

A SIMULATION APPROACH TO THE VALUATION OF CAPITAL BUDGETING PROJECTS INCORPORATING A DEFER OPTION*

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ABSTRACT

Techniques applied to determine the value of derivatives have been recently exported in the field of investment valuation. This paper aims to provide some light to the use of a new technique in the investment valuation literature, aiming to take into account the value of flexibility. This technique, designed by Longstaff and Schwartz, combines Monte Carlo simulation and the Ordinary Least Squares in order to value American-style derivatives with different specifications. We show that this method can easily be incorporated to value capital budgeting projects in the framework of the real options theory and provides coherent results from an economic point of view. We do this by estimating the value of several cases of an investment project that incorporates an option to defer the initial investment or layout through time. We estimate these values by using the Ox programming language.

Key words: Least Squares Monte Carlo, defer option, geometric Brownian motion, investment project.

RESUMEN

Algunas técnicas aplicadas para determinar el valor de derivados han sido recientemente exportadas en el campo de la valoración de inversiones. Este trabajo tiene como objetivo clarificar el uso de una nueva técnica dentro de la literatura de valoración de inversiones, teniendo en cuenta el valor de flexibilidad. Esta técnica, diseñada por Longstaff y Schwartz, combina la simulación de Monte Carlo y los Mínimos Cuadrados Ordinarios con el objetivo de valorar derivados de tipo americano con distintas especificaciones. En este trabajo demostramos que este método puede ser fácilmente incorporado para valorar proyectos de inversión en el marco de la teoría de opciones reales y muestra resultados coherentes desde el punto de vista económico. Lo hacemos estimando el valor de distintos casos de un proyecto de inversión que incorpora una opción de posponer la inversión inicial en el tiempo. Estimamos estos valores usando el lenguaje de programación Ox.

Palabras clave: Least Squares Monte Carlo, opción de espera, movimiento geométrico Browniano, proyecto de inversión.

1 Introduction

There are many methods available in the literature to price derivatives. The most famous of them is the analytical expression developed by Black, Merton and Scholes (see Black and Scholes (1973) and Merton (1973)). Unfortunately, the Black-Scholes model is only suitable for pricing European-style options.

In the case of pricing options with American features, it is very difficult to find analytical solutions, and it is necessary to use numerical methods. Numerical methods like the Binomial method suggested by Cox, Ross and Rubinstein (1979) or finite difference methods can be used to price American options. However, these methods are not feasible when increasing the number of stochastic factors that the option could incorporate.

An alternative to price options with complex features like multiple stochastic processes, is the use of Monte Carlo simulation introduced to value options by Boyle (1977). The Monte Carlo approach consists in simulating trajectories for the underlying asset. However, this technique is normally used for pricing European-style options.

Recently, a new approach to price American-style options has been developed. The Least Squares Monte Carlo algorithm created by Longstaff and Schwartz (2001) approximates the value of the option by assuming that it can be exercised in a subset of times $t = 1, 2 \dots T$. This method combines the simulation techniques with least squares regression to determine the convenience to exercise or not the option in each discretization by backward induction. The Least Squares Monte Carlo method provides a stopping rule for the exercise of the option. Through this stopping rule, by discounting and averaging, the price of the option is found.

On the other hand, the real options theory provides an alternative to price capital budgeting projects by considering those from a dynamic point of view. The firm that estimates the value of a project today with the purpose to decide whether it is worth to undertake it or not, has the possibility to take decisions throughout the life of the project. Those decisions can be labeled as *options* that the firm possesses in the project. This flexibility increases the value of the project and can modify the decision to undertake it or not. Those options can incorporate a high number of stochastic factors, and it is worth to point that methods used to value financial derivatives can be easily imported to value real options.

A project can incorporate many kinds of options, and even options that depend on other options. This work deals mainly with the option to defer an irreversible investment through time. An example of this is as follows: suppose a firm that is considering to build a plant. To take a decision, the firm calculates the discounted expected benefits and the necessary costs to build this plant, undertaking this project if the discounted expected benefits are greater than the costs. The firm incurs the irreversible cost of building it once the decision is taken. However, the opportunity to defer those costs allows to postpone them to the future, creating an additional value and making an incentive in accepting the decision to build the plant. The

additional value can change radically the decision concerning the profitability of projects.

