\mathbb{N}}$ with

$$T(M_1) \subset \bigcup_{n\in\mathbb{N}} B(A_n, \varepsilon_n).$$

Let $y_1 \in M_1$ be arbitrary. We set $\gamma^0(1) := 1, N_0 := 0$. We take a sequence of non negative integers (N_k) such that $N_k > N_{k-1}$ and $\varepsilon_{N_k} < \varepsilon(\gamma^k(1))$ for every $k \geq 1$. We claim that we can choose inductively a sequence (y_k) such that

$$y_{k+1} \in B(y_k, \frac{\gamma^{k-1}(1)}{2})$$

and

$$T(B(y_{k+1}, \gamma^k(1))) \cap \bigcup_{N_{k-1} < n \leq N_k} B(A_n, \varepsilon_n) = \emptyset.$$

Indeed, assume that for all $y_{k+1} \in B(y_k, \frac{\gamma^{k-1}(1)}{2})$ there is $zB(y_{k+1}, \gamma^k(1))$ and $N_{k-1} < n \leq N_k$ with $T(z) \in B(A_n, \varepsilon_n)$. Then $d_2(T(y_{k+1}), A_n) \leq d_2(T(y_{k+1}), T(z)) + \varepsilon_n < 2\varepsilon(\gamma^{k-1}(1))$. This would imply

$$T(B(y_k, \frac{\gamma^{k-1}(1)}{2})) \subset B(\bigcup_{N_{k-1} < n \leq N_k} A_n, 2\varepsilon(\gamma^{k-1}(1))),$$

a contradiction to (1).

From $\gamma(\gamma^{k-1}(1)) \leq \frac{\gamma^{k-1}(1)}{2}$ we obtain $B(y_{k+1}, \gamma^k(1)) \subset B(y_k, \gamma^{k-1}(1))$ for all $k \in \mathbb{N}$. The completeness of (M_1, d_1) implies that there is $y \in \bigcap_{n\in\mathbb{N}} B(y_n, \gamma^{n-1}(1))$ and we obtain

$$T(y) \notin \bigcup_{k\in\mathbb{N}_0} \bigcup_{N_{k-1} < n \leq N_k} B(A_n, \varepsilon_n),$$

a contradiction to $T(M_1) \subset \bigcup_{n\in\mathbb{N}} B(A_n, \varepsilon_n)$. ■

Our goal is to apply this characterization of small ball properties to the context of Fréchet spaces. We refer the reader to [4] for notation and general properties of Fréchet spaces. Let us recall that any Fréchet space F admits a translation invariant metric that generates the topology of F. On the other hand it is easy to see that small ball properties do not depend on the translation invariant metric that is fixed on F, generating its topology. From now on all Fréchet spaces will be assumed to be endowed with a translation invariant metric. Natural maps between Fréchet spaces are linear and continuous, and within this setting we can obtain a nicer characterization of small ball properties.

Lemma 1 *Let $T : E \to F$ be a continuous linear map between Fréchet spaces and let $\mathscr{A} \subset 2^F$ such that*
i) for all $A \in \mathscr{A}$ and $\lambda \geq 0$ we have $\lambda A \in \mathscr{A}$, and
ii) for all $A \in \mathscr{A}$ and $x \in E$ there is $B \in \mathscr{A}$ such that $T(x) + A \subset B$.
Then $T(E)$ has the $\mathscr{A}$-sbp if and only if there exists a zero neighbourhood V in E such that for every zero neighbourhood U in F there is $A \in \mathscr{A}$ with

$$(*) \quad T(V) \subset A + U.$$

PROOF. Using i) and the fact that neighbourhoods are absorbing it is easy to see that condition $(*)$ is sufficient.

