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POLINOMIOS DE APPELL Y FUNCIONES ESPECIALES EN LA TEORÍA DE DUNKL

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Esta tesis doctoral se ha realizado por compendio de publicaciones siguiendo la normativa de la Universidad de La Rioja,[1] dentro del «Programa de Doctorado en Matemáticas y Computación» (Plan 782D, según el R.D. 99/2011), y esta memoria está estructura en apartados tal como se detalla en la citada normativa.

La publicaciones que incluimos en este compendio son tres artículos con fecha de publicación posterior al de inicio del programa de doctorado (curso 2021/22). Los artículos en cuestión, puestos en el orden en el que los elaboramos temporalmente hablando, son los siguientes:

- A. Gil Asensi, J. L. Varona, Appell-Dunkl sequences and Hurwitz-Dunkl zeta functions, *J. Math. Anal. Appl.* **520** (2023), no. 1, artículo no. 126870, 40 pp.
  DOI: `https://doi.org/10.1016/j.jmaa.2022.126870`.
- A. Gil Asensi, J. L. Varona, A general method to find special functions that interpolate Appell polynomials, with examples, *J. Math. Anal. Appl.* **531** (2024), no. 2, artículo no. 127825, 18 pp.
  DOI: `https://doi.org/10.1016/j.jmaa.2023.127825`.
- A. Gil Asensi, E. Labarga, J. Mínguez Ceniceros, J. L. Varona, Boole-Dunkl polynomials and generalizations, *Rev. R. Acad. Cienc. Exactas Fís. Nat. Ser. A Mat. RACSAM* **118** (2024), no. 1, artículo no. 16, 18 pp.
  DOI: `https://doi.org/10.1007/s13398-023-01518-3`.

Las dos revistas en las que están publicados estos artículos —*Journal of Mathematical Analysis and Applications* (editada por Elsevier) los dos primeros, y *Revista de la Real Academia de Ciencias Exactas, Físicas y Naturales. Serie A. Matemáticas* (editada por Springer) el tercero— tienen gran tradición matemática y su objetivo es publicar artículos originales de calidad que supongan un claro avance en la investigación matemática.

Ambas revistas están indexadas, entre otros sitios, en el *Journal Citation Reports/Science Edition* (Clarivate Analytics), SCImago y Scopus. En concreto, ambas aparecen habitualmente en los primeros cuartiles de la clasificación según el *Impact Factor* del JCR, en la categoría *Mathematics*. Los índices de 2023 y 2024 (los años de publicación de los artículos) no han sido publicados aún.

Además, tanto *MathSciNet* (también conocida como *Mathematical Reviews*), de la American Mathematical Society (AMS) como *zbMATH* (*Zentralblatt MATH*), de la European Mathematical Society (EMS) —los dos sitios referencia para la investigación en matemáticas— publican reseñas de los artículos de ambas revistas. En nuestro caso, los tres artículos, en el mismo orden en el que aparecen citados antes, están reseñados aquí:

[1] «Criterios de formato de la tesis doctoral en la Universidad de La Rioja», aprobados por el Comité de Dirección de Doctorado el 10 de mayo de 2022.

- *MathSciNet*:
  - `https://mathscinet.ams.org/article?mr=4513856`
  - `https://mathscinet.ams.org/article?mr=4655704`
  - `https://mathscinet.ams.org/article?mr=4660286`
- *zbMATH*:
  - `https://zbmath.org/7634997`
  - reseña pendiente
  - `https://zbmath.org/7762423`

En matemáticas, la propagación de los resultados, lo que tardan los artículos en ser publicados desde que se envían y la posibilidad de que los artículos sean citados por otros investigadores en otras revistas es muy lenta en comparación con lo que ocurre en otras disciplinas —por el contrario, su vigencia en el tiempo es mucho más duradera, «*un teorema es para siempre*»—. Aunque otros grupos de investigación se han interesado en nuestros resultados, este interés todavía no se ha podido plasmar en citas de otros artículos ya publicados (presentamos artículos publicados en 2023 y 2024, así que no ha habido tiempo para ello), más allá de las citas hechas por nuestro grupo de investigación en artículos sobre el mismo tema.

Índice general

Capítulo 1

Resumen

La tesis está dedicada a profundizar en el conocimiento de las sucesiones de Appell-Dunkl como extensión al contexto de Dunkl de lo que ocurre en el caso clásico (polinomios de Appell). Nuestro grupo de investigación es pionero en el estudio de los operadores de Dunkl en la recta real, y en su relación con los polinomios de Appell-Dunkl (ver, por ejemplo, [1, 2, 10, 12, 13, 14, 15, 16, 17, 23, 25, 42, 43, 44]), y lo que ahora presentamos supone algunos pasos más en este camino.

Si tenemos una familia de polinomios de Appell $\{a_n(x)\}_{n=0}^{\infty}$, es un tema clásico encontrar una función especial de variable compleja, $F(s,x)$, tal que $F(-n,x) = a_n(x)$, $n = 1, 2, \ldots$, es decir, que $F(s,x)$ interpola los polinomios de Appell. El caso más conocido es el de los polinomios de Bernoulli y la función zeta de Hurwitz. En la literatura existen diferentes métodos y estrategias para conseguir este tipo de funciones especiales; entre ellos [45, Theorem 1], donde se aplica la transformada de Mellin (añadiendo un factor $1/\Gamma(s)$) a la función generatriz de los polinomios de Appell.

En la tesis, nuestro trabajo se enmarca dentro de la teoría de Dunkl en la recta real. En la teoría de Dunkl la derivada clásica es reemplazada por la derivada de Dunkl, Λ_α, y la función exponencial, e^t, por la exponencial de Dunkl, $E_\alpha(t)$, por citar un par de diferencias (entraremos en más detalles en el capítulo 2). Más en concreto, y de manera muy resumida, la tesis se dedica al estudio de los polinomios de Appell-Dunkl y a su relación con las funciones especiales que aparecen en ese contexto.

En este campo hemos conseguido diversos avances. En primer lugar, hemos extendido el método de [45, Theorem 1] a nuestro contexto de Dunkl, lo cual presenta considerables dificultades: no sólo el operador de Dunkl es más complicado que la derivada ordinaria, sino que la exponencial de Dunkl tiene un comportamiento asintótico mucho peor que la exponencial ordinaria. En nuestro primer artículo [28], el objetivo era obtener generalizaciones, en un sentido de Dunkl, de algunas de las funciones especiales más importantes, como son las función zeta de Riemann $\zeta(s)$ y las funciones de Hurwitz $\zeta(s,x)$, y pensamos que lo hemos conseguido de manera bastante satisfactoria. En particular, hemos podido aplicar estas técnicas a los polinomios de Bernoulli-Dunkl y a los de Euler-Dunkl, para encontrar las correspondientes funciones zeta-Dunkl y demostrar muchas de sus propiedades.

Continuando con este camino, buscábamos conseguir resultados de ese tipo para algunas familias de polinomios de Appell-Dunkl adicionales en las que no sólo no se podía aplicar lo desarrollado en [28], sino que ni siquiera era posible aplicar los resultados de [45, Theorem 1] a los casos clásicos (no de Dunkl). Por supuesto, esto era así porque había dificultades añadidas aún no solventadas. Como resultado parcial en esta dirección, en [29] demostramos un teorema más general que [45, Theorem 1] y en el que permitimos, por ejemplo, que la función $A(t)$ tenga singularidades en la recta real (hecho que [45] no contemplaba), y mostramos cómo se aplica a diversos ejemplos. De momento, no hemos conseguido encontrar cómo extender ese resultado a las correspondientes familias de polinomios de Appell-Dunkl, y en [45] lo hemos propuesto como problema abierto, explicando dónde están las dificultades técnicas para lograrlo; esperamos poder continuar trabajando en ello en el futuro.

También hemos estudiado propiedades de algunos polinomios de Appell-Dunkl concretos (ya no en relación con las funciones especiales que los interpolan). Así, en [27] hemos definido y estudiado una nueva familia de polinomios Appell-Dunkl: los polinomios de Boole-Dunkl. En concreto, son una familia de polinomios de Appell-Dunkl discretos que generalizan los polinomios de Boole clásicos. Las dificultades de esto son, de nuevo, numerosas, esta vez basadas, sobre todo, en que la traslación de Dunkl es un operador τ_y mucho más sofisticado que la traslación clásica $x \mapsto x + y$, y eso hace que las familias de Appell-Dunkl discretas (que fueron definidas en [25]) tengan una definición que no es sencilla porque involucra las traslaciones de Dunkl.

Capítulo 2

Introducción

La tesis se enmarca dentro del estudio de la teoría de Dunkl en la recta real y los polinomios de Appell-Dunkl. En los siguientes apartados damos las definiciones básicas que se necesitan para poder seguir el contenido de la tesis, así como las explicaciones pertinentes sobre las herramientas que hemos utilizado.

2.1. Definiciones básicas de la teoría de Dunkl

La teoría de Dunkl debe su nombre a Charles F. Dunkl, quien la inició en el año 1989 con la publicación de [22]. Allí aparecen operadores en $\mathbb{R}^n$ asociados a grupos de reflexión pero, para nuestros propósitos, será suficiente restringirnos a la recta real $\mathbb{R}$ y al grupo $\mathbb{Z}_2$. Llamaremos Λ_α al *operador de Dunkl* (o *derivada de Dunkl*) definido como

$$\Lambda_\alpha f(x) = \frac{d}{dx} f(x) + \frac{2\alpha+1}{2}\left(\frac{f(x)-f(-x)}{x}\right),$$

donde $\alpha > -1$. En inglés, estos operadores se conocen como *differential-difference operators* porque tienen una parte diferencial y otra en diferencias.

Se conoce como *núcleo de Dunkl* (o *exponencial de Dunkl*) y se denota como $E_\alpha(x)$ a la función definida como

$$E_\alpha(x) = \mathcal{I}_\alpha(x) + \frac{x}{2(\alpha+1)}\,\mathcal{I}_{\alpha+1}(x),$$

donde

$$\mathcal{I}_\alpha(z) = 2^\alpha \Gamma(\alpha+1)\,\frac{J_\alpha(iz)}{(iz)^\alpha} = \Gamma(\alpha+1)\sum_{n=0}^{\infty} \frac{(z/2)^{2n}}{n!\,\Gamma(n+\alpha+1)}.$$

La función $\mathcal{I}_\alpha$ es una variación de la función Bessel modificada de orden α (que normalmente se denota como I_α; ver, por ejemplo, [57] o [47]). Por otro lado, la función $E_\alpha(t)$ satisface

$$\Lambda_\alpha E_\alpha(\lambda x) = \lambda E_\alpha(\lambda x),$$

así que da lugar a los autovalores del operador Λ_α.

Cuando $\alpha = -1/2$ se recupera el caso clásico, es decir,

$$\Lambda_{-1/2} = \frac{d}{dx} \qquad \text{y} \qquad E_{-1/2}(x) = e^x.$$

Además, $\mathcal{I}_{-1/2}(ix) = \cos(x)$ e $\mathcal{I}_{1/2}(ix) = \operatorname{sen}(x)/x$, luego la fórmula de Euler $e^{ix} = \cos(x) + i\operatorname{sen}(x)$ es un caso particular de

$$E_\alpha(ix) = \mathcal{I}_\alpha(ix) + \frac{ix}{2(\alpha+1)}\mathcal{I}_{\alpha+1}(ix);$$

en otras palabras, $\mathcal{I}_\alpha(ix)$ desempeña el papel de $\cos(x)$, y $\frac{x}{2(\alpha+1)}\mathcal{I}_{\alpha+1}(ix)$ el papel de la parte $\operatorname{sen}(x)$.

Hay otras formas de expresar estas funciones; puede hacerse, por ejemplo, utilizando funciones hipergeométricas

$$\begin{aligned} E_\alpha(x) &= e^x\, {}_1F_1(\alpha+1/2, 2\alpha+2, -2x),\\ \mathcal{I}_\alpha(x) &= {}_0F_1(\alpha+1, x^2/4), \end{aligned}$$

o mediante series de potencias

$$\begin{aligned} E_\alpha(z) &= \sum_{n=0}^\infty \frac{z^n}{\gamma_{n,\alpha}},\\ \mathcal{I}_\alpha(x) &= \sum_{n=0}^\infty \frac{z^{2n}}{\gamma_{2n,\alpha}}. \end{aligned}$$

Los coeficientes $\gamma_{n,\alpha}$ que ahí aparecen se conocen como *factoriales de Dunkl* y están definidos como

$$\gamma_{n,\alpha} = \begin{cases} 2^{2k}k!\,(\alpha+1)_k, & \text{si } n = 2k,\\ 2^{2k+1}k!\,(\alpha+1)_{k+1}, & \text{si } n = 2k+1, \end{cases}$$

donde $(a)_n$ denota el símbolo de Pochhammer

$$(a)_n = a(a+1)(a+2)\cdots(a+n-1) = \frac{\Gamma(a+n)}{\Gamma(a)}$$

(con $a \neq -1, -2, -3, \dots$). Por supuesto, cuando $\alpha = -1/2$ se tiene que $\gamma_{n,-1/2} = n!$. En relación con lo anterior, es interesante definir (pues haremos referencia a ello más adelante)

$$\theta_{n,\alpha} = \frac{\gamma_{n,\alpha}}{\gamma_{n-1,\alpha}} = n + (\alpha+1/2)(1-(-1)^n),$$

y *el coeficiente binomial de Dunkl*

$$\binom{n}{j}_\alpha = \frac{\gamma_{n,\alpha}}{\gamma_{j,\alpha}\gamma_{n-j,\alpha}}.$$

En el caso $\alpha = -1/2$, se tiene que $\theta_{n,-1/2} = n$ y el coeficiente binomial de Dunkl se trataría simplemente del coeficiente binomial clásico. Algunas veces simplificamos la notación $\gamma_n = \gamma_{n,\alpha}$ y $\theta_n = \theta_{n,\alpha}$.

Otro importante operador de Dunkl es *la traslación de Dunkl*, que se define como

$$\tau_y f(x) = \sum_{n=0}^{\infty} \Lambda_\alpha^n f(x) \frac{y^n}{\gamma_{n,\alpha}}, \qquad \alpha > -1, \tag{2.1}$$

donde Λ_α^0 es el operador identidad y $\Lambda_\alpha^{n+1} = \Lambda_\alpha(\Lambda_\alpha^n)$. A veces utilizaremos $\tau_{y,x}$ para remarcar que la traslación actúa sobre la variable x. En el caso clásico, $\alpha = -1/2$, la traslación de Dunkl $\tau_y f$ es simplemente el desarrollo de Taylor de una función f en torno al punto x, esto es,

$$f(x+y) = \sum_{n=0}^{\infty} f^{(n)}(x) \frac{y^n}{n!}.$$

Conviene mencionar que el operador traslación τ_y tiene muchas propiedades relevantes, como se puede ver en [39, 50, 52, 55]. Además, τ_y también se puede escribir por medio de expresiones integrales que se pueden aplicar a una clase más amplia de funciones que el que aparece en (2.1) (que requiere que la función f sea infinitamente diferenciable).

La traslación de Dunkl τ_y tiene un operador inverso (que, en general, no es una traslación de Dunkl, salvo en el caso $\alpha = -1/2$), y está dada por

$$\tau_y^{-1} f(x) = \sum_{n=0}^{\infty} \frac{c_n y^n}{\gamma_{n,\alpha}} \Lambda_\alpha^n f(x), \tag{2.2}$$

donde $c_0 = 1$ y c_n, para $n \geq 1$, se define recursivamente mediante $c_n = -\sum_{j=0}^{n-1} \binom{n}{j}_\alpha c_j$ (puede encontrarse una prueba en [13, Lemma 4.4]).

La traslación de Dunkl se puede utilizar para generalizar, por ejemplo, el binomio de Newton $(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} y^k x^{n-k}$ al contexto de Dunkl:

$$\tau_y((\cdot)^n)(x) = \sum_{k=0}^{n} \binom{n}{k}_\alpha y^k x^{n-k}.$$

También podemos encontrar propiedades relacionadas con la exponencial de Dunkl, por ejemplo, que

$$\tau_y(E_\alpha(t\,\cdot))(x) = E_\alpha(xt)E_\alpha(yt);$$

esta propiedad generaliza el hecho de que $e^{t(x+y)} = e^{tx}e^{ty}$.

En este punto nos gustaría destacar otra interesante propiedad que demostramos durante el desarrollo de la tesis, y que no era conocida antes (ver [26]). Se trata de

$$\lim_{n\to\infty} \tau_1((\cdot)^n)(t/n) = E_\alpha(t),$$

que extiende al contexto de Dunkl la definición clásica de función exponencial como límite:

$$\lim_{n\to\infty} \left(1 + \frac{t}{n}\right)^n = e^t.$$

Son muchos los desarrollos matemáticos clásicos que se han extendido al contexto de Dunkl, bien sobre la recta real (con el grupo de reflexión $\mathbb{Z}_2$, como aquí hacemos), o sobre otros espacios con grupos de reflexión más sofisticados. Existe una transformada de Dunkl que generaliza la transformada de Fourier utilizando el núcleo $E_\alpha(-ixy)$ en lugar de e^{-ixy} [32, 50], resultados sobre multiplicadores y transplantación [10, 15, 46], teoremas de Paley-Wiener [4, 19, 54, 56] o teoremas de muestreo [16]. Otros conceptos y resultados que se han extendido y estudiado en el contexto de Dunkl son las series de Fourier-Dunkl [12, 16], las funciones maximales [54] y las transformadas de Riesz [55], por citar unos pocos ejemplos. Por supuesto, en lo que a nosotros respecta cabe destacar el estudio los polinomios de Appell-Dunkl y funciones zeta-Dunkl, en las que se centra esta tesis doctoral.

2.2. Polinomios de Appell y de Appell-Dunkl

Los polinomios de Appell-Dunkl son una generalización al contexto de Dunkl de los polinomios de Appell. Aunque hay diferentes maneras equivalentes de definir los polinomios de Appell, $\{a_n(x)\}_{n=0}^\infty$, posiblemente la manera más habitual es hacerlo mediante una función generatriz, en cuyo desarrollo de Taylor aparecen los polinomios:

$$G(x,t) = A(t)e^{xt} = \sum_{n=0}^\infty a_n(x)\frac{t^n}{n!}, \tag{2.3}$$

donde $A(t)$ es una función analítica en $t = 0$ y tal que $A(0) \neq 0$ (esto garantiza que $a_n(x)$ sea un polinomio de grado n). Es fácil comprobar que, en este caso, $\frac{d}{dx}a_n(x) = na_{n-1}(x)$; de manera alternativa, se puede empezar con una familia de polinomios que satisfacen esa relación y llegar a que también se pueden expresar en la forma (2.3).

Algunos ejemplos clásicos de polinomios de Appell son los polinomios de Bernoulli $\{B_n(x)\}_{n=0}^\infty$ (donde $A(t) = t/(e^t - 1)$), los polinomios de Euler $\{E_n(x)\}_{n=0}^\infty$ (donde $A(t) = 2/(e^t+1)$), y los polinomios de Hermite probabilísticos $\{\mathrm{He}_n(x)\}_{n=0}^\infty$ (donde $A(t) = e^{-t^2/2}$).

La forma natural de extender los polinomios de Appell al contexto de Dunkl es reemplazar e^{xt} por $E_\alpha(xt)$ y $n!$ por $\gamma_{n,\alpha}$ en (2.3). Es decir, se dice que $\{a_{n,\alpha}(x)\}_{n=0}^\infty$ es una familia de *polinomios de Appell-Dunkl* si

$$A(t)E_\alpha(xt) = \sum_{n=0}^\infty a_{n,\alpha}(x)\frac{t^n}{\gamma_{n,\alpha}}, \tag{2.4}$$

donde, de nuevo, $A(t)$ tiene que ser analítica en $t = 0$ con $A(t) \neq 0$ para garantizar que los $a_{n,\alpha}(x)$ no son nulos (y, de hecho, $a_{n,\alpha}(x)$ es un polinomio de grado n). En este caso, todos los polinomios de Appell-Dunkl cumplen

$$\Lambda_\alpha a_{n,\alpha}(x) = \theta_{n,\alpha}a_{n-1,\alpha}(x).$$

Los polinomios de Appell-Dunkl más sencillos son los monomios $\{x^n\}_{n=0}^\infty$ (para $A(t) = 1$) y, por tanto, se tiene que $\Lambda_\alpha((\cdot)^n) = \theta_{n,\alpha}x^{n-1}$. Los primeros polinomios de Appell-Dunkl que encontramos en la literatura son los polinomios de Hermite generalizados (ver [50]), seguidos por los polinomios de Bernoulli-Dunkl y los de Euler-Dunkl (ver [12, 13, 23]); el artículo [21] proporciona la definición (2.4), pero no estudia ningún ejemplo concreto. En esta tesis doctoral, resultarán de mucho interés los *polinomios de Bernoulli-Dunkl* $\{\mathfrak{B}_{n,\alpha}(x)\}_{n=0}^{\infty}$, que se definen mediante la función generatriz

$$\frac{E_\alpha(xt)}{\mathcal{I}_{\alpha+1}(t)} = \sum_{n=0}^{\infty} \frac{\mathfrak{B}_{n,\alpha}(x)}{\gamma_{n,\alpha}}\, t^n,$$

y los *polinomios de Euler-Dunkl* $\{\mathfrak{E}_{n,\alpha}(x)\}_{n=0}^{\infty}$, definidos con la función generatriz

$$\frac{E_\alpha(xt)}{\mathcal{I}_{\alpha}(t)} = \sum_{n=0}^{\infty} \frac{\mathfrak{E}_{n,\alpha}(x)}{\gamma_{n,\alpha}}\, t^n.$$

Los polinomios de Bernoulli y los de Euler clásicos están definidos en el intervalo $[0,1]$, pero los polinomios de Bernoulli-Dunkl y Euler-Dunkl, por las simetrías inherentes al contexto de Dunkl, están definidos en el intervalo $[-1,1]$. Por tanto, para recuperar los polinomios clásicos, además de tomar $\alpha = -1/2$ se deben hacer los cambios $x \mapsto 2x-1$ (que transforma el intervalo $[0,1]$ en el $[-1,1]$) y $t \mapsto t/2$. Con esto,

$$\frac{\mathfrak{B}_{n,-1/2}(2x-1)}{2^n} = B_n(x) \quad \text{y} \quad \frac{\mathfrak{E}_{n,-1/2}(2x-1)}{2^n} = E_n(x).$$

Todos las familias de polinomios de Appell-Dunkl $\{a_{n,\alpha}(x)\}_{n=0}^{\infty}$ satisfacen

$$\tau_y(a_{k,\alpha})(x) = \sum_{j=0}^{k} \binom{k}{j}_{\alpha} a_{j,\alpha}(x) y^{k-j}, \tag{2.5}$$

que, de nuevo, cuando $\alpha = -1/2$ resulta en una conocida propiedad de los polinomios de Appell: $a_k(x+y) = \sum_{j=0}^{k} \binom{k}{j} a_j(x) y^{k-j}$. Así pues, tanto los polinomios de Bernoulli-Dunkl como los de Euler-Dunkl cumplen (2.5) por ser polinomios de Appell-Dunkl. Más propiedades interesantes como, por ejemplo,

$$\Lambda_\alpha((\cdot)^k)(x) = (\alpha+1)(\tau_1\mathfrak{B}_k(x) - \tau_{-1}\mathfrak{B}_k(x)),$$

que es una generalización de

$$kx^{k-1} = B_k(x+1) - B_k(x),$$

se pueden encontrar en [12, 13].

Algunas veces utilizaremos polinomios en los que la función $A(t)$ que aparece en (2.4) incorpora un parámetro r real o complejo. Esto permite relacionar los polinomios que surgen al cambiar de parámetro. Se trata de los *polinomios de Appell-Dunkl generalizados de orden* r, $\{a_{n,\alpha}^{(r)}(x)\}_{n=0}^{\infty}$, definidos como

$$A(t)^r E_\alpha(xt) = \sum_{n=0}^{\infty} a_{n,\alpha}^{(r)}(x) \frac{t^n}{\gamma_{n,\alpha}}.$$

Por su relación con los «polinomios de Appell-Dunkl discretos» que veremos más adelante, particularmente interesantes son los *polinomios de Bernoulli-Dunkl generalizados* $\{\mathfrak{B}_{n,\alpha}^{(r)}(x)\}_{n=0}^{\infty}$, que se definen como

$$\frac{E_\alpha(xt)}{\mathcal{I}_{\alpha+1}(t)^r} = \sum_{n=0}^{\infty} \frac{\mathfrak{B}_{n,\alpha}^{(r)}(x)}{\gamma_{n,\alpha}}\, t^n, \tag{2.6}$$

y los *polinomios de Euler-Dunkl generalizados* $\{\mathfrak{E}_{n,\alpha}^{(r)}(x)\}_{n=0}^{\infty}$,

$$\frac{E_\alpha(xt)}{\mathcal{I}_\alpha(t)^r} = \sum_{n=0}^{\infty} \frac{\mathfrak{E}_{n,\alpha}^{(r)}(x)}{\gamma_{n,\alpha}}\, t^n; \tag{2.7}$$

ambos fueron definidos en [13].

2.3. Polinomios de Appell-Dunkl discretos

Por otro lado, también es necesario definir los polinomios de Appell-Dunkl discretos (que generalizan los polinomios de Appell discretos clásicos). Se pueden encontrar por primera vez en [25]. La familia de polinomios $\{p_{k,\alpha}(x)\}_{n=0}^{\infty}$ se dice que son *polinomios de Appell-Dunkl discretos* si se pueden expresar mediante el desarrollo de la función generatriz

$$A(t)E_\alpha(xG_\alpha^{-1}(t)) = \sum_{k=0}^{\infty} p_{k,\alpha}(x)\frac{t^k}{\gamma_{k,\alpha}}, \quad \alpha > -1,$$

donde $A(t)$ es analítica en un entorno de 0 con $A(0) \neq 0$ y $G_\alpha(t) = t\mathcal{I}_{\alpha+1}(t)$. Cuando $\alpha = -1/2$ es fácil comprobar que $G_{-1/2}^{-1}(t) = \log(t + \sqrt{1+t^2})$ y, por lo tanto, $E_{-1/2}(xG_{-1/2}^{-1}(t)) = (t + \sqrt{1+t^2})^x$. Por supuesto, esto significa que son una extensión de los polinomios de Appell discretos clásicos $\{p_k(x)\}_{k=0}^{\infty}$ definidos como

$$A(t)(t + \sqrt{1+t^2})^x = \sum_{k=0}^{\infty} p_k(x)\frac{t^k}{k!}.$$

Además, la familia de polinomios $\{p_{k,\alpha}(x)\}_{k=0}^{\infty}$ es Appell-Dunkl discreta si y sólo si

$$(\alpha+1)(\tau_1 - \tau_{-1})p_{k,\alpha}(x) = \theta_{k,\alpha}p_{k-1,\alpha}(x),$$

que en el caso clásico se reduce a

$$\frac{p_k(x+1) - p_k(x-1)}{2} = kp_{k-1}(x), \qquad k \geq 1.$$

El término «discreto» en el caso clásico proviene de que se ha reemplazado la derivada usual por la derivada en diferencias (central) $\Delta f(x) = (f(x+1) - f(x-1))/2$. La forma en la que esto se hace en el caso de Dunkl es mediante traslaciones de Dunkl (2.1), como veremos más adelante.

Los primeros polinomios de Appell-Dunkl discretos estudiados fueron los *polinomios factoriales de Dunkl* $\{f_{k,\alpha}\}_{k=0}^{\infty}$, definidos simplemente mediante (ver [25, Theorem 3.1])

$$E_\alpha(xG_\alpha^{-1}(t)) = \sum_{k=0}^{\infty} f_{k,\alpha}(x)\frac{t^k}{\gamma_{k,\alpha}}.$$

En el caso clásico, $\alpha = -1/2$, los factoriales de Dunkl se convierten en los factoriales descendentes

$$x^{\underline{n}} = x(x-1)(x-2)\cdots(x-n+1) = \prod_{j=0}^{n-1}(x-j).$$

Los factoriales descendentes (como su contraparte, los factoriales ascendentes) tienen diferentes notaciones. Nosotros seguimos la que se utiliza en [38] o en [30, § 2.6, p. 47]), por ejemplo.

Los polinomios factoriales de Dunkl se pueden calcular explícitamente utilizando los «números de Bernoulli-Dunkl generalizados» (ver [44, (17)])

$$f_{k,\alpha}(x) = \sum_{j=0}^{k} \frac{j}{k} \binom{k}{j}_{\alpha} \mathfrak{B}^{(k)}_{k-j,\alpha}(0) x^j, \quad k = 1, 2, \dots. \tag{2.8}$$

Los polinomios factoriales de Dunkl son, en cierto modo, los polinomios de Appell-Dunkl discretos más importantes. Esto se debe a que aparecen en muchas de las propiedades de los polinomios de Appell-Dunkl discretos como en (ver [25, Theorem 3.1])

$$\tau_y(p_{k,\alpha}(\cdot))(x) = \sum_{j=0}^{k} \binom{k}{j}_{\alpha} p_{j,\alpha}(x) f_{k-j,\alpha}(y),$$

que es, en cierta forma, similar a (2.5). En el caso clásico se reduce a

$$p_k(x+y) = \sum_{j=0}^{k} p_k(x) y^{\underline{k-j}}.$$

Además, a través de los polinomios factoriales de Dunkl, en [44] se definieron los números de Stirling-Dunkl de primera y segunda especie (denotados como $s^{\alpha}(n,k)$ y $S_{\alpha}(n,k)$ respectivamente). Estos números son una generalización de los números de Stirling de primera y segunda especie clásicos (denotados como $s(n,k)$ y $S(n,k)$ respectivamente) que, aunque tienen su origen en la combinatoria, pueden ser descritos utilizando el factorial descendente (clásico) mediante las fórmulas

$$x^{\underline{n}} = \sum_{k=0}^{n} s(n,k) x^k, \tag{2.9}$$

y

$$x^n = \sum_{k=0}^{n} S(n,k) x^{\underline{k}}. \tag{2.10}$$

Reemplazando los polinomios factoriales descendentes clásicos por los polinomios factoriales de Dunkl, los *números de Stirling-Dunkl de primera especie* se definen como

$$s^{\alpha}(n,k) = \frac{k}{n} \binom{n}{k}_{\alpha} \mathfrak{B}^{(n)}_{n-k,\alpha}(0).$$

Por tanto, utilizando (2.8) se llega a que

$$f_{n,\alpha}(x) = \sum_{k=0}^{n} s^{\alpha}(n,k) x^k, \quad k = 1, 2, \dots,$$

que extiende (2.9) al caso de Dunkl. Puesto que, para $k > n$, $s^{\alpha}(n,k)$ no aparece en esta fórmula, debemos definir $s^{\alpha}(n,k) = 0$ en estos casos.

En su lugar, los *números de Stirling-Dunkl de segunda especie*, $S_\alpha(n,k)$, se pueden definir, motivándonos en (2.10), como

$$x^n = \sum_{k=0}^{n} S_\alpha(n,k) f_{k,\alpha}(x),$$

donde, de nuevo, $S_\alpha(n,k) = 0$ para $k > n$. Para más propiedades de los números de Stirling-Dunkl, consultar [44].

Volviendo al estudio de los polinomios de Appell discretos, continuamos mencionando la versión discreta de los polinomios de Bernoulli, conocidos como polinomios de Bernoulli de segunda especie, $\{b_k(x)\}_{k=0}^{\infty}$ (ver [11]). Estos polinomios fueron introducidos independientemente tanto por Jordan [33] como por Rey Pastor [49] en 1929 (de hecho, en algunos sitios son conocidos como polinomios de Rey Pastor [8]). Están definidos, de nuevo, mediante el desarrollo de una función generatriz,

$$\frac{t}{\log(1+t)}(1+t)^x = \sum_{k=0}^{\infty} b_k(x)\,\frac{t^k}{k!}. \tag{2.11}$$

Cuando $x = 0$, los números $b_k(0)$ (o, a veces, $b_k(0)\cdot k!$) se conocen como números de Bernoulli de segunda especie (ver [20, § 24.16] o [58]), números logarítmicos, coeficientes de Gregory o números de Cauchy de primera especie [41, 59]).

La versión de Dunkl de estos polinomios, estudiada por primera vez en [25], son los llamados polinomios de Bernoulli-Dunkl de segunda especie, $\{b_{k,\alpha}(x)\}_{k=0}^{\infty}$. Para pasar (2.11) al caso de Dunkl es más conveniente tratar de estudiar una versión «centrada» de (2.11), es decir, donde se utilicen diferencias centradas en vez de diferencias finitas. Para ello, se busca generalizar los polinomios de Bernoulli centrados de segunda especie, $\{b_k^{II}(x)\}_{k=0}^{\infty}$, cuya definición por medio de una función generatriz es

$$\frac{t}{\log(t+\sqrt{1+t^2})}\big(t+\sqrt{1+t^2}\big)^x = \sum_{k=0}^{\infty} b_k^{II}(x)\frac{t^k}{k!}.$$

(ver [53]). Por tanto, como $G_\alpha^{-1}(t)$ va a jugar el papel de $\log\big(t+\sqrt{1+t^2}\big)$, la definición de los *polinomios de Bernoulli-Dunkl de segunda especie* (ver [25, ecuación (6.3)]) viene dada por

$$\frac{t}{G_\alpha^{-1}(t)}E_\alpha(xG_\alpha^{-1}(t)) = \sum_{k=0}^{\infty} b_{k,\alpha}(x)\frac{t^k}{\gamma_{k,\alpha}}, \qquad x\in\mathbb{R}.$$

Los primeros polinomios de Bernoulli-Dunkl de segunda especie son

$$\begin{aligned}
&b_{0,\alpha}(x) = 1, \qquad b_{1,\alpha}(x) = x, \qquad b_{2,\alpha}(x) = x^2 + \frac{\alpha+1}{\alpha+2},\\
&b_{3,\alpha}(x) = x^3, \qquad b_{4,\alpha}(x) = x^4 - 2x^2 - \frac{(\alpha+1)(3\alpha+10)}{(\alpha+2)(\alpha+3)},\\
&b_{5,\alpha}(x) = x^5 - \frac{(4\alpha+12)x^3}{\alpha+2},\\
&b_{6,\alpha}(x) = x^6 - \frac{9(\alpha+3)x^4}{\alpha+2} + \frac{3(5\alpha+16)x^2}{\alpha+2} + \frac{(\alpha+1)(25\alpha^2+190\alpha+364)}{(\alpha+2)^2(\alpha+4)},
\end{aligned}$$

$$b_{7,\alpha}(x) = x^7 - \frac{12(\alpha+4)x^5}{\alpha+2} + \frac{6(\alpha+4)(6\alpha+19)x^3}{(\alpha+2)^2}.$$

En nuestro artículo [27] continuamos el estudio de los polinomios de Appell-Dunkl discretos iniciado en [25]. Al igual que existe una contraparte discreta de los polinomios de Bernoulli, también hay una versión discreta de los polinomios de Euler. Estos polinomios se conocen como polinomios de Boole (en vez de polinomios de Euler de segunda especie). Nuestro propósito era generalizar estos polinomios en un sentido de Dunkl y demostrar su propiedades; por descontado, a estos nuevos polinomios los hemos denominado polinomios de Boole-Dunkl. Entraremos en detalles en el siguiente capítulo.