This work aims to estimate the value of capital budgeting projects by applying the Least Squares Monte Carlo algorithm. In this paper, we show that the Least Squares Monte Carlo method can easily be incorporated to value capital budgeting projects in the framework of the real options theory. In fact, we estimate the value of several projects that incorporate an option to defer the initial investment or layout through time.

This work is divided as follows. In Section 2 we explain some aspects about the real options theory, making special emphasis in the option to defer investment. In Section 3 we estimate the value of projects for different specifications incorporating the flexibility of a defer option. In Section 4 we analyze the implications of programming the valuations performed in Ox, a programming language similar to C and C++. Finally, we conclude in Section 5.

2 Real options framework.

Traditionally, the value of a project or an investment undertaken by a firm is determined by the Net Present Value, which is equal to the sum of the expected future cash flows CF_t generated by the project at time t , discounted at the discount rate r applicable to the project, minus the required initial investment or outlay I . The expression of the Net Present Value (NPV) is the following:

$$NPV = \sum_t \frac{CF_t}{(1+r)^t} - I. \quad (1)$$

The cash flows of a project are generally expected cash flows, and what is more important, very often the initial decision taken when valuing a project can be changed.

The NPV criterion is useful when valuing simple projects, as it does not capture the evolution of a highly uncertain project, and the ways in which the firm can influence in it. In this sense, trusting in this method as an indicator of decisions, could lead to wrong actions. If the firm decides to reject a project based on a negative NPV, a critical feature of the investment opportunity can be missed by disregarding that throughout the life of the project the firm can have the right, but not the obligation, to face the cost of the investment. Therefore, the traditional NPV does not take into account the decision flexibility of complex projects by assuming no active actions by the firm throughout the life of the project.

The theory of real options incorporates the value of flexibility into the project, and allows firms to benefit from uncertainty, receiving additional profits out of it.

In this section, we focus mainly in the possibility to postpone investment decisions, known in the real option literature as a defer option, option to postpone the investment or option to wait. A short review of other kinds of real options is also

presented. This section ends with the basic assumptions done when valuing real options in continuous time.

2.1 The option to defer investment.

The possibility to postpone the starting of a project is known in the literature as an option to defer investment or option to wait, as noted in Trigeorgis (1998) and Amran and Kulatilaka (1999). This option is particularly valuable when high uncertainties and long investment horizons are present like resource extraction industries and real estate development. The option to postpone an investment is exercised if the value that can be achieved by postponing the decision is larger than the value of immediate exercise.

Incorporating the decision flexibility of postponing an investment when valuing a project, the real options theory imports the notion of a call option borrowed from securities markets. A call option is the right, but not the obligation, to buy an underlying asset at a certain price (the exercise or strike price) and at a certain date (date of expiration). This definition corresponds to an European-style option in contrast to American-style option, which are options that can be exercised at any time until the date of expiration. Obviously, in the case of an European call option, it would be exercised only if the value of the underlying asset is above the strike price on the expiration date.

We start in a discrete time framework to show the understanding of the concept of the defer option. Let us see an example that reflects the value of flexibility in the NPV criterion. Suppose a pharmaceutical firm that wants to analyze if the development of a pharmaceutical product will be profitable. The success of the product depends on the trials addressed to obtain the FDA approval, which needs a capital outlay of 5 millions of euros. Past experience suggests a 25% probability of success to the trials. Success is defined as obtaining the FDA approval with broad market potential. If those trials are successful, the value in one year from now of expected cash-flows is estimated to be 45 millions. The trials are unsuccessful with a 75% probability, which means that an approval with limited potential will result leading to a value of 3 millions. Additionally, full-scale production requires the construction of a new plant with an estimated cost of 12 millions.