Assume now that $T(E)$ has the $\mathscr{A}$-sbp. By Proposition 1, there is a zero neighbourhood V in E such that for all zero neighbourhoods U in F there is $x(U) \in E$ and $A_0(U) \in \mathscr{A}$ such that

$$T(x(U) + V) \subset A_0(U) + U.$$

Condition $(*)$ follows by taking $A \in \mathscr{A}$ such that $T(-x(U)) + A_0(U) \subset A$. ■

Examples for systems $\mathscr{A}$ satisfying the above assumptions $i)$ and $ii)$ are the system $\mathscr{E}(F)$ of all finite subsets of F, the system $\mathscr{B}(F)$ of all bounded subsets of F, and the system $\Sigma(F)$ of all absolutely convex and $\sigma(F, F')$-compact subsets of F. Applying Lemma 1 to these examples we obtain

Theorem 1 *Let $T : E \to F$ be a continuous linear map between Fréchet spaces. Then*
i) T is compact iff $T(E)$ has the $\mathscr{E}(F)$-sbp,
ii) T is bounded iff $T(E)$ has the $\mathscr{B}(F)$-sbp,
iii) T is weakly compact iff $T(E)$ has the $\Sigma(F)$-sbp.

PROOF. We will just show $iii)$: It is obvious that weak compactness is sufficient. Conversely, let $T(E)$ have the $\Sigma(F)$-sbp. If $(U_n)_{n\in\mathbb{N}}$ is an arbitrary zero basis in F, Lemma 1 ensures the existence of an absolutely convex zero neighborhood V in E and a sequence $(A_n)_{n\in\mathbb{N}}$ of absolutely convex and weakly compact sets in F such that

$$T(V) \subset \bigcap_{n\in\mathbb{N}} (A_n + U_n).$$

This implies that $T(V)$ is bounded and that

$$T(V)^{\circ\circ} \subset \bigcap_{n\in\mathbb{N}} (A_n + U_n)^{\circ\circ},$$

where the first polar is taken with respect to the dual system $< F, F' >$ and the second polar is taken with respect to $< F', F'' >$. Since A_n is absolutely convex and weakly compact we have $(A_n + U_n)^{\circ\circ} = A_n + U_n^{\circ\circ}$ and hence

$$T(V)^{\circ\circ} \subset \bigcap_{n\in\mathbb{N}} (F + U_n^{\circ\circ}) \subset F,$$

since F is a closed subset of its strong bidual F''. Together with the boundedness of $T(V)$, this shows that $T(V)$ is relatively weakly compact. ■

Now we can characterize several topological properties for Fréchet spaces in terms of small ball properties. Let us recall the most fundamental properties:

Given a Fréchet space E, we say that

- E is a *Schwartz* space if, for any absolutely convex zero neighbourhood U in E, we have that the canonical map $\phi_U : E \to E_U$ into the local Banach space E_U is compact,
- E is a *Montel* space if every bounded subset of E is relatively compact,
- E is *quasinormable* if, for any zero neighbourhood U in E, there is a zero neighbourhood V contained in U such that, for each $\varepsilon > 0$, there is a bounded set B satisfying

$$V \subset B + \varepsilon U.$$

Assertion $i)$ in the following theorem is already proved by Behrends and Kadets [1, Corollary 3.7] in the particular case when E is a Banach space.

Theorem 2 *Let E be a Fréchet space. Then*
i) E is finite dimensional iff E has the $\mathscr{E}(E)$-sbp.
ii) E is a Banach space iff E has the $\mathscr{B}(E)$-sbp.
iii) E is a reflexive Banach space iff E has the $\Sigma(E)$-sbp.
iv) E is a Schwartz space iff for every continuous linear map T from E into any Banach space X the set $T(E)$ has the $T(\mathscr{E}(E))$-sbp (or, equivalently, the $\mathscr{E}(F)$-sbp).
v) E is quasinormable iff for every continuous linear map T from E into any Banach space X the set $T(E)$ has the $T(\mathscr{B}(E))$-sbp.
vi) E is reflexive and quasinormable iff for every continuous linear map T from E into any Banach space X the set $T(E)$ has the $T(\Sigma(E))$-sbp.
vii) E is Montel iff for every continuous linear map T from any Banach space X into E the set $T(X)$ has the $\mathscr{E}(E)$-sbp.
viii) E is reflexive iff for every continuous linear map T from any Banach space X into E the set $T(X)$ has the $\Sigma(E)$-sbp.