2.4. Transformada de Mellin

En el siglo XVIII, Euler halló los valores de las siguientes series:

$$\sum_{m=1}^{\infty} \frac{1}{m^{2k}} = \frac{(-1)^{k-1}2^{2k-1}\pi^{2k}}{(2k)!} B_{2k}, \quad k = 1, 2, \dots, \tag{2.12}$$

$$\sum_{m=1}^{\infty} \frac{(-1)^m}{(2j-1)^{2k+1}} = \frac{(-1)^{k+1}\pi^{2k+1}}{4(2k)!} E_{2k}, \quad k = 0, 1, 2, \dots, \tag{2.13}$$

donde B_{2k} y E_{2k} son los números de Bernoulli y de Euler, respectivamente (estos son valores particulares de los polinomios de Bernoulli y de Euler, concretamente $B_{2k} = B_{2k}(0)$ y $E_{2k} = E_{2k}(1/2)$). Cuando se toma $k = 1$ en (2.12) se llega a la famosa serie $\sum_{m=1}^{\infty} 1/m^2 = \pi^2/6$ (evaluar esta serie es lo que se denomina *problema de Basilea*).

Que en esa serie aparezcan los números de Bernoulli no es casualidad, puesto que los polinomios de Bernoulli guardan una fuerte relación con la función Hurwitz

$$\zeta(s,x) = \sum_{n=0} \frac{1}{(n+x)^s}, \quad \mathrm{Re}(s) > 1 \text{ y } x \neq 0, -1, -2, \dots, \tag{2.14}$$

y, por tanto, con la función zeta de Riemann (que resulta de tomar $x = 1$ en la función de Hurwitz)

$$\zeta(s) = \sum_{n=1} \frac{1}{n^s}, \quad \mathrm{Re}(s) > 1;$$

en concreto, (2.12) se trata, simplemente, de $\zeta(2k)$, con $k = 1, 2, \dots$. Un forma de establecer la conexión entre estas funciones y los polinomios de Bernoulli es usar la *transformada de Mellin*, un operador que transforma una función $f(t)$ definida en $(0, \infty)$ mediante la integral

$$\mathcal{M}\{f(t)\}(s) = \int_0^{\infty} f(t)t^{s-1}\,dt.$$

Con una ligera variación de la transformada de Mellin, la función de Hurwitz $\zeta(s,x)$ se puede obtener a partir de la función $G(x,-t) = A(-t)e^{-xt}$, donde $G(x,t)$ es la función generatriz de los polinomios de Bernoulli, es decir, $A(t) = t/(e^t - 1)$. En concreto,

$$\zeta(s+1,x) = \frac{1}{\Gamma(s)} \int_0^\infty G(x,-t) t^{s-1}\, dt = \frac{1}{\Gamma(s)} \int_0^\infty \frac{e^{-xt}}{e^{-t}-1} t^s\, dt.$$

Nótese el cambio de signo en $G(x,-t)$ y el factor añadido $1/\Gamma(s)$. Como veremos a continuación, la función $\zeta(s,x)$, como función de s, puede extenderse de manera analítica al plano complejo.

Esta técnica se puede utilizar de forma bastante general. De hecho, es conocida como *teorema máster de Ramanujan* (ver, por ejemplo, [3]). Especialmente nos resultaba interesante el método de [45] donde se dan condiciones para que este método funcione cuando se aplica a series de Appell. Estas condiciones son que (i): $A(t)$ esté definida en $(-\infty, 0]$, (ii): $A(t)$ no sea constante en un entorno de 0 y (iii): $A(-t)$ es continua en $[0,\infty)$ y con crecimiento exponencial en $+\infty$.

Con estas condiciones, se tiene [45, Theorem 1]:

Teorema 2.1. *Sea $G(x,t) = A(t)e^{xt}$ la función generatriz de ciertos polinomios de Appell, $\{a_n(x)\}_{n=0}^\infty$, y supongamos que $A(t)$ tiene un cero de orden k en $t = 0$. Entonces,*

$$F(s,x) = \frac{1}{\Gamma(s)} \int_0^\infty G(x,-t) t^{s-1}\, dt \tag{2.15}$$

converge en el semiplano complejo $\sigma = \mathrm{Re}(s) > -k$ a una función holomorfa que admite extensión analítica a una función entera sobre la variable s que satisface

$$F(-n,x) = a_n(x), \quad n = 0,1,2,\dots \tag{2.16}$$

Nos referiremos a (2.15) también como transformada de Mellin (a pesar del factor $1/\Gamma(s)$). La potencia del método radica en que, en el caso de los polinomios de Bernoulli, a partir de (2.16) se obtiene (2.12). Del mismo modo, si se aplicara al caso de los polinomios de Euler, ahora, en vez de la función de Hurwitz, aparece una función similar a la de Hurwitz (a la que hemos llamado «de tipo Euler»)

$$\zeta_E(s,x) = \sum_{n=0}^\infty \frac{(-1)^{n-1}}{(n+x)^s},$$

de la cual, cuando $x = 0$, se obtiene la función zeta «de tipo Euler»

$$\zeta_E(s) = \sum_{n=1}^\infty \frac{(-1)^{n-1}}{n^s}$$

(a la función $\zeta_E(s)$ también se la conoce como función eta de Dirichlet, $\eta(s)$). Aplicando [45, Theorem 1] a la función generatriz de los polinomios de Euler, $G(x,t) = A(t)e^{xt} = 2e^{xt}/(e^t+1)e^x t$, lo que se tiene es

$$\zeta_E(s+1,x) = \frac{1}{\Gamma(s)} \int_0^\infty A(-t) t^{s-1}\, dt = \frac{1}{\Gamma(s)} \int_0^\infty \frac{2e^{-xt}}{e^{-t}+1} t^{s-1}\, dt,$$

y de aquí, por (2.16), se llega a (2.13).

Capítulo 3

Objetivos

El objetivo de esta tesis ha sido profundizar en el conocimiento de las sucesiones de Appell-Dunkl, como extensión al contexto de Dunkl de lo que ocurre en el caso clásico (polinomios de Appell), y estudiar su comportamiento. Nuestro grupo de investigación es pionero en el estudio de los operadores de Dunkl en la recta real, y en su relación con los polinomios de Appell-Dunkl (ver, por ejemplo [1, 2, 10, 12, 13, 14, 15, 16, 17, 23, 25, 42, 43, 44]), y lo que aquí presentamos supone algunos pasos más en este camino.

En lo que sigue, vamos a explicar, someramente, qué es lo que hacemos en cada uno de los artículos de esta tesis, que se presenta como compendio de artículos.

3.1. Primer artículo

El objetivo principal de nuestro primer artículo, [28],

> A. Gil Asensi, J. L. Varona, Appell-Dunkl sequences and Hurwitz-Dunkl zeta functions, *J. Math. Anal. Appl.* **520** (2023), no. 1, artículo no. 126870, 40 pp.

consistía en encontrar expresiones para funciones zeta en un contexto de Dunkl. Para ello, nuestra idea era intentar proceder de forma similar a lo que ocurre en el caso clásico. Conocemos la función generatriz de los polinomios de Bernoulli-Dunkl y de los polinomios de Euler-Dunkl, así que cabía esperar que, siguiendo [45] (o con muy ligeras modificaciones), llegáramos a los resultados buscados. Sin embargo, nos encontramos con algunas dificultades. En primer lugar, en vez del teorema 2.1, en la versión de Dunkl logramos demostrar el siguiente resultado (ver [28, Theorem 4.1]):

Teorema 3.1. *Sea* $\{\mathcal{P}_n(x)\}_{n=0}^{\infty}$ *una familia de polinomios de Appell-Dunkl con función generatriz* $G(x,t) = A(t)E_\alpha(xt)$ *y supongamos que* $A(t)$ *tiene un cero de orden* k *en* $t = 0$*. Supongamos además que, para* $x \in (a,b)$*, la integral*

$$H(s,x) = \frac{1}{\Gamma(s)} \int_0^\infty G(x,-t)\, t^{s-1}\, dt = \frac{1}{\Gamma(s)} \int_0^\infty A(-t)E_\alpha(-xt)t^{s-1}\, dt$$

converge a una función holomorfa en el semiplano complejo $\mathrm{Re}(s) > -k$*. Entonces,* $H(s,x)$ *se puede extender analíticamente a una función entera en la variable* s *que satisface*

$$H(-n,x) = \frac{n!}{\gamma_{n,\alpha}}\mathcal{P}_n(x), \quad n = 0,1,2,\dots$$

Obsérvese que en la versión de Dunkl la convergencia de la integral ya no está garantizada con las mismas condiciones que en el teorema 2.1 y, por tanto, tuvimos que incluirla como hipótesis. Esto se debe a que, en el caso clásico, el factor e^{-xt} era de mucha ayuda para que la integral convergiese cuando $t \to \infty$, pero, en la versión de Dunkl, como $E_\alpha(xt)$ se comporta de manera similar a $e^{|t|}$ (ver, por ejemplo, [51, p. 128]), el término $E_\alpha(-xt)$ lejos de ayudar, perjudica.

No obstante, el teorema se puede aplicar al caso de Bernoulli-Dunkl y se obtiene que

$$H(s,x) = \frac{1}{\Gamma(s)}\int_0^\infty G(x,-t)t^{s-1}\,dt = \frac{1}{\Gamma(s)}\int_0^\infty \frac{E_\alpha(-xt)}{\mathcal{I}_{\alpha+1}(t)}t^{s-1}\,dt \tag{3.1}$$

converge para $\mathrm{Re}(s) > 0$ y $x \in (-1,1)$. En el caso de Euler-Dunkl se obtienen expresiones análogas, y, a partir de este punto, en el artículo se hace un estudio similar. Pero, en lo sigue, sólo explicaremos el caso de Bernoulli-Dunkl.

Con algunas modificaciones, podemos afirmar que $H(s,x)$ juega el papel de una función «de Hurwitz-Dunkl». En primer lugar, buscamos escribir lo anterior como una serie, similar a (2.14). Para ello, observamos que

$$A(t) = \frac{1}{\mathcal{I}_{\alpha+1}(t)} = \frac{t}{\alpha+1}\frac{1}{E_\alpha(t)}\frac{1}{1-\frac{E_\alpha(-t)}{E_\alpha(t)}} = \frac{t}{\alpha+1}\frac{1}{E_\alpha(t)}\sum_{n=0}^\infty\left(\frac{E_\alpha(-t)}{E_\alpha(t)}\right)^n,$$

por lo cual denominamos *función zeta de Hurwitz-Dunkl* (para $x \in (-1,1)$ y $\mathrm{Re}(s) > 1$) a

$$\begin{aligned}\zeta_\alpha(s,x) &= \frac{1}{\Gamma(s)}\int_0^\infty \frac{E_\alpha(-xt)}{\mathcal{I}_{\alpha+1}(t)}t^{s-1}\,dt \\ &= \frac{1}{\Gamma(s)}\frac{1}{\alpha+1}\sum_{n=0}^\infty\int_0^\infty \frac{E_\alpha(-xt)}{E_\alpha(t)}\left(\frac{E_\alpha(-t)}{E_\alpha(t)}\right)^n t^s\,dt.\end{aligned}$$

Cuando $n = 0$, diremos que

$$d_\alpha(s,x) = \frac{1}{\Gamma(s)}\int_0^\infty \frac{E_\alpha(-xt)}{E_\alpha(t)}t^{s-1}\,dt$$

es el *sumando básico de Hurwitz-Dunkl*. Para obtener algo todavía más parecido a una expresión en serie, lo que hicimos fue observar que, para $n = 0,1,2,3,\dots$, se cumple (ver [28, Lemma 3.1])

$$\tau_y^n(E_\alpha(t\cdot))(x) = E_\alpha(tx)E_\alpha(ty)^n,$$

y

$$\tau_y^{-n}(E_\alpha(t\cdot))(x) = E_\alpha(tx)/E_\alpha(ty)^n,$$

donde τ_y es la traslación de Dunkl (2.1) y τ_y^{-1} su inversa (2.2). De esta forma, podemos expresar $\zeta_\alpha(s,x)$ mediante la traslación de Dunkl simétrica

$$\sigma_1 = \tau_1 \tau_{-1}^{-1}$$

que, con otros objetivos, ya había sido definida en [13] (nótese que, puesto que las traslaciones de Dunkl y sus inversas conmutan, podemos usar $\sigma_1^n = \tau_1^n \tau_{-1}^{-n}$ sin tener en cuenta el orden de los operadores). El resultado fue el siguiente teorema (ver [28, Theorem 4.5]):

Teorema 3.2. *Para* $\operatorname{Re}(s) > 1$*, la función zeta de Hurwitz-Dunkl puede escribirse como*

$$\zeta_\alpha(s,x) = \frac{1}{\Gamma(s)} \sum_{n=0}^{\infty} \sigma_1^n \int_0^\infty \frac{E_\alpha(-xt)}{E_\alpha(t)} t^{s-1}\, dt = \sum_{n=0}^{\infty} \sigma_1^n d_\alpha(s,x).$$

Cuando tomamos $\alpha = -1/2$ tenemos que

$$d_{-1/2}(s,x) = \frac{1}{\Gamma(s)} \int_0^\infty e^{-t(x+1)} t^{s-1}\, dt = \frac{1}{(x+1)^s}.$$

Además, tanto $\tau_1^n(f)(x) = f(x+n)$ como $\tau_{-1}^{-n}(f)(x) = f(x+n)$ y, por tanto,

$$\zeta_{-1/2}(s,x) = \sum_{n=0}^{\infty} \frac{1}{(x+1+2n)^s}.$$

Finalmente, con el cambio $x \mapsto 2x-1$ (que, recordemos, transforma el intervalo $(0,1)$ en el $(-1,1)$) se tiene que

$$\zeta_{-1/2}(s,2x-1) = \sum_{n=0}^{\infty} \frac{1}{(2x+1-1+2n)^s} = \frac{1}{2^s} \sum_{n=0}^{\infty} \frac{1}{(x+n)^s} = \frac{1}{2^s} \zeta(s,x).$$

Tras obtener algunos resultados sobre la función zeta de Hurwitz-Dunkl, nuestro próximo objetivo era obtener una «función zeta de Riemann-Dunkl». Aquí aparecieron algunas complicaciones más. La primera dificultad inmediata es que no podemos simplemente sustituir $x = 1$ en (3.1), pues la expresión integral sólo es válida en $x \in (-1,1)$. Para sortear esta dificultad, tuvimos que utilizar resultados de integración compleja para demostrar la fórmula

$$\zeta_\alpha(1-s,x) = \frac{\Gamma(s)}{2} \left(e^{-\pi s i/2} \mathcal{F}(x,s) + e^{\pi s i/2} \mathcal{F}(-x,s) \right), \tag{3.2}$$

donde

$$\mathcal{F}(x,s) = \sum_{m=1}^{\infty} \frac{E_\alpha(x i s_m)}{\mathcal{I}_\alpha(i s_m)} \frac{1}{s_m^s}$$

y $\{s_m\}_{m=1}^\infty$ son los ceros positivos de la función de Bessel $J_{\alpha+1}$ (ver [28, Theorem 5.4]). Conviene aquí mencionar que la aparición de los ceros de la función $J_{\alpha+1}$ no es algo sorprendente, puesto que, en el caso clásico $\alpha = -1/2$, esa función de Bessel es $J_{1/2}(x) = 2^{1/2}(\pi x)^{-1/2} \operatorname{sen}(x)$, cuyos ceros positivos son $s_m = m\pi$, $m = 1, 2, 3, \dots$. De hecho, la serie clásica (2.12), que Euler sumó cuando $s = 2k$

usando los polinomios de Bernoulli B_{2k}, se extiende al contexto de Dunkl cambiando los sumandos $1/m^{2k}$ (lo cual equivale a tener $1/(\pi m)^{2k}$ pues la constante multiplicativa π^{2k} no causa problemas) por $1/s_m^{2k}$; y estas sumas se evalúan usando los polinomios de Bernoulli-Dunkl $\mathfrak{B}_{2k}$, tal como se puede ver en [12].

A la expresión (3.2) la hemos llamado *fórmula de Hurwitz-Dunkl*, porque es análoga a la fórmula de Hurwitz (ver [5, § 12.7, Theorem 12.6] o [6, 25.13.3]),

$$\zeta(1-s,x) = \frac{\Gamma(s)}{(2\pi)^s}\left(e^{-\pi si/2}F(x,s) + e^{\pi si/2}F(-x,s)\right),$$

y a $\mathcal{F}(x,s)$ la hemos llamado *función zeta de Lerch-Dunkl*, porque es análoga a la función zeta de Lerch clásica

$$F(x,s) = \sum_{n=1}^{\infty}\frac{e^{2\pi inx}}{n^s}, \quad \operatorname{Re}(s) > 1$$

(cabe mencionar que $F(x,s)$ es periódica, pero $\mathcal{F}(x,s)$ no). El verdadero interés de la fórmula (3.2) es que $\mathcal{F}(x,s)$ converge para todo $x \in \mathbb{R}$ y $\operatorname{Re}(s) > 1$; así pues, la parte derecha de (3.2) también, y aquí podemos considerar (3.2) como la extensión analítica de $\zeta_\alpha(s,x)$ y, por tanto, sustituir $x = 1$ en esa expresión. De esta forma obtuvimos (ver [28, Theorem 5.5]), para $\operatorname{Re}(s) > 1$,

$$\zeta_\alpha(1-s) = \Gamma(s)\cos\left(\frac{\pi s}{2}\right)\sum_{m=1}^{\infty}\frac{1}{s_m^s},$$

o equivalentemente —con el cambio $1-s \mapsto s$ y, por tanto, para $\operatorname{Re}(s) < 0$—,

$$\zeta_\alpha(s) = \Gamma(1-s)\operatorname{sen}\left(\frac{\pi s}{2}\right)\sum_{m=1}^{\infty}\frac{1}{s_m^{1-s}}.$$

Estás fórmulas pueden extenderse a todo $\mathbb{C}$ fácilmente. Tanto la función Γ como cos y sen están definidas en todo $\mathbb{C}$, y la función $Z_\alpha(s) = \sum_{m=1}^{\infty}\frac{1}{s_m^s}$ —que se conoce como *función zeta de Bessel*— también admite una extensión analítica (ver [31]). Cabe destacar que, evidentemente, estas fórmulas generalizan las clásicas

$$\zeta(1-s) = 2(2\pi)^{-s}\Gamma(s)\cos\left(\frac{\pi s}{2}\right)\zeta(s), \qquad s \in \mathbb{C},$$

$$\zeta(s) = 2(2\pi)^{s-1}\Gamma(1-s)\operatorname{sen}\left(\frac{\pi s}{2}\right)\zeta(1-s), \qquad s \in \mathbb{C}.$$

Concluimos nuestro artículo [28] con un estudio sobre las propiedades de $\zeta_\alpha(s)$.

3.2. Segundo artículo

En nuestro segundo artículo, [29],

> A. Gil Asensi, J. L. Varona, A general method to find special functions that interpolate Appeal polynomials, with examples, *J. Math. Anal. Appl.* **531** (2024), no. 2, artículo no. 127825, 18 pp.

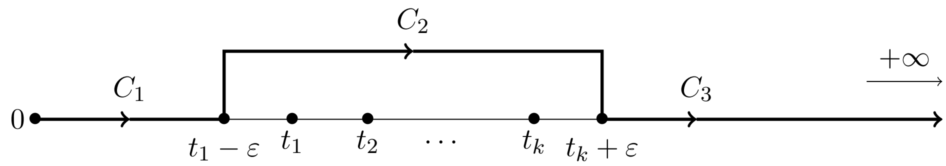

Figura 3.1: Ejemplo de cómo el camino C «esquiva» las singularidades $t_1, t_2, \ldots, t_k$ de $A(-t)$ del teorema 3.3 (el radio de convergencia $A(t)$ debe satisfacer $R > t_1 - \varepsilon$).

nos propusimos buscar un método más general para obtener funciones especiales que el de [45]. Como hemos ya hemos mencionado, al intentar obtener resultados más generales en el caso de polinomios de Appell-Dunkl nos encontramos con problemas de convergencia de la integral del método debido al mal comportamiento de $E_\alpha(-xt)$ en $t \to \infty$ (salvo en el caso $\alpha = -1/2$, claro). Esto significa que la convergencia pasa de ser tesis del teorema a hipótesis. Para acercarnos a una solución a esta problemática, comenzamos por investigar distintas maneras de obtener condiciones más generales que las del teorema 2.1.

Esto nos lleva a nuestro principal teorema en ese artículo (ver [29, Theorem 2.1]), que es el siguiente:

Teorema 3.3. *Sea $A(-t)$ una función meromorfa, continua en $[0, +\infty)$ excepto por ciertas singularidades aisladas $t = t_1, t_2, \ldots, t_k$ (ordenadas de la forma $t_1 < t_2 < \ldots < t_k$). Más aún, supongamos que $A(-t)$ es analítica en el rectángulo perforado*

$$T = \{t \in \mathbb{C} : t_1 - \eta < \operatorname{Re}(t) < t_k + \eta,\ -\eta < \operatorname{Im}(t) < \eta\} \setminus \{t_1, \ldots, t_k\}$$

para cierto $\eta > 0$ y tal que $A(-t)$ tenga crecimiento polinomial en $t \to +\infty$. Considérese la familia de polinomios de Appell $\{P_n(x)\}_{n=0}^\infty$ definida mediante

$$G(x,t) = A(t)e^{xt} = \sum_{n=0}^{\infty} P_n(x)\frac{t^n}{n!}, \quad |t| < R,$$

con cierto radio de convergencia R que cumple $R > t_1 - \eta$. Entonces la integral

$$H(s,x) = \frac{1}{\Gamma(s)}\int_C G(x,-t)t^{s-1}\,dt = \frac{1}{\Gamma(s)}\int_C A(-t)e^{-xt}t^{s-1}\,dt$$

(donde el camino C va desde $t = 0$ hasta $t = \infty$ evitando las singularidades t_j como se muestra en la figura 3.1, con $0 < \varepsilon < \eta$) converge en el semiplano complejo $\operatorname{Re}(s) > 0$ a una función holomorfa sobre la variable s que puede ser extendida analíticamente a una función entera que satisface

$$H(-n,x) = P_n(x), \quad n = 0, 1, 2, \ldots$$

En resumidas cuentas, hemos debilitado las condiciones de [45, Theorem 1] en el sentido de que la función $A(-t)$ ya no tiene por qué ser continua en $[0, \infty)$, sino que puede presentar un número finito arbitrario de singularidades aisladas en ese intervalo. Comúnmente, dichas singularidades se tratarán de polos. Por otro lado, cabe mencionar que en la prueba del teorema 3.3 no tiene ninguna importancia esquivar los polos utilizando el rectángulo de la figura 3.1, y puede utilizarse cualquier camino (por ejemplo una semicircunferencia) que salte del primer polo al último.

Este nuevo método más general lo acompañamos de diferentes ejemplos. Es interesante comenzar con $A(t) = 1/(1-t)^r$ que corresponde con los polinomios $\{P_n^{(r-)}(x)\}_{n=0}^{\infty}$ definidos por

$$\frac{1}{(1-t)^r} e^{xt} = \sum_{n=0}^{\infty} P_n^{(r-)}(x) \frac{t^n}{n!}, \qquad |t| < 1.$$

Claramente, $A(-t)$ no tiene polos en $[0, \infty)$; sin embargo, tiene mucha relación con el caso análogo $A(t) = 1/(1-t)^r$ (del que hablaremos un poco más adelante), donde sí que aparecerán polos.

El caso $r = 1$ es interesante en sí mismo ya que, si $|t| < 1$, tenemos

$$\frac{1}{1-t} e^{xt} = \left(\sum_{j=0}^{\infty} t^j \right) \left(\sum_{j=0}^{\infty} \frac{(xt)^j}{j!} \right) = \sum_{n=0}^{\infty} \frac{t^n}{n!} \left(\sum_{k=0}^{n} \frac{x^{n-k}}{(n-k)!} k! \right)$$

y, por tanto, llegamos a que

$$P_n^{(1-)}(x) = n! \left(1 + x + \frac{x}{2} + \cdots + \frac{x^n}{n!} \right) = n!\, e_n(x).$$

Los polinomios $e_n(x)$ se llaman *polinomios exponenciales truncados* (sobre polinomios trucados, ver, por ejemplo, [18, 35, 36]).

Para un r general, tenemos que calcular

$$H^{(r-)}(s, x) = \frac{1}{\Gamma(s)} \int_0^{\infty} e^{-xt} (1+t)^{-r} t^{s-1}\, dt, \quad \operatorname{Re}(s) > 0,\ \operatorname{Re}(x) > 0,$$

Lo anterior puede expresarse mediante la función de Tricomi $\Psi(a, c; x)$ (también denotada como $U(a, c, x)$, ver [24, § 6.5, ecuación (2)] o [40, p. 242 en § 5.5.2]) de la siguiente forma:

$$\Psi(a, c; x) = \frac{1}{\Gamma(a)} \int_0^{\infty} e^{-xt} (1+t)^{c-a-1} t^{a-1}\, dt, \quad \operatorname{Re}(a) > 0,\ \operatorname{Re}(x) > 0.$$

Es decir, $H^{(r-)}(s, x) = \Psi(s, s-r+1; x)$ y $H^{(r-)}(-n, x) = P_n^{(r-)}(x)$. Cuando $s = -n$, en efecto, por [24, § 6.9.2, ecuación (36)] tenemos

$$\Psi(-n, -n-r+1; x) = (-1)^n n!\, L_n^{(-n-r)}(x),$$

donde

$$L_n^{(\alpha)}(x) = \sum_{k=0}^{n} \binom{n+\alpha}{n-k} \frac{(-x)^k}{k!}$$

denota los polinomios de Laguerre (generalizados) de grado n y orden α (aquí, $\binom{n+\alpha}{n-k}$ es el coeficiente binomial generalizado). En consecuencia,

$$P_n^{(r-)}(x) = (-1)^n n!\, L_n^{(-n-r)}(x).$$

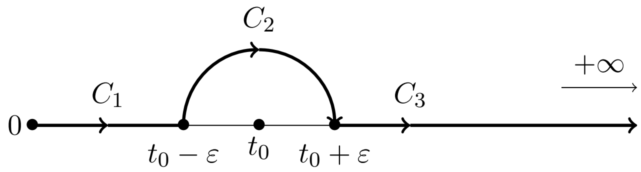

Figura 3.2: Un ejemplo de cómo el camino C «evita» una única singularidad t_0 de $A(-t)$.

Utilizando propiedades de la función de Tricomi se puede probar que en el caso $r = 1$ se tiene

$$H^{(1-)}(s,x) = e^x \Gamma(1-s,x),$$

y, por tanto,

$$H^{(1-)}(-n,x) = e^x \Gamma(n+1,x) = n!\, e_n(x);$$

aquí, $\Gamma(s,x)$ denota la función gamma incompleta

$$\Gamma(s,x) = \int_x^{\infty} e^{-t} t^{s-1}\, dt.$$

Nótese que hemos recuperado una bonita propiedad de la función gamma incompleta (ver, por ejemplo, [48, 8.4.8]), esto es:

$$\Gamma(n,x) = (n-1)!\, e^{-x} e_{n-1}(x), \quad n \in \mathbb{N}.$$

A continuación, se aborda el caso análogo para los polinomios $\{P_n^{(r+)}(x)\}_{n=0}^{\infty}$ definidos por

$$\frac{1}{(1+t)^r} e^{xt} = \sum_{n=0}^{\infty} P_n^{(r-)}(x) \frac{t^n}{n!}, \qquad |t| < 1.$$

Es evidente que, en este caso, $A(-t) = 1/(1-t)^r$ tiene un polo de orden r en $t_0 = 1$. Para hallar la función especial que interpola a los $P_n^{(r+)}(x)$, calculamos (utilizando el teorema 3.3)

$$H^{(r+)}(s,x) = \frac{1}{\Gamma(s)} \int_C e^{-xt} (1-t)^{-1} t^{s-1}\, dt,$$

donde el camino C va de 0 hasta ∞ saltando el polo en $t_0 = 1$ (como se muestra en la figura 3.2).

Para hallar $H^{(r+)}(s,x)$, partimos de su expresión análoga

$$H^{(r-)}(s,x) = \Psi(s, s-r+1; x) = \frac{1}{\Gamma(s)} \int_0^{\infty} e^{-xt} (1+t)^{-r} t^{s-1}\, dt,$$

definida para $\mathrm{Re}(s) > 0$ y $\mathrm{Re}(x) > 0$. Utilizando diferentes argumentos analíticos que se explican con detalle en [29] y un cambio de variable, se puede probar que lo anterior es también es igual a

$$H^{(r-)}(s,x) = \frac{1}{\Gamma(s)} \int_0^{-\infty} e^{-xt} (1+t)^{-r} t^{s-1}\, dt, \tag{3.3}$$

donde el camino evita el polo en $t_0 = -1$, como en la figura 3.2, pero en el eje real negativo. Finalmente, aplicando el cambio de variable $z = -t = e^{i\pi}t$ en (3.3), se llega a que

$$H^{(r-)}(s, xe^{i\pi}) = \Psi(s, s-r+1; xe^{i\pi}) = \frac{e^{i\pi s}}{\Gamma(s)} \int_C e^{-xz}(1-z)^{-r} z^{s-1}\, dz,$$

donde C ahora es el camino de la figura 3.2 (con $z_0 = 1$). Por ende,

$$H^{(r+)}(s, x) = \frac{1}{\Gamma(s)} \int_C e^{-xz}(1-z)^{-r} z^{s-1}\, dz = e^{-\pi i s}\Psi(s, s-r+1; xe^{\pi i})$$

es la función que buscábamos que satisface $H^{(r+)}(-n, x) = P_n^{(r+)}(x)$.

Con esta idea en mente, se pueden encontrar otras funciones especiales en casos más complicados. En nuestro artículo [29] estudiamos algunos ejemplos en los que, por ejemplo, la función especial es una *función G de Meijer*, un tipo de función especial muy general que está definida como

$$G_{p,q}^{m,n}\left(\begin{matrix} a_1, \dots, a_p \\ b_1, \dots, b_q \end{matrix} \,\middle|\, z\right) = \frac{1}{2\pi i} \int_L \frac{\prod_{j=1}^m \Gamma(b_j - t) \prod_{j=1}^n \Gamma(1 - a_j + t)}{\prod_{j=m+1}^q \Gamma(1 - b_j + t) \prod_{j=n+1}^p \Gamma(a_j - t)} z^t\, dt$$

(los detalles de la notación y la descripción del camino de integración L, se pueden ver en [7, 9]).

Concluimos este artículo con el caso de Dunkl como problema abierto. Resulta interesante pensar en qué se podría hacer para obtener la función especial que interpole los polinomios definidos con esta función generatriz:

$$\frac{1}{1 \pm t} E_\alpha(xt) = \sum_{n=0}^{\infty} P_{n,\alpha}^{(1\pm)}(x) \frac{t^n}{n!}.$$

Estos son la versión de Dunkl de los polinomios $\{P_n^{(r\pm)}\}_{n=0}^{\infty}$ antes comentados, y fueron introducidos muy recientemente [42]. Como se espera, $P_{n,\alpha}^{(1-)}(x) = \gamma_{n,\alpha} e_{n,\alpha}(x)$ y $P_{n,\alpha}^{(1+)}(x) = (-1)^n \gamma_{n,\alpha} e_{n,\alpha}(-x)$, donde, ahora,

$$e_{n,\alpha}(x) = 1 + \frac{x}{\gamma_{1,\alpha}} + \frac{x^2}{\gamma_{2,\alpha}} + \cdots + \frac{x^n}{\gamma_{n,\alpha}}$$

es el n-ésimo *polinomio exponencial de Dunkl truncado*. Si intentamos utilizar nuestro teorema en este caso con $\alpha \neq -1/2$, tenemos que

$$\int_0^\infty \frac{E_\alpha(-xt)}{1+t} t^{s-1}\, dt$$

no converge para ningún x, porque el comportamiento asintótico de $E_\alpha(-xt)$ cuando $t \to \infty$ es, esencialmente, como el de $e^{|xt|}$. ¿Hay alguna forma de solventar este problema?

3.3. Tercer artículo

Finalmente, en nuestro tercer artículo de esta tesis por compendio de publicaciones, [27],

> A. Gil Asensi, E. Labarga, J. Mínguez Ceniceros, J. L. Varona, Boole-Dunkl polynomials and generalizations, *Rev. R. Acad. Cienc. Exactas Fís. Nat. Ser. A Mat. RACSAM* **118** (2024), no. 1, artículo no. 16, 18 pp.

volvemos al estudio de polinomios de Appell-Dunkl; en este caso, discretos. En concreto, buscamos definir y encontrar buenas propiedades de la versión «Euler» de los polinomios de Bernoulli-Dunkl de segunda especie. En el caso clásico, estos polinomios podrían haberse llamado *polinomios de Euler de segunda especie*, como puede verse en el título de [11], pero son más conocidos como *polinomios de Boole* [34, § 113, p. 317] (o polinomios de Changhee [37]), $\{e_k(x)\}_{k=0}^\infty$, y están definidos mediante

$$\frac{2}{2+t}(1+t)^x = \sum_{k=0}^\infty e_k(x)\frac{t^k}{k!}.$$

De hecho, se pueden definir de forma más general, es decir mediante

$$\left(\frac{2}{2+t}\right)^r (1+t)^x = \sum_{k=0}^\infty \frac{e_k^{(r)}(x)}{k!}t^k;$$

en este caso, los $\{e_k^r(x)\}_{k=0}^\infty$ se conocen como *polinomios de Boole generalizados.* Estos polinomios satisfacen la relación

$$\Delta e_k^{(r)}(x) = k e_{k-1}^{(r)}(x),$$

donde $\Delta f(x) = f(x+1) - f(x)$ es el operador en diferencias. Además,

$$M e_k^{(r)}(x) = e_k^{(r-1)}(x),$$

donde $Mf(x) = \frac{1}{2}(f(x+1) + f(x))$ es el operador media. Esto se deduce de comprobar que

$$M_x\left(\left(\frac{2}{2+t}\right)^r (1+t)^x\right) = \left(\frac{2}{2+t}\right)^{r-1} (1+t)^x \tag{3.4}$$

(en este caso, escribimos M_x en vez de M para dejar claro que el operador actúa sobre la variable x, al igual que hemos hecho con otros operadores).