Applying the NPV technique is quite straightforward. The firm has to pay today 5 millions for the trials and 12 millions for the construction of a new plant, and will receive the probability weighted average of the payoffs in one year. Assuming that the required rate of return of this project is 15%, the NPV calculation is as follows:

$$NPV = -17 + \frac{0.25 * 45 + 0.75 * 3}{1 + 0.15} = -5.26 \text{ millions.}$$

According to the result, this project should be rejected, as the NPV is negative. However, when calculating the NPV, the value of flexibility has been disregarded, as the calculation assumes that the plant will be constructed today, before the outcome

of the trial is known. Let us assume that the plant can be constructed in one year from now for an investment of 13 millions of euros. As the trials begin today, the outcome would be known *before* taking the decision of constructing the plant. Obviously, it would be efficient to construct the plant if the trial is successful and not to construct the plant if it is not successful, therefore avoiding the investment of 12 millions.

Taking this new perspective into account, then the project will generate $\max(45-13,0)=32$ millions with a probability of success of 0.25 and $\max(3-13,0)=0$ million with a probability of no success of 0.75. Note that those cash-flows are similar to the cash flows generated by an European option with a strike price of 13. The value of the project today, modified to reflect the option to defer is as follows:

$$\text{modified } NPV = -5 + \frac{0.25 * 32 + 0.75 * 0}{1 + 0.15} = 1.95 \text{ millions.}$$

For simplicity, we have assumed that the discount rate appropriate for the project remains at 15%. However, incorporating this flexibility means a change of the risk of the project, and therefore a change in the discount rate. The risk is now reduced because capital investment in the future will be made only if favourable outcomes occur.

Assuming a risk-free rate of 0.06, we can calculate the risk neutral probabilities, apply them in the calculation of the NPV and solve this problem. The use of the risk neutral probabilities result in a risk-free expected value.

$$\frac{0.25 * 45 + 0.75 * 3}{1 + 0.15} = \frac{p * 45 + (1 - p) * 3}{1 + 0.06}.$$

The risky expected value is equal to the left part of the equality. Then we equalize it to the cash-flows discounted by the risk-free rate. The probabilities of those discounted cash-flows are the risk neutral probabilities. Thus, the probability of the up state is 22.4% and the probability of the down state is 77.6%. Having found those probabilities, we can use them jointly with the risk-free rate to give a more correct value of the project, incorporating the flexibility of the defer option:

$$\text{corrected } NPV = -5 + \frac{0.224 * 32 + 0.776 * 0}{1 + 0.06} = 1.76 \text{ millions.}$$

Note that there is a positive relationship between uncertainty and the value of the option. This is because the option allows to capture the upper value while eliminating the down value. In the example, if the value of success will be greater and the value of no success lower, the cash-flow of success will be greater although the cash-flow of no success will remain zero. That is how the project benefits from uncertainty.

Incorporating to the calculations the value of flexibility, the decision changes completely. When the option value is disregarded in calculating the NPV, we find that some projects can be refused although it would be interesting to undertake

them. Early investment implies the sacrifice of the value of the option to wait. The NPV tool assumes early investment, not taking into account this value. Therefore, the NPV increases when taking this value into consideration. In fact:

$$\text{Corrected NPV} = \text{Standard NPV} + \text{Option Premium.}$$

In continuous time, the real options approach incorporates the uncertainty from the cash flows, as well as the possibility to postpone the starting of a project by using stochastic processes describing its value, and then evaluating the option to invest. Imagine the previous example, but instead of having two times, *today* and *tomorrow*, the firm can postpone the outlay of the project at any time through a certain period. This is the new context in which we value a defer option in this work. The option to defer investment is now analogous to an American call option on the expected value of the project, with an exercise price equal to the required outlay.

Just as the owner of an American call option on a financial asset possess the right, but not the obligation, to acquire the asset by paying the exercise price before the maturity date and will exercise the option if it is in his interest, so will the holder of an option on real assets.

As the following table illustrates, there exist a very close analogy between options on a financial asset and options on a real asset.

| Call option on stock | Real option on project |
|-----------------------------|--------------------------------------|
| Current value of stock | Present value of expected cash flows |
| Exercise price | Investment cost |
| Time to expiration | Time until opportunity disappears |
| Stock uncertainty | Project uncertainty |
| Riskless interest rate | Riskless interest rate |

Table 1: Comparison between a call option on a stock and a real option on a project. Source: Trigeorgis (1998)

Thanks to this analogy, the techniques developed to value options on financial assets can be easily imported to value options on real assets.