PROOF. $i)$ is a consequence of property $i)$ in Theorem 1 for $T = I$ and the fact that a non-empty open set in a topological vector space is relatively compact if and only if the space is finite dimensional.
$ii)$ is a direct consequence of property $ii)$ in Theorem 1 for $T = I$ and the fact that an open set in a Fréchet space is bounded if and only if the space is Banach.
$iii)$ is a consequence of the previous equivalence $ii)$, Theorem 1 $iii)$, and the fact that a Banach space is reflexive if and only if its closed unit ball is weakly compact.
$iv)$: If E is Schwartz, X is Banach space and $T : E \to X$ is linear and continuous, then it is obviously compact and we obtain that $T(E)$ has the $T(\mathscr{E}(E))$-sbp by Theorem 1 $i)$. Conversely, if for each local Banach space $X := E_U$ we have that the canonical map $T : E \to X$ satisfies that $T(E)$ has the $T(\mathscr{E}(E))$-sbp, then Theorem 1 $i)$ implies that T is compact, and therefore E is Schwartz.
$v)$: E is quasinormable if and only if, for each absolutely convex zero neighbourhood U, there is a zero neighbourhood V contained in U such that for every $\varepsilon > 0$ there is a bounded set B satisfying

$$T(V) \subset T(B) + \varepsilon\tilde{U},$$

where $T : E \to E_U$ is the canonical map, and $\tilde{U}$ is the unit ball of E_U. Therefore, by Lemma 3, quasinormability is a necessary condition. Conversely, if E is quasinormable and $T : E \to X$ is a continuous linear map into an arbitrary Banach space X, then there is a local Banach space E_U such that T factorizes in the canonical way trough the map $S : E \to E_U$. We have that $S(E)$ has the $S(\mathscr{B}(E))$-sbp, thus $T(E)$ has the $T(\mathscr{B}(E))$-sbp.
$vi)$ is a consequence of the previous equivalence $v)$ and the fact that a Fréchet space is reflexive if and only

if the bounded sets are relatively weakly compact.
$vii)$: If E is Montel, X is a Banach space and $T : X \to E$ is linear and continuous, then $T(B_X)$ is bounded and thus relatively compact. Property $i)$ in Theorem 1 implies that $T(X)$ has the $\mathscr{E}(E)$-sbp. Conversely, if B is an absolutely convex and closed bounded subset of E, then the continuous inclusion $T : E_B \to E$ defined on the Banach space E_B associated to B satisfies that, by Theorem 1 $i)$, T is compact. This implies that B is relatively compact.
$viii)$: A Fréchet space is reflexive if and only if for each B bounded and for every zero neighbourhood U, there is a weakly compact subset K of E such that

$$(*)' \quad B \subset K + U.$$

(See [3, 1.9] and [2, Lemma 2.2] for a detailed proof in a more general case).

If $T(E_B)$ has the $\Sigma(E)$-sbp for each absolutely convex and closed bounded subset B of E, then Lemma 3 gives $(*)'$, so E is reflexive. Conversely, if E is reflexive, X is a Banach space, and $T : X \to E$ is continuous and linear, then T is weakly compact and we conclude by Theorem 1 $iii)$. ■

References

[1] Behrends, E. and Kadets, V.M. (2001). Metric spaces with the small ball property, *Studia Math.*, **148** (3), 275–287.

[2] Bonet, J. and Friz, M. (2002). Weakly compact composition operators on locally convex spaces, *Math. Nachr.*, **245**, 26–44.

[3] Díaz, J . C. and Domański, P. (1998). Reflexive operators with domain in Köthe spaces, *Manuscripta Math.*, **97**, 189–204.

[4] Meise, R. and Vogt, D. (1997). *Introduction to Functional Analysis*, Clarendon Press, Oxford.

L. Frerick
Bergische Universität Wuppertal
FB Mathematik
Gaußstr. 20
D–42097 Wuppertal, German
frerick@math.uni-wuppertal.de

A. Peris
E. T. S. Arquitectura
Departament de Matemàtica Aplicada
Universitat Politècnica de València
E–46022 València, Spain
aperis@mat.upv.es