A la hora de pasar al contexto de Dunkl, es conveniente sustituir el operador en diferencias Δ por el operador en diferencias central, Δ_{c}, dado por

$$\Delta_{\mathrm{c}} f(x) = \frac{f(x+1) - f(x-1)}{2},$$

y el operador media M por su versión centrada

$$M_{\mathrm{c}} f(x) = \frac{1}{2}(f(x+1) + f(x-1)),$$

a la que llamaremos operador media centrado. A continuación, por analogía con (3.4),buscamos una función $A(t)$ tal que

$$M_{c,x}\big(A(t)^r\big(t+\sqrt{1+t^2}\big)^x\big) = A(t)^{r-1}\big(t+\sqrt{1+t^2}\big)^x.$$

Esto es,

$$\begin{aligned} M_{c,x}\big(\big(t+\sqrt{1+t^2}\big)^x\big) &= \frac{1}{2}\Big(\big(t+\sqrt{1+t^2}\big)^{x+1} + \big(t+\sqrt{1+t^2}\big)^{x-1}\Big) \\ &= \frac{1}{2}\big(t+\sqrt{1+t^2}\big)^x\left(t+\sqrt{1+t^2}+\frac{1}{t+\sqrt{1+t^2}}\right) \\ &= \frac{t^2+1+t\sqrt{1+t^2}}{t+\sqrt{1+t^2}}\big(t+\sqrt{1+t^2}\big)^x. \end{aligned} \tag{3.5}$$

A esta variación de los polinomios de Boole la llamamos *polinomios centrados de Boole generalizados de orden* r, $\{e_{k,c}^{(r)}(x)\}_{k=0}^{\infty}$, y vendrán dados por

$$\left(\frac{t+\sqrt{1+t^2}}{t^2+1+t\sqrt{1+t^2}}\right)^r\big(t+\sqrt{1+t^2}\big)^x = \sum_{k=0}^{\infty} e_{k,c}^{(r)}(x)\frac{t^k}{k!}.$$

Por supuesto, estos polinomios satisfacen

$$\Delta_c e_{k,c}^{(r)}(x) = k e_{k-1,c}^{(r)}(x), \tag{3.6}$$

y

$$M_c(e_{k,c}^{(r)})(x) = e_{k,c}^{(r-1)}(x). \tag{3.7}$$

Una vez descrito el caso clásico centrado, ya estamos en condiciones de buscar una extensión de Dunkl de los polinomios (centrados) de Boole. Para ello, reemplazaremos el operador en diferencias central Δ_c por su versión de Dunkl

$$\Delta_\alpha f(x) = (\alpha+1)(\tau_1-\tau_{-1})f(x),$$

y el operador media central M_c por

$$M_\alpha f(x) = \frac{1}{2}(\tau_1+\tau_{-1})f(x).$$

Teniendo en cuenta (3.5) escogeremos la función $1/\mathcal{I}_\alpha(G_\alpha^{-1}(t))$ como aquella que juegue el papel de la versión de Dunkl de $(t+\sqrt{1+t^2})/(t^2+1+t\sqrt{1+t^2})$. Con esto, definiremos los *polinomios de Boole-Dunkl generalizados de orden* r, $\{e_{k,\alpha}^{(r)}(x)\}_{k=0}^{\infty}$, mediante la función generatriz

$$\frac{E_\alpha(xG_\alpha^{-1}(t))}{\mathcal{I}_\alpha(G_\alpha^{-1}(t))^r} = \sum_{k=0}^{\infty} e_{k,\alpha}^{(r)}(x)\frac{t^k}{\gamma_{k,\alpha}}.$$

Puesto que la función $\mathcal{I}_\alpha(G_\alpha^{-1}(t))^r$ es par, se deduce fácilmente que $e_{2k,\alpha}^{(r)}(x)$ es un polinomio par para $k \geq 0$ y que $e_{2k+1,\alpha}^{(r)}(x)$ es un polinomio impar para $k \geq 0$ (y, por tanto, se anula en $x = 0$).

Los primeros polinomios de Boole-Dunkl generalizados de orden r son

$$\begin{aligned}
e_{0,\alpha}^{(r)}(x) &= 1, \qquad e_{1,\alpha}^{(r)}(x) = x, \qquad e_{2,\alpha}^{(r)}(x) = x^2 - r, \\
e_{3,\alpha}^{(r)}(x) &= x^3 - \frac{1+\alpha+r(2+\alpha)}{1+\alpha}x, \\
e_{4,\alpha}^{(r)}(x) &= x^4 - \frac{2(2+2\alpha+r(2+\alpha))}{1+\alpha}x^2 + \frac{r(5+4\alpha+r(2+\alpha))}{1+\alpha}, \\
e_{5,\alpha}^{(r)}(x) &= x^5 - \frac{2(3+\alpha)(3+3\alpha+r(2+\alpha))}{(1+\alpha)(2+\alpha)}x^3 \\
&\qquad + \left(5 + \frac{6}{2+\alpha} + \frac{r(3+\alpha)(7+6\alpha+r(2+\alpha))}{(1+\alpha)^2}\right)x.
\end{aligned}$$

El artículo prosigue con el estudio de algunas propiedades de estos polinomios. Por ejemplo, las propiedades clásicas (3.6) y (3.7) en el caso de Dunkl se convierten en (ver [27, Theorem 3.1])

$$\Delta_\alpha e_{k,\alpha}^{(r)}(x) = \theta_{k,\alpha} e_{k-1,\alpha}^{(r)}(x), \qquad k \geq 1,$$

y

$$M_\alpha e_{k,\alpha}^{(r)}(x) = e_{k,\alpha}^{(r-1)}(x).$$

Por otro lado, también nos resultaba muy interesante generalizar [11, (3.18)], donde se expresan los polinomios de Boole generalizados en función de los polinomios generalizados de Euler y de Bernoulli:

$$e_k^{(r)}(x) = \sum_{j=0}^{k} \binom{k}{j} E_j^{(r)}(x) B_{k-j}^{(k+1)}(1).$$

Cuando se trata de los polinomios de Boole-Dunkl generalizados, obtenemos, para $\alpha > -1$ y $r \geq 0$ un entero (ver [27, Theorem 3.4]),

$$e_{k,\alpha}^{(r)}(x) = \sum_{l=0}^{k} \frac{l}{k} \binom{k}{l}_\alpha \mathfrak{E}_{l,\alpha}^{(r)}(x) \mathfrak{B}_{k-l,\alpha}^{(k)}(0),$$

y, también,

$$\Lambda_\alpha e_{k,\alpha}^{(r)}(x) = \frac{\theta_{k,\alpha}}{k} \sum_{l=0}^{k-1} (l+1) \binom{k-1}{l}_\alpha \mathfrak{E}_{l,\alpha}^{(r)}(x) \mathfrak{B}_{k-l-1,\alpha}^{(k)}(0),$$

donde $\mathfrak{E}_{l,\alpha}^{(r)}(x)$ son los polinomios de Euler-Dunkl generalizados de orden r (2.6) y $\mathfrak{B}_{l,\alpha}^{(k)}(x)$ los polinomios de Bernoulli-Dunkl generalizados de orden k (2.7).

Otra interesante propiedad de los polinomios de Boole generalizados es que cualquier polinomio $P_n(x)$ de grado n puede expresarse como combinación lineal de polinomios de Boole mediante la fórmula

$$P_n(x) = c_{n,0} + c_{n,1}e_1(x) + c_{n,2}e_2(x) + \cdots + c_{n,n}e_n(x),$$

donde

$$c_{n,k} = \frac{M\Delta^k P_n(0)}{k!}, \qquad k = 0, 1, \ldots, n.$$

Como cabría esperar, en el caso de Dunkl también podemos escribir un polinomio $P_n(x)$ de grado n como combinación de polinomios de Boole-Dunkl generalizados con una expresión del mismo tipo (ver [27, Theorem 4.1]):

$$P_n(x) = c_{n,0} + c_{n,1}e_{1,\alpha}(x) + c_{n,2}e_{2,\alpha}(x) + \cdots + c_{n,n}e_{n,\alpha}(x),$$

donde

$$c_{n,k} = \frac{M_\alpha \Delta_\alpha^k P_n(0)}{\gamma_{k,\alpha}}, \qquad k = 0, 1, \ldots, n.$$

Por último, destacamos las fórmulas de conexión, que relacionan los polinomios de Boole generalizados y los polinomios de Euler generalizados a través de los números de Stirling, de primera y segunda especie. Dentro de la teoría de Dunkl (y tal como se ve en [44, Remark 3.6]), en su lugar aparecen los números de Stirling-Dunkl de primera y segunda especie y los polinomios de Euler-Dunkl, resultando las hermosas fórmulas

$$e_{k,\alpha}^{(r)}(x) = \sum_{n=0}^{k} s^\alpha(k,n)\mathfrak{E}_{n,\alpha}^{(r)}(x), \qquad \mathfrak{E}_{k,\alpha}^{(r)}(x) = \sum_{n=0}^{k} S_\alpha(k,n)e_{n,\alpha}^{(r)}(x).$$

Capítulo 4

Copia completa de los artículos

En las siguientes páginas se incluyen, en el orden en el que los elaboramos, los artículos desarrollados durante esta tesis por compendio de publicaciones que —recordamos— son los siguientes:

- A. Gil Asensi, J. L. Varona, Appell-Dunkl sequences and Hurwitz-Dunkl zeta functions, *J. Math. Anal. Appl.* **520** (2023), no. 1, artículo no. 126870, 40 pp.
- A. Gil Asensi, J. L. Varona, A general method to find special functions that interpolate Appell polynomials, with examples, *J. Math. Anal. Appl.* **531** (2024), no. 2, artículo no. 127825, 18 pp.
- A. Gil Asensi, E. Labarga, J. Mínguez Ceniceros, J. L. Varona, Boole-Dunkl polynomials and generalizations, *Rev. R. Acad. Cienc. Exactas Fís. Nat. Ser. A Mat. RACSAM* **118** (2024), no. 1, artículo no. 16, 18 pp.

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Regular Articles

Appell-Dunkl sequences and Hurwitz-Dunkl zeta functions ☆

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ABSTRACT

Dunkl theory on the real line involves some tools such as the Dunkl derivative

$$\Lambda_\alpha f(x) = \frac{d}{dx} f(x) + \frac{2\alpha+1}{2} \frac{f(x)-f(-x)}{x}$$

or the Dunkl exponential $E_\alpha(z)$ that is defined in terms of the Bessel functions. Taking $\alpha = -1/2$ we get $\Lambda_{-1/2} = d/dx$ and $E_{-1/2}(z) = e^z$, hence, the classic derivative and exponential are particular cases. In recent years, some papers have generalized, in a Dunkl sense, number theoretic concepts such as Appell sequences, and then they are called Appell-Dunkl sequences; in particular, the so called Bernoulli-Dunkl and Euler-Dunkl polynomials have been defined, among others. Here we generalize, also in a Dunkl sense, some Hurwitz or Lerch zeta functions such as $\zeta(s,x) = \sum_{n=0}^\infty 1/(n+x)^s$ and, in addition, we get properties that relate those functions, extended to the s-complex plane and evaluated at negative integers s, with Bernoulli-Dunkl and Euler-Dunkl polynomials. One of the results we get for the "Dunkl zeta function" $\zeta_\alpha(s)$ is

$$\zeta_\alpha(1-s) = \Gamma(s)\cos\left(\frac{\pi s}{2}\right)\sum_{n=1}^\infty \frac{1}{s_n^s}, \qquad \mathrm{Re}(s) > 1$$

(where s_n are the positive zeros of the Bessel function $J_{\alpha+1}(x)$). This equation provides a generalization of the reflection formula of the Riemann zeta function, where the function $\sum_{n=1}^\infty 1/s_n^s$ is playing a similar role as $\sum_{n=1}^\infty 1/n^s$.

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1. Introduction

An Appell sequence $\{P_n(x)\}_{n=0}^\infty$ is a sequence of polynomials defined by a Taylor generating expansion

$$A(t)e^{xt} = \sum_{n=0}^{\infty} P_n(x)\frac{t^n}{n!}, \tag{1.1}$$

where $A(t)$ is a function analytic at $t = 0$ with $A(0) \neq 0$. Since the exponential function e^x is invariant under the differential operator d/dx, it is easy to show that $P_n(x)$ is a polynomial of degree n and $P_n'(x) = nP_{n-1}(x)$. Typical examples of Appell sequences are the Bernoulli polynomials $\{B_n(x)\}_{n=0}^\infty$, the Euler polynomials $\{E_n(x)\}_{n=0}^\infty$, or the probabilistic Hermite polynomials $\{\mathrm{He}_n(x)\}_{n=0}^\infty$ that are defined by taking $A(t) = \frac{te^{xt}}{e^t-1}$, $\frac{2e^{xt}}{e^t+1}$ or $e^{-t^2/2}$ respectively (a slight variation is the physicists' Hermite polynomials $\{H_n(x)\}_{n=0}^\infty$ defined by $e^{-t^2}e^{2xt} = \sum_{n=0}^\infty H_n(x)\frac{t^n}{n!}$).

The Appell sequences of polynomials have been extended in many ways. One of them consists of changing the derivative operator by operators in the context of Dunkl. In [16] and [13], the derivative operator was replaced by

$$\Lambda_\alpha f(x) = \frac{d}{dx}f(x) + \frac{2\alpha+1}{2}\left(\frac{f(x)-f(-x)}{x}\right),$$

where $\alpha > -1$ is a fixed parameter (see [17,29]); observe that the case $\alpha = -1/2$ recovers the classical case $\Lambda_{-1/2} = \frac{d}{dx}$. In that setting, an Appell-Dunkl sequence $\{P_n\}_{n=0}^\infty$ is a sequence of polynomials that satisfies

$$\Lambda_\alpha P_n(x) = \big(n + (\alpha+1/2)(1-(-1)^n)\big)P_{n-1}(x)$$

(instead of $\Lambda_\alpha P_n = nP_{n-1}$, the previous definition with a different multiplicative constant in the place of n is used for convenience with the notation). The Appell-Dunkl sequences can be written as a generating expansion similar to (1.1), namely

$$A(t)E_\alpha(xt) = \sum_{n=0}^{\infty} \mathcal{P}_n(x)\frac{t^n}{\gamma_{n,\alpha}},$$

for a certain function E_α and certain constants $\gamma_{n,\alpha}$ (with $E_{-1/2} = \exp$ and $\gamma_{n,-1/2} = n!$); we will see the details in Section 2. The first Appell-Dunkl sequence of polynomials studied in the mathematical literature were the so called generalized Hermite polynomials; see [29]. In recent years, also the Bernoulli and the Euler polynomials (among other Appell families) have been extended to the Dunkl context; see, for instance, [13, 14,18]. These polynomials have proved to be very useful to extend some classical properties to a more general context. For instance, the Bernoulli polynomials can be used to find the values of the series $\sum_{m=1}^\infty 1/m^{2k}$, and the Bernoulli-Dunkl polynomials can be used to compute the Rayleigh series $\sum_{m=1}^\infty 1/s_m^{2k}$, where $\{s_m\}_{m=1}^\infty$ are the positive zeros of a Bessel function (note that, essentially, the sine function is a particular case of a Bessel function, and the positive zeros of the sine are $s_m = \pi m$, $m \geq 1$, so in this case the corresponding Rayleigh series reduces to $\sum_{m=1}^\infty 1/m^{2k}$).

In the classical case, there is a large class of Appell sequences $\{P_n(x)\}_{n=0}^\infty$ for which there is a function $H(s,x)$, entire in s for fixed x with $\operatorname{Re} x > 0$, and satisfying $H(-n,x) = P_n(x)$ for $n = 0,1,2,\dots$ For example, in the case of Bernoulli polynomials, H is essentially the Hurwitz zeta function $\zeta(s,x)$ that for $\operatorname{Re}(s) > 1$ is defined as $\zeta(s,x) = \sum_{m=0}^\infty (m+x)^{-s}$, and whose analytic extension to the s-complex plane satisfies $-n\zeta(1-n,x) = B_n(x)$. Another well-known example is the Apostol-Bernoulli polynomials, whose corresponding function H is, essentially, the Lerch transcendent function (see [3]). More examples can be found in [8,9,22]. The papers [24,25] show how this can be done, in a very general way, with the help of the Mellin transform $\int_0^\infty f(t)t^{s-1}\,dt$, and provide many additional examples.

The aim of this paper is to show how to do it in the context of Appell-Dunkl sequences. Here, there are two important difficulties. The first one is the size of $E_\alpha(t)$ when $t \to \pm\infty$. Although $E_\alpha(t)$ is a generalization of e^t to the Dunkl context, it is not true that $E_\alpha(t) \sim e^t$ when $t \to \pm\infty$, but, roughly speaking, $E_\alpha(t) \sim e^{|t|}$ (except for $\alpha = -1/2$). In the above mentioned Mellin transform, a factor e^{-t} in $f(t)$ greatly contributes to the convergence of the integral; however, this does not happen with $E_\alpha(-t)$. In the second place, the classical translation $f(x) \mapsto f(x+m)$ becomes a complicate operator in the Dunkl context, and this affects the summands of type $(x+m)^{-s}$ of the classical Hurwitz zeta function, which are not so simple in the new context.

The organization of this paper is as follows. In Section 2 we give the details of the Dunkl context, and the precise definitions of the Appell-Dunkl sequences. Section 3 gives the details of the Dunkl translation. In Section 4 we give a general procedure, based on the Mellin transform, to extend an Appell-Dunkl sequence $\{\mathcal{P}_n(x)\}_{n=0}^\infty$ to an analytic function $H(s,x)$ such that $H(-n,x) = \mathcal{P}_n(x)$ (actually, it is a bit different); due to the above mentioned difficulties, this is not as general as in the classical case studied in [24], is not valid in the whole range of x, and requires some additional hypotheses. This section also studies several particular cases of Appell-Dunkl polynomials (Bernoulli-Dunkl, Euler-Dunkl, generalized Bernoulli-Dunkl, generalized Euler-Dunkl, and generalized Hermite), giving their corresponding Hurwitz-Dunkl zeta functions. In Section 5 we study some additional properties of these Hurwitz-Dunkl zeta functions. In particular, we show how these functions are connected with series of type $\sum_{m=1}^\infty 1/j_{m,\alpha}^s$ (where $\{j_{m,\alpha}\}_{m=1}^\infty$ are the positive zeros of the Bessel function of order α), by means of some formulas that resembles Riemann's functional equation for the classical $\zeta(s)$ function: if we use ζ_α to denote the function associated to de Bernoulli-Dunkl polynomials, we have

$$\zeta_\alpha(1-s) = \Gamma(s)\cos\left(\frac{\pi s}{2}\right)\sum_{m=1}^\infty \frac{1}{j_{m,\alpha+1}^s}, \qquad \operatorname{Re}(s) > 1$$

(see the details in that section). In Section 6 we study the connection of our results with the analytic continuation to the s-complex plane of $Z_\alpha(s) = \sum_{m=1}^\infty 1/j_{m,\alpha}^s$, which was studied by Hawkins [21]. Finally, Section 7 includes some of the technical proofs of the results presented in Section 5.

2. Appell-Dunkl sequences

For $\alpha > -1$, let J_α denote the Bessel function of order α and, for complex values of the variable z, let

$$\mathcal{I}_\alpha(z) = 2^\alpha\Gamma(\alpha+1)\,\frac{J_\alpha(iz)}{(iz)^\alpha} = \Gamma(\alpha+1)\sum_{n=0}^\infty \frac{(z/2)^{2n}}{n!\,\Gamma(n+\alpha+1)} = {}_0F_1(\alpha+1, z^2/4)$$

(the function $\mathcal{I}_\alpha$ is a small variation of the so-called modified Bessel function of the first kind and order α, usually denoted by I_α; see [35] or [28]). Also, again for $z \in \mathbb{C}$, take

$$E_\alpha(z) = \mathcal{I}_\alpha(z) + \frac{z}{2(\alpha+1)}\,\mathcal{I}_{\alpha+1}(z) = e^z\,{}_1F_1(\alpha+1/2, 2\alpha+2, -2z). \tag{2.1}$$

Following [17] for $\alpha \ge -1/2$ and [29] for $\alpha > -1$, in the real line and with the reflection group $\mathbb{Z}_2$, the Dunkl operator Λ_α is defined as

$$\Lambda_\alpha f(x) = \frac{d}{dx}f(x) + \frac{2\alpha+1}{2}\left(\frac{f(x)-f(-x)}{x}\right), \tag{2.2}$$

where f is a suitable function on $\mathbb{R}$. If we want to specify that the variable involved in the Dunkl operator is x, we will use $\Lambda_{\alpha,x}$. For any $\lambda \in \mathbb{C}$, we have

$$\Lambda_\alpha E_\alpha(\lambda x) = \Lambda_{\alpha,x} E_\alpha(\lambda x) = \lambda E_\alpha(\lambda x). \tag{2.3}$$

Let us note that, when $\alpha = -1/2$, we have $\Lambda_{-1/2} = d/dx$ and $E_{-1/2}(\lambda x) = e^{\lambda x}$.

From the definition, it is easy to check that

$$E_\alpha(z) = \sum_{n=0}^\infty \frac{z^n}{\gamma_{n,\alpha}}, \qquad \mathcal{I}_\alpha(x) = \sum_{n=0}^\infty \frac{x^{2n}}{\gamma_{2n,\alpha}},$$

with

$$\gamma_{n,\alpha} = \begin{cases} 2^{2k} k!\,(\alpha+1)_k, & \text{if } n = 2k, \\ 2^{2k+1} k!\,(\alpha+1)_{k+1}, & \text{if } n = 2k+1, \end{cases} \tag{2.4}$$

and where $(a)_n$ denotes the Pochhammer symbol

$$(a)_n = a(a+1)(a+2)\cdots(a+n-1) = \frac{\Gamma(a+n)}{\Gamma(a)}$$

(with a a non-negative integer); of course, $\gamma_{n,-1/2} = n!$. From (2.4), we have

$$\frac{\gamma_{n,\alpha}}{\gamma_{n-1,\alpha}} = n + (\alpha + 1/2)(1 - (-1)^n) =: \theta_{n,\alpha}. \tag{2.5}$$

We also define

$$\binom{n}{j}_\alpha = \frac{\gamma_{n,\alpha}}{\gamma_{j,\alpha}\gamma_{n-j,\alpha}},$$

which becomes the ordinary binomial coefficient in the case $\alpha = -1/2$. To simplify the notation we sometimes write $\gamma_n = \gamma_{n,\alpha}$ and $\theta_n = \theta_{n,\alpha}$. For each function $A(t)$ analytic in a neighborhood of $t = 0$ and with $A(0) \neq 0$, we define an Appell-Dunkl sequence $\{P_n(x)\}_{n=0}^\infty$ by means of the generating function

$$A(t)E_\alpha(xt) = \sum_{n=0}^\infty P_n(x)\frac{t^n}{\gamma_n} \tag{2.6}$$

(additionally to the papers [2,29] cited in the introduction, Appell-Dunkl sequences have been also considered, for instance, in [10,11,16]). From this definition, it is not difficult to prove that $P_{n,\alpha}(x)$ is a polynomial of degree n and, moreover, $\Lambda_\alpha P_n(x) = \frac{\gamma_n}{\gamma_{n-1}} P_{n-1}(x)$ (when $\alpha = -1/2$, this becomes the classical $P_n'(x) = nP_{n-1}(x)$ in the Appell sequences).

Besides the generalized Hermite polynomials that, in the Dunkl context, were studied in [29], we will use the so called Bernoulli-Dunkl polynomials, Euler-Dunkl polynomials, and their corresponding generalization with an extra parameter.

2.1. Bernoulli-Dunkl polynomials

Following [13], we define the Bernoulli-Dunkl polynomials $\{\mathfrak{B}_{n,\alpha}\}_{n=0}^\infty$ by means of the generating function

$$\frac{E_\alpha(xt)}{\mathcal{I}_{\alpha+1}(t)} = \sum_{n=0}^\infty \frac{\mathfrak{B}_{n,\alpha}(x)}{\gamma_{n,\alpha}} t^n. \tag{2.7}$$

Table 1
Scheme that describes the process to transform the definition of the classical Bernoulli and Euler polynomials into the definition of the Bernoulli-Dunkl and Euler-Dunkl polynomials (and their generalizations of order r). In the classical case, we use the "basic" interval $[0, 1]$, the function exp and the factorial $n!$; in the Dunkl case with $\alpha > -1$, we must use the "basic" interval $[-1, 1]$, the function E_α and $\gamma_{n,\alpha}$.

	Bernoulli $\mapsto$ Bernoulli-Dunkl	Euler $\mapsto$ Euler-Dunkl
Classical	$\frac{te^{xt}}{e^t-1} = \sum_{n=0}^{\infty} B_n(x)\frac{t^n}{n!}$	$\frac{2e^{xt}}{e^t+1} = \sum_{n=0}^{\infty} E_n(x)\frac{t^n}{n!}$
$x \mapsto \frac{x+1}{2}$	$\frac{te^{xt/2}e^{t/2}}{e^t-1} = \sum_{n=0}^{\infty} B_n(\frac{x+1}{2})\frac{t^n}{n!}$	$\frac{2e^{xt/2}e^{t/2}}{e^t+1} = \sum_{n=0}^{\infty} E_n(\frac{x+1}{2})\frac{t^n}{n!}$
$t \mapsto 2t$	$\frac{2te^{xt}e^t}{e^{2t}-1} = \sum_{n=0}^{\infty} B_n(\frac{x+1}{2})\frac{2^n t^n}{n!}$	$\frac{2e^{xt}e^t}{e^{2t}+1} = \sum_{n=0}^{\infty} E_n(\frac{x+1}{2})\frac{2^n t^n}{n!}$
rewrite	$\frac{2te^{xt}}{e^t-e^{-t}} = \sum_{n=0}^{\infty} B_n(\frac{x+1}{2})\frac{2^n t^n}{n!}$	$\frac{2e^{xt}}{e^t+e^{-t}} = \sum_{n=0}^{\infty} E_n(\frac{x+1}{2})\frac{2^n t^n}{n!}$
$\exp \mapsto E_\alpha$	$\frac{2tE_\alpha(xt)}{E_\alpha(t)-E_\alpha(-t)} = \sum_{n=0}^{\infty} B_n^*(\frac{x+1}{2})\frac{2^n t^n}{\gamma_{n,\alpha}}$	$\frac{2E_\alpha(xt)}{E_\alpha(t)+E_\alpha(-t)} = \sum_{n=0}^{\infty} E_n^*(\frac{x+1}{2})\frac{2^n t^n}{\gamma_{n,\alpha}}$
rewrite	$\frac{2(\alpha+1)E_\alpha(xt)}{\mathcal{I}_{\alpha+1}(t)} = \sum_{n=0}^{\infty} B_n^*(\frac{x+1}{2})\frac{2^n t^n}{\gamma_{n,\alpha}}$	$\frac{E_\alpha(xt)}{\mathcal{I}_\alpha(t)} = \sum_{n=0}^{\infty} E_n^*(\frac{x+1}{2})\frac{2^n t^n}{\gamma_{n,\alpha}}$
Dunkl	$\frac{E_\alpha(xt)}{\mathcal{I}_{\alpha+1}(t)} = \sum_{n=0}^{\infty} \mathfrak{B}_{n,\alpha}(x)\frac{t^n}{\gamma_{n,\alpha}}$	$\frac{E_\alpha(xt)}{\mathcal{I}_\alpha(t)} = \sum_{n=0}^{\infty} \mathfrak{E}_{n,\alpha}(x)\frac{t^n}{\gamma_{n,\alpha}}$
Generalized	$\frac{E_\alpha(xt)}{(\mathcal{I}_{\alpha+1}(t))^r} = \sum_{n=0}^{\infty} \mathfrak{B}_{n,\alpha}^{(r)}(x)\frac{t^n}{\gamma_{n,\alpha}}$	$\frac{E_\alpha(xt)}{(\mathcal{I}_\alpha(t))^r} = \sum_{n=0}^{\infty} \mathfrak{E}_{n,\alpha}^{(r)}(x)\frac{t^n}{\gamma_{n,\alpha}}$

To simplify the notation we sometimes write $\mathfrak{B}_n = \mathfrak{B}_{n,\alpha}$ (and $\gamma_n = \gamma_{n,\alpha}$).

The first few Bernoulli-Dunkl polynomials are

$$\mathfrak{B}_0(x) = 1, \qquad \mathfrak{B}_1(x) = x,$$

$$\mathfrak{B}_2(x) = x^2 - \frac{\alpha+1}{\alpha+2}, \qquad \mathfrak{B}_3(x) = x^3 - x,$$

$$\mathfrak{B}_4(x) = x^4 - 2x^2 + \frac{(\alpha+4)(\alpha+1)}{(\alpha+3)(\alpha+2)}, \qquad \mathfrak{B}_5(x) = x^5 - 2\,\frac{\alpha+3}{\alpha+2}\,x^3 + \frac{\alpha+4}{\alpha+2}\,x.$$

Some of the properties of these polynomials can be seen in [13].

Before we continue, let us explain why we use "Bernoulli-Dunkl" to name these polynomials. The first reason is that

$$\frac{\mathfrak{B}_{n,-1/2}(2x-1)}{2^n} = B_n(x), \tag{2.8}$$

where $\{B_n\}_{n=0}^{\infty}$ are the Bernoulli polynomials (for the definition and properties of the Bernoulli polynomials see, for instance, [15] or [20]). Indeed, taking into account that

$$E_{-1/2}(x) = e^x, \quad \mathcal{I}_{1/2}(x) = \frac{\sin(ix)}{ix},$$

the relation (2.8) can be deduced substituting x for $2x-1$, t for $t/2$ and α for $-1/2$ in the definition (2.7). Here, we must note that the change $x \mapsto 2x-1$ in (2.8) is very natural, because in the reflection group $\mathbb{Z}_2$, which is key in the standard definition of the Dunkl operator (2.2), the symmetry plays an important role, and thus the role of $x = 0$ and $x = 1$ on the classical Bernoulli polynomials must be translated to the points -1 and 1. In fact, this is the process that is explained in Table 1 (extracted from [14]) to define Bernoulli-Dunkl polynomials as an extension to the Dunkl case of the classical Bernoulli polynomials. As is shown in the table, this process can be used for other classical polynomials.

Another reason to use the name Bernoulli-Dunkl polynomials for $\mathfrak{B}_n$ is the role that they play in certain sums involving the zeros of the Bessel functions (see [13]), which is a generalization of what happens in the case $\alpha = -1/2$ with the Bernoulli polynomials. This will appear again later in this paper; see Corollary 5.6.

2.2. Generalized Bernoulli-Dunkl polynomials

In the classical case, the generalized Bernoulli polynomials of order r are $\{B_n^{(r)}(x)\}_{n=0}^{\infty}$, defined by

$$\left(\frac{t}{e^t-1}\right)^r e^{xt} = \sum_{n=0}^{\infty} B_n^{(r)}(x)\frac{t^n}{n!}.$$

They were introduced by Nørlund in 1922 (see [26,27]).

When $\alpha > -1$ we can also define the generalized Bernoulli-Dunkl polynomials $\{\mathfrak{B}_{n,\alpha}^{(r)}\}_{n=0}^{\infty}$ (or $\mathfrak{B}_n^{(r)}$) of order r by means of the generating function

$$\frac{E_\alpha(xt)}{\mathcal{I}_{\alpha+1}(t)^r} = \sum_{n=0}^{\infty} \frac{\mathfrak{B}_{n,\alpha}^{(r)}(x)}{\gamma_{n,\alpha}}\, t^n. \tag{2.9}$$

In this case, the generalized Bernoulli polynomials and the generalized Bernoulli-Dunkl polynomials are related by

$$\mathfrak{B}_{n,-1/2}^{(r)}(2x-r) = 2^n B_n^{(r)}(x).$$

In the recent paper [19] we can see how the polynomials $\mathfrak{B}_{n,\alpha}^{(r)}$ can be used in the context of Appell-Dunkl discrete sequences, in the same way that $B_n^{(r)}$ appear in the context of Appell discrete sequences and falling factorial polynomials.

2.3. Euler-Dunkl polynomials

We define the Euler-Dunkl polynomials $\{\mathfrak{E}_{n,\alpha}\}_{n=0}^{\infty}$ of order $\alpha > -1$ by means of the generating function

$$\frac{E_\alpha(xt)}{\mathcal{I}_\alpha(t)} = \sum_{n=0}^{\infty} \frac{\mathfrak{E}_{n,\alpha}(x)}{\gamma_{n,\alpha}}\, t^n.$$

As usual, we will sometimes denote it only by $\mathfrak{E}_n$, without specifying α. The first few Euler-Dunkl polynomials are

$$\mathfrak{E}_0(x) = 1, \qquad \mathfrak{E}_1(x) = x,$$

$$\mathfrak{E}_2(x) = x^2 - 1, \qquad \mathfrak{E}_3(x) = x^3 - \frac{\alpha+2}{\alpha+1}\,x,$$

$$\mathfrak{E}_4(x) = x^4 - 2\,\frac{\alpha+2}{\alpha+1}\,x^2 + \frac{\alpha+3}{\alpha+1}, \qquad \mathfrak{E}_5(x) = x^5 - 2\,\frac{\alpha+3}{\alpha+1}\,x^3 + \frac{(\alpha+3)^2}{(\alpha+1)^2}\,x.$$

These polynomials are related to the classical Euler polynomials $\{E_n\}_{n=0}^{\infty}$ by

$$\frac{\mathfrak{E}_{n,-1/2}(2x-1)}{2^n} = E_n(x) \tag{2.10}$$

(for the definition and properties of the Euler polynomials see, for instance, [15]). This process has been sketched in Table 1.