One of the most important formulas to price options on financial assets is the one developed by Black and Scholes (1973) and Merton (1973). This is a very simple and easy-to-use formula, but it is only useful to price European-style options. In the case of American options, there exists some analytical expressions, but there are no easily computable formulas currently available. Then, numerical methods are usually required.

2.2 Different kinds of real options.

This paper is centered in the valuation of a option to defer investment. However, it is valuable to make a general overview of the different kinds of option on assets, in order to have a wider idea of the scope of the real option theory.

The most common categories of encountered real options are the following:

- **Option to alter operating scale.** During the life of a project already undertaken by a firm, if market conditions are more favorable than expected, the firm can expand the scale of production or accelerate resource utilization. Conversely, if conditions are less favorable than expected, the firm can reduce the scale of operations.
- **Option to abandon.** If market conditions decline severely, the firm can abandon operations permanently and sell the capital equipment and other assets on the market.
- **Option to switch inputs or outputs.** When the prices or demand change, the firm can change the output that it is producing in response of these changes. Alternatively, the firm could change the inputs to produce the same outputs when input prices fluctuate. This option arises normally in taking purchasing decisions on the acquisition of technology or machinery that facilitates this change.
- **Growth options.** This option is present when making an early investment, which opens interrelated projects that bring future growth opportunities. Examples of those investments are present in *R&D* projects, a strategic acquisition or an information network.

Jointly with the option to wait, those are the more common real options. Additionally, in the real-life projects we can find that a same project involves a collection of the options analyzed. In this case, the combined value of those may differ from the sum of their separate values due to interaction among those, because one option can depend on other(s). For example, it might be necessary to exercise a defer option in a project in order to exercise an abandonment option in that project.

3 Valuing projects that involve a defer option.

In this section, we value capital budgeting projects that involves an option to wait. Let us assume a certain firm who desires to value a project that possess an option to defer an investment through time and can exercise it at any moment.

The discounted benefits of the project behave randomly following a geometric Brownian motion path, following this expression:

$$dV_t = rV_t dt + \sigma V_t dW_t.$$

Additionally, we assume that the layout follows also a geometric Brownian motion path given by:

$$dI_t = \tilde{r}I_t dt + \tilde{\sigma}I_t d\tilde{W}_t,$$

obviously with the same form but with other parameters. The risk that affects the value of the project is different from the risk that affects the layout. Thus, the risk parameters must be different. We assume no correlation between both processes.

We fix $r = 0.06$ and $\tilde{r} = 0.04$, which are the drift under the risk-neutral probability measure. We vary the values of σ and $\tilde{\sigma}$ when performing the simulation as we indicate in the tables.

We simulate the value of the option to wait and the NPV using Ox. In each simulation, we use a total of 100,000 paths and 60 discretizations.³ We denote the paths by M and the total of time steps by T . Therefore, we fix T as 60.

We approximate the conditional expectation by the three weighted Laguerre polynomials, generated by the following expression⁴:

$$L_n(X) = \exp(-X/2) \frac{e^x}{n!} \frac{d^n}{dX^n} (X^n e^{-X}). \quad (2)$$

Thus, we perform 59 (T-1) regressions on the following polynomial at each time step $t < T$:

$$\begin{aligned} E(Y_t|X_t) = & \beta_0 + \beta_1 \exp(-X_t/2) + \beta_2 \exp(-X_t/2)(1 - X_t) \\ & + \beta_3 \exp(-X_t/2)(1 - 2X_t + X_t^2/2) + u_t. \end{aligned} \quad (3)$$

In table 2 we show our results. We suppose a constant riskless interest rate equal to 6%. For simplicity, we assume that the initial value of the project is equal to the initial value of the layout at $t = 0$, which means that the NPV of the project without considering flexibility is equal to zero. This assumption can be easily relaxed. We change the risk parameter of the project and the initial value of the project and the layout. The value of the estimated NPV increases as we increase the uncertainty of the project, because it benefits from the increasing risk as explained in Section 2.1. On average, the estimated NPV is a 90.813% bigger than the NPV of the project if the company decides to undertake the project at $t = T$, as if it were an European option. This percentage infers the value of flexibility of the project with respect to the last exercise date on average.