2.4. Generalized Euler-Dunkl polynomials

When $\alpha > -1$ we can also define the generalized Euler-Dunkl polynomials $\{\mathfrak{E}^{(r)}_{n,\alpha}\}_{n=0}^{\infty}$ (or $\mathfrak{E}^{(r)}_{n}$) of order r by means of the generating function

$$\frac{E_\alpha(xt)}{\mathcal{I}_\alpha(t)^r} = \sum_{n=0}^{\infty} \frac{\mathfrak{E}^{(r)}_{n,\alpha}(x)}{\gamma_{n,\alpha}} t^n.$$

In the classical case, the generalized Euler polynomials of order r are $\{E^{(r)}_n(x)\}_{n=0}^{\infty}$ defined by

$$\left(\frac{2}{e^t+1}\right)^r e^{xt} = \sum_{n=0}^{\infty} E^{(r)}_n(x) \frac{t^n}{n!}.$$

The generalized Euler polynomials and the generalized Euler-Dunkl polynomials are related by

$$\mathfrak{E}^{(r)}_{n,-1/2}(2x-r) = 2^n E^{(r)}_n(x).$$

3. The Dunkl translation: definition and some properties

The Dunkl translation operator of a function f is defined by

$$\tau_y f(x) = \sum_{n=0}^{\infty} \Lambda^n_\alpha f(x) \frac{y^n}{\gamma_{n,\alpha}}, \qquad \alpha > -1, \tag{3.1}$$

where Λ^0_α is the identity operator and $\Lambda^{n+1}_\alpha = \Lambda_\alpha(\Lambda^n_\alpha)$. As in the case of $\Lambda_{\alpha,x} = \Lambda_\alpha$, we sometimes use $\tau_{y,x}$ if we want to indicate that the translation τ_y is acting on a function whose variable is x. In the case $\alpha = -1/2$, the translation $\tau_y f$ is just the Taylor expansion of a function f around a fixed point x, that is,

$$f(x+y) = \sum_{n=0}^{\infty} f^{(n)}(x) \frac{y^n}{n!}.$$

Of course, definition (3.1) is valid only for C^∞ functions, and assuming also that the series on the right is convergent. In particular, this can be guaranteed when f is a polynomial, because the operator Λ_α applied to a polynomial of degree k generates a polynomial of degree $k-1$, so the series (3.1) has only a finite number of nonzero summands. Other properties of the translation operator τ_y can be found in [29], [31], [34] and [23], including some integral expressions that can be applied to a wider class of functions than (3.1).

From the definition (3.1), it is clear that τ_y commutes with the Dunkl operator Λ_α. In what follows, we are going to see some other basic properties. It is not difficult to prove these properties, and here we state most of them without a proof; in most cases, more details can be found in [14].

A nice property of the Dunkl translation, which resembles the Newton binomial $(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} y^k x^{n-k}$, is the following:

$$\tau_y((\cdot)^n)(x) = \sum_{k=0}^{n} \binom{n}{k}_\alpha y^k x^{n-k}. \tag{3.2}$$

More generally, and in relation to the Appell-Dunkl sequences $\{P_n(x)\}_{n=0}^{\infty}$ defined as in (2.6), the Dunkl translation satisfies

$$\tau_y(P_k)(x) = \sum_{j=0}^{k} \binom{k}{j}_\alpha P_j(x) y^{k-j},$$

which in the classical case $\alpha = -1/2$ becomes $P_k(x+y) = \sum_{j=0}^{k} \binom{k}{j} P_j(x) y^{k-j}$.

Another important property is the fact

$$\tau_y f(x) = \tau_x f(y). \tag{3.3}$$

This is a direct consequence of the above mentioned integral expressions for the Dunkl translation. Moreover, at least for polynomials, it can be easily checked starting from (3.1) using the linearity of τ_y and its behavior on $f(x) = x^n$, $n = 0, 1, 2, \dots$ Indeed, using $\binom{n}{k}_\alpha = \binom{n}{n-k}_\alpha$ and (3.2) we have $\tau_y((\cdot)^n)(x) = \tau_x((\cdot)^n)(y)$, and this proves (3.3).

The inverse operator of τ_y defined as in (3.1) is

$$\tau_y^{-1} f(x) = \sum_{n=0}^{\infty} \frac{c_n y^n}{\gamma_{n,\alpha}} \Lambda_\alpha^n f(x), \tag{3.4}$$

where $c_0 = 1$ and c_n for $n \geq 1$ is defined by the recurrence $c_n = -\sum_{j=0}^{n-1} \binom{n}{j}_\alpha c_j$ (a proof can be found in [14, Lemma 4.4]). The operator τ_y^{-1} is not, in general, a translation (in particular, it is not τ_{-y} except when $\alpha = -1/2$).

Moreover, it is not difficult to check that the operators of type τ_a, τ_b, τ_c^{-1} and τ_d^{-1} commute; for instance, $\tau_a\tau_b = \tau_b\tau_a$, $\tau_c^{-1}\tau_d^{-1} = \tau_d^{-1}\tau_c^{-1}$, $\tau_a\tau_c^{-1} = \tau_c^{-1}\tau_a$ and so on. Note that, in general (except when $\alpha = -1/2$), $\tau_a\tau_b$ is not a new translation, even if $a = b$.

In relation to E_α, the Dunkl translation has a nice behavior that resembles the classical $e^{t(x+y)} = e^{tx}e^{ty}$, namely

$$\tau_y(E_\alpha(t\cdot))(x) = E_\alpha(tx)E_\alpha(ty). \tag{3.5}$$

Indeed, using $\Lambda_{\alpha,x}(E_\alpha(tx)) = tE_\alpha(tx)$ (this is $\frac{d}{dx}e^{tx} = te^{tx}$ in the classical case), the proof of (3.5) is a simple consequence of the definition (3.1):

$$\tau_y\big(E_\alpha(t\cdot)\big)(x) = \sum_{m=0}^{\infty} \Lambda_{\alpha,x}^m E_\alpha(tx) \frac{y^m}{\gamma_m} = \sum_{m=0}^{\infty} E_\alpha(tx) \frac{(ty)^m}{\gamma_m} = E_\alpha(tx)E_\alpha(ty).$$

It is also easy to check that

$$\tau_y^{-1}\big(E_\alpha(t\cdot)\big)(x) = E_\alpha(tx)/E_\alpha(ty).$$

From these relations, we can easily state the following lemmas, which we will use later in this paper:

Lemma 3.1. *Let τ_y be the Dunkl translation operator. Then the identities*

$$\tau_y^n\big(E_\alpha(t\cdot)\big)(x) = E_\alpha(tx)E_\alpha(ty)^n$$

and

$$\tau_y^{-n}\big(E_\alpha(t\cdot)\big)(x) = E_\alpha(tx)/E_\alpha(ty)^n$$

holds for all $n = 0, 1, 2, 3, \dots$

Lemma 3.2. *Let τ_y and τ_z be Dunkl translations and let n and m be two non-negative integers. Then*

$$\tau_y^n \tau_z^{-m}\big(E_\alpha(t\cdot)\big) = E_\alpha(t\cdot)E_\alpha(ty)^n/E_\alpha(tz)^m.$$

There are still a couple of technical lemmas about the behavior of the Dunkl translation which we will use later in the paper (Subsections 4.1 and 4.2). We will apply these results only to functions like E_α, so we can use the Dunkl translation operator (3.1), which is valid only for functions in $\mathcal{C}^\infty$. Then, we can assume in the lemmas and in the proofs that the functions are in $\mathcal{C}^\infty$ (this could be weakened using integral expressions for the translation).

Lemma 3.3. *Let $\Lambda_{\alpha,x}$ be the Dunkl operator acting over the variable x and let $g(t,x)$ be a function such as the integral $\int_0^\infty g(t,x)\,dt$ converges and $\Lambda_{\alpha,x} g(t,x)$ exists. Then,*

$$\Lambda_{\alpha,x}\int_0^\infty g(t,x)\,dt = \int_0^\infty \Lambda_{\alpha,x} g(t,x)\,dt.$$

Proof. Using the definition of $\Lambda_{\alpha,x}$, we have

$$\begin{aligned}\Lambda_{\alpha,x}\int_0^\infty g(t,x)\,dt &= \frac{d}{dx}\int_0^\infty g(t,x)\,dt + \frac{2\alpha+1}{2}\,\frac{\int_0^\infty g(t,x)\,dt - \int_0^\infty g(-t,x)\,dt}{x}\\ &= \int_0^\infty \frac{d}{dx} g(t,x)\,dt + \frac{2\alpha+1}{2}\int_0^\infty \frac{g(t,x)-g(-t,x)}{x}\,dt\\ &= \int_0^\infty \Lambda_{\alpha,x} g(t,x)\,dt. \qquad \square\end{aligned}$$

Lemma 3.4. *Let $g(t,x)$ be a function in $\mathcal{C}^\infty$ such that the integral $\int_0^\infty g(t,x)\,dt$ converges, and let $\tau_{y,x}$ be the Dunkl translation operator. Then*

$$\tau_{y,x}\int_0^\infty g(t,x)\,dt = \int_0^\infty \tau_{y,x} g(t,x)\,dt.$$

Proof. By the previous lemma,

$$\begin{aligned}\tau_{y,x}\int_0^\infty g(t,x)\,dt &= \sum_{n=0}^\infty \Lambda_{\alpha,x}^n\Big(\int_0^\infty g(t,x)\,dt\Big)\frac{y^n}{\gamma_n} = \sum_{n=0}^\infty \Big(\int_0^\infty \Lambda_{\alpha,x}^n g(t,x)\,dt\Big)\frac{y^n}{\gamma_n}\\ &= \int_0^\infty \sum_{n=0}^\infty \Lambda_{\alpha,x}^n g(t,x)\frac{y^n}{\gamma_n}\,dt = \int_0^\infty \tau_{y,x} g(t,x)\,dt. \qquad \square\end{aligned}$$

4. The Mellin transform to get Appell-Dunkl polynomials as values of Hurwitz-Dunkl zeta functions

In this section, we define a special function, $H(s,x)$, which generalizes the Appell-Dunkl polynomials in such way that $H(-n,x)$ will give us the n-th Appell-Dunkl polynomial $P_n(x)$ multiplied by some constant. We express $H(s,x)$ in terms of the well-known Mellin transform

$$\mathcal{M}(f)(s) = \frac{1}{\Gamma(s)} \int_0^\infty f(t) t^{s-1}\, dt.$$

Here we have a very general result, and later we study particular cases of generating functions that involve Bernoulli-Dunkl and Euler-Dunkl polynomials (and also their respective generalized families) and obtain particular special functions, $H(s,x)$ for each one. In Theorem 4.3 we relate this $H(s,x)$ with a function which we call Hurwitz-Dunkl zeta function, $\zeta_\alpha(s,x)$ (see Definition 4.4), because it plays a similar role as the traditional Hurwitz zeta function $\zeta(s,x) = \sum_{n=0}^\infty 1/(x+n)^s$ and, in addition, it generalizes $\zeta(s,x)$ when changing $\alpha = -1/2$, $x \mapsto 2x-1$ and $t \mapsto t/2$, as we explain with more detail later. On the other hand, we obtain a similar function in the Euler-Dunkl case, $\zeta_{E,\alpha}(s,x)$ (see Definition 4.8), which generalizes the so-called Hurwitz zeta function of Euler type, $\zeta_E(s,x) = \sum_{n=0}^\infty (-1)^n/(x+n)^s$. Note that in this theorem we could assume $k = 0$ (which corresponds to the usual case $A(0) \neq 0$), but we allow a more general case.

Theorem 4.1. *Let $\{\mathcal{P}_n(x)\}_{n=0}^\infty$ be an Appell-Dunkl sequence with generating function $G(x,t) = A(t)E_\alpha(xt)$ and suppose that $A(t)$ has a zero of order k at $t = 0$. We also assume that, for all $x \in (a,b)$, the integral*

$$H(s,x) = \frac{1}{\Gamma(s)} \int_0^\infty G(x,-t)\, t^{s-1}\, dt = \frac{1}{\Gamma(s)} \int_0^\infty A(-t)E_\alpha(-xt) t^{s-1}\, dt \tag{4.1}$$

converges in the right plane $\operatorname{Re}(s) > -k$ to a holomophic function. Then, $H(s,x)$ may be analytically continued to an entire function of s satisfying

$$H(-n,x) = \frac{n!}{\gamma_{n,\alpha}} \mathcal{P}_n(x), \quad n = 0,1,2,\dots$$

Proof. Suppose $H(s,x)$ converges in the right plane $\operatorname{Re}(s) > -k$ for all $x \in (a,b)$, as was stated in the hypothesis of the theorem. Given $N \in \mathbb{N} \cup \{0\}$ with $N \geq k$, the Mellin integral can be analytically continued to the half plane $\operatorname{Re}(s) > -N-1$ as follows. Fix r with $0 < r < R$ and x with $a < x < b$ and separate the complete integral into three parts:

$$\begin{aligned} H(s,x) &= \frac{1}{\Gamma(s)} \int_r^\infty A(-t)E_\alpha(-xt) t^{s-1}\, dt \\ &\quad + \frac{1}{\Gamma(s)} \int_0^r \left(A(-t)E_\alpha(-xt) - \sum_{n=0}^N \mathcal{P}_n(x) \frac{(-t)^n}{\gamma_{n,\alpha}} \right) t^{s-1}\, dt \\ &\quad + \frac{1}{\Gamma(s)} \int_0^r \sum_{n=0}^N \mathcal{P}_n(x) \frac{(-t)^n}{\gamma_{n,\alpha}} t^{s-1}\, dt. \end{aligned}$$

In the first part, the integrand is $E_\alpha(-xt)A(-t)t^{s-1}$. Since $a < x < b$, it converges when $t \to \infty$, hence the integral is an entire function of s, dominated on arbitrary closed vertical strips of finite width. We may conclude that the integral is an entire function of s.

In the second part, the integrand is the product of t^{s-1} with the tail of the generating series, $\sum_{n=N+1}^\infty \mathcal{P}_n(x)(-t)^n/\gamma_{n,\alpha}$, which, since $|t| \leq r < R$, is $\mathcal{O}(t^{N+1})$ at $t = 0$. Thus, for $\operatorname{Re}(s) > -N-1$, the complete integrand is $\mathcal{O}(t^{N+\operatorname{Re}(s)})$ at $t = 0$ (with the order constant depending only on x) and hence is integrable on $[0,r]$ and dominated on closed vertical sub-strips of finite width of this section of the s-plane. Therefore the second integral is a holomorphic function of s for $\operatorname{Re}(s) > -N-1$.

In the third part, we have

$$\frac{1}{\Gamma(s)}\int_0^r \sum_{n=0}^{N} \mathcal{P}_n(x)\frac{(-t)^n}{\gamma_{n,\alpha}} t^{s-1}\,dt = \frac{1}{\Gamma(s)}\sum_{n=0}^{N} \mathcal{P}_n(x)\frac{(-1)^n}{\gamma_{n,\alpha}}\int_0^r t^{s+n-1}\,dt$$
$$= \frac{1}{\Gamma(s)}\sum_{n=0}^{N} \mathcal{P}_n(x)\frac{(-1)^n}{\gamma_{n,\alpha}}\frac{r^{s+n}}{s+n},$$

which is an entire function of s because of the simple pole of $\Gamma(s)$ at $s = -n$ cancels the simple zero of $s+n$ for $n = 0, 1, 2, \dots$, leaving the non-zero residue $(-1)^n/n!$.

Finally, if $s = -n$ with $0 \le n \le N$, the $1/\Gamma(s)$ factors in front of the first two terms vanish, as well as every term in the sum except the one corresponding to n, where the remaining value is $\mathcal{P}_n(x)n!/\gamma_{n,\alpha}$ because of the residue of $\Gamma(s)$ at $-n = 0, 1, 2, \dots$ Thus $H(-n, x) = \mathcal{P}_n(x)n!/\gamma_{n,\alpha}$ for these n and, as $N \ge k$ was arbitrary, this completes the proof. □

The previous theorem is very general but needs the convergence of (4.1). In the classical case $\alpha = -1/2$ stated in [24], we have $E_{-1/2}(-t) = e^{-t}$, which tends very quickly to 0 when $t \to \infty$. This allows us to prove the convergence of (4.1) with very weak hypothesis for $A(t)$. For instance, when $G(x, t)$ is the generating function for the Bernoulli polynomials, (4.1) becomes

$$H(s, x) = \frac{1}{\Gamma(s)}\int_0^\infty G(x, -t)\, t^{s-1}\,dt = \frac{1}{\Gamma(s)}\int_0^\infty \frac{t}{1-e^{-t}} e^{-xt} t^{s-1}\,dt$$

and it is clear that this integral is convergent for every $x > 0$ when s is in right half-plane $\mathrm{Re}(s) > 0$.

But this is no longer true when $\alpha \ne -1/2$. Actually, let us recall that $E_\alpha(z) = e^z\, {}_1F_1(\alpha+1/2, 2\alpha+2, -2z)$. For proving the convergence of the integral we need to estimate the size of the integrand in (4.1); in particular, the size of the factor $E_\alpha(-xt)$.

With this aim, let us use the asymptotic expansions of the Kummer confluent hypergeometric function ${}_1F_1(\cdot, \cdot, z)$ for $|z| \to \infty$ in the sectors

$$S_+ = \{z \in \mathbb{C} : -\pi/2 < \arg(z) < 3\pi/2\},$$
$$S_- = \{z \in \mathbb{C} : -3\pi/2 < \arg(z) < \pi/2\}$$

(see, for instance [30, p. 128]). In our case, these asymptotic expansions are, respectively, of the form

$$\begin{aligned}{}_1F_1\left(\frac{2\alpha+1}{2}, 2\alpha+2, z\right) &= \frac{\Gamma(2\alpha+2)}{\Gamma(\frac{2\alpha+1}{2})} e^z z^{-\alpha-3/2}\left(1 + \mathcal{O}\Big(\frac{1}{|z|}\Big)\right) \\ &\quad + \frac{\Gamma(2\alpha+2)}{\Gamma(\frac{2\alpha+3}{2})} e^{\pm(2\alpha+1)i\pi/2} z^{-\alpha-1/2}\left(1 + \mathcal{O}\Big(\frac{1}{|z|}\Big)\right).\end{aligned} \tag{4.2}$$

Notice that, in the case $\alpha = -1/2$, the coefficient of the first summand is $\Gamma(1)/\Gamma(0) = 0$, so the first summand vanishes. Otherwise (for simplicity, let us assume here that the variable z is real), the "exponential parts" for $E_\alpha(z) = e^z\, {}_1F_1(\alpha+1/2, 2\alpha+2, -2z)$ in (4.2) appears as e^{-z} in the first summand, and as e^z in the second summand. Then, the asymptotic size e^{-t} (for $t \to \infty$) of the classical case $\alpha = -1/2$ becomes something similar to $E_\alpha(-t) \sim e^{|t|}$ for $\alpha \ne -1/2$. In this way, instead of "a help" to prove the convergence of (4.1), the factor $E_\alpha(-xt)$ is a handicap, and a further analysis will be necessary to state the convergence of (4.1).

On the other hand, we would like to rewrite the function $H(s, x)$ that appears in Theorem 4.1 as a series, just as it occurs in the classical zeta function.

For the generating function of the Bernoulli polynomials, the Mellin transform of the $G(x,-t)$ is, for $x > 0$ and $\operatorname{Re}(s) > 0$,

$$\begin{aligned} H(s,x) &= \frac{1}{\Gamma(s)} \int_0^\infty G(x,-t)\, e^{-xt}\, t^{s-1}\, dt = \frac{1}{\Gamma(s)} \int_0^\infty \frac{t}{1-e^{-t}}\, e^{-xt}\, t^{s-1}\, dt \\ &= \frac{1}{\Gamma(s)} \int_0^\infty \sum_{n=0}^\infty e^{-nt}\, e^{-xt}\, t^s\, dt = \frac{1}{\Gamma(s)} \sum_{n=0}^\infty \int_0^\infty e^{-(n+x)t}\, t^s\, dt \\ &= \frac{\Gamma(s+1)}{\Gamma(s)} \sum_{n=0}^\infty \frac{1}{(n+x)^{s+1}} = s \sum_{n=0}^\infty \frac{1}{(n+x)^{s+1}} \end{aligned} \tag{4.3}$$

(a similar method can be followed for the Euler polynomials, as well as other Appell sequences; see, for instance, [24]). Then, $H(s,x)$ can be given, for $x > 0$ and $\operatorname{Re}(s) > 0$, in terms of the Hurwitz zeta function

$$\zeta(s,x) = \sum_{n=0}^\infty \frac{1}{(x+n)^s}, \tag{4.4}$$

so $H(s,x)$ is the analytic continuation (in the variable s) of $s\zeta(s+1,x)$ for ζ defined in (4.4) by means of a series that converges in a certain domain. Or, with the same meaning, we can say that the analytic continuation of the function $\zeta(s,x)$ defined in (4.4) is $H(s-1,x)/(s-1)$.

Let us finally note that, in the previous example related to the Bernoulli polynomials, $A(-t)$ was written, essentially, as a geometric series $\sum_{n=0}^\infty (e^{-t})^n$, and then $H(s,x)$ was computed as a series where there was a way to compute each summand. The analogous behavior for the Dunkl case is much more cumbersome. Not only is it not possible to express the integrals by means of well-known standard functions, but also the summands $1/(x+n)$ of the series become Dunkl translations instead of ordinary translations.

4.1. The Bernoulli-Dunkl case

To adapt Theorem 4.1 to the case of the Bernoulli-Dunkl polynomials defined in (2.7), let us first note that $\mathcal{I}_\alpha(t)$ is an even function, so the denominator in the left hand side of (2.7) can be written as

$$\mathcal{I}_{\alpha+1}(t) = \frac{\alpha+1}{t}(E_\alpha(t) - E_\alpha(-t)). \tag{4.5}$$

Then, concerning (4.1) for the Bernoulli-Dunkl case, we have the following:

Lemma 4.2. *For $\alpha > -1$ and $x \in (-1,1)$, the integral*

$$H(s,x) = \frac{1}{\Gamma(s)} \int_0^\infty \frac{E_\alpha(-xt)}{\mathcal{I}_{\alpha+1}(t)} t^{s-1}\, dt = \frac{1}{(\alpha+1)\Gamma(s)} \int_0^\infty \frac{E_\alpha(-xt)}{E_\alpha(t) - E_\alpha(-t)} t^s\, dt$$

converges in the right plane $\operatorname{Re}(s) > 0$ to a holomorphic function.

Proof. The convergence of the integral for t near 0 is clear, so let us analyze what happens when $t \to \infty$. By using (4.2), we have, for $|z| \to \infty$,

$$
\begin{aligned}
E_\alpha(-z) &= e^{-z}\,{}_1F_1\left(\frac{2\alpha+1}{2}, 2\alpha+2, 2z\right) \\
&= \frac{\Gamma(2\alpha+2)}{\Gamma(\frac{2\alpha+1}{2})} e^{z}(2z)^{-\alpha-3/2}\left(1+\mathcal{O}\Big(\frac{1}{|z|}\Big)\right) \\
&\qquad + \frac{\Gamma(2\alpha+2)}{\Gamma(\frac{2\alpha+3}{2})} e^{\pm(2\alpha+1)i\pi/2} e^{-z}(2z)^{-\alpha-1/2}\left(1+\mathcal{O}\Big(\frac{1}{|z|}\Big)\right).
\end{aligned}
$$

For simplicity, let us write it as

$$
E_\alpha(-z) = C_1 e^{z} z^{-\alpha-3/2}\left(1+\mathcal{O}\Big(\frac{1}{|z|}\Big)\right) + C_2^{\pm} e^{-z} z^{-\alpha-1/2}\left(1+\mathcal{O}\Big(\frac{1}{|z|}\Big)\right).
$$

When $\alpha = -1/2$ then $C_1 = 0$; but this case is well-known and we do not need to analyze it. Then, let us assume that $\alpha \neq -1/2$.

Now, let us suppose that $x > 0$. Then we have, for $t \to \infty$ (without lost of generality we can assume $|xt| > 1$),

$$
\Big|\frac{E_\alpha(-xt)}{E_\alpha(t) - E_\alpha(-t)} t^{s}\Big| = C_3 \frac{e^{xt}(xt)^{-\alpha-3/2} t^{\mathrm{Re}(s)}(1+\mathcal{O}(|xt|^{-1}))}{e^{t} t^{-\alpha-1/2}(1+\mathcal{O}(|t|^{-1}))}; \tag{4.6}
$$

this guarantees the convergence of the integral for $0 \le x < 1$. For $x < 0$ we have

$$
\Big|\frac{E_\alpha(-xt)}{E_\alpha(t) - E_\alpha(-t)} t^{s}\Big| = C_4 \frac{e^{|xt|}|xt|^{-\alpha-1/2} t^{\mathrm{Re}(s)}(1+\mathcal{O}(|xt|^{-1}))}{e^{t} t^{-\alpha-1/2}(1+\mathcal{O}(|t|^{-1}))}, \tag{4.7}
$$

and this guarantees the convergence of the integral for $-1 < x \le 0$.

By standard arguments on differentiation of parametric integrals, together with the above estimates, the function $H(s,x)$ is holomorphic on s. □

The above lemma proves the hypothesis of Theorem 4.1 for $x \in (-1,1)$. Then, we have the following:

Theorem 4.3. *Let $E_\alpha(xt)/\mathcal{I}_{\alpha+1}(t)$ be the generating function of Bernoulli-Dunkl polynomials. Then for $x \in (-1,1)$, the integral*

$$
H(s,x) = \frac{1}{\Gamma(s)} \int_0^\infty \frac{E_\alpha(-xt)}{\mathcal{I}_{\alpha+1}(-t)} t^{s-1}\,dt = \frac{1}{(\alpha+1)\Gamma(s)} \int_0^\infty \frac{E_\alpha(-xt)}{E_\alpha(t) - E_\alpha(-t)} t^{s}\,dt \tag{4.8}
$$

converges in the right plane $\mathrm{Re}(s) > 0$ *to a holomorphic function, which may be analytically continued to an entire function of s satisfying*

$$
H(-n,x) = \frac{n!}{\gamma_{n,\alpha}} \mathfrak{B}_{n,\alpha}(x), \quad n = 0,1,2,\dots
$$

The next step is to try to write $H(s,x)$, for $\mathrm{Re}(s) > 0$, as a kind of Hurwitz function similar to (4.4), as in the classical Bernoulli case.

In order to compute $H(s,x)$ we may write $A(t) = 1/\mathcal{I}_{\alpha+1}(t)$ as a geometric series. To do that, we use the fact that $\mathcal{I}_{\alpha+1}(t)$ is an even function, and we use the definition of $E_\alpha(t)$. By (4.5) we have that

$$
A(t) = \frac{1}{\mathcal{I}_{\alpha+1}(t)} = \frac{t}{\alpha+1}\frac{1}{E_\alpha(t)}\frac{1}{1-\frac{E_\alpha(-t)}{E_\alpha(t)}} = \frac{t}{\alpha+1}\frac{1}{E_\alpha(t)}\sum_{n=0}^{\infty}\left(\frac{E_\alpha(-t)}{E_\alpha(t)}\right)^{n}.
$$

This is valid for all $t \geq 0$ and it is enough for our purposes since we just need convergence for $t \in [0, \infty)$.

Finally, we have

$$\begin{aligned} H(s,x) &= \frac{1}{\Gamma(s)} \int_0^\infty \frac{E_\alpha(-xt)}{\mathcal{I}_{\alpha+1}(t)} t^{s-1}\, dt \\ &= \frac{1}{\Gamma(s)} \frac{1}{\alpha+1} \sum_{n=0}^\infty \int_0^\infty \frac{E_\alpha(-xt)}{E_\alpha(t)} \left(\frac{E_\alpha(-t)}{E_\alpha(t)}\right)^n t^s\, dt. \end{aligned} \tag{4.9}$$

Notice that, in a similar way to the proof of Lemma 4.2, we can easily check that all the integrals in (4.9) are convergent for $x \in (-1, 1)$, and the interchange of the sum and the integral is justified. However, (4.9) is more complicated than (4.3); the integrals cannot be written in a closed form and we don't obtain something as simple as (4.4).

In any case, we can define a kind of Hurwitz function related to the Bernoulli-Dunkl case in the following way (observe that, with the notation of (4.9) and (4.9), now we are changing s to $s-1$):

Definition 4.4. For $x \in (-1, 1)$ and $\operatorname{Re}(s) > 1$, we define the Hurwitz-Dunkl zeta function as

$$\zeta_\alpha(s,x) = \frac{1}{\Gamma(s)} \sum_{n=0}^\infty \int_0^\infty \frac{E_\alpha(-xt)}{E_\alpha(t)} \left(\frac{E_\alpha(-t)}{E_\alpha(t)}\right)^n t^{s-1}\, dt. \tag{4.10}$$

Also, we call

$$d_\alpha(s,x) = \frac{1}{\Gamma(s)} \int_0^\infty \frac{E_\alpha(-xt)}{E_\alpha(t)} t^{s-1}\, dt \tag{4.11}$$

the basic Hurwitz-Dunkl term.

Then we have, for $x \in (-1, 1)$ and $\operatorname{Re}(s) > 0$,

$$H(s,x) = \frac{s}{\alpha+1}\, \zeta_\alpha(s+1, x), \tag{4.12}$$

so we can say that the function $H(s, x)$ of Theorem 4.1 (which exists for $s \in \mathbb{C}$) is the analytic extension to the s-complex plane of the function $\frac{s}{\alpha+1}\, \zeta_\alpha(s + 1, x)$; equivalently, we can define the analytic extension of

$$\zeta_\alpha(s,x) = \frac{1}{\Gamma(s)} \int_0^\infty \frac{E_\alpha(-xt)}{E_\alpha(t) - E_\alpha(-t)} t^{s-1}\, dt, \quad \operatorname{Re}(s) > 1 \tag{4.13}$$

(which corresponds to (4.10)) as

$$\zeta_\alpha(s,x) = \frac{\alpha+1}{s-1} H(s-1, x), \quad s \in \mathbb{C}, \tag{4.14}$$

valid for $x \in (-1, 1)$.

Finally, let us see how it is possible to give an expression for $\zeta_\alpha(s, x)$ (valid in the half plane $\operatorname{Re}(s) > 1$) which, in some sense, is very similar to the series (4.4) for the classical Hurwitz zeta function $\zeta(s, x)$, where we have a series of summands translated by means of $x \mapsto x + n$. In the Dunkl case, we are going to find an expression for $\zeta_\alpha(s, x)$ that, in the place of classical translations, use the Dunkl transform defined in (3.1).

The following theorem provides an expression of the Hurwitz-Dunkl zeta function by using Dunkl translations. For simplicity, we have defined, here and in what follows, a "symmetric translation" σ_1 as

$$\sigma_1 = \tau_1 \tau_{-1}^{-1}$$

(or $\sigma_{1,x} = \tau_{1,x}\tau_{-1,x}^{-1}$ to clarify that it is applied to the variable x). Notice that the composition of translation operators is commutative, and also the composition with inverse translations (see its expression in (3.4)), so we can use $\sigma_1^n = \tau_1^n \tau_{-1}^{-n}$ without paying attention to the order of the operators.

Theorem 4.5. *For* $\operatorname{Re}(s) > 1$*, the Hurwitz-Dunkl zeta function can be written as*

$$\zeta_\alpha(s,x) = \frac{1}{\Gamma(s)} \sum_{n=0}^{\infty} \sigma_1^n \int_0^\infty \frac{E_\alpha(-xt)}{E_\alpha(t)} t^{s-1}\,dt = \sum_{n=0}^{\infty} \sigma_1^n d_\alpha(s,x). \tag{4.15}$$

Proof of Theorem 4.5. From Lemma 3.1, we have that

$$\tau_1^n \big(E_\alpha(-t\cdot)\big)(x) = E_\alpha(-xt)E_\alpha(-t)^n,$$
$$\tau_{-1}^{-(n+1)} \big(E_\alpha(-t\cdot)\big)(x) = E_\alpha(-xt)/E_\alpha(t)^{n+1}$$

hold. This can be easily proved as it was stated in Lemma 3.1 by changing $E_\alpha(t\cdot)$ for $E_\alpha(-t\cdot)$. So, from Lemma 3.2 we can conclude that

$$\begin{aligned}\tau_1^n \tau_{-1}^{-(n+1)} \big(E_\alpha(t\cdot)\big)(-x) = \tau_1^n \left(\frac{E_\alpha(-t\cdot)}{E_\alpha(t)^{n+1}}\right)(x) &= E_\alpha(-xt)\frac{E_\alpha(-t)^n}{E_\alpha(t)^{n+1}} \\ &= \frac{E_\alpha(-xt)}{E_\alpha(t)}\left(\frac{E_\alpha(-t)}{E_\alpha(t)}\right)^n,\end{aligned}$$

and Lemma 3.4 gives us that

$$\begin{aligned}H(s,x) &= \frac{1}{\Gamma(s)}\frac{1}{\alpha+1}\sum_{n=0}^{\infty}\int_0^\infty \tau_1^n\tau_{-1}^{-(n+1)}\big(E_\alpha(-t\cdot)\big)(x)t^s\,dt \\ &= \frac{1}{\Gamma(s)}\frac{1}{\alpha+1}\sum_{n=0}^{\infty}\sigma_1^n\int_0^\infty \tau_{-1}^{-1}\big(E_\alpha(-t\cdot)\big)(x)t^s\,dt \\ &= \frac{1}{\Gamma(s)}\frac{1}{\alpha+1}\sum_{n=0}^{\infty}\sigma_1^n\int_0^\infty \frac{E_\alpha(-xt)}{E_\alpha(t)}t^s\,dt,\end{aligned}$$

and the proof is concluded. □

Let us see that the role of $\zeta_\alpha(s,x)$ with the Mellin transform of Appell-Dunkl sequences is the same as the role of $\zeta(s,x)$ with the Mellin transform of Appell sequences. In fact, it generalizes the traditional Hurwitz zeta function. To see that, we observe that to transform Bernoulli polynomials into Bernoulli-Dunkl polynomials, we had to change $x \mapsto (x+1)/2$ and $t \mapsto 2t$. For that, we need to undo the change to recover the classical Hurwitz zeta function, that means, to take $\alpha = -1/2$, $x \mapsto 2x-1$ and $t \mapsto t/2$ (although many times we will not change t).

Now, let $\alpha = -1/2$. Then,

$$d_{-1/2}(s,x) = \frac{1}{\Gamma(s)} \int_0^\infty e^{-t(x+1)} t^{s-1}\, dt = \frac{1}{(x+1)^s}$$

if $x \in (-1,1)$. In this case, $\tau_1^n(f)(x) = f(x+n)$ and $\tau_{-1}^{-n}(f)(x) = f(x+n)$ and hence,

$$\zeta_{-1/2}(s,x) = \sum_{n=0}^\infty \frac{1}{(x+1+2n)^s}.$$

And finally, as we are considering Bernoulli-Dunkl polynomials, we need to change $x \mapsto 2x-1$. Hence,

$$\zeta_{-1/2}(s,2x-1) = \sum_{n=0}^\infty \frac{1}{(2x+1-1+2n)^s} = \frac{1}{2^s} \sum_{n=0}^\infty \frac{1}{(x+n)^s} = \frac{1}{2^s}\zeta(s,x).$$

Furthermore, (4.14) is an integral representation of $\zeta_\alpha(s,x)$ which, as expected, generalizes under these changes the classical integral representation of $\zeta(s,x)$:

$$\zeta(s,x) = \frac{1}{\Gamma(s)} \int_0^\infty \frac{e^{-xt}}{1-e^{-t}} t^{s-1}\, dt.$$

Next we summarize some of the properties of $\zeta_\alpha(s,x)$ that generalize the ones for $\zeta(s,x)$; we have already proved most of them in the preceding sections, or they are direct consequences:

Proposition 4.6 *(Properties of $\zeta_\alpha(s,x)$). For $\alpha > -1$ and $x \in (-1,1)$, the function $\zeta_\alpha(s,x)$ satisfies the following:*

(i) *Recurrence identities: let $\sigma_1 = \tau_1 \tau_{-1}^{-1}$; then for $\mathrm{Re}(s) > 0$,*

$$\zeta_\alpha(s,x) - \sigma_1(\zeta_\alpha(s,\cdot))(x) = d_\alpha(s,x), \tag{4.16}$$

$$\zeta_\alpha(s,x) - \sigma_1^m(\zeta_\alpha(s,\cdot))(x) = \sum_{n=0}^{m-1} \sigma_1^n d_\alpha(s,x). \tag{4.17}$$

(ii) *The Dunkl derivative of ζ_α:*

$$\Lambda_{\alpha,x}(\zeta_\alpha(s,x)) = -s\zeta_\alpha(s+1,x). \tag{4.18}$$

(iii) *Relation of ζ_α with Bernoulli-Dunkl polynomials: for $n = 0,1,2,\dots,$ we have*

$$\zeta_\alpha(-n,x) = -\mathfrak{B}_{n+1,\alpha}(x)\frac{(\alpha+1)n!}{\gamma_{n+1,\alpha}}. \tag{4.19}$$

Now we show that when $\alpha = -1/2$ and $x \mapsto 2x-1$ we get the corresponding properties of the classical $\zeta(s,x)$. First, for the recurrence identities, we have $\sigma_1^n(f(x)) = f(x+2n)$ so $\sigma_1^n(\zeta_\alpha(s,\cdot))(x) = \zeta_\alpha(s,x+2n)$ and hence, (4.16) transforms into

$$\zeta(s,x) = \zeta(s,x+1) + x^{-s}$$

and (4.17) transforms into

$$\zeta(s,x) = \zeta(s,x+m) + \sum_{n=0}^{m-1} (x+n)^{-s}$$

(see, for instance [28, 25.11.3 and 25.11.4]). Basically, the Dunkl translation σ_1 is playing the role of $x+1$ in the Hurwitz function.

In the case $\alpha = -1/2$, we have $\Lambda_{-1/2,x} = d/dx$, so (4.18) transforms into (see, for instance, [28, 25.11.17])

$$\frac{d}{dx}\zeta(s,x) = -s\zeta(s+1,x).$$

We also get the classical relation with Bernoulli polynomials since

$$\zeta_{-1/2}(-n, 2x-1) = -\mathfrak{B}_{n+1,-1/2}(2x-1)\frac{n!}{(n+1)!}\frac{1}{2} = -B_{n+1}(x)2^{n+1}\frac{1}{n+1}\frac{1}{2}.$$

Since $\zeta_{-1/2}(-n, 2x-1) = 2^n \zeta(-n,x)$, we get (see [28, 25.11.14])

$$\zeta(-n,x) = -\frac{B_{n+1}(x)}{n+1}. \tag{4.20}$$

4.2. The Euler-Dunkl case

This is similar to the Bernoulli-Dunkl case, but with $A(t) = 1/\mathcal{I}_\alpha(t)$.

Theorem 4.7. *Let $E_\alpha(xt)/\mathcal{I}_\alpha(t)$ be the generating function of Euler-Dunkl polynomials. Then for $x \in (-1,1)$, the integral*

$$H(s,x) = \frac{1}{\Gamma(s)}\int_0^\infty \frac{E_\alpha(-xt)}{\mathcal{I}_\alpha(-t)} t^{s-1}\,dt = \frac{2}{\Gamma(s)}\int_0^\infty \frac{E_\alpha(-xt)}{E_\alpha(t)+E_\alpha(-t)} t^{s-1}\,dt$$

converges in the right plane $\operatorname{Re}(s) > 0$ *to a holomorphic function, which may be analytically continued to an entire function of s satisfying*

$$H(-n,x) = \frac{n!}{\gamma_{n,\alpha}}\mathfrak{E}_{n,\alpha}(x), \quad n = 0,1,2,\dots$$

Proof. The statement that $H(s,x)$ is convergent for $-1 < x < 1$ holds by the same reasoning we made in the Bernoulli-Dunkl case. Hence, by Theorem 4.1, we have $H(-n,x) = n!\,\mathfrak{E}_{n,\alpha}(x)/\gamma_{n,\alpha}$, for $n = 0,1,2,\dots$ By the same argument as in the Bernoulli-Dunkl case, we can write $A(t)$ as a geometric series as

$$A(t) = \frac{1}{\mathcal{I}_\alpha(t)} = \frac{2}{E_\alpha(t)}\frac{1}{1+\frac{E_\alpha(-t)}{E_\alpha(t)}} = \frac{2}{E_\alpha(t)}\sum_{n=0}^\infty \left(\frac{-E_\alpha(-t)}{E_\alpha(t)}\right)^n.$$

The special function $H(s,x)$ is giving (when $-1 < x < 1$) by

$$H(s,x) = \frac{1}{\Gamma(s)}\int_0^\infty \frac{E_\alpha(-xt)}{\mathcal{I}_\alpha(t)} t^{s-1}\,dt = \frac{2}{\Gamma(s)}\sum_{n=0}^\infty \int_0^\infty \frac{E_\alpha(-xt)}{E_\alpha(t)}\left(\frac{-E_\alpha(-t)}{E_\alpha(t)}\right)^n t^{s-1}\,dt$$

$$= \frac{2}{\Gamma(s)} \sum_{n=0}^{\infty} (-1)^n \sigma_1^n \int_0^\infty \frac{E_\alpha(-xt)}{E_\alpha(t)} t^{s-1}\, dt,$$

which concludes the proof. □

Definition 4.8. For $x \in (-1,1)$ and $\mathrm{Re}(s) > 0$, we define the Hurwitz-Dunkl zeta function of Euler type as

$$\zeta_{E,\alpha}(s,x) = \frac{1}{\Gamma(s)} \sum_{n=0}^{\infty} (-1)^n \sigma_1^n \int_0^\infty \frac{E_\alpha(-xt)}{E_\alpha(t)} t^{s-1}\, dt. \tag{4.21}$$

Finally, notice that the function $H(s,x)$ may be extended to the entire complex s-plane and we have, for $x \in (-1,1)$ and $\mathrm{Re}(s) > 0$,

$$H(s,x) = 2\zeta_{E,\alpha}(s,x).$$

Hence, we can consider, equivalently, that the analytic extension of

$$\zeta_{E,\alpha}(s,x) = \frac{1}{\Gamma(s)} \int_0^\infty \frac{E_\alpha(-xt)}{E_\alpha(t) + E_\alpha(-t)} t^{s-1}\, dt, \quad \mathrm{Re}(s) > 0 \tag{4.22}$$

(which corresponds to (4.21)) is

$$\zeta_{E,\alpha}(s,x) = \frac{1}{2} H(s,x), \quad s \in \mathbb{C}. \tag{4.23}$$

Again, when $\alpha - 1/2$ and $x \mapsto 2x - 1$ (now by (2.10) we make the changes to recover Euler polynomials from Euler-Dunkl polynomials, as we did with the Hurwitz-Dunkl zeta function) we get the function

$$\zeta_{E,-1/2}(s, 2x-1) = \frac{1}{2^s} \sum_{n=0}^{\infty} \frac{(-1)^n}{(x+n)^s} = \frac{1}{2^s} \zeta_E(s,x),$$

where $\zeta_E(s,x)$ is called the Hurwitz-type Euler zeta function (see, for instance, [22]). Again, $\zeta_{\alpha,E}$ generalized many properties of $\zeta_{E,\alpha}$ by the changes $\alpha = -1/2$ and $x \mapsto 2x - 1$ (and sometimes also $t \mapsto t/2$). There is also the recurrence identity

$$\zeta_{E,\alpha}(s,x) + \sigma_1(\zeta_{E,\alpha}(s,\cdot))(x) = d_\alpha(s,x)$$

which generalizes [22, (2.1)] and (4.23) is an integral representation that generalizes [36, (3.1)]. Moreover, it is easy to prove that the relation of $\zeta_{E,\alpha}$ with Euler-Dunkl polynomials

$$\zeta_{E,\alpha}(-n,x) = \frac{1}{2} \mathfrak{E}_{n,\alpha}(x) \frac{n!}{\gamma_{n,\alpha}}$$

holds for all $x \in (-1,1)$ and $n = 0,1,2,\dots$, which give, when we recover the classical Hurwitz-type Euler zeta function, the identity (see [22, (2.7)])

$$\zeta_E(-n,x) = \frac{1}{2} E_n(x).$$

4.3. The generalized Bernoulli-Dunkl case

In the Bernoulli-Dunkl case we had

$$A(t) = \frac{1}{\mathcal{I}_{\alpha+1}(t)} = \frac{t}{\alpha+1}\frac{1}{E_\alpha(t)}\frac{1}{1-\frac{E_\alpha(-t)}{E_\alpha(t)}}.$$

For r a positive integer, the generalized Bernoulli-Dunkl polynomials are defined as $A(t)^r E_\alpha(xt) = \sum_{n=0}^\infty \mathfrak{B}_{n,\alpha}^{(r)}(x)t^n/\gamma_{n,\alpha}$, and we have

$$A(t)^r = \left(\frac{t}{\alpha+1}\frac{1}{E_\alpha(t)}\sum_{n=0}^\infty \left(\frac{E_\alpha(-t)}{E_\alpha(t)}\right)^n\right)^r.$$

Theorem 4.9. *Let $E_\alpha(xt)/(\mathcal{I}_{\alpha+1}(t))^r$ be the generating function of Bernoulli-Dunkl polynomials of order $r = 1, 2, \dots$ Then for each $x \in (-r, r)$ the integral*

$$H(s,x) = \frac{1}{\Gamma(s)}\int_0^\infty \frac{E_\alpha(-xt)}{(\mathcal{I}_{\alpha+1}(t))^r} t^{s-1}\,dt \tag{4.24}$$

converges in the right plane $\operatorname{Re}(s) > r$ to a holomorphic function, which may be analytically continued to an entire function of s satisfying

$$H(-n,x) = \frac{n!}{\gamma_{n,\alpha}}\mathfrak{B}_{n,\alpha}^{(r)}(x), \quad n = 0,1,2,\dots$$

The theorem can be easily proved by the same arguments as in Theorem 4.1 and in Subsection 4.1. The only thing left to prove is the convergence of $H(s,x)$ in $x \in (-r,r)$.

Proof of Theorem 4.9. Let us first analyze the convergence of the integral (4.24). We use the asymptotic behavior of the Kummer confluent hypergeometric function given in (4.2), and proceed as in the proof of Lemma 4.2. If $x > 0$, the "exponential part" of the integrand of $H(s,x)$ has size $e^{-t(x+r)}$, so the integral converges if $x < r$. Repeating the argument for $x < 0$, we get the convergence in $x \in (-r,r)$.

Let us use that, for $r = 1, 2, \dots$ and $|z| < 1$,

$$\left(\sum_{n=0}^\infty z^n\right)^r = \frac{1}{(1-z)^r} = \sum_{n=0}^\infty \binom{r+n-1}{n} z^n.$$

Then,

$$\begin{aligned} A(t)^r &= \left(\frac{t}{\alpha+1}\frac{1}{E_\alpha(t)}\sum_{n=0}^\infty\left(\frac{E_\alpha(-t)}{E_\alpha(t)}\right)^n\right)^r \\ &= \frac{t^r}{(\alpha+1)^r}\frac{1}{E_\alpha(t)^r}\sum_{n=0}^\infty \binom{r+n-1}{n}\left(\frac{E_\alpha(-t)}{E_\alpha(t)}\right)^n. \end{aligned}$$

Hence,

$$H(s,x) = \frac{1}{\Gamma(s)}\int_0^\infty \frac{E_\alpha(-xt)}{(\mathcal{I}_{\alpha+1}(t))^r} t^{s-1}\,dt$$

$$= \frac{1}{\Gamma(s)} \frac{1}{(\alpha+1)^r} \sum_{n=0}^{\infty} \binom{r+n-1}{n} \int_0^{\infty} \frac{E_\alpha(-xt)}{(E_\alpha(t))^r} \left(\frac{E_\alpha(-t)}{E_\alpha(t)}\right)^n t^{s+r-1}\, dt$$

$$= \frac{\Gamma(s+r)}{\Gamma(s)} \frac{1}{(\alpha+1)^r} \sum_{n=0}^{\infty} \sigma_1^n \left(\binom{r+n-1}{n} \int_0^{\infty} \frac{E_\alpha(-xt)}{(E_\alpha(t))^r} t^{s+r-1}\, dt \right),$$

which proves the theorem. □

Definition 4.10. For $\mathrm{Re}(s) > 1$ we define the Hurwitz-Dunkl zeta function of order $r = 1, 2, 3, \dots$ as

$$\zeta_\alpha^{(r)}(s,x) = \frac{1}{\Gamma(s)} \sum_{n=0}^{\infty} \binom{r+n-1}{n} \sigma_1^n \left(\int_0^{\infty} \frac{E_\alpha(-xt)}{(E_\alpha(t))^r} t^{s-1}\, dt \right).$$

As in (4.11) (that is, the case $r = 1$), we can define the basic term

$$d_\alpha^{(r)}(s,x) = \frac{1}{\Gamma(s)} \int_0^{\infty} \frac{E_\alpha(-xt)}{(E_\alpha(t))^r} t^{s-1}\, dt$$

and then write

$$\zeta_\alpha^{(r)}(s,x) = \sum_{n=0}^{\infty} \binom{r+n-1}{n} \sigma_1^n d_\alpha^{(r)}(s,x).$$

Notice that, as the function $H(s,x)$ is extended to the entire complex s-plane, and for $\mathrm{Re}(s) > 0$ we have

$$H(s,x) = \frac{(s)_r}{(\alpha+1)^r} \zeta_\alpha^{(r)}(s+r,x) \quad \text{for} \quad \mathrm{Re}(s) > 1-r.$$

Hence, we can define the extension of $\zeta_\alpha^{(r)}(s,x)$ to the complex plane by using $\zeta_\alpha^{(r)}(s+r,x) = (\alpha+1)^r H(s,x)/(s)_r$, i.e., by taking

$$\zeta_\alpha^{(r)}(s,x) = (\alpha+1)^r H(s-r,x)/(s-r)_r, \qquad -r < x < r, \quad s \in \mathbb{C},$$

which generalizes (4.14).

In the case $\alpha = -1/2$, this kind of zeta functions for the classical generalized Bernoulli polynomials has been studied in [8, §4.4]; see also [8, §4.1] for the classical generalized Euler polynomials.

4.4. The generalized Euler-Dunkl case

Again, as we did with the generalized Bernoulli-Dunkl case, by using the generation function of the generalized Euler-Dunkl polynomials we have $A(t) = 1/\mathcal{I}_\alpha(t)$ and

$$A(t)^r = \left(\frac{2}{E_\alpha(t)} \sum_{n=0}^{\infty} \left(-\frac{E_\alpha(-t)}{E_\alpha(t)} \right)^n \right)^r.$$

Theorem 4.11. *Let $E_\alpha(xt)/(\mathcal{I}_\alpha(t))^r$ be the generating function of Euler-Dunkl polynomials of order $r = 1, 2, \dots$ Then for each $x \in (-r, r)$ the integral*

$$H(s,x) = \frac{1}{\Gamma(s)} \int_0^\infty \frac{E_\alpha(-xt)}{(\mathcal{I}_\alpha(t))^r} t^{s-1}\,dt$$

converges in the right plane $\mathrm{Re}(s) > 0$ *to a holomorphic function which may be analytically continued to an entire function of* s *satisfying*

$$H(-n,x) = \frac{n!}{\gamma_{n,\alpha}} \mathfrak{E}_{n,\alpha}^{(r)}(x), \quad n = 0,1,2,\dots$$

Proof. We only need to notice that

$$A(t)^r = \frac{2^r}{E_\alpha(t)^r} \sum_{n=0}^\infty (-1)^n \binom{r+n-1}{n} \left(\frac{E_\alpha(-t)}{E_\alpha(t)}\right)^n,$$

and proceed as in Theorem 4.9. □

Definition 4.12. For $\mathrm{Re}(s) > 0$ we define the Hurwitz-Dunkl zeta function of Euler type and order $r \in \mathbb{N}$ as

$$\zeta_{E,\alpha}^{(r)}(s,x) = \sum_{n=0}^\infty (-1)^n \binom{r+n-1}{n} \sigma_1^n d_\alpha(s,x).$$

Finally, as the function $H(s,x)$ is extended to the entire complex s-plane, we have

$$H(s,x) = 2^r \zeta_{E,\alpha}^{(r)}(s,x)$$

for $\mathrm{Re}(s) > 0$. Hence, we can define the extension of $\zeta_{E,\alpha}^{(r)}(s,x)$ to the complex plane by

$$\zeta_{E,\alpha}^{(r)}(s,x) = H(s,x)/2^r, \qquad -r < x < r, \quad s \in \mathbb{C}.$$

4.5. The generalized Hermite case

The classical Hermite polynomials $H_n(x)$ are giving by the generating function e^{-t^2+2tx}, and they are orthogonal on the real line with respect to the weight e^{-x^2}. A well known generalization of these polynomials is the so-called generalized Hermite polynomials of order $\mu > -1/2$, which are orthogonal on the real line with respect to the weight $\omega_\mu(x) = |x|^{2\mu} e^{-x^2}$, that is, they are polynomials $\{H_n^\mu(x)\}_{n=0}^\infty$ satisfying

$$\int_{-\infty}^\infty H_m^\mu(x) H_n^\mu(x) \omega_\mu(x)\,dx = 0;$$

see, for instance, [2], [12, Chapters 1 and 5] or [33, p. 380, problem 25].

In [29], Rosenblum shows that these polynomials can be studied in the context of the Dunkl transform on the real line. This is done by means of

$$e^{-t^2} E_\mu(2xt) = \sum_{n=0}^\infty H_n^\mu(x) \frac{t^n}{n!} \tag{4.25}$$

with $\mu = \alpha + 1/2$. Except by a simple change of variable, this is an Appell-Dunkl sequence in the sense of (2.6).

For these polynomials, it is easy to find the analytic extension $H(s,x)$ such that, for n a negative integer, the corresponding value is $H_n^\mu(x)$, except for a multiplicative constant. Due to the factor e^{-t^2}, which appears in (4.25), the extension given in Theorem 4.1 does not present any problem and is valid for $x \in \mathbb{R}$. The same happens with the integral

$$\frac{1}{\Gamma(s)} \int_0^\infty e^{-t^2} E_\mu(-2xt)\, dt,$$

which is similar to the ones that appear in Theorems 4.3 or 4.7.

That leads us to the following result.

Theorem 4.13. *Let $G(-t,x) = e^{-t^2} E_\mu(-2xt)$, with $\mu > -1/2$. Then, for $x \in \mathbb{R}$,*

$$\begin{aligned} H(s,x) &= \frac{1}{\Gamma(s)} \int_0^\infty e^{-t^2} E_\mu(-2xt)\, dt \\ &= \frac{\sqrt{\pi}}{2^s \Gamma(\frac{s+1}{2})}\, {}_1F_1\Big(\frac{s}{2}, \mu + \frac{1}{2}, x^2\Big) - \frac{\sqrt{\pi}}{2^s \Gamma(\frac{s}{2})} \frac{x}{\mu + \frac{1}{2}}\, {}_1F_1\Big(\frac{s}{2}, \mu + \frac{3}{2}, x^2\Big) \end{aligned}$$

is an entire function of s and satisfies $H(-n,x) = H_n^\mu(x)$ for $n = 0, 1, 2, \dots$

In fact, we have $1/\Gamma(\frac{s}{2}) = 0$ when $s = -2n$, and $1/\Gamma(\frac{s+1}{2}) = 0$ when $s = -2n-1$ for $n = 0, 1, 2, \dots$ Furthermore, $\Gamma(-n + \frac{1}{2}) = \frac{(-1)^n\, 2^{2n} (2n)!}{n!} \sqrt{\pi}$, which means

$$\begin{aligned} H(-2n,x) = H_{2n}^\mu(x) &= (-1)^n \frac{(2n)!}{n!}\, {}_1F_1\Big(-n, \mu + \frac{1}{2}, x^2\Big), \\ H(-2n-1,x) = H_{2n+1}^\mu(x) &= (-1)^n \frac{(2n+1)!}{n!} \frac{x}{\mu + \frac{1}{2}}\, {}_1F_1\Big(-n, \mu + \frac{3}{2}, x^2\Big). \end{aligned}$$

This, as expected, is the same as [29, (2.1.1) and (2.2.1)].

5. Properties of the Hurwitz-Dunkl zeta functions

The aim of this section is, firstly, to provide generalization of the Riemann zeta function $\zeta(s) = \sum_{n=1}^\infty 1/n^s$ and the Euler-type zeta function $\zeta_E(s) = \sum_{n=1}^\infty (-1)^{n+1}/n^s$ (also known as Dirichlet eta function $\eta(s)$) in a Dunkl sense through the functions $\zeta_\alpha(s,x)$ and $\zeta_{E,\alpha}(s,x)$, respectively. We also provide a generalization in a Dunkl sense of the analytic continuation of $\zeta(s)$ (and $\zeta_E(s)$), as well as the so-called reflection formula, and other properties concerning our Hurwitz-Dunkl zeta functions $\zeta_\alpha(s,x)$, $\zeta_{E,\alpha}(s,x)$, $\zeta_\alpha(s)$ and $\zeta_{E,\alpha}(s)$. A connection appears here between these functions and the function $Z_\alpha(s) = \sum_{n=1}^\infty 1/j_n^s$ (and also with $Z_{\alpha+1}(s)$), where j_n are the positive zeros of the Bessel function $J_\alpha(x)$. We study $Z_\alpha(s)$ in Section 6.

In this section we will state the main results. The proofs are rather technical and require several lemmas. We will postpone them to Section 7.

5.1. Theorems for $\zeta_\alpha(s,x)$ and $\zeta_\alpha(s)$

In this section we are going to give another way of expressing the analytic continuation of $\zeta_\alpha(s,x)$ for $\mathrm{Re}(s) < 1$, and we will give some consequences that involve the zeros of $J_{\alpha+1}(x)$.

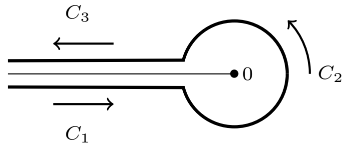

Fig. 1. The contour C from Theorem 5.1.

It is well known (see [35, Chapter 15] or [28, §10.21]) that, for any $\alpha > -1$, the zeros of the Bessel function $J_\alpha(x)/x^\alpha$ can be written as $j_{m,\alpha}$, $m \in \mathbb{Z} \setminus \{0\}$, with $j_{m,\alpha} = -j_{-m,\alpha}$ and $0 < j_{m,\alpha} < j_{m+1,\alpha}$, $m \geq 1$. Moreover, $j_{m,\alpha} \sim (m + \alpha/2 - 1/4)\pi + o(1/m)$ when $m \to \infty$ (see, for instance, [28, 10.21.19]).

Now, we are interested in the zeros of $J_{\alpha+1}(x)/x^{\alpha+1}$ so, to avoid confusion, we will denote $s_{m,\alpha} = j_{m,\alpha+1}$; in this way, we will often use s_m for $s_{m,\alpha}$. Again, with this notation we have $s_{m,\alpha} = -s_{-m,\alpha}$ and $0 < s_{m,\alpha} < s_{m+1,\alpha}$, $m \geq 1$, where $is_{m,\alpha}$, $m \in \mathbb{Z} \setminus \{0\}$, are the zeros of $\mathcal{I}_{\alpha+1}(x)$ (or the zeros of $J_{\alpha+1}(ix)/(ix)^{\alpha+1}$). For $\alpha = -1/2$ we have $s_{m,-1/2} = \pi m$. Let us also note that $\mathcal{I}_\alpha(is_{m,\alpha})$ provides a generalization of the sign sequence $(-1)^m$ because $\mathcal{I}_{-1/2}(is_{m,-1/2}) = (-1)^m$.

The first result is the following, which is similar to the classical case that can be found, for instance, in [4, §12.4, Theorem 12.3], and the proof follows the same scheme. However, we have now the functions $E_\alpha(t)$, which are much more complicated than e^t, and then the proof needs some additional details. In particular, we require the use of our Lemma 7.1. Actually, the zeros $s_{m,\alpha}$ do not explicitly appear in the statement of this theorem, but they will be crucial in the proof of the lemma.

Theorem 5.1. *Let $x \in (-1, 1)$ and define*

$$I(s,x) = \frac{1}{2\pi i} \int_C \frac{E_\alpha(xt)}{E_\alpha(-t) - E_\alpha(t)} \, t^{s-1} \, dt, \tag{5.1}$$

where C is the contour shown in Fig. 1. Then $I(s,x)$ is an entire function of s and satisfies

$$\zeta_\alpha(s,x) = \Gamma(1-s) I(s,x) \qquad \text{if } \operatorname{Re}(s) > 1, \tag{5.2}$$

where $\zeta_\alpha(s,x)$ is the function defined in (4.13).

Taking into account that (5.1) is valid in the entire s-plane, and that $\zeta_\alpha(s,x)$ satisfies (5.2) for $\operatorname{Re}(s) > 1$, we can define the following analytic continuation for $\zeta_\alpha(s,x)$ in the entire s-plane:

$$\zeta_\alpha(s,x) = \Gamma(1-s) I(s,x), \tag{5.3}$$

valid for $-1 < x < 1$. Of course, the analytic continuation of a function is unique, so this function $\zeta_\alpha(s,x)$ is the same that we defined in (4.14). From this and by Cauchy's residue theorem it is also possible to prove Theorems 5.2 and 5.3.

Theorem 5.2. *The function $\zeta_\alpha(s,x)$ defined in (5.3) is analytic for $s \in \mathbb{C}$ except for a simple pole at $s = 1$ with residue $\alpha + 1$.*

Taking $\alpha = -1/2$ and $x \mapsto 2x - 1$, since $\zeta_{-1/2}(s, 2x-1) = \zeta(s,x)/2^s$ we get that $\zeta(s,x)$ has a simple pole at $s = 1$ with residue 1, which is what happens in the classical case (see [4, §12.5, Theorem 12.4]).

The next result was already proved in Proposition 4.6, but later we provide another way to show it, this time starting from (5.3) and using Cauchy's residues theorem:

Theorem 5.3. *The function $\zeta_\alpha(s,x)$ defined in (5.3) satisfies, for $x \in (-1,1)$,*

$$\zeta_\alpha(-n,x) = -\mathfrak{B}_{n+1}(x)\frac{n!\,(\alpha+1)}{\gamma_{n+1,\alpha}}, \qquad n = 0,1,2,\dots \tag{5.4}$$

Another classical result in analytic number theory is the so-called Hurwitz formula (see [4, §12.7, Theorem 12.6] or [5, 25.13.3]), namely

$$\zeta(1-s,x) = \frac{\Gamma(s)}{(2\pi)^s}\left(e^{-\pi si/2}F(x,s) + e^{\pi si/2}F(-x,s)\right), \tag{5.5}$$

where

$$F(x,s) = \sum_{n=1}^{\infty}\frac{e^{2\pi inx}}{n^s}, \quad \operatorname{Re}(s) > 1, \tag{5.6}$$

is known as the Lerch (or periodic) zeta function (see [4, §12.7, equation (9), p. 257] or [5, 25.13.1]).

In the Dunkl context, this formula can be generalized as follows. The convergence of the series (5.6) is clear, but to prove the convergence of the corresponding series $\mathcal{F}(x,s)$, which we will use in the Dunkl context, will require some effort.

Theorem 5.4 *(Hurwitz-Dunkl formula). Let $\alpha > -1$ and $\{s_m\}_{m=1}^\infty$ be the positive zeros of $J_{\alpha+1}$. For $\operatorname{Re}(s) > 1$, the function*

$$\mathcal{F}(x,s) = \sum_{m=1}^{\infty}\frac{E_\alpha(xis_m)}{\mathcal{I}_\alpha(is_m)}\frac{1}{s_m^s} \tag{5.7}$$

converges for every $x \in \mathbb{R}$. Moreover, for $x \in (-1,1)$ and $\operatorname{Re}(s) > 1$, the Hurwitz-Dunkl zeta function $\zeta_\alpha(s,x)$ satisfies

$$\zeta_\alpha(1-s,x) = \frac{\Gamma(s)}{2}\left(e^{-\pi si/2}\mathcal{F}(x,s) + e^{\pi si/2}\mathcal{F}(-x,s)\right). \tag{5.8}$$

We call $\mathcal{F}(x,s)$ the Lerch-Dunkl zeta function since it plays a similar role as the Lerch zeta function $F(x,s)$ (but $\mathcal{F}(x,s)$ is not periodic). In fact, when $\alpha = -1/2$ and $x \mapsto 2x-1$, we have $\mathcal{F}(2x-1,s) = \pi^{-s}F(x,s)$, so (5.8) becomes (5.5).

Now, although the identity (5.8) is valid only for $x \in (-1,1)$, the right hand side is valid for $x \in \mathbb{R}$, so we can extend the definition of $\zeta_\alpha(1-s,x)$ for $\operatorname{Re}(s) > 1$ by taking

$$\zeta_\alpha(1-s,x) = \frac{\Gamma(s)}{2}\left(e^{-\pi si/2}\mathcal{F}(x,s) + e^{\pi si/2}\mathcal{F}(-x,s)\right), \quad x \in \mathbb{R}. \tag{5.9}$$

Replacing $1-s$ by s, we can also define, for $\operatorname{Re}(s) < 0$,

$$\zeta_\alpha(s,x) = \frac{\Gamma(1-s)}{2}\left(-ie^{\pi si/2}\mathcal{F}(x,1-s) + ie^{-\pi si/2}\mathcal{F}(-x,1-s)\right), \quad x \in \mathbb{R}.$$

The Hurwitz-Dunkl formula gives us an expression for $\zeta_\alpha(s,x)$ and $x \in \mathbb{R}$ free of the intricate integrals. With that, we can easily prove a “reflection formula” (but in this case that isn't a suitable name) for $\zeta_\alpha(s)$ in a Dunkl sense that can be seen as a generalization of the reflection formula for $\zeta(s)$.

Using the notation $\zeta(s) = \zeta(s,1)$, the reflection formulas of the classical zeta function (also known as “Riemann's functional equation”)

$$\zeta(1-s) = 2(2\pi)^{-s}\Gamma(s)\cos\left(\frac{\pi s}{2}\right)\zeta(s), \qquad s \in \mathbb{C},$$

$$\zeta(s) = 2(2\pi)^{s-1}\Gamma(1-s)\sin\left(\frac{\pi s}{2}\right)\zeta(1-s), \qquad s \in \mathbb{C},$$

can be proved by taking $x = 1$ in the Hurwitz formula (5.5) (see, for instance, [4, §12.8, Theorem 12.7]); for the first formula, the result is clear for $\mathrm{Re}(s) > 1$, and is then valid for $s \in \mathbb{C}$ by analytic continuation. Actually, many properties of $\zeta(s)$ and $\zeta(s, x)$ can be seen as consequences of (5.5).

In our case, taking $x = \pm 1$ in (5.7), we get $\mathcal{F}(\pm 1, s) = \sum_{m=1}^{\infty} 1/s_m^s$, since $E_\alpha(\pm i s_m) = \mathcal{I}_\alpha(i s_m)$. Thus, we can define, for $\mathrm{Re}(s) > 1$,

$$\zeta_\alpha(1-s) = \zeta_\alpha(1-s, 1), \tag{5.10}$$

where $\zeta_\alpha(1-s, 1)$ is given in (5.9); of course, the same can be done for $\zeta_\alpha(s)$ with $\mathrm{Re}(s) < 0$ (with this notation, $\zeta_{-1/2}(s) = \zeta_{-1/2}(s, 1) = \zeta(s, 1)/2^s = \zeta(s)/2^s$). Then, we have the following:

Theorem 5.5. *Let $\alpha > -1$ and $\{s_m\}_{m=1}^{\infty}$ be the positive zeros of $J_{\alpha+1}$. For $\mathrm{Re}(s) > 1$ we have*

$$\zeta_\alpha(1-s) = \Gamma(s)\cos\left(\frac{\pi s}{2}\right)\sum_{m=1}^{\infty}\frac{1}{s_m^s}, \tag{5.11}$$

or equivalently, for $\mathrm{Re}(s) < 0$,

$$\zeta_\alpha(s) = \Gamma(1-s)\sin\left(\frac{\pi s}{2}\right)\sum_{m=1}^{\infty}\frac{1}{s_m^{1-s}}. \tag{5.12}$$

When $\alpha = -1/2$, we get $\sum_{m=1}^{\infty} 1/s_m^s = \pi^{-s}\sum_{m=1}^{\infty} 1/m^s = \pi^{-s}\zeta(s)$ so, in fact, $\sum_{m=1}^{\infty} 1/s_m^s$ is playing the role of $\zeta(s)$. Hence, when $\alpha = -1/2$, Theorem 5.5 provides a generalization in a Dunkl sense of classical reflection formulas. However, there is an important difference if we compare Theorem 5.5 with the classical case: the sums $\sum_{m=1}^{\infty}$ in (5.11) and (5.12) are not the functions $\zeta_\alpha(1-s)$ and $\zeta_\alpha(s)$, respectively.

Finally, by taking $s = n + 1$ in (5.9) we have

$$\zeta_\alpha(-n, x) = \frac{n!}{2}\left(e^{-\pi(n+1)i/2}\mathcal{F}(x, n+1) + e^{\pi(n+1)i/2}\mathcal{F}(-x, n+1)\right), \quad x \in \mathbb{R};$$

and, on the other hand, by Theorem 5.3,

$$\zeta_\alpha(-n, x) = -\mathfrak{B}_{n+1}(x)\frac{n!\,(\alpha+1)}{\gamma_{n+1,\alpha}}, \quad x \in (-1, 1).$$

Then, for $x \in (-1, 1)$,

$$-\mathfrak{B}_{n+1}(x)\frac{(\alpha+1)}{\gamma_{n+1,\alpha}} = \frac{1}{2}\left(e^{-\pi(n+1)i/2}\mathcal{F}(x, n+1) + e^{\pi(n+1)i/2}\mathcal{F}(-x, n+1)\right).$$

Since the above function is a polynomial, the limit as $x \to 1^-$ exists and we have

$$-\mathfrak{B}_{n+1}(1)\frac{(\alpha+1)}{\gamma_{n+1,\alpha}} = \cos\left(\frac{\pi(n+1)}{2}\right)\sum_{m=1}^{\infty}\frac{1}{s_m^{n+1}}.$$

Letting $n = 2k - 1$, we have the following:

Corollary 5.6. *Let $\alpha > -1$ and $\{s_m\}_{m=1}^{\infty}$ be the positive zeros of $J_{\alpha+1}$. Then,*

$$\sum_{m=1}^{\infty} \frac{1}{s_m^{2k}} = \frac{\mathfrak{B}_{2k}(1)(-1)^{k+1}}{2^{2k} k!\,(\alpha+2)_{k-1}}, \quad k = 1, 2, 3, \dots$$

The previous expression for $\sum_{m=1}^{\infty} 1/s_m^{2k}$ in terms of $\mathfrak{B}_{2k}(1)$ was proved in [13, Theorem 4.1] by other methods.