The estimated NPV today is indicated in the table as p_m . We see that all calculated values are always positive as an effect to incorporate the option to wait. As the NPV without the flexibility of waiting to invest is equal to zero, the values of the option to wait are equal to the values reported on the table.

In table 3 we change the risk of the layout given by the parameter $\tilde{\sigma}$. We note that the more risk the bigger is the estimated price. Thus, the project benefits from the risk of the layout. The more it fluctuates, the more differences $V_t - I_t$ are created, so the ITM paths are bigger.

³Half of the simulations performed are antithetic as a way to reduce the variance of the estimator.

⁴All the different kinds of basis functions and its properties are described in Chapter 22 of Abramowitz and Stegun (1970). For deeper details of these functions see Walter (1994)

| V=I | σ | p_m | p_{eu} |
|-----|----------|--------|----------|
| 40 | 0.2 | 1.9014 | 0.1765 |
| 40 | 0.4 | 2.5872 | 0.2379 |
| 42 | 0.2 | 1.9987 | 0.1852 |
| 42 | 0.4 | 2.7196 | 0.2497 |
| 44 | 0.2 | 2.0963 | 0.1941 |
| 44 | 0.4 | 2.8628 | 0.2616 |
| 46 | 0.2 | 2.1951 | 0.2029 |
| 46 | 0.4 | 3.0053 | 0.2735 |
| 48 | 0.2 | 2.2926 | 0.2117 |
| 48 | 0.4 | 3.1416 | 0.28548 |
| 50 | 0.2 | 2.3900 | 0.2205 |
| 50 | 0.4 | 3.2792 | 0.2973 |

Table 2: This table represents the NPV of the project (p_m), and its value if it were exercised at T (p_{eu}) for different values of the initial value of the project represented by V , and the initial value of the layout represented by I . We assume $V = I$. The riskless interest rate is equal to 6%. The constant part of the layout process \tilde{r} is fixed and equal to 0.04. The stochastic part of the layout $\tilde{\sigma}$ is fixed and equal to 0.3. Thus, $dI = 0.04I dt + 0.3I d\tilde{W}$. The risk of the value of the project is represented by σ in the table.

Finally, in table 4 we relax the assumption of the initial value of the project equal to the initial value of the layout $V = I$ and report the values of the option to wait for different combinations of V and I . The bigger is V with respect to I , the bigger is the probability to generate positive net values for each path and discretization. Note that in each case it is possible to calculate the value of the option as the difference between the NPV incorporating flexibility, and the NPV without flexibility. Those results are also shown in the table. We note that when the NPV is negative, the value of flexibility is bigger. Economically it makes sense, as with a positive NPV the probability of exercising now instead of waiting should be bigger.

4 Comments about the performance of the computations.

All calculations performed to estimate the NPV of the different projects in Section 3 have been made in Ox⁵, a programming language similar to C and C++. We highlight the main aspects of the computations performed in the appendix.

We have chosen Ox between other programs because it is fast and easy to use in every platform. The computation for the projects involving the defer option with stochastic layout takes around 10 minutes. Those have been performed in a

⁵For more information about Ox, visit www.oxmetrics.net

| V=I | σ | $\tilde{\sigma}$ | p_m | p_{eu} |
|-----|----------|------------------|--------|----------|
| 40 | 0.2 | 0.3 | 1.9014 | 0.1764 |
| 40 | 0.2 | 0.6 | 2.5872 | 0.2379 |
| 40 | 0.4 | 0.3 | 2.5872 | 0.2379 |
| 40 | 0.4 | 0.6 | 3.7506 | 0.3351 |
| 50 | 0.2 | 0.3 | 2.3900 | 0.2205 |
| 50 | 0.2 | 0.6 | 4.1595 | 0.3715 |
| 50 | 0.4 | 0.3 | 3.2792 | 0.2973 |
| 50 | 0.4 | 0.6 | 4.7054 | 0.4194 |
| 60 | 0.2 | 0.3 | 2.8728 | 0.2646 |
| 60 | 0.2 | 0.6 | 4.9895 | 0.4460 |
| 60 | 0.4 | 0.3 | 3.9391 | 0.3568 |
| 60 | 0.4 | 0.6 | 5.6365 | 0.5027 |

Table 3: This table represents the NPV of the project (p_m), and its value if it were exercised at T (p_{eu}) for different values of the initial value of the project represented by V , and the initial value of the layout represented by I . We assume $V = I$. The riskless interest rate is equal to 6%. The stochastic part of the value of the project σ and the stochastic part of the layout $\tilde{\sigma}$ are as indicated in the table.