Corollary 5.7. *The function $\zeta_\alpha(s)$ defined in Theorem 5.5 satisfies*

$$\zeta_\alpha(-n) = -\mathfrak{B}_{n+1}(1) \frac{n!\,(\alpha+1)}{\gamma_{n+1,\alpha}}, \quad n = 1, 2, \dots \tag{5.13}$$

Proof. Taking $s = n = 2k$, $k = 1, 2, \dots$, in (5.11) and using Corollary 5.6 we get (5.13) for n odd. Taking $s = n = 2k - 1$, $k = 1, 2, \dots$ in (5.11) we then get $\cos(\pi s/2) = 0$ and hence $\zeta_\alpha(1-n)$. Since $-\mathfrak{B}_{n+1}(1) = 0$ for n even, this completes the proof. $\square$

5.2. Theorems for $\zeta_{E,\alpha}(s,x)$ and $\zeta_{E,\alpha}(s)$

Here, we are going state some results for $\zeta_E(s,x)$, that will be similar to the results for $\zeta_E(s,x)$ in Subsection 5.1. Let us recall that $j_m = j_{m,\alpha}$, $m \in \mathbb{Z} \setminus \{0\}$, are the zeros of the Bessel function $J_\alpha(x)/x^\alpha$, and that they can be ordered so that $j_{m,\alpha} = -j_{-m,\alpha}$ and $0 < j_{m,\alpha} < j_{m+1,\alpha}$, $m \geq 1$. Moreover, $ij_{m,\alpha}$, $m \in \mathbb{Z} \setminus \{0\}$, are the zeros of $\mathcal{I}_\alpha(x)$ and for $\alpha = -1/2$, $j_{m,-1/2}$ are the zeros of $\mathcal{I}_{-1/2}(it)$, namely the zeros of the cosine. Hence, $j_{m,-1/2} = (m - 1/2)\pi$ for $m \geq 1$.

We begin with a result that is similar to Theorem 5.1:

Theorem 5.8. *Let $x \in (-1, 1)$ and*

$$I_E(s,x) = \frac{1}{2\pi i} \int_C h(t) t^{s-1}\,dt = \frac{1}{2\pi i} \int_C \frac{E_\alpha(xt)}{E_\alpha(-t) + E_\alpha(t)}\, t^{s-1}\,dt,$$

where C is again the contour shown in Fig. 1 of Theorem 5.1. Then $I_E(s,x)$ is an entire function of s and satisfies

$$\zeta_{E,\alpha}(s,x) = \Gamma(1-s) I_E(s,x) \quad \text{if } \operatorname{Re}(s) > 0,$$

where $\zeta_{E,\alpha}(s,x)$ is the function defined in (4.22).

Of course, this theorem again allows us to give the analytic extension for $\zeta_{E,\alpha}(s,x)$ to the entire s-plane, valid for $-1 < x < 1$.

Now, the "Hurwitz-Dunkl formula of Euler type" is the following:

Theorem 5.9. *Let $\alpha > -1$ and $\{j_m\}_{m=1}^{\infty}$ be the positive zeros of J_α. For $\operatorname{Re}(s) > 1$, the function*

$$\mathcal{F}_E(x,s) = \sum_{m=1}^{\infty} \frac{E_\alpha(ij_m x)}{\mathcal{I}_{\alpha+1}(ij_m)} \frac{1}{j_m^{s+1}}$$

converges for every $x \in \mathbb{R}$. Moreover, for $x \in (-1, 1)$ and $\operatorname{Re}(s) > 1$ we have

$$\zeta_{E,\alpha}(1-s,x) = -(\alpha+1)\Gamma(s) \left(e^{-\frac{\pi i}{2}(s+1)} \mathcal{F}_E(x,s) + e^{\frac{\pi i}{2}(s+1)} \mathcal{F}_E(-x,s) \right). \tag{5.14}$$

In the particular case $\alpha = -1/2$, we get $j_m = (m - 1/2)\pi$ for $m \geq 1$. Also, $J_{1/2}(x) = \sqrt{2/(\pi x)}\sin(x)$, which leads to $\mathcal{I}_{1/2}(ij_m) = (-1)^{m+1}/j_m$. That means, when taking $x \mapsto 2x - 1$, we have that

$$\mathcal{F}_E(2x-1,s) = \frac{2^s}{\pi^s} i \sum_{m=1}^{\infty} \frac{e^{(2m-1)i\pi x}}{(2m-1)^s} = \frac{2^s}{\pi^s} i\ell_E(s,x),$$

where the notation $\ell_E(s,x)$ for the above series has already been used in [22, (7.1)]. Furthermore, with these changes, and noticing that $\mathcal{F}_E(-x,s) = -\mathcal{F}_E(1-x,s)$, (5.14) transforms into

$$\frac{1}{2^{1-s}}\zeta_E(1-s,x) = \zeta_{E,-1/2}(1-s,2x-1) = \frac{2^s\Gamma(s)}{\pi^s}\left(e^{-\frac{i\pi s}{2}}\ell_E(s,x) - e^{\frac{i\pi s}{2}}\ell_E(s,1-x)\right),$$

which is just [22, (7.2)].

As in the case of ζ_α, we can use (5.14) to define $\zeta_\alpha(1-s,x)$ for $\operatorname{Re}(s) > 1$ and $x \in \mathbb{R}$, as well as $\zeta_\alpha(s,x)$ for $\operatorname{Re}(s) < 0$ and $x \in \mathbb{R}$. In particular, taking $x = 1$ and defining $\zeta_{E,\alpha}(s) = \zeta_{E,\alpha}(s,1)$, we have the following:

Theorem 5.10. *Let $\alpha > -1$ and $\{j_m\}_{m=1}^{\infty}$ be the positive zeros of J_α. For $\operatorname{Re}(s) > 1$ we have*

$$\zeta_{E,\alpha}(1-s) = -\Gamma(s)\cos\left(\frac{\pi s}{2}\right)\sum_{m=1}^{\infty}\frac{1}{j_m^s} \tag{5.15}$$

or equivalently, for $\operatorname{Re}(s) < 0$,

$$\zeta_{E,\alpha}(s) = -\Gamma(1-s)\sin\left(\frac{\pi s}{2}\right)\sum_{m=1}^{\infty}\frac{1}{j_m^{1-s}}.$$

When $\alpha = -1/2$, (5.15) transforms into (see [22, (7.4)])

$$\zeta_E(1-s) = -2\pi^{-s}\Gamma(s)\cos\left(\frac{\pi s}{2}\right)\sum_{m=1}^{\infty}\frac{1}{(2m-1)^s}.$$

Finally, the equivalent result of Corollary 5.6 for $\sum_{m=1}^{\infty} 1/j_m^{2k}$ is the following:

Corollary 5.11. *Let $\alpha > -1$ and $\{j_m\}_{m=1}^{\infty}$ be the positive zeros of J_α. Then,*

$$\sum_{m=1}^{\infty}\frac{1}{j_m^{2k}} = \frac{\mathfrak{E}_{2k-1}(1)(-1)^{k+1}}{2^{2k}(k-1)!\,(\alpha+1)_k}, \qquad k = 1,2,3,\dots$$

This can be easily proved from (5.15), as in Subsection 5.1. Note that this result was also proved in [18] in a different way.

Corollary 5.12. *The function $\zeta_{E,\alpha}(s)$ defined in Theorem 5.10 satisfies*

$$\zeta_{E,\alpha}(-n) = \frac{1}{2}\mathfrak{E}_n(1)\frac{n!}{\gamma_{n,\alpha}}, \quad n = 1,2,\dots$$

6. Analytic continuation of $\zeta_\alpha(s)$ and $\zeta_{E,\alpha}(s)$

Finally, let us define, for $\alpha > -1$,

$$Z_\alpha(s) = \sum_{m=1}^{\infty} \frac{1}{j_m^s}, \qquad \mathrm{Re}(s) > 1. \tag{6.1}$$

Of course, in a like manner we get

$$Z_{\alpha+1}(s) = \sum_{m=1}^{\infty} \frac{1}{s_m^s}, \qquad \mathrm{Re}(s) > 1,$$

hence, $Z_\alpha(s)$ is related with $\zeta_{E,\alpha}(s)$ and $Z_{\alpha+1}(s)$ with $\zeta_\alpha(s)$. This function is similar to the classical Riemann zeta function $\sum_{m=1}^\infty 1/m^s$ where the positive zeros $\{\pi m\}_{m=1}^\infty$ of the sine have been changed by the zeros of the positive zeros of a Bessel function. Then, we will call $Z_{\alpha+1}(s)$ the "Riemann-Bessel zeta function".

In his thesis [21], Hawkins provides an analytic continuation of $Z_\alpha(s)$. To do so, he first gets easily the analytic continuation for $\mathrm{Re}(s) > 0$ by integration by parts, and repeating the process he is able to continue the function to $\mathrm{Re}(s) > -1$. However, he does not go forward by this method and, instead, uses other tools. He ends up proving that there exists an analytic continuation of $Z_\alpha(s)$ to the entire s-plane with simple poles at $s = 1, -1, -3, -5, \dots$ but he didn't get an explicit formula. Due to its simplicity, we now show how to continue $Z_\alpha(s)$ to the region $\mathrm{Re}(s) > 0$.

Theorem 6.1. *The function* $Z_\alpha(s) - \frac{\pi^{-s}}{s-1}$ *extends analytically to the region* $\mathrm{Re}(s) > 0$.

Proof. We start from (6.1), valid for $\mathrm{Re}(s) > 1$, and decompose

$$\begin{aligned} Z_\alpha(s) - \frac{\pi^{-s}}{s-1} &= \sum_{m=1}^{\infty} \frac{1}{j_m^s} - \int_1^\infty \frac{\pi^{-s}\,dx}{x^s} = \sum_{m=1}^{\infty} \frac{1}{j_m^s} - \sum_{m=1}^{\infty} \int_m^{m+1} \frac{dx}{(\pi x)^s} \\ &= \sum_{m=1}^{\infty} \left(\frac{1}{j_m^s} - \int_m^{m+1} \frac{dx}{(\pi x)^s} \right) = \sum_{m=1}^{\infty} \int_m^{m+1} \left(\frac{1}{j_m^s} - \frac{1}{(\pi x)^s} \right) dx, \end{aligned}$$

again valid for $\mathrm{Re}(s) > 1$. Now, let us denote

$$f_m(s) = \int_m^{m+1} \left(\frac{1}{j_m^s} - \frac{1}{(\pi x)^s} \right) dx.$$

If we prove that $\sum_{m=1}^\infty f_m(s)$ is analytic in $\mathrm{Re}(s) > 0$, we will have the analytic extension of $Z_\alpha(s) - \pi^{-s}/(s-1)$.

Every function $f_m(s)$ is analytic in $\mathrm{Re}(s) > 0$, so it is enough to see that the series converges uniformly on compacts in that region. Now, let us recall that the zeros $\{j_m\}_{m=1}^\infty$ of $J_\alpha(t)$ satisfy $j_m \sim (m + \alpha/2 - 1/4)\pi + o(1/m)$ (see [28, 10.21.19]), so $\pi m - c \le j_m \le \pi m + c$ for a positive constant c independent of m. Then, because

$$\int_{j_m}^{\pi x} u^{-s-1}\,du = \frac{1}{s} \left(\frac{1}{j_m^s} - \frac{1}{(\pi x)^s} \right),$$

we have

$$\begin{aligned}\left|\int_m^{m+1}\left(\frac{1}{j_m^s}-\frac{1}{(\pi x)^s}\right)dx\right| &= \left|s\int_m^{m+1}\int_{j_m}^{\pi x}\frac{du}{u^{s+1}}\,dx\right| \\ &\le |s|\int_m^{m+1}\int_{\pi m-c_1}^{\pi m+c_2}\frac{du}{|u^{s+1}|}\,dx = |s|\int_m^{m+1}\int_{\pi m-c_1}^{\pi m+c_2}\frac{du}{u^{1+\operatorname{Re}(s)}}\,dx \\ &\le \frac{|s|}{(s_m-c_2-c_1)^{1+\operatorname{Re}(s)}}\int_m^{m+1}\int_{\pi m-c_1}^{\pi m+c_2}du\,dx = \frac{(c_2+c_1)|s|}{(j_m-c_2-c_1)^{1+\operatorname{Re}(s)}}.\end{aligned}$$

Consequently,

$$\sum_{m=1}^{\infty}\left|\int_m^{m+1}\left(\frac{1}{j_m^s}-\frac{1}{(\pi x)^s}\right)dx\right| \le \sum_{m=1}^{\infty}\frac{C|s|}{j_m^{1+\operatorname{Re}(s)}} < \infty \quad \text{for } \operatorname{Re}(s)>0,$$

and the Weierstrass M-test ensures the uniform convergence on compacts in $\operatorname{Re}(s) > 0$. □

Using the analytic continuation in the entire s-plane [21], some of the above identities can also be analytically continued to the entire s-plane; see also [32]. Since Hawkins didn't get any explicit formula for $Z_\alpha(s)$, we won't get an explicit formula for $\zeta_\alpha(s)$ either. Furthermore, in this way we do not obtain $\zeta_\alpha(s)$ as our $\zeta_\alpha(s,1)$, because $\zeta_\alpha(s,x)$ does not exist for $x = 1$ in the half-plane $\operatorname{Re}(s) \ge 1$ (see Definition 4.4). In the same way, contrary to what happen in the classical case, we do not have $\zeta_\alpha(s,1) = Z_{\alpha+1}(s)$ for $s \ge 1$.

Hawkins also computed the residues of $Z_\alpha(s)$ at $s = 1,-1,-3,\dots$ and the values of $Z_\alpha(-2k)$ for $k = 0,1,2,\dots$ They are given by (see [21, Theorem 3.5])

$$\operatorname{Res}_{s=-2k-1}(Z_\alpha(s)) = \frac{(-1)^{k+1}}{\pi}c_{2k}, \qquad Z_\alpha(-2k) = \frac{(-1)^k}{2}c_{2k-1},$$

where $c_k := c_{k,\alpha}$ are given by the identity

$$\left(\sum_{k=0}^{\infty}\frac{(-1)^k}{(2x)^k}(\alpha,k)\right)\left(\sum_{k=0}^{\infty}\frac{c_k}{x^{k+2}}\right) = \sum_{k=0}^{\infty}\frac{(-1)^k}{(2x)^{k+1}}(\alpha,k), \tag{6.2}$$

with $(\alpha,k) = \Gamma(\alpha+k+1/2)/(k!\,\Gamma(\alpha-k+1/2))$ (see [21, Lemma 3.4]). To extend $\zeta_\alpha(s)$, we need to change $\alpha \mapsto \alpha+1$ in order to correspond the coefficients c_k with our $Z_{\alpha+1}(s)$. Finally, Hawkins proved that c_k are polynomials of α which vanish at $\alpha = -1/2$ and $\alpha = 1/2$ (see, for instance, [21, Proposition 4.2]).

With all this, we can now study the poles of $\zeta_\alpha(s)$ with $s \in \mathbb{C}$ and $-1/2 \ne \alpha > -1$.

Theorem 6.2. *We get*

$$\zeta_\alpha(1-s) = \Gamma(s)\cos\left(\frac{\pi s}{2}\right)Z_{\alpha+1}(s), \qquad s \in \mathbb{C}, \tag{6.3}$$

or equivalently

$$\zeta_\alpha(s) = \Gamma(1-s)\sin\left(\frac{\pi s}{2}\right)Z_{\alpha+1}(1-s), \qquad s \in \mathbb{C}. \tag{6.4}$$

In particular, $\zeta_\alpha(s)$ can be analytically continued to the entire s-plane with simple poles at $s = n = 1, 2, 3, \dots$ (for $\alpha \neq -1/2$) whose residues are equal to $d_{n-1}/(2n!)$, where $d_n := d_{n,\alpha} = c_{n,\alpha+1}$ with the notation of (6.2). Moreover, $\zeta_\alpha(0) = -1/2$.

Proof. We can analytically continue equations (5.11) and (5.12) by considering the analytic continuation of $Z_{\alpha+1}(s)$. From the equation (6.3) the only possible poles are the ones of $\Gamma(s)$, at $s = 0, -1, -2, \dots$, and the ones of $Z_{\alpha+1}(s)$, at $s = 1, -1, -3, \dots$ It is straightforward to see that when $s = -2k = 0, -2, -4, \dots$ we get $\cos(-\pi k)Z_{\alpha+1}(-2k) \neq 0$ and $\Gamma(-2k)$ has a pole. Hence, at those values there are simple poles. The residue at $s = -2k$ is

$$\lim_{s\to-2k}(s+2k)\Gamma(s)\cos\left(\frac{\pi s}{2}\right)Z_{\alpha+1}(s) = \frac{(-1)^k}{(2k)!}Z_{\alpha+1}(-2k) = \frac{d_{2k-1}}{2(2k)!}.$$

When $s = -(2k+1) = -1, -3, -5, \dots$ we prove that $\zeta_\alpha(1-s)$ has a pole at those values by a little trick and the L'Hôpital rule as follows:

$$\begin{aligned}
&\lim_{s\to-2k-1}\Gamma(s)\cos\left(\frac{\pi s}{2}\right)Z_{\alpha+1}(s)\\
&= \lim_{s\to-2k-1}(s+2k+1)\Gamma(s)(s+2k+1)Z_{\alpha+1}(s)\frac{\cos\left(\frac{\pi s}{2}\right)}{(s+2k+1)^2}\\
&= \operatorname{Res}_{s=-2k-1}(\Gamma(s))\operatorname{Res}_{s=-2k-1}(Z_{\alpha+1}(s))\lim_{s\to-2k-1}\frac{-\pi}{4}\frac{\sin\left(\frac{\pi s}{2}\right)}{s+2k+1}.
\end{aligned}$$

Hence, there is a pole at those values. To calculate its residues we compute

$$\begin{aligned}
&\lim_{s\to-2k-1}(s+2k+1)\Gamma(s)\cos\left(\frac{\pi s}{2}\right)Z_{\alpha+1}(s)\\
&= \lim_{s\to-2k-1}(s+2k+1)\Gamma(s)(s+2k+1)Z_{\alpha+1}(s)\frac{\cos\left(\frac{\pi s}{2}\right)}{s+2k+1}\\
&= \operatorname{Res}_{s=-2k-1}(\Gamma(s))\operatorname{Res}_{s=-2k-1}(Z_{\alpha+1}(s))\lim_{s\to-2k-1}\frac{-\pi}{2}\sin\left(\frac{\pi s}{2}\right)\\
&= \frac{(-1)^{k+1}}{(2k+1)!}\frac{\pi}{2}\operatorname{Res}_{s=-2k-1}(Z_{\alpha+1}(s)) = \frac{d_{2k}}{2(2k+1)!}.
\end{aligned}$$

Finally, we consider the case $s = 1$. We use $\operatorname{Res}_{s=1}(Z_{\alpha+1}(s)) = 1/\pi$. So,

$$\begin{aligned}
\lim_{s\to1}\Gamma(s)\cos\left(\frac{\pi s}{2}\right)Z_{\alpha+1}(s) &= \lim_{s\to1}(s-1)Z_{\alpha+1}(s)\Gamma(s)\frac{\cos\left(\frac{\pi s}{2}\right)}{s-1}\\
&= \operatorname{Res}_{s=1}(Z_{\alpha+1}(s))\lim_{s\to1}\frac{\cos\left(\frac{\pi s}{2}\right)}{s-1} = -1/2. \qquad \square
\end{aligned}$$

Once we have extended $\zeta_\alpha(s)$ to the entire s-plane (with simple poles at $s = 1, 2, 3, \dots$), we use the continuations of equations (6.3) and (6.4), both valid for $s \in \mathbb{C}$, in order to get

$$Z_{\alpha+1}(s)Z_{\alpha+1}(1-s) = \frac{\zeta_\alpha(1-s)\zeta_\alpha(s)}{\Gamma(1-s)\Gamma(s)\sin(\pi s/2)\cos(\pi s/2)} = \frac{2}{\pi}\zeta_\alpha(1-s)\zeta_\alpha(s). \tag{6.5}$$

From that, a simple verification leads us to the following functional equation.

Corollary 6.3. *The function*

$$\Phi(s) = \sqrt{\frac{2}{\pi}} \frac{\zeta_\alpha(s)}{Z_{\alpha+1}(s)}$$

satisfies the functional equation $\Phi(s) = 1/\Phi(1-s)$.

Next we study the analytic continuation of $\zeta_{E,\alpha}(s)$ which is rather similar to $\zeta_\alpha(s)$.

Theorem 6.4. *We get*

$$\zeta_{E,\alpha}(1-s) = -\Gamma(s) \cos\left(\frac{\pi s}{2}\right) Z_\alpha(s), \qquad s \in \mathbb{C}, \tag{6.6}$$

or equivalently

$$\zeta_{E,\alpha}(s) = -\Gamma(1-s) \sin\left(\frac{\pi s}{2}\right) Z_\alpha(1-s), \qquad s \in \mathbb{C}. \tag{6.7}$$

In particular, $\zeta_{E,\alpha}(s)$ *can be analytically continued to the entire* s*-plane with simple poles at* $s = n = 1, 2, 3, \dots$ *(for* $\alpha \neq \pm 1/2$*) whose residues are equal to* $-c_{n-1}/(2n!)$*. Moreover,* $\zeta_{E,\alpha}(0) = -1/2$*.*

Finally, let us mention that, having $Z_\alpha(s)$ defined in $0 < \mathrm{Re}(s) < 1$ (Theorem 6.1, whose proof provides a convergent series to evaluate $Z_\alpha(s)$ in this region), one can wonder where the zeros of these functions are. Hawkins did an analysis of the zeros of $Z_\alpha(s)$ and provided some results involving zero free regions [21, Section 2] (see also [1]). In addition, many of the graphical or numerical methods for finding the zeros of $\zeta(s)$ in the critical strip (see, for instance, [6,7] and the references therein) can be adapted to the case of $Z_\alpha(s)$. It is then easy to find zeros of $Z_\alpha(s)$ that do not satisfy $\mathrm{Re}(s) = 1/2$. However, as far as we know, a further analysis of the zeros of $Z_\alpha(s)$ is yet to be done, but doesn't seem to be straightforward at first glance. Is there a deeper theory behind this problem?

7. Proofs of the results of Section 5

In this section we prove the theorems of Subsections 5.1 and 5.2. In Subsection 7.1, we begin by proving results concerning the Hurwitz-Dunkl zeta function $\zeta_\alpha(s,x)$ stated in Subsection 5.1; in Subsection 7.2, and with less details, we prove the corresponding results for $\zeta_{E,\alpha}(s,x)$ stated in Subsection 5.2.

7.1. The Dunkl zeta function case

Our goal is to prove Theorem 5.1, and then use it to prove Theorems 5.2, 5.4 and 5.5. For that, some preliminary results are needed.

Lemma 7.1. *Let* $\{s_m\}_{m=1}^\infty$ *be the positive zeros of* $J_{\alpha+1}(t)$ *and let* $S = \mathbb{C} \setminus \{0, \pm is_1, \pm is_2, \dots\}$ *denote the region that remains when we remove from the* t*-plane the origin and all zeros of* $\mathcal{I}_{\alpha+1}(t)$*, as in Fig. 2. Then for* $x \in [-1,1] \setminus \{0\}$*, the function*

$$g(t) = \frac{E_\alpha(xt)}{E_\alpha(-t) - E_\alpha(t)}$$

is bounded on compact subsets of S *and compact subsets of* $x \in [-1,1] \setminus \{0\}$*. Furthermore, if* $\alpha < 1 + 1/2$*, then for* $x = 0$ *the function* $g(t)$ *is bounded on compact subsets of* S*.*

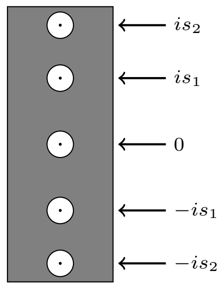

Fig. 2. A compact subset of the region S from Lemma 7.1.

Proof. We use arguments similar to those of [18, §2], and reproduce most of them for the sake of completeness (actually, here it is somewhat simpler because [18] uses $\alpha \in \mathbb{C}$ and here we have the standard $\alpha > -1$ of the Dunkl context). To get started, let us take a large circle $D = \{z \in \mathbb{C} : |z| = A\}$ of radius A with the condition that none of the points is_m, $m \in \mathbb{Z} \setminus \{0\}$, must lie on D. The poles of $g(t)$ inside D are is_m, with $|s_m| < A$, and all of them are simple. Now, we prove that the value of A can be chosen arbitrarily large and such that there exists some constant $c > 0$ independent of A (but depending on α) satisfying

$$|J_\alpha(t)| \geq c\, e^{\operatorname{Im}(t)}/|t|^{1/2} \tag{7.1}$$

for $t \in D$. For that, we proceed based on what is done in [35, §15.41, p. 498]. First, we denote $H_\alpha^{(1)}(t)$ and $H_\alpha^{(2)}(t)$ as the Bessel functions of the third kind. We use the equality

$$2J_\alpha(t) = H_\alpha^{(1)}(t) + H_\alpha^{(2)}(t), \tag{7.2}$$

and, in addition, the fact that the Bessel functions of the third kind satisfy the estimates

$$H_\alpha^{(1)}(t) = \left(\frac{2}{\pi t}\right)^{1/2} e^{i(t-\frac{1}{2}\alpha\pi-\frac{1}{4}\pi)}\Big(1 + \eta_{1,\alpha}(t)\Big), \tag{7.3}$$

$$H_\alpha^{(2)}(t) = \left(\frac{2}{\pi t}\right)^{1/2} e^{-i(t-\frac{1}{2}\alpha\pi-\frac{1}{4}\pi)}\Big(1 + \eta_{2,\alpha}(t)\Big), \tag{7.4}$$

were $\eta_{1,\alpha}(t)$ and $\eta_{2,\alpha}(t)$ are functions of order $\mathcal{O}(1/t)$ for large $|t|$ (see [35, §15.4, p. 496]). Therefore,

$$\frac{1}{2}\left(\frac{2}{\pi|t|}\right)^{1/2} e^{-\operatorname{Im}(t)} \leq \left|H_\alpha^{(1)}(t)\right| \leq 2\left(\frac{2}{\pi|t|}\right)^{1/2} e^{-\operatorname{Im}(t)},$$
$$\frac{1}{2}\left(\frac{2}{\pi|t|}\right)^{1/2} e^{\operatorname{Im}(t)} \leq \left|H_\alpha^{(2)}(t)\right| \leq 2\left(\frac{2}{\pi|t|}\right)^{1/2} e^{\operatorname{Im}(t)},$$

for $|t|$ large enough. This, together with (7.2), gives

$$\begin{aligned} 2|J_\alpha(t)| &\geq \frac{1}{2}\left(\frac{2}{\pi|t|}\right)^{1/2} e^{|\operatorname{Im}(t)|} - 2\left(\frac{2}{\pi|t|}\right)^{1/2} e^{-|\operatorname{Im}(t)|} \\ &= \frac{1}{2}\left(\frac{2}{\pi|t|}\right)^{1/2} e^{|\operatorname{Im}(t)|}\Big(1 - 4e^{-2|\operatorname{Im}(t)|}\Big) \end{aligned}$$

for $|t|$ large enough, which proves (7.1) if $|\operatorname{Im}(t)| \geq 1$. On the two arcs of D with $|\operatorname{Im}(t)| \leq 1$, according to (7.2), (7.3) and (7.4), the problem reduces essentially to get a lower bound for $|\cos(t - \frac{1}{2}\alpha\pi - \frac{1}{4}\pi)|$, which can be done by simply choosing A so that to avoid the zeros of the cosine function. This proves (7.1). Furthermore, (7.2), (7.3) and (7.4) also give

$$|J_\alpha(t)| \le C\, e^{|\operatorname{Im}(t)|}/|t|^{1/2}$$

for $|t|$ large enough, with a constant $C > 0$ depending only on α. Therefore, for any compact set $K \subset [-1,1] \setminus \{0\}$ the radius A can be chosen with the additional property that there exists $C > 0$ such that, for any $t \in D$ and any $x \in K$,

$$|J_\alpha(tx)| \le C\, e^{|\operatorname{Im}(tx)|}/|tx|^{1/2}, \tag{7.5}$$

$$|J_{\alpha+1}(tx)| \le C\, e^{|\operatorname{Im}(tx)|}/|tx|^{1/2}. \tag{7.6}$$

Using (7.1), (7.5) and (7.6), we get, for $x \in K$ and $t \in D$,

$$\begin{aligned} \left|\frac{E_\alpha(xt)}{E_\alpha(-t) - E_\alpha(t)}\right| &= \left|\frac{E_\alpha(tx)\,(\alpha+1)}{\mathcal{I}_{\alpha+1}(t)t}\right| = \left|\frac{(\alpha+1)\mathcal{I}_\alpha(tx)}{\mathcal{I}_{\alpha+1}(t)t} + \frac{x\mathcal{I}_{\alpha+1}(tx)}{2\mathcal{I}_{\alpha+1}(t)}\right| \\ &= \left|\frac{J_\alpha(itx)i + J_{\alpha+1}(itx)}{2x^\alpha J_{\alpha+1}(it)}\right| \le \tilde{c}\,\frac{e^{|\operatorname{Im}(xit)|}/|xit|^{1/2}}{|x|^\alpha e^{|\operatorname{Im}(it)|}/|it|^{1/2}} = \tilde{c}\,\frac{e^{(|x|-1)|\operatorname{Re}(t)|}}{|x|^{\alpha-1/2}} \end{aligned}$$

for some constant $\tilde{c}$ depending only on α and K. This proves the result for $x \in [-1,1] \setminus \{0\}$. Finally, let us study the particular case $x = 0$. As $E_\alpha(0) = 1$, it follows that

$$|g(t)| = \left|\frac{\alpha+1}{\mathcal{I}_{\alpha+1}(t)t}\right| \le \tilde{c}\,\frac{|t|^{\alpha-1-1/2}}{e^{|\operatorname{Re}(t)|}}.$$

Since we can choose t such as $|t| \to \infty$ and $\operatorname{Re}(t)$ is constant, to ensure that $g(t)$ is bounded at $x = 0$ on compact subsets of S we have to consider that $\alpha < 1 + 1/2$. □

We now have the tools for the next step:

Proof of Theorem 5.1. For simplicity, let us denote

$$g(t) = \frac{E_\alpha(xt)}{E_\alpha(-t) - E_\alpha(t)}.$$

The contour C in Fig. 1 is composed of three parts, C_1, C_2 and C_3. We take C_2 as a positively oriented circle of radius $0 < c < s_1$ (where s_1 is the first zero of $J_{\alpha+1}(x)/x^{\alpha+1}$) about the origin. This avoids C_2 passing through a zero of $g(t)$. On the other hand, C_1 and C_3 are the lower and upper edges of a "cut" in the t-plane along the negative real axis, traversed as shown in Fig. 1. Then,

$$2\pi i I(s,x) = \left(\int_{C_1} + \int_{C_2} + \int_{C_3}\right) g(t)t^{s-1}\,dt. \tag{7.7}$$

We consider an arbitrary compact disk $|s| \le M$ and prove that the integrals along C_1 and C_3 converge uniformly on every such disk. Since the integrand is an entire function of s, this will prove that $I(s,x)$ is entire.

We have $t = re^{-\pi i}$ on C_1, $t = re^{\pi i}$ on C_3 (with r varying from $c > 0$ to ∞) and $g(t) = g(-r)$. Also, let us denote $\sigma = \operatorname{Re}(s)$. Along C_1 and C_3, for $r \ge 1$,

$$|t^{s-1}| = r^{\sigma-1}|e^{\pm\pi i(\sigma-1+iy)}| = r^{\sigma-1}e^{\pm\pi y} \le r^{M-1}e^{\pi M}.$$

Hence on either C_1 or C_3, for $r \ge 1$,

$$\left|g(t)t^{s-1}\right| \le r^{M-1}e^{\pi M}\left|g(-r)\right|.$$

Following the proof of Lemma 7.1, we find that $g(-r)$ is bounded by

$$\tilde{c}\,\frac{e^{(|x|-1)r}}{|x|^{\alpha-1/2}}.$$

That means $|g(t)t^{s-1}| \le Ar^M e^{(|x|-1)|r|}$ for some constant A depending on M and x. Since the integral $\int_c^\infty r^M e^{(|x|-1)r}\,dr$ converges when $c > 0$ and $-1 < x < 1$, this shows the convergence along C_1 and C_3 and hence, $I(s,x)$ is entire.

Now, we compute $I(s,x)$ by (7.7), taking into account that $t = ce^{i\theta}$ (with $-\pi \le \theta \le \pi$) on C_2. Let us take

$$\begin{aligned}2\pi i I(s,x) = &\int_\infty^c r^{s-1}e^{-\pi is}g(-r)\,dr\\ &+\int_{-\pi}^{\pi} c^{s-1}e^{i\theta(s-1)}g(ce^{i\theta})ice^{i\theta}\,d\theta + \int_c^\infty r^{s-1}e^{\pi is}g(-r)\,dr.\end{aligned}$$

The sum of the integrals along C_1 and C_3 is equal to

$$\begin{aligned}\int_c^\infty r^{s-1}g(-r)(e^{\pi is}-e^{-\pi is})\,dr &= 2i\sin(s\pi)\int_c^\infty r^{s-1}g(-r)\,dr\\ &=: 2i\sin(s\pi)I_1(s,c),\end{aligned}$$

and the integral along C_2 is equal to

$$ic^s\int_{-\pi}^{\pi} e^{i\theta s}g(ce^{i\theta})\,d\theta =: ic^s I_2(s,c).$$

Dividing by $2i$, we get

$$\pi I(s,x) = \sin(s\pi)I_1(s,c) + \frac{c^s}{2}I_2(s,c).$$

If we take $c \to 0$, we notice that $I_1(s,c) \to \Gamma(s)\zeta_\alpha(s,x)$ if $\sigma > 1$, where $\zeta_\alpha(s,x)$ is the function defined in (4.13), so it only remains to prove that $I_2(s,c) \to 0$ as $c \to 0$.

Notice that $g(t)$ is analytic in $|t| < s_1$ except on the simple pole at $t = 0$. Hence $g(t)t$ is analytic everywhere on $|t| < s_1$ and so it is bounded here, say $g(t) \le A/|t|$ for some constant $A > 0$ and $|t| = c > 0$. Therefore we have

$$|I_2(s,c)| \le \frac{c^\sigma}{2}\int_{-\pi}^{\pi} e^{-y\theta}\,\frac{A}{c}\,d\theta \le Ae^{\pi|y|}c^{\sigma-1}.$$

If $\sigma > 1$ and $c \to 0$, we find $I_2(s,c) \to 0$. In conclusion, for $\sigma > 1$,

$$\pi I(s,x) = \sin(s\pi)\Gamma(s)\zeta_\alpha(s,x),$$

and finally, using that $\Gamma(s)\Gamma(1-s) = \pi/\sin(\pi s)$, we get (5.2). $\square$

Let us give the proof of Theorem 5.2, where we show that the unique singularity of $\zeta_\alpha(s,x)$, such as defined in (5.3), is a simple pole at $s = 1$.

Proof of Theorem 5.2. Since $I(s,x)$ is entire, the only possible singularities of $\zeta_\alpha(s,x)$ are the poles of $\Gamma(1-s)$, that is, the points $s = 1, 2, 3, \dots$ But $\zeta_\alpha(s,x)$ is analytic for $s > 1$, so $s = 1$ is the only possible pole of $\zeta_\alpha(s,x)$.

If s is any integer, say $s = n$, the integrand in the contour integral for $I(s,x)$ takes the same values on C_1 as on C_3, and hence the integrals along C_1 and C_3 cancel, leaving, by Cauchy's residue theorem,

$$I(n,x) = \frac{1}{2\pi i}\int_{C_2} \frac{E_\alpha(xt)}{E_\alpha(-t) - E_\alpha(t)} t^{n-1}\,dt = \operatorname{Res}_{t=0}\Big(\frac{E_\alpha(xt)}{E_\alpha(-t) - E_\alpha(t)} t^{n-1}\Big).$$

In particular, when $s = 1$ we have

$$I(1,x) = \operatorname{Res}_{t=0}\Big(\frac{E_\alpha(xt)}{E_\alpha(-t) - E_\alpha(t)}\Big) = -\lim_{t\to 0} \frac{tE_\alpha(xt)(\alpha+1)}{\mathcal{I}_{\alpha+1}(t)t} = -(\alpha+1).$$

To find the residue of $\zeta_\alpha(s,x)$ at $s = 1$ we compute the limit

$$\lim_{s\to 1}(s-1)\zeta_\alpha(s,x) = -\lim_{s\to 1}(1-s)\Gamma(1-s)I(s,x) = -I(1,x)\lim_{s\to 1}\Gamma(2-s) = \alpha + 1.$$

This proves that $\zeta_\alpha(s,x)$ has a simple pole at $s = 1$ with residue $\alpha + 1$. $\square$

Now that Theorem 5.1 is proved, we can obtain the expression of $\zeta_\alpha(-n,x)$, for $n = 0, 1, 2, \dots$ related to the Bernoulli-Dunkl polynomials:

Proof of Theorem 5.3. Evaluating at $s = -n$ in (5.1) we get $\zeta_\alpha(-n,x) = n!\,I(-n,x)$. Applying Cauchy's residue theorem, we have

$$\begin{aligned} I(-n,x) &= \operatorname{Res}_{t=0}\left(\frac{E_\alpha(xt)}{E_\alpha(-t) - E_\alpha(t)} t^{-n-1}\right) = \operatorname{Res}_{t=0}\left(-(\alpha+1)\frac{E_\alpha(xt)}{\mathcal{I}_{\alpha+1}(t)} t^{-n-2}\right) \\ &= -(\alpha+1)\operatorname{Res}_{t=0}\left(t^{-n-2}\sum_{m=0}^{\infty} \frac{\mathfrak{B}_{m,\alpha}(x)}{\gamma_{m,\alpha}} t^m\right) \\ &= -(\alpha+1)\lim_{t\to 0}\left(t^{-n-1}\sum_{m=0}^{\infty} \frac{\mathfrak{B}_{m,\alpha}(x)}{\gamma_{m,\alpha}} t^m\right) = -(\alpha+1)\frac{\mathfrak{B}_{n+1,\alpha}(x)}{\gamma_{n+1,\alpha}}. \quad \square \end{aligned}$$

Now we are ready to prove the convergence of the Lerch-Dunkl zeta function $\mathcal{F}(x,s)$ defined in (5.7), and the Hurwitz-Dunkl formula:

Proof of Theorem 5.4. We begin by proving the convergence of (5.7), with $\operatorname{Re}(s) > 1$, for $x \in \mathbb{R}$. For real values of the variable, we have

$$J_\alpha(t)^2 + J_{\alpha+1}(t)^2 = \frac{2}{\pi t}(1 + o(1)), \quad t \to \infty,$$

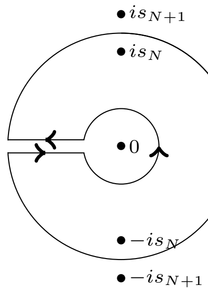

Fig. 3. The contour $C(N)$ from (7.8).

so $\lim_n s_n^{1/2}|J_\alpha(s_n)| = \sqrt{2/\pi}$, and consequently

$$|\mathcal{I}_\alpha(is_m)| \sim Cs_m^{-\alpha-1/2}, \quad t \to \infty.$$

Moreover, $|E_\alpha(xis_m)| \le C|xs_m|^{-\alpha-1/2}$ by (4.2). Then,

$$\left| \frac{E_\alpha(xis_m)}{\mathcal{I}_\alpha(is_m)} \frac{1}{s_m^s} \right| \le C|x|^{-\alpha-1/2} s_m^{-s},$$

and this guarantees the absolute convergence of (5.7).

To prove (5.8), let us consider the contour integral

$$I_N(s,x) = \frac{1}{2\pi i} \int\limits_{C(N)} \frac{E_\alpha(xt)}{E_\alpha(-t) - E_\alpha(t)} t^{s-1}\, dt, \tag{7.8}$$

where $C(N)$ is the loop shown in Fig. 3. We now denote $\sigma = \mathrm{Re}(s)$.

First we prove that $\lim_{N\to\infty} I_N(s,x) = I(s,x)$ if $\sigma < 0$. For this it suffices to show that the integral along the outer circle tends to 0 as $N \to \infty$.

On the outer circle we have $t = Re^{i\theta}$, $-\pi \le \theta \le \pi$, hence

$$|t^{s-1}| = |R^{\sigma-1} e^{\pi i(\sigma+iy)}| = R^{\sigma-1}e^{-y\pi} \le R^{\sigma-1}e^{\pi|y|}.$$

Since the outer circle lies in the set S of Lemma 7.1, the integrand is bounded by $AR^{\sigma-1}e^{\pi|y|}$, where A is the bound for $g(t)$ implied by Lemma 7.1; hence, the integral is bounded by $2\pi AR^\sigma e^{\pi|y|}$. This tends to 0 as $R \to \infty$ if $\sigma < 0$. Therefore, replacing s by $1-s$, we see that

$$\lim_{N\to\infty} I_N(1-s,x) = I(1-s,x), \qquad \text{if } \sigma > 1.$$

Since

$$\frac{E_\alpha(xt)}{E_\alpha(-t) - E_\alpha(t)} t^{-s} = -\frac{E_\alpha(xt)(\alpha+1)}{\mathcal{I}_{\alpha+1}(t)} t^{-s-1},$$

the poles of $g(t)t^{-s}$ are just the zeros of $\mathcal{I}_{\alpha+1}(t)$, say is_m, $m \in \mathbb{Z} \setminus \{0\}$ (we don't take into account the pole at $t = 0$ because $C(N)$ doesn't contain it). We compute $I_N(1-s,x)$ explicitly by Cauchy's residue theorem. We have

$$I_N(1-s,x) = -\sum_{\substack{m=-N\\ m\neq 0}}^{m=N} R(m) = -\sum_{\substack{m=-N\\ m\neq 0}}^{m=N} \mathrm{Res}_{t=is_m} \left(\frac{E_\alpha(xt)}{E_\alpha(-t) - E_\alpha(t)} t^{-s} \right). \tag{7.9}$$

Now, if $m > 0$,

$$\begin{aligned} -R(m) &= \lim_{t \to is_m} (t - is_m) \frac{E_\alpha(xt)(\alpha+1)}{\mathcal{I}_{\alpha+1}(t)} t^{-s-1} \\ &= E_\alpha(xis_m)(is_m)^{-s-1}(\alpha+1) \lim_{t \to is_m} \frac{(t - is_m)}{\mathcal{I}_{\alpha+1}(t)} \\ &= E_\alpha(xis_m)(is_m)^{-s-1}(\alpha+1) \frac{1}{\mathcal{I}'_{\alpha+1}(is_m)}. \end{aligned}$$

Now, we compute $\mathcal{I}'_{\alpha+1}(is_m)$ as follows. First, let us write $\mathcal{I}_\alpha(z) = 2^\alpha \Gamma(\alpha+1) I_\alpha(z)/z^\alpha$, where I_α is the modified Bessel function of the first kind and order α, see [35,28]. We will use the identities (see, for instance, [28, 10.29.2])

$$I'_\alpha(z) = I_{\alpha+1}(z) + \frac{\alpha}{z} I_\alpha(z) \tag{7.10}$$

and

$$I'_\alpha(z) = I_{\alpha-1}(z) - \frac{\alpha}{z} I_\alpha(z). \tag{7.11}$$

By (7.10) we have

$$\mathcal{I}'_\alpha(z) = 2^\alpha \Gamma(\alpha+1) \left(\frac{I'_\alpha(z)}{z^\alpha} - \alpha \frac{I_\alpha(z)}{z^{\alpha+1}} \right) = \frac{z}{2(\alpha+1)} \mathcal{I}_{\alpha+1}(z), \tag{7.12}$$

and by (7.11) (with $\alpha+1$ instead of α) we deduce that

$$\begin{aligned} \mathcal{I}'_{\alpha+1}(z) &= 2^{\alpha+1} \Gamma(\alpha+2) \left(\frac{I'_{\alpha+1}(z)}{z^{\alpha+1}} - (\alpha+1) \frac{I_{\alpha+1}(z)}{z^{\alpha+2}} \right) \\ &= 2^{\alpha+1} \Gamma(\alpha+2) \left(\frac{I_\alpha(z)}{z^{\alpha+1}} - 2(\alpha+1) \frac{I_{\alpha+1}(z)}{z^{\alpha+2}} \right) \\ &= \frac{2(\alpha+1)}{z} (\mathcal{I}_\alpha(z) - \mathcal{I}_{\alpha+1}(z)). \end{aligned}$$

Hence, $\mathcal{I}'_{\alpha+1}(is_m) = \frac{2(\alpha+1)}{is_m} \mathcal{I}_\alpha(is_m)$. Therefore we get, for $m = 1, 2, \dots,$

$$-R(m) = \frac{1}{2} \frac{E_\alpha(xis_m)}{\mathcal{I}_\alpha(is_m)} (is_m)^{-s}. \tag{7.13}$$

Analogously, for $m = -1, -2, \dots,$ we can compute $-R(m)$ the same way as before, but taking into account that $s_{-m} = -s_m$ and knowing that $\mathcal{I}_\alpha(t)$ is an even function of t. In this case, we get

$$-R(m) = \frac{1}{2} \frac{E_\alpha(-xis_m)}{\mathcal{I}_\alpha(is_m)} (-is_m)^{-s}. \tag{7.14}$$

By (7.13) and (7.14) we are able to compute (7.9). Indeed,

$$I_N(1-s, x) = \frac{i^{-s}}{2} \sum_{m=1}^N \frac{E_\alpha(xis_m)}{\mathcal{I}_\alpha(is_m) s_m^s} + \frac{(-i)^{-s}}{2} \sum_{m=1}^N \frac{E_\alpha(-xis_m)}{\mathcal{I}_\alpha(is_m) s_m^s}.$$

Writing $i^{-s} = e^{-\pi s/2}$ and $(-i)^{-s} = e^{\pi s/2}$ and taking $N \to \infty$ we get

$$I(1-s,x) = \frac{1}{2}\left(e^{-\pi si/2}\sum_{m=1}^{\infty}\frac{E_\alpha(xis_m)}{\mathcal{I}_\alpha(is_m)s_m^s} + e^{\pi si/2}\sum_{m=1}^{\infty}\frac{E_\alpha(-xis_m)}{\mathcal{I}_\alpha(is_m)s_m^s}\right).$$

Since $\zeta_\alpha(1-s,x) = \Gamma(s)I(1-s,x)$, if we call $\mathcal{F}(x,s) = \sum_{m=1}^{\infty}\frac{E_\alpha(xis_m)}{\mathcal{I}_\alpha(is_m)s_m^s}$ we finally get the Hurwitz-Dunkl formula (5.8). □

The Hurwitz-Dunkl formula gives us an expression for $\zeta_\alpha(s,x)$ free of the intricate integrals. With it, we can easily prove Theorem 5.5.

Proof of Theorem 5.5. Taking $x = 1$ in the Hurwitz-Dunkl formula (5.8), we get $\mathcal{F}(1,s) = \sum_{m=1}^{\infty} 1/s_m^s$, since $E_\alpha(\pm is_m) = \mathcal{I}_\alpha(is_m)$. Hence,

$$\zeta_\alpha(1-s) = \zeta_\alpha(1-s,1) = \frac{\Gamma(s)}{2}\sum_{m=1}^{\infty}\frac{1}{s_m^s}\left(e^{-\pi si/2}+e^{\pi si/2}\right) = \Gamma(s)\cos\left(\frac{\pi s}{2}\right)\sum_{m=1}^{\infty}\frac{1}{s_m^s}.$$

Changing $1-s$ for s we get the equivalent expression in terms of $\sin(x)$. □

7.2. The Euler-type Dunkl zeta function case

Now we prove the analogous results for $\zeta_{E,\alpha}(s,x)$ and $\zeta_{E,\alpha}(s)$.

Lemma 7.2. *Let $\{j_m\}_{m=1}^{\infty}$ be the positive zeros of $J_\alpha(t)$ and let $S = \mathbb{C}\setminus\{0,\pm ij_1,\pm ij_2,\dots\}$. Then for $x\in[-1,1]\setminus\{0\}$, the function*

$$h(t) = \frac{E_\alpha(xt)}{E_\alpha(-t)+E_\alpha(t)}$$

is bounded on compact subsets of S and compact subsets of $x\in[-1,1]\setminus\{0\}$. Furthermore, if $\alpha<1/2$, then for $x=0$ the function $h(t)$ is bounded on compact subsets of S.

Proof. We start again by taking a large circle $D = \{z\in\mathbb{C} : |z| = A\}$ of radius A with the only condition that none of the points j_m, $m\in\mathbb{Z}\setminus\{0\}$, must lie on D. The poles of $h(t)$ inside D are j_m, with $|j_m| < A$, and all of them are simple. For any compact set $K\subset[-1,1]\setminus\{0\}$ the radius A can be chosen with the additional property that there exist $c, C > 0$ such that, for any $t\in D$ and $x\in K$, equations (7.1), (7.5) and (7.6) are satisfied. Hence, we have

$$\begin{aligned}\left|\frac{E_\alpha(xt)}{E_\alpha(-t)+E_\alpha(t)}\right| &= \left|\frac{E_\alpha(tx)}{2\mathcal{I}_\alpha(t)}\right| = \left|\frac{\mathcal{I}_\alpha(tx)}{2\mathcal{I}_\alpha(t)} + \frac{xt\mathcal{I}_{\alpha+1}(tx)}{4(\alpha+1)\mathcal{I}_\alpha(t)}\right| \\ &= \left|\frac{J_\alpha(itx)i + J_{\alpha+1}(itx)}{2x^\alpha J_\alpha(it)}\right| \le \tilde{c}\,\frac{e^{(|x|-1)|\operatorname{Re}(t)|}}{|x|^{\alpha-1/2}}\end{aligned}$$

for some constant $\tilde{c}$ depending only on α and K. This proves the result for $x\in[-1,1]\setminus\{0\}$. Finally, we consider the particular case $x=0$. As $E_\alpha(0)=1$, it follows that

$$|h(t)| = \left|\frac{1}{2\mathcal{I}_\alpha(t)}\right| \le \tilde{c}\,\frac{|t|^{\alpha+1/2}}{e^{|\operatorname{Re}(t)|}},$$

which is bounded, when $|t|\ge A$, if $\alpha<1/2$. Hence, for $x=0$, $h(t)$ is bounded on S if $\alpha<1/2$. □

Proof of Theorem 5.8. The proof is identical to the one of Theorem 5.1 but using the bound of Lemma 7.2 instead. □

Proof of Theorem 5.9. The convergence of $\mathcal{F}_E(x,s)$, for $x \in \mathbb{R}$, can be proved as in the case of Theorem 5.4, this time with $|\mathcal{I}_{\alpha+1}(j_m)| \sim C|j_m|^{-\alpha-1-1/2}$, so

$$\left| \frac{E_\alpha(ij_m x)}{\mathcal{I}_{\alpha+1}(ij_m)} \frac{1}{j_m^{s+1}} \right| \le C|x|^{-\alpha-1/2} j_m^{-s},$$

and the converges is again for $\operatorname{Re}(s) > 1$.

To prove (5.14), let us now consider

$$I_N(s,x) = \frac{1}{2\pi i} \int_{C(N)} \frac{E_\alpha(xt)}{E_\alpha(-t) + E_\alpha(t)} t^{s-1}\, dt$$

with $C(N)$ the loop of Fig. 3. On the outer circle the integrand is bounded by $AR^{\sigma-1}e^{\pi|y|}$, where A is the bound for $h(t)$ implied by Lemma 7.2; hence, the integral is bounded by $2\pi AR^{\sigma}e^{\pi|y|}$. If $\sigma < 0$ the integral $I_N(s,x) \to 0$ along the outer circle of $C(N)$ when $R \to \infty$. Hence, replacing s for $1-s$, we get $\lim_{N\to\infty} I_N(1-s,x) = I_E(1-s,x)$ for $\sigma > 1$. We compute $I_N(1-s,x)$ by Cauchy's residue theorem. Let $m = 1, 2, \dots$ We compute the residue at $t = ij_m$ using (7.12):

$$\begin{aligned} -R(m) &= -\operatorname{Res}_{t=ij_m} \left(\frac{E_\alpha(xt)}{2\mathcal{I}_\alpha(t)} t^{-s} \right) = -\lim_{t\to ij_m} (t - ij_m) \left(\frac{E_\alpha(xt)}{2\mathcal{I}_\alpha(t)} t^{-s} \right) \\ &= -\frac{E_\alpha(ij_m x)}{2\mathcal{I}'_\alpha(ij_m)} (ij_m)^{-s} = -\frac{(\alpha+1)E_\alpha(ij_m x)}{\mathcal{I}_{\alpha+1}(ij_m)} (ij_m)^{-s-1}. \end{aligned}$$

Also, when $m = -1, -2, \dots$, we have

$$-R(m) = -\frac{(\alpha+1)E_\alpha(-ij_m x)}{\mathcal{I}_{\alpha+1}(-ij_m)} (-ij_m)^{-s-1}.$$

Then,

$$I_N(1-s,x) = -\sum_{\substack{m=-N \\ m\neq 0}}^{N} R(m) = -\sum_{\substack{m=-N \\ m\neq 0}}^{N} \frac{(\alpha+1)E_\alpha(ij_m x)}{\mathcal{I}_{\alpha+1}(ij_m)} (ij_m)^{-s-1}.$$

Letting $N \to \infty$, we get (5.14). □

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Regular Articles

A general method to find special functions that interpolate Appell polynomials, with examples ☆

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A B S T R A C T

Given an Appell sequence $\{P_n(x)\}_{n=0}^\infty$ defined by means of a generating function

$$A(t)e^{xt} = \sum_{n=0}^\infty P_n(x)\frac{t^n}{n!},$$

we discuss a general procedure for constructing a complex function $F(s, x)$, which is entire in s for each fixed x with $\operatorname{Re} x > 0$, and satisfies $F(-n, x) = P_n(x)$ at $n = 0, 1, 2, \dots$. The method is based on the Mellin transform and allows $A(-t)$ to have isolated singularities on the half-line $(0, \infty)$, in contrast with other general methods that appear in the mathematical literature. We illustrate our procedure with some elucidatory examples. However, our approach cannot be used for analogously defined Appell-Dunkl sequences, a fact which has led us to include an open problem related to this case.

1. Introduction

An Appell sequence $\{P_n(x)\}_{n=0}^\infty$ is defined formally by an exponential generating function of the form

$$G(x, t) = A(t)e^{xt} = \sum_{n=0}^\infty P_n(x)\frac{t^n}{n!}, \tag{1.1}$$

where x, t are indeterminates and $A(t)$ is a formal power series.

It is easily seen that (1.1) implies $P_n(x)$ is a polynomial of the form

$$P_n(x) = A(0)x^n + \cdots .$$

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Thus, the assumption $A(0) \neq 0$ (which is usually given together with (1.1)) means that $P_n(x)$ has degree n. In addition, it is straightforward to verify that any such generating function has polynomial coefficients satisfying $P_n'(x) = nP_{n-1}(x)$ for all $n \geq 1$, and conversely, that this condition on a polynomial sequence is equivalent to having a generating function of the given form.

The members of an Appell sequence are called Appell polynomials. Typical examples of Appell sequences are the Bernoulli, Euler, and Hermite polynomials, whose generating functions $G(x,t)$ are respectively $te^{xt}/(e^t - 1)$, $2e^{xt}/(e^t + 1)$ and $e^{-t^2/2}$.

In a series of recent papers [21–23], the authors give a method to build transcendental functions whose values at the negative integers are the polynomials defined by (1.1), requiring only a few easy conditions on the function $A(t)$, and provide many examples and properties. This method uses a slight modification of the Mellin transform of the generating function $G(x, -t)$ (note the sign change) and conditions on $A(t)$ that ensure the integral defining the transform converges. For instance, for the Bernoulli polynomials $\{B_n(x)\}_{n=0}^{\infty}$ the corresponding function is $s\zeta(s + 1, x)$, where $\zeta(s, x)$ is the Hurwitz zeta function. This is not surprising, since a well-known property of $\zeta(s, x)$ is $\zeta(-n + 1, x) = -B_n(x)/n$. Many other examples, such as those coming from some generalizations of the Bernoulli and Euler polynomials, the classical Hermite and Laguerre polynomials, and the Bell numbers, are discussed there.

However, although the conditions on $A(t)$ given in [21] are rather general, one of them requires that $A(-t)$ be continuous on $[0, +\infty)$, thus excluding complex analytic functions $A(-t)$ with singularities on $(0, +\infty)$; indeed, the Mellin transform does not converge in this case. The purpose of this paper is to extend these kinds of results by allowing the existence of isolated singularities, and to give some additional examples.

This article is organized as follows. In Section 2, we present the general method for obtaining an entire function $s \mapsto H(s, x)$ that satisfies $H(-n, x) = P_n(x)$ for a given Appell sequence $\{P_n(x)\}_{n=0}^{\infty}$, modifying the hypotheses of the main theorem of [21] in order to be able to apply it in cases where the Mellin transform fails to converge because of the appearance of an isolated singularity (usually, a pole). The extended result is contained in Theorem 2.1. This section also includes many comments regarding the use of the theorem. In Sections 3, 4 and 5, we apply this method to the Appell sequences that arise when we take $A(t) = 1/(1 \pm t)^r$ and $A(t) = 1/(1 \pm t^k)$, constructing the corresponding transcendental functions $H(s, x)$ that satisfy $H(-n, x) = P_n(x)$ for such Appell polynomials. Some of these sequences are related to the truncated exponential polynomials

$$e_n(x) = \sum_{k=0}^{n} \frac{1}{k!} x^k,$$

and have been recently studied in [7,17,20]. Section 6 includes some additional examples. Finally, in Section 7, we conclude by presenting an open problem related to Appell-Dunkl sequences, which are a generalization of Appell sequences where the exponential e^{xt} has been replaced by the Dunkl exponential $E_\alpha(xt) = e^{xt}\,{}_1F_1(\alpha + 1/2, 2\alpha + 2, -2xt)$. The details are given in that section.

2. Appell-Mellin sequences

The Mellin transform has been widely used in number theory as well as other fields of mathematics. For a given function $f(t)$, it is defined by the integral

$$\mathcal{M}\{f(t)\}(s) = \int_0^\infty f(t) t^{s-1}\, dt. \tag{2.1}$$

We will often add the factor $1/\Gamma(s)$ in front of the integral while still referring to it as a Mellin transform, but we reserve the symbol $\mathcal{M}$ to always denote (2.1). We apply the Mellin transform to a generating function of an Appell sequence (with a sign change). This provides an entire function that, when restricted to negative integer values, yields those Appell polynomials. Mellin transforms of generating functions have also been used in the recent papers [3] and [4] for not too dissimilar purposes.

In [21] a subclass of Appell sequences, the so-called Appell-Mellin sequences, were introduced. These are sequences $\{P_n(x)\}_{n=0}^{\infty}$ defined by a generating function of the form (1.1), where $A(t)$ is a complex function defined on the union of a neighborhood of the origin with $(-\infty, 0)$, satisfying

(a) $A(t)$ is non-constant and analytic around 0;
(b) $A(-t)$ is continuous on $[0, +\infty)$ and has polynomial growth at $+\infty$.

Following [21, Theorem 1] for an Appell-Mellin sequence and a fixed $x > 0$, we consider the integral

$$H(s,x) = \frac{1}{\Gamma(s)} \int_0^\infty G(x,-t)t^{s-1}\,dt = \frac{1}{\Gamma(s)} \int_0^\infty A(-t)e^{-xt}t^{s-1}\,dt. \tag{2.2}$$

This converges for $\mathrm{Re}(s) > 0$ to a holomorphic function of s having an analytic continuation to an entire function satisfying $H(-n, x) = P_n(x)$ for $n = 0, 1, 2, \dots$.

Let R denote the radius of convergence of the Taylor series of $A(t)$ at $t = 0$. Note that it does not depend on x and that the generating series in (1.1) converges for all $x \in \mathbb{C}$ and $|t| < R$.

The above integral is an example of a parametric integral with a holomorphic integrand in the s-domain. The condition $x > 0$ is stated for simplicity, but it is not really necessary and can be replaced in most of the results by $x \in \mathbb{C}$ with $\mathrm{Re}(x) > 0$.

As we pointed out in the introduction, this result can be extended to functions such that $A(-t)$ has poles on $[0, \infty)$; in other words, we are going to replace condition (b) by a weaker one. This is quite useful since it allows us to obtain a special function satisfying $H(-n, x) = P_n(x)$, $n = 0, 1, 2, \dots$ in some remarkable cases where the integral $\int_0^\infty G(x,-t)t^{s-1}\,dt$ doesn't converge, for example, when $G(x,-t) = e^{-xt}/(1-t^2)$. The conditions are given in the following theorem.

Theorem 2.1. *Let $A(-t)$ be a meromorphic function, continuous on $[0, +\infty)$ except for isolated singularities at $t = t_1, t_2, \dots, t_k$ (ordered by $t_1 < t_2 < \cdots < t_k$). Furthermore, suppose that $A(-t)$ is analytic in the k-punctured rectangle*

$$T = \{t \in \mathbb{C} : t_1 - \eta < \mathrm{Re}(t) < t_k + \eta,\ -\eta < \mathrm{Im}(t) < \eta\} \setminus \{t_1, \dots, t_k\}$$

for some $\eta > 0$ and that $A(-t)$ has polynomial growth for $t \to +\infty$. Consider the Appell sequence $\{P_n(x)\}_{n=0}^{\infty}$ defined by

$$G(x,t) = A(t)e^{xt} = \sum_{n=0}^{\infty} P_n(x)\frac{t^n}{n!}, \quad |t| < R,$$

with radius of convergence R satisfying $R > t_1 - \eta$. Then the integral

$$H(s,x) = \frac{1}{\Gamma(s)} \int_C G(x,-t)t^{s-1}\,dt = \frac{1}{\Gamma(s)} \int_C A(-t)e^{-xt}t^{s-1}\,dt \tag{2.3}$$

(where the path C goes from $t = 0$ to $t = \infty$ avoiding the singularities t_j as shown in Fig. 1, with $0 < \varepsilon < \eta$) converges in the right plane $\mathrm{Re}(s) > 0$ to a holomorphic function of s which may be analytically continued to an entire function satisfying

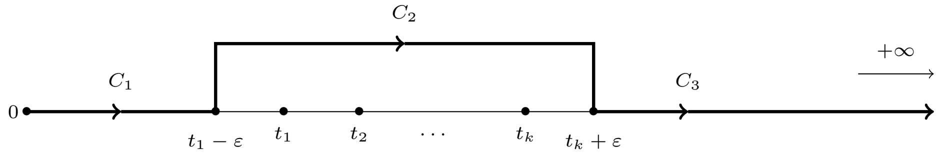

Fig. 1. An example of how the path C "avoids" the singularities $t_1, t_2, \dots, t_k$ of $A(-t)$, from Theorem 2.1 (the radius of convergence of $A(t)$ must satisfy $R > t_1 - \varepsilon$).

$$H(-n,x) = P_n(x), \quad n = 0,1,2,\dots$$

Proof. The technique of the proof is somewhat similar to that in [21], but more care must be taken to avoid the singularities.

Given $N \in \mathbb{N} \cup \{0\}$, the Mellin integral can be analytically continued to the half-plane $\mathrm{Re}(s) > -N-1$ as follows. For a fixed ε such that $t_1 - R < \varepsilon < \min\{\eta, R\}$ (note that necessarily $R \le t_1$ since t_1 is a singularity), separate the complete integral into three parts, each following the integration paths C_1, C_2 and C_3 (see Fig. 1; the upper corners of C_2 are $t_1 - \varepsilon + i\varepsilon$ and $t_k + \varepsilon + i\varepsilon$, in order for C_2 to lie inside the k-punctured rectangle T where $A(-t)$ is analytic). Next, further divide the integral along C_1 into two parts, so now we have four parts as follows:

$$\begin{aligned} H(s,x) &= \frac{1}{\Gamma(s)} \int_{t_k+\varepsilon}^{\infty} A(-t)e^{-xt}t^{s-1}\,dt \\ &\quad + \frac{1}{\Gamma(s)} \int_{C_2} A(-t)e^{-xt}t^{s-1}\,dt \\ &\quad + \frac{1}{\Gamma(s)} \int_0^{t_1-\varepsilon} \left(A(-t)e^{-xt} - \sum_{n=0}^{N} P_n(x)\frac{(-t)^n}{n!} \right) t^{s-1}\,dt \\ &\quad + \frac{1}{\Gamma(s)} \int_0^{t_1-\varepsilon} \sum_{n=0}^{N} P_n(x)\frac{(-t)^n}{n!} t^{s-1}\,dt. \end{aligned} \tag{2.4}$$

In the first part, the integrand $e^{-xt}A(-t)t^{s-1}$ converges exponentially to 0 when $t \to \infty$; it is dominated on arbitrary closed vertical strips of finite width, hence the integral is an entire function of s. Since $1/\Gamma(s)$ is entire, the complete first term is also.

In the second part, the integrand is again $e^{-xt}A(-t)t^{s-1}$ and since the path C_2 is finite and the integrand is analytic there, we again conclude that the integral is an entire function of s.

In the third part, note that the radius of convergence of $A(t)$ is $R > t_1 - \varepsilon$. The integrand is the product of t^{s-1} with the tail of the generating series, $\sum_{n=N+1}^{\infty} P_n(x)(-t)^n/n!$, which is $\mathcal{O}(t^{N+1})$ at $t = 0$. Thus, for $\mathrm{Re}(s) > -N-1$, the complete integrand is $\mathcal{O}(t^{N+\mathrm{Re}(s)})$ at $t = 0$ (with the order constant depending only on x) and hence is integrable on $[0, t_1 - \varepsilon]$ and dominated on closed vertical sub-strips of finite width of this section of the s-plane. Therefore the third integral is a holomorphic function of s for $\mathrm{Re}(s) > -N-1$.

In the fourth part, we get

$$\begin{aligned} \frac{1}{\Gamma(s)} \int_0^{t_1-\varepsilon} \sum_{n=0}^{N} P_n(x)\frac{(-t)^n}{n!} t^{s-1}\,dt &= \frac{1}{\Gamma(s)} \sum_{n=0}^{N} P_n(x)\frac{(-1)^n}{n!} \int_0^{t_1-\varepsilon} t^{s+n-1}\,dt \\ &= \frac{1}{\Gamma(s)} \sum_{n=0}^{N} P_n(x)\frac{(-1)^n}{n!} \frac{(t_1-\varepsilon)^{s+n}}{s+n}, \end{aligned} \tag{2.5}$$

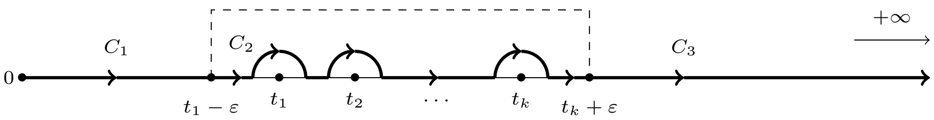

Fig. 2. An alternative for the path C_2, as described in Remark 1.

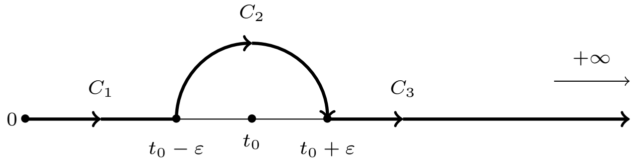

Fig. 3. An example of how the path C that "avoids" a unique singularity t_0 of $A(-t)$.

which is an entire function of s because the simple pole of $\Gamma(s)$ at $s=-n$ cancels the simple zero of $s+n$ for $n=0,1,2,\dots$, leaving the non-zero residue $(-1)^n/n!$.

Finally, if $s=-n$ with $0\le n\le N$, the factor $1/\Gamma(s)$ in front of every integral is zero, so the first, second, and third parts in (2.4) vanish, while in the fourth part, the only non-zero summand in (2.5) corresponds to n, and yields the value $P_n(x)$ because of the residue of $\Gamma(s)$ at $-n=0,1,2,\dots$, which is equal to $(-1)^n/n!$. Thus $H(-n,x)=P_n(x)$ for these n, and this completes the proof. $\square$

In the next sections we show how to apply Theorem 2.1 to specific Appell sequences. In most cases, the singularities reduce to a single pole. Before we proceed, we make some observations about the theorem.

Remark 1. We have taken the C_2 part of the path C as the upper part of a rectangle. This is not essential since for functions which are analytic in a simply connected domain, the integral is independent of the path. For instance, we could also take the C_2 part as in Fig. 2 (again with $R>t_1-\varepsilon$). The most common (and easy) case of the previous theorem is when $A(-t)$ has only one singularity, which we call t_0. In this case, we usually describe the C_2 part of the path as a semicircle of radius ε centered on t_0, as in Fig. 3. In many cases, the radius of convergence of $A(-t)$ is $R=t_0$, and then we can take any $\varepsilon>0$.