Silicon Graphics Fuel Workstation with a 500Mhz MIPS R 14000(IP35) processor with RAM 512 Mb.

It worths to mention in this section about the random number generator in Ox. The default random number generator is the Park & Miller random number generator, which requires only one seed. Thus, when we simulate the standard Normal distribution in order to simulate the geometric Brownian motion, we always obtain the same values as output. To obtain different values, we have to change the seed by the command `ranseed()`.

In this work we have used a seed of 100,000 for the defer options valued for the discounted benefits. Obviously, the seed used to generate the value of the project must be different to the seed used to generate the layout in order to use different values. The seed of the layout we consider in our valuations is the default one.

In table 5 we show the different NPV of the projects including the option to wait, changing the seed of the discounted benefits:

As for the defer option with stochastic layout, we consider in the table above $V = K = 40$, $\sigma = 0.2$ and $\tilde{\sigma} = 0.3$. According to the results, both the price of the American defer option and the European defer option does not fructuate significantly, as the number of simulations have increased enough.

According to our estimations in Section 3, the application of the LSM method never leads to negative estimations as it focuses in the ITM paths. Does it mean that it is always profitable to undertake any project? Of course not. Projects that cannot generate any cash-flows will never have a positive net value, but obviously negative. Unfortunately, as the algorithm only focus in positive net values, we cannot estimate how much negative this value is.

| V | I | p_m | opc |
|----|----|--------|--------|
| 40 | 35 | 5.8619 | 0.8619 |
| 40 | 40 | 3.7506 | 3.7506 |
| 40 | 45 | 2.5607 | 7.5607 |
| 45 | 40 | 6.2613 | 1.2613 |
| 45 | 45 | 4.2777 | 4.2777 |
| 45 | 50 | 3.0024 | 8.0024 |
| 50 | 45 | 6.6905 | 1.6905 |
| 50 | 50 | 4.7054 | 4.7054 |
| 50 | 55 | 3.4489 | 8.4489 |

Table 4: This table represents the NPV of the project (p_m), and the value of the option (opc) for different values of the initial value of the project represented by V , and the initial value of the layout represented by I . We let V be different to I . The riskless interest rate is equal to 6%. The stochastic part of the value of the project σ is fixed and equal to 0.4. The stochastic part of the layout $\tilde{\sigma}$ is fixed and equal to 0.6.

| ranseed() | LSM price. | European price. |
|----------------|------------|-----------------|
| 100,000 | 1.9014 | 0.1764 |
| 200,000 | 1.9035 | 0.1763 |
| 300,000 | 1.9116 | 0.1762 |
| 400,000 | 1.9062 | 0.1763 |
| 500,000 | 1.8987 | 0.1768 |
| 600,000 | 1.9040 | 0.1766 |
| 700,000 | 1.9095 | 0.1760 |
| 800,000 | 1.8946 | 0.1761 |
| mean | 1.9036 | 0.1763 |
| std. deviation | 0.0055 | 0.0002 |

Table 5: Different prices of the option considering several values of the seed.

When uncertainty is reduced and the initial value of the project is too small compared to the initial layout, the probability of exercising the project is reduced and the value of flexibility is near zero. But what happens when there is no value of flexibility, or it is too reduced compared to the expected losses? In this case, the program provides no value, as it is no possible to estimate the value of continuation at some step. This is the case when we set, for instance, $V_0 = 45$, $I_0 = 60$, $\sigma = 0.4$ and $\tilde{\sigma} = 0.3$, where the NPV without value of flexibility is $V_0 - I_0 = -15$ million of euros, which is a big loss.

5 Conclusions.

In this work, we analyze the Least Squares Monte Carlo algorithm and we apply it in the context of the real options theory. In particular, we estimate the net value at $t = 0$ of several cases of an investment project including a defer option or option to wait that can be exercised at any time.

We demonstrate that the algorithm can be perfectly used to value the flexibility embedded to those projects and estimate its Net Present Value. We note that for projects in which the initial Net Present Value is more reduced, the value of flexibility is bigger. Additionally, the projects with a higher volatility in the value process and in the layout reflect increasing values of flexibility.

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A Computations for valuing projects with defer options.

We would like to highlight the main aspects of the program to value defer options in Section 3. We calculate the values of the defer option with $T = 60$, which supposes to perform 59 regressions, 59 comparisons, etc. Thus, a way to abbreviate the writing of the program is crucial, because otherwise the extension would be too big.

After declaring the parameters and variables, the first step is to simulate the path of the value of the project. For that, we construct the matrix of random numbers from a normal distribution with dimension $M \times T$, where M stands for the simulation paths (8 in this case) and T stands for the discretizations (3 in the example). We add a first column of zeros, which will be the initial value of the process. We apply the antithetic technique when doing the simulations to reduce the variance of the price. Thus, we construct the matrix *ran* with the first column full of zeros and for the rest of the columns, the first $M/2$ rows composed by simulated values of a standard normal distribution and the following (and last) $M/2$ rows composed by the same values with opposite sign. Then we fill this matrix, substituting the zero values with the simulated values generated by the matrix of random values:

```
antit= rann(M/2,T);

ran = 0 ~ (antit | -antit);

v = V ~ zeros(M,T);

for (j=0; j < T; ++j)

{
v[] [j+1]= v[] [j] .*exp((r - sig^2 / 2) *del + sig *
sqrt(del) .* ( ran[] [j+1]) );
}
```

Note that to generate the values, we have used the *for* loop. The *for* loop consists of three parts, an initialization part, a termination check, and an incrementation part. So j starts at 0 until $T - 1$, incrementing his value by one unit.

Additionally, the outlay is stochastic, so it is also necessary to generate its values, in a similar way as the values of the project:

```
for (j=0; j < T; ++j)

{
i[] [j+1]= i[] [j] .*exp((alf - isig^2 / 2) *del
+ isig * sqrt(del) .* ( ran[] [j+1]) );
}
```

We made the first regression and comparison at $t = 59$. We also draw the last provisional columns of the matrix mc . The regression function is taken from the Laguerre polynomial in formula (19).

```
eyx = b[0] + b[1]*exp(-xt[] [t] ./2) + b[2]*exp(-xt[] [t] ./2).*(1-xt[] [t])
+ b[3]*exp(-xt[] [t] ./2).*(1 -2 .*xt[] [t] + xt[] [t].^2 ./2);
```

Before this initial step, to make the analysis from $t = 58$ to $t = 1$, we introduce a *for* loop that fructuates t down to $t = 1$, and performs the operations of regressing, comparing and designing the matrix mc automatically.

```
for(t=T-2; t>0; --t)
{
}
```

Inside of this loop, that we call *root loop* because it contains other loops, we include all operations related to regress and compare the calculated value of continuation with the value of immediate exercise.

A matrix h is designed as an intermediate tool to calculate the matrix yt of discounted prices. We introduce another *for* loop inside of the previous loop to vary the position of the column in matrix h . This matrix is composed by the discounted elements from mc . Each row of matrix h has one element different to zero or all elements different to zero, so we sum all rows and reduce it to the vector yth with dimension $1 \times N$. After calculating this, we filtrate the elements of this vector, placing `.NaN` when the value of the project is not ITM at that time step t :

```
h = zeros(M,T);

for(j=T; j>t; --j){

h[] [j-1] = mc[] [j-1].>0 .? mc[] [j-1]*exp(-(j-t)*r) .: 0;

}

decl yth =sumr(h);

yt = itm[] [t].>0 .? yth .: .NaN;
```

In order to change the previous decisions taken at the previous comparisons, we include another *for* loop inside of the root loop with the following expression:

```
for(j = T-1; j > t-1; --j){  
  mc[][j] = mc[][t-1] .>0 .? 0 .: mc[][j];  
}
```

After this, we discount the prices and calculate the LSM price.