Remark 2. For simplicity, assume that we have a unique singularity at t_0, as described in the previous remark, with the path C passing above the singularity as shown in Fig. 3, and C_2 equal to a semicircle. Clearly, we could also take the path C passing under the singularity t_0, obtaining two different functions $H(s,x)$ satisfying $H(-n,x)=P_n(x)$. Let C^+ denote the path going over t_0 (as in the figure) and C^- the path going under t_0, and let us denote by $H^+(s,x)$ and $H^-(s,x)$ the corresponding functions $H(s,x)$. Then, by Cauchy's residue theorem,

$$\begin{aligned} H^-(s,x) &= \frac{1}{\Gamma(s)}\int_{C^-} A(-t)e^{-xt}t^{s-1}\,dt \\ &= \frac{1}{\Gamma(s)}\int_{C^+} A(-t)e^{-xt}t^{s-1}\,dt + \frac{1}{\Gamma(s)}\int_{|t-t_0|=\varepsilon} A(-t)e^{-xt}t^{s-1}\,dt \\ &= H^+(s,x) + \frac{2\pi i}{\Gamma(s)}\operatorname{Res}(A(-t)e^{-xt}t^{s-1}, t=t_0) \end{aligned}$$

in the right plane $\operatorname{Re}(s)>0$. By analytic continuation, this relation between $H^-(s,x)$ and $H^+(s,x)$ is also true in the complex s-plane. If $A(-t)$ has a pole of order 1 at t_0, let us write $A(-t)=\sum_{k=-1}^{\infty} a_k(t-t_0)^k$; then, it is easy to see that $\operatorname{Res}(A(-t)e^{-xt}t^{s-1}, t=t_0) = a_{-1}e^{-t_0x}t_0^{s-1}$.

In the case of more than one singularity, as in Fig. 2, we can also conceive of paths that zigzag between singularities, some going over and some under (for instance, with semicircles above and below the horizontal axis). This will generate many different functions $H(s,t)$ but, again, they will be related by the residues at the singularities.

Remark 3. For certain generating functions $A(t)$ such that $A(-t)$ has a pole at a certain $t_0 > 0$, Theorem 2.1 might be superfluous. For instance, let us assume given $A(t)$ and the corresponding Appell polynomials $P_n(x)$, and let $\widetilde{A}(t) = A(-t)$, with corresponding Appell polynomials $\widetilde{P}_n(x)$; if $A(-t)$ has a pole at $t_0 > 0$, it becomes $-t_0$ (negative) for $\widetilde{A}(-t)$, and perhaps $\widetilde{A}(t)$ satisfies the hypotheses of [21, Theorem 1]. With this notation,

$$\sum_{n=0}^{\infty} P_n(x)\frac{t^n}{n!} = A(t)e^{xt} = \widetilde{A}(-t)e^{(-x)(-t)} = \sum_{n=0}^{\infty} \widetilde{P}_n(-x)\frac{(-t)^n}{n!},$$

so $P_n(x) = (-1)^n \widetilde{P}_n(-x)$. Consequently, if $\widetilde{H}(s,x)$ is the s-entire function that satisfies $\widetilde{H}(-n,x) = \widetilde{P}_n(x)$, the new s-entire function $H(s,x) = e^{i\pi s}\widetilde{H}(s,-x)$ (or $e^{-i\pi s}\widetilde{H}(s,-x)$) satisfies

$$H(-n,x) = e^{-i\pi n}\widetilde{H}(-n,-x) = (-1)^n \widetilde{P}_n(-x) = P_n(x).$$

In any case, the trick above mention cannot be used if, for instance, $A(-t)$ has poles both at t_0 and at $-t_0$. In particular, this happens for $A(t) = 1/(1-t^2)$.

Remark 4. Sometimes, a clever use of Theorem 2.1 allows us to find in a simple way the function $H(s,x)$ corresponding to (1.1) if $A(-t)$ has more than one pole on $(0,\infty)$. To exemplify it, let us assume that we have t_1 and t_2 satisfying $0 < t_1 < t_2$ and

$$A(t) = \frac{\widetilde{A}(t)}{(t+t_1)(t+t_2)},$$

for a function $\widetilde{A}(t)$ without singularities. Let us take the partial fraction decomposition

$$\frac{1}{(t+t_1)(t+t_2)} = \frac{k_1}{t+t_1} + \frac{k_2}{t+t_2}, \qquad k_1 = (t_2-t_1)^{-1}, \quad k_2 = (t_1-t_2)^{-1},$$

as well as the Appell sequences

$$\frac{\widetilde{A}(t)}{t+t_j}e^{xt} = \sum_{n=0}^{\infty} P_n^j(x)\frac{t^n}{n!}, \qquad j = 1, 2.$$

Then,

$$\sum_{n=0}^{\infty} P_n(x)\frac{t^n}{n!} = A(t)e^{xt} = k_1\frac{\widetilde{A}(t)}{t+t_1}e^{xt} + k_2\frac{\widetilde{A}(t)}{t+t_2}e^{xt} = k_1\sum_{n=0}^{\infty} P_n^1(x)\frac{t^n}{n!} + k_2\sum_{n=0}^{\infty} P_n^2(x)\frac{t^n}{n!},$$

so $P_n(x) = k_1P_n^1(x) + k_2P_n^2(x)$.

It is clear that the radius of convergence of $\frac{\tilde{A}(t)}{t+t_j}$ is t_j, for $j = 1, 2$, and we can apply Theorem 2.1 in both cases, obtaining s-entire functions $H_j(s,x)$ such that $H_j(-n,x) = P_n^j(x)$. Consequently, taking

$$H(s,x) = k_1H_1(s,x) + k_2H_2(s,x)$$

we have $H(-n,x) = P_n(x)$, as desired.

3. The case $A(t) = 1/(1-t)^r$

For completeness, and to compare it with the case $A(t) = 1/(1+t)^r$ that will be explored in the next section, we begin with an example that does not require the extension given in Theorem 2.1 of this paper because $A(-t) = 1/(1+t)^r$ does not have singularities on $[0, \infty)$. However, perhaps the most interesting case, which corresponds to $r = 1$, was not studied in [21].

Let $\{P_n^{(r-)}(x)\}_{n=0}^{\infty}$ be the polynomials defined by

$$\frac{1}{(1-t)^r} e^{xt} = \sum_{n=0}^{\infty} P_n^{(r-)}(x) \frac{t^n}{n!}, \qquad |t| < 1. \tag{3.1}$$

The case $r = 1$ is of interest on its own. In this particular case, for $|t| < 1$ we have

$$\frac{1}{1-t} e^{xt} = \left(\sum_{j=0}^{\infty} t^j \right) \left(\sum_{j=0}^{\infty} \frac{(xt)^j}{j!} \right) = \sum_{n=0}^{\infty} \frac{t^n}{n!} \left(\sum_{k=0}^{n} \frac{x^{n-k}}{(n-k)!} k! \right)$$

and hence, equating coefficients in (3.1), we get

$$P_n^{(1-)}(x) = n! \left(1 + x + \frac{x}{2} + \cdots + \frac{x^n}{n!} \right) = n!\, e_n(x). \tag{3.2}$$

The $e_n(x)$ are called the truncated exponential polynomials.

For general r, we have

$$H^{(r-)}(s,x) = \frac{1}{\Gamma(s)} \int_0^{\infty} e^{-xt} (1+t)^{-r} t^{s-1}\, dt, \quad \operatorname{Re}(s) > 0,\ \operatorname{Re}(x) > 0,$$

and [21, Theorem 1] (and also Theorem 2.1 of this paper, of course) implies that this function can be analytically continued to a s-entire function satisfying $H^{(r-)}(-n,x) = P_n^{(r-)}(x)$. Actually, $H^{(r-)}(s,x)$ is an old well-known function in the mathematical literature, as we now show.

Let us recall that Tricomi's confluent hypergeometric function is

$$\Psi(a,c;x) = \frac{\Gamma(1-c)}{\Gamma(a+1-c)}\, {}_1F_1(a,c;t) + \frac{\Gamma(b-1)}{\Gamma(a)} t^{1-c}\, {}_1F_1(a+1-c, 2-c; t)$$

(it is also denoted by $U(a,c,x)$, see [12, § 6.5, equation (2)] or [19, p. 242 in § 5.5.2]). This function, that will appear several other times in this paper, is often defined as the Mellin transform

$$\Psi(a,c;x) = \frac{1}{\Gamma(a)} \int_0^{\infty} e^{-xt} (1+t)^{c-a-1} t^{a-1}\, dt, \quad \operatorname{Re}(a) > 0,\ \operatorname{Re}(x) > 0,$$

and then extended by analytic continuation.

With our notation, $H^{(r-)}(s,x) = \Psi(s, s-r+1; x)$ and $H^{(r-)}(-n,x) = P_n^{(r-)}(x)$. But, when $s = -n$, by [12, § 6.9.2, equation (36)] we have

$$\Psi(-n, -n-r+1; x) = (-1)^n n!\, L_n^{(-n-r)}(x),$$

where

$$L_n^{(\alpha)}(x) = \sum_{k=0}^{n} \binom{n+\alpha}{n-k} \frac{(-x)^k}{k!}$$

denotes the (generalized) Laguerre polynomial of degree n and order α (here, $\binom{n+\alpha}{n-k}$ is the generalized binomial coefficient). Consequently, $P_n^{(r-)}(x) = (-1)^n n!\, L_n^{(-n-r)}(x)$.

The case $r = 1$ yields

$$H^{(1-)}(s,x) = \frac{1}{\Gamma(s)} \int_0^\infty e^{-xt}(1+t)^{-1} t^{s-1}\, dt = \Psi(s,s;x),$$

and the function $H^{(1-)}(s,x)$ analytically continued to the s-plane satisfies $H(-n,x) = n!\, e_n(x)$.

Now, note that one of the properties of the incomplete Gamma function

$$\Gamma(s,x) = \int_x^\infty e^{-t} t^{s-1}\, dt, \tag{3.3}$$

is the following relation (see, for instance, [13, § 9.1, equation (4)]):

$$\Gamma(s,x) = e^{-x}\Psi(1-s, 1-s; x). \tag{3.4}$$

Hence,

$$H^{(1-)}(s,x) = e^x \Gamma(1-s, x), \tag{3.5}$$

and consequently

$$H^{(1-)}(-n,x) = e^x \Gamma(n+1, x) = n!\, e_n(x). \tag{3.6}$$

Thus we recover a nice property of the incomplete Gamma function (see, for instance, [24, 8.4.8]):

$$\Gamma(n,x) = (n-1)!\, e^{-x} e_{n-1}(x), \quad n \in \mathbb{N}. \tag{3.7}$$

3.1. McBride polynomials

Let us finish this section by mentioning some polynomials related to what has been shown and that have been studied in the mathematical literature. They are the McBride polynomials $\{e_n^\lambda(x)\}_{n=0}^\infty$, defined by (see, for instance, [7, equation (12)])

$$\sum_{n=0}^\infty t^n e_n^\lambda(x) = e^{xt} \frac{\Gamma(\lambda+1)}{(1-t)^\lambda}.$$

These polynomials are an easy variation of $\{P_n^{(\lambda-)}(x)\}_{n=0}^\infty$, and thus we have that $\Psi(s, s-\lambda+1; x)$ satisfies

$$\Psi(-n, -n-\lambda+1; x) = n!\, \Gamma(\lambda+1) e_n^\lambda(x).$$

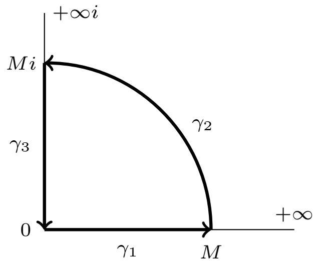

Fig. 4. Path for transforming an integral on $(0, +\infty)$ into an integral on $(0, +\infty i)$.

4. The case $A(t) = 1/(1+t)^r$

Let $\{P_n^{(r+)}(x)\}_{n=0}^{\infty}$ be the polynomials defined by

$$\frac{1}{(1+t)^r} e^{xt} = \sum_{n=0}^{\infty} P_n^{(r+)}(x) \frac{t^n}{n!}, \qquad |t| < 1.$$

Note that in this case the integral

$$H^{(r+)}(s,x) = \frac{1}{\Gamma(s)} \int_0^{\infty} e^{-xt}(1-t)^{-r} t^{s-1}\, dt$$

does not converge because $A(-t) = (1-t)^{-r}$ has a pole at $t_0 = 1$. However, we can apply Theorem 2.1, and thus we can consider

$$H^{(r+)}(s,x) = \frac{1}{\Gamma(s)} \int_C e^{-xt}(1-t)^{-1} t^{s-1}\, dt,$$

where the path C goes from 0 to ∞ but jumps over the pole $t_0 = 1$ (as shown in Fig. 3), and Theorem 2.1 guarantees that $H^{(r+)}(s,x)$ can be analytically continued to an s-entire function that satisfies $H^{(r+)}(-n,x) = P_n^{(r+)}(x)$.

To identify the function $H^{(r+)}(s,x)$, let us start by noticing that for $H^{(r-)}(s,x)$ we had, for $\mathrm{Re}(s) > 0$ and $\mathrm{Re}(x) > 0$,

$$H^{(r-)}(s,x) = \Psi(s, s-r+1; x) = \frac{1}{\Gamma(s)} \int_0^{\infty} e^{-xt}(1+t)^{-r} t^{s-1}\, dt.$$

If we change the path of integration of $H^{(r-)}(s,x)$ to the positive imaginary axis, we have that

$$\frac{1}{\Gamma(s)} \int_0^{\infty i} e^{-xt}(1+t)^{-r} t^{s-1}\, dt \tag{4.1}$$

is convergent for $\mathrm{Re}(s) > 0$ and $-\pi < \arg(x) < 0$. Let $f(t) = e^{-xt}(1+t)^{-r}t^{s-1}$ and consider the integral $\int_\gamma f(t)\, dt$, where γ is the composition of the paths γ_1, γ_2 and γ_3 given in Fig. 4. Since $f(t)$ has no singularities inside the contour γ, $\int_\gamma f(t)\, dt = \int_{\gamma_1} f(t)\, dt + \int_{\gamma_2} f(t)\, dt + \int_{\gamma_3} f(t)\, dt = 0$, and it is easy to prove that $\int_{\gamma_2} f(t)\, dt = 0$ when $M \to \infty$ (because of the term e^{-xt} in the integrand). Hence $\int_0^{\infty} f(t)\, dt = \int_0^{\infty i} f(t)\, dt$, so $H^{(r-)}(s,x)$ is equal to (4.1) in the overlapping domain $-\pi/2 < \arg(x) < 0$. By the same reasoning, we can change the path to the negative real axis $t \in [0, -\infty)$, and thus

$$H^{(r-)}(s,x) = \frac{1}{\Gamma(s)} \int_0^{-\infty} e^{-xt}(1+t)^{-r} t^{s-1}\, dt, \tag{4.2}$$

where the path passes above the pole at $t_0 = -1$, similarly to Fig. 3, but on the negative real axis.

Finally, substituting $z = -t = e^{i\pi} t$ in (4.2), we get

$$H^{(r-)}(s, xe^{i\pi}) = \Psi(s, s-r+1; xe^{i\pi}) = \frac{e^{i\pi s}}{\Gamma(s)} \int_C e^{-xz}(1-z)^{-r} z^{s-1}\, dz,$$

with C as in Fig. 3 (with $z_0 = 1$). Then,

$$H^{(r+)}(s,x) = \frac{1}{\Gamma(s)} \int_C e^{-xz}(1-z)^{-r} z^{s-1}\, dz = e^{-\pi i s}\Psi(s, s-r+1; xe^{\pi i}) \tag{4.3}$$

is a function that satisfies $H^{(r+)}(-n,x) = P_n^{(r+)}(x)$.

Note that if we substitute $z = te^{-i\pi}$ instead of $z = te^{i\pi}$ we obtain a different special function $H^{(r+)}(s,x)$ which satisfies $H^{(r+)}(-n,x) = P_n^{(r+)}(x)$ (see, for instance, [26])

$$\widetilde{H}^{(r+)}(s,x) = e^{\pi i s}\Psi(s, s-r+1; e^{-i\pi}x).$$

In this case C passes under the pole $t_0 = 1$, which is related to Remark 2. We will use this reasoning for obtaining $H^{(r+)}(s,x)$ and $\widetilde{H}^{(r+)}(s,x)$ many other times in this paper, but we will omit the details from now on.

The case $r = 1$ is interesting for its own sake. We get

$$\frac{1}{1+t} e^{xt} = \sum_{n=0}^{\infty} P_n^{(1+)}(x) \frac{t^n}{n!}, \qquad |t| < 1, \tag{4.4}$$

and it is easy to prove that $P_n^{(1+)}(x) = (-1)^n n!\, e_n(-x)$. Of course, this can be easily deduced from (3.1) and (3.2) changing x to $-x$ and t to $-t$.

A direct consequence of (4.3) together with (3.4) is

$$\Gamma(-s+1, xe^{\pm\pi i}) = e^x e^{\pm\pi i s} \frac{1}{\Gamma(s)} \int_C \frac{e^{-xt}}{1-t} t^{s-1}\, dt$$

and

$$H^{(1+)}(s,x) = \Gamma(-s+1, xe^{\pm\pi i}) e^{-x} e^{\mp\pi i s}. \tag{4.5}$$

4.1. Appell-type Changhee polynomials

As in Section 3, let us finish this section by mentioning another family of Appell polynomials with a proper name. The so-called Appell-type Changhee polynomials, $\{\mathrm{Ch}_n^*(x)\}_{n=0}^{\infty}$, introduced in [18], are defined by

$$\frac{2}{2+t}e^{xt} = \sum_{n=0}^{\infty} \mathrm{Ch}_n^*(x)\frac{t^n}{n!}, \qquad |t| < 2.$$

The generating function $\frac{2}{2+t}e^{xt}$ is closely related to $\frac{1}{1+t}e^{xt}$ by means of the change of variables $t \mapsto 2t$ and $x \mapsto x/2$. It is easy to check that the Appell-type Changhee polynomials are related to the truncated exponential polynomials by the equation

$$\mathrm{Ch}_n^*(x) = \frac{(-1)^n n!}{2^n} e_n(-2x).$$

To find the special function that satisfies $H(-n,x) = \mathrm{Ch}_n^*(x)$ for $n = 0, 1, 2, \dots$, let us take the integral

$$H(s,x) = \frac{2}{\Gamma(s)} \int_C \frac{e^{-xt}}{2-t} t^{s-1}\, dt,$$

where now C is a path avoiding the pole at $t_0 = 2$. By changing $t \mapsto 2t$ in the integral and recalling (4.5), we get

$$H(s,x) = 2^s \Gamma(-s+1, 2xe^{\pm\pi i}) e^{-2x} e^{\mp\pi i s}.$$

5. The cases $A(t) = 1/(1-t^k)$ and $A(t) = 1/(1+t^k)$

In this section we are going to study together the two cases $A(t) = 1/(1-t^k)$ and $A(t) = 1/(1+t^k)$, whose corresponding Appell polynomials will be denoted by $P_n^{[k]}(x)$ and $Q_n^{[k]}(x)$. We study these cases simultaneously because we will obtain two special functions $H_j^{[k]}(s,x)$ such that $H_j^{[k]}(-n,x)$ (both for $j = 1, 2$) is equal to $P_n^{[k]}(x)$ or $Q_n^{[k]}(x)$ depending on k being even or odd.

First, let us denote by $\{P_n^{[k]}(x)\}_{n=0}^{\infty}$ the polynomials defined by

$$\frac{1}{1-t^k}e^{xt} = \sum_{n=0}^{\infty} P_n^{[k]}(x)\frac{t^n}{n!}, \qquad |t| < 1.$$

Of course, the case $k = 1$ gives us $P_n^{(1)}(x) = n!\, e_n(x)$, and they are given by

$$P_n^{[k]}(x) = n! \sum_{j=0}^{\lfloor n/k \rfloor} \frac{x^{n-kj}}{(n-kj)!}$$

(these polynomials are studied, for instance, in [16, (1.19)], where they are denoted by ${}_{[k]}e_n(x)$).

In this case, we get that

$$H^{[k]}(s,x) = \frac{1}{\Gamma(s)} \int_0^{\infty} \frac{e^{-xt}}{1-(-t)^k} t^{s-1}\, dt = \begin{cases} \frac{1}{\Gamma(s)} \int_0^\infty e^{-xt}(1-t^k)^{-1} t^{s-1}\, dt, & \text{if } k \text{ even}, \\ \frac{1}{\Gamma(s)} \int_0^\infty e^{-xt}(1+t^k)^{-1} t^{s-1}\, dt, & \text{if } k \text{ odd}. \end{cases}$$

On the other hand, if $\{Q_n^{[k]}(x)\}_{n=0}^{\infty}$ are the polynomials defined by

$$\frac{1}{1+t^k}e^{xt} = \sum_{n=0}^{\infty} Q_n^{[k]}(x)\frac{t^n}{n!}, \qquad |t| < 1,$$

where

$$Q_n^{[k]}(x) = n! \sum_{j=0}^{\lfloor n/k \rfloor} \frac{(-1)^j x^{n-kj}}{(n-kj)!},$$

we get that

$$H^{[k]}(s,x) = \frac{1}{\Gamma(s)} \int_0^\infty \frac{e^{-xt}}{1+(-t)^k} t^{s-1}\,dt = \begin{cases} \frac{1}{\Gamma(s)} \int_0^\infty e^{-xt}(1+t^k)^{-1} t^{s-1}\,dt, & \text{if } k \text{ even}, \\ \frac{1}{\Gamma(s)} \int_0^\infty e^{-xt}(1-t^k)^{-1} t^{s-1}\,dt, & \text{if } k \text{ odd}. \end{cases}$$

Then we could argue that, for $k = 1, 2, 3, \dots$, the function

$$H_1^{[k]}(s,x) = \frac{1}{\Gamma(s)} \int_0^\infty e^{-xt}(1+t^k)^{-1} t^{s-1}\,dt$$

can be analytically continued to the s-complex plane (in this case [21, Theorem 1] is enough to do it) and satisfies $H_1^{[k]}(-n,x) = P_n^{[k]}(x)$ if k is odd and $H_1^{[k]}(-n,x) = Q_n^{[k]}(x)$ if k is even.

On the other hand, we notice that the integral

$$\int_0^\infty e^{-xt}(1-t^k)^{-1} t^{s-1}\,dt$$

does not converge for any $k = 1, 2, 3, \dots$. However, we can consider instead

$$H_2^{[k]}(s,x) = \frac{1}{\Gamma(s)} \int_C e^{-xt}(1-t^k)^{-1} t^{s-1}\,dt,$$

which can be analytically continued to the s-complex plane as described in Theorem 2.1. Then, $H_2^{[k]}(-n,x) = Q_n^{[k]}(x)$ if k is odd and $H_2^{[k]}(-n,x) = P_n^{[k]}(x)$ if k is even.

Next, we show how to express $H_1^{[k]}(s,x)$ and $H_2^{[k]}(s,x)$ in terms of previously known special functions. Both of them are particular cases of Meijer G-functions, a remarkable family of functions of one variable, each of them determined by finitely many indices. By definition, via the Mellin-Barnes integral representation, the Meijer G-function is

$$G_{p,q}^{m,n}\left(\begin{matrix} a_1, \dots, a_p \\ b_1, \dots, b_q \end{matrix} \,\middle|\, z \right) = \frac{1}{2\pi i} \int_L \frac{\prod_{j=1}^m \Gamma(b_j - t) \prod_{j=1}^n \Gamma(1 - a_j + t)}{\prod_{j=m+1}^q \Gamma(1 - b_j + t) \prod_{j=n+1}^p \Gamma(a_j - t)} z^t\,dt$$

where the integration path L separates the poles of the factors $\Gamma(b_j - t)$ from those of the factors $\Gamma(1 - a_j + t)$, with three possible choices for this path (for details, see [1, § 16.17] or [2] and the references therein).

Meijer G-functions have proved useful for generalizing a huge class of functions, including elementary functions, gamma functions, Bessel functions, hypergeometric functions, and so on. If $A(t)$ can be written as a Meijer G-function as $A(-t) = G_{p,q}^{m,n}\left(\begin{smallmatrix} a_1, \dots, a_p \\ b_1, \dots, b_q \end{smallmatrix} \,\middle|\, \eta t \right)$ for some constant η, then

$$\int_0^\infty G_{p,q}^{m,n}\left(\begin{matrix} a_1, \dots, a_p \\ b_1, \dots, b_q \end{matrix} \,\middle|\, \eta t \right) e^{-xt} t^{s-1}\,dt = x^{-s} G_{p+1,q}^{m,n+1}\left(\begin{matrix} 1-s, a_1, \dots, a_p \\ b_1, \dots, b_q \end{matrix} \,\middle|\, \frac{\eta}{x} \right) \tag{5.1}$$

(see [12, § 5.5.2]). However, in our case we have $1/(1+t^k) = G_{1,1}^{1,1}\left(\begin{smallmatrix} 0 \\ 0 \end{smallmatrix} \,\middle|\, t^k \right)$, which is not of this form. Hence, the identity (5.1) alone isn't enough to compute the integral (2.2), and for this reason we need some auxiliary results involving Mellin transforms.

Theorem 5.1. *For $k \in \mathbb{N}$ we have that*

$$\frac{1}{\Gamma(s)} \int_0^\infty \frac{e^{-xt}}{1+t^k} t^{s-1}\, dt = \frac{(2\pi)^{\frac{1-k}{2}}}{\sqrt{k}\,\Gamma(s)}\, G^{1,k+1}_{k+1,1}\left(\begin{matrix} 1, \frac{k-1}{k}, \dots, \frac{1}{k}, \frac{s}{k} \\ s/k \end{matrix} \,\middle|\, \left(\frac{k}{x}\right)^k \right),$$

where the function $G^{1,k+1}_{k+1,1}$ is a Meijer G-function.

We need the following lemmas to prove the above theorem. We refer Lemma 5.2 to [14, Chapter VI], and Lemma 5.3 to [8, § 8.2 and § 8.3] or [27, Theorem 73, p. 95].

Lemma 5.2. *Let $\mathcal{M}\{f(t)\}(s)$ denote the Mellin transform of a suitable function $f(t)$. Then*

$$\mathcal{M}\{e^{-xt}\}(s) = \Gamma(s)x^{-s}, \quad 0 < \operatorname{Re}(s),$$

and

$$\mathcal{M}\left\{\frac{1}{1+t^k}\right\}(s) = \frac{\pi}{k}\csc\left(\frac{\pi}{k}s\right) = \frac{1}{k}\Gamma\left(\frac{s}{k}\right)\Gamma\left(1-\frac{s}{k}\right), \quad 0 < \operatorname{Re}(s) < k.$$

Lemma 5.3 *(Parseval's formula for Mellin transforms). Let $f_1(t)$, $f_2(t)$ be two functions with Mellin transforms $\tilde{f}_j(t) = \mathcal{M}\{f_j(t)\}(s)$, $j = 1, 2$, in the strips $\alpha_{1,2} < \operatorname{Re}(s) < \beta_{1,2}$, respectively. Take $c \in \mathbb{R}$ such that $\alpha_1 < c < \beta_1$ and suppose that $f_1(t)t^c$ and $f_2(t)t^{\operatorname{Re}(s)-c}$ belong to $L^2((0,\infty))$. Then, for $\alpha_2 + c < \operatorname{Re}(s) < \beta_2 + c$, we have*

$$\mathcal{M}\{f_1(t)f_2(t)\}(s) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \tilde{f}_1(r)\tilde{f}_2(s-r)\, dr.$$

Proof of Theorem 5.1. We start by computing the Mellin transform of $e^{-xt}/(1+t^k)$ for $k \in \mathbb{N}$. We apply Parseval's formula (Lemma 5.3 with $0 < c < \infty$ and $c < \operatorname{Re}(s) < c + k$) to e^{-xt} and $1/(1+t^k)$. This, together with Lemma 5.2, gives

$$\begin{aligned}
\int_0^\infty e^{-xt}(1+t^k)^{-1}t^{s-1}\, dt &= \frac{1}{2\pi i k} \int_{c-i\infty}^{c+i\infty} \Gamma(r)\Gamma\left(1-\frac{s-r}{k}\right)\Gamma\left(\frac{s-r}{k}\right)x^{-r}\, dr \\
&= \frac{1}{2\pi i} \int_{\tilde{c}-i\infty}^{\tilde{c}+i\infty} \Gamma(kt)\Gamma\left(1-\frac{s}{k}+t\right)\Gamma\left(\frac{s}{k}-t\right)x^{-tk}\, dt \\
&= \frac{(2\pi)^{\frac{1-k}{2}}}{\sqrt{k}} \frac{1}{2\pi i} \int_{\tilde{c}-i\infty}^{\tilde{c}+i\infty} \prod_{j=1}^{k} \Gamma\left(\frac{j-1}{k}+t\right)\Gamma\left(1-\frac{s}{k}+t\right)\Gamma\left(\frac{s}{k}-t\right)\left(\frac{k}{x}\right)^{tk} dt,
\end{aligned}$$

where in the last step we have applied the multiplication formula for the gamma function (see [12, § 1.2, equation (11)]):

$$\Gamma(ks) = (2\pi)^{(1-k)/2} k^{ks-1/2} \prod_{j=0}^{k-1} \Gamma\left(s+\frac{j}{k}\right), \qquad ks \neq -1, -2, \dots.$$

The last integral, together with the factor $1/(2\pi i)$, is a Meijer-G function $G_{k+1,1}^{1,k+1}$ with coefficients $a_j = 1-(j-1)/k$ for $j=1,\dots,k$, $a_{k+1}=s/k$ and $b_1 = s/k$, and $z=(k/x)^k$. Here, the path L in $\int_{\tilde{c}-i\infty}^{\tilde{c}+i\infty}$ is one of the three possible types of path in the integral representation of the Meijer G-function $G_{p,q}^{m,n}$; namely, L runs from $-i\infty$ to $+i\infty$ in such a way that all poles of $\Gamma(b_j - s)$, $j=1,2,\dots,m$, are to the right of the path, while all poles of $\Gamma(1-a_k+s)$, $k=1,2,\dots,n$, are to the left (case (i) with the notation of [1, § 16.17], that can be used when $p+q<2(m+n)$). $\square$

Now, to obtain $H_2^{[k]}(s,x)$, we follow the reasoning of Section 4. We start from

$$H_1^{[k]}(s,x) = \frac{1}{\Gamma(s)} \int_0^\infty \frac{e^{-xt}}{1+t^k} t^{s-1}\,dt$$

and we get

$$H_1^{[k]}(s, xe^{\pm i\pi/k}) = \frac{1}{\Gamma(s)} \int_0^\infty \frac{e^{-xe^{\pm i\pi/k}t}}{1+t^k} t^{s-1}\,dt.$$

By changing variables to $e^{\pm i\pi/k}t = u$, and using the path avoiding the pole at $t_0 = 1$ as usual, we get

$$H_1^{[k]}(s, xe^{\pm i\pi/k}) = \frac{e^{\frac{\pm i\pi}{k}s}}{\Gamma(s)} \int_C \frac{e^{-xu}}{1-u^k} u^{s-1}\,dt.$$

Hence

$$H_2^{[k]}(s,x) = e^{\mp\frac{i\pi}{k}s} H_1^{[k]}(s, xe^{\pm i\pi/k}).$$

6. An example with 2-variable-truncated Appell polynomials

Let us assume that we have an Appell sequence defined as in (1.1). Then, following [17, § 2], the corresponding 2-variable-truncated Appell polynomials are

$$\frac{A(t)}{1-yt^r} e^{xt} = \sum_{n=0}^\infty P_n^{[r]}(x,y) \frac{t^n}{n!}. \tag{6.1}$$

The polynomials $P_n^{[r]}(x,y)$ are equal to

$$P_n^{[r]}(x,y) = n! \sum_{j=0}^{\lfloor n/r \rfloor} \frac{y^j A_{n-rj}(x)}{(n-rj)!},$$

where the sequencce $\{A_n(x)\}_{n=0}^\infty$ is determined by the generating function

$$A(t)e^{xt} = \sum_{n=0}^\infty A_n(x) \frac{t^n}{n!}.$$

In this kind of Appell sequence, it is clear that the denominator $1-yt^r$ introduces a pole on $(0,\infty)$ when $y<0$ (recall that in the Mellin transform, the generating function appears as $G(x,-t)$). Thus, finding $H^{[r]}(s,x,y)$ such that $H^{[r]}(-n,x,y) = P_n^{[r]}(x,y)$ could be done with the help of Theorem 2.1.

Of course, it is not feasible to give general formulas for this, because the integrals that appear in the process depend strongly on the function $A(t)$ in (6.1). Here, we are going to study some simple cases.

Let us consider the polynomials $\{P_n(x,y)\}_{n=0}^\infty$ defined by

$$\frac{1}{(1-yt)(1+t)}e^{xt} = \sum_{n=0}^\infty P_n(x,y)\frac{t^n}{n!}, \qquad |t| < \min\{1, 1/|y|\}.$$

Starting from

$$\frac{e^{zu}}{1-u} = \sum_{n=0}^\infty n!\, e_n(z)\frac{u^n}{n!}$$

(recall (3.1) and (3.2)) the substitutions $z = x/y$ and $u = yt$ give

$$\frac{e^{xt}}{1-yt} = \sum_{n=0}^\infty n!\, y^n e_n(x/y)\frac{t^n}{n!};$$

moreover (see (4.4)),

$$\frac{e^{xt}}{1+t} = \sum_{n=0}^\infty (-1)^n n!\, e_n(-x)\frac{t^n}{n!}.$$

Let us separate $A(-t)$ into partial fractions as

$$\frac{1}{(1+yt)(1-t)} = \frac{y}{y+1}\cdot\frac{1}{1+yt} + \frac{1}{y+1}\cdot\frac{1}{1-t}, \qquad y \neq -1.$$

Then, it is easy to check that

$$P_n(x,y) = \frac{y}{y+1}n!\, y^n e_n(x/y) + \frac{1}{y+1}(-1)^n n!\, e_n(-x) = \frac{n!}{y+1}\big(y^{n+1}e_n(x/y) + (-1)^n e_n(-x)\big).$$

For completeness, let us observe that $y^{n+1}e_n(x/y) + (-1)^n e_n(-x)$, which is a polynomial of degree $n+1$ in y, vanishes when $y = -1$, so it is divisible by $y+1$. Consequently, $P_n(x,y)$ is, as expected, a polynomial of degree n both in x and in y.

The special function $H(s,x,y)$ such that $H(-n,x,y) = P_n(x,y)$ is found by separating the integral corresponding to (2.3) into two previously studied integrals. For $y \neq -1$ we have

$$\begin{aligned} H(s,x,y) &= \frac{1}{\Gamma(s)}\int_C \frac{e^{-xt}}{(1+yt)(1-t)}t^{s-1}\,dt \\ &= \frac{y}{y+1}\frac{1}{\Gamma(s)}\int_C \frac{e^{-xt}}{1+yt}t^{s-1}\,dt + \frac{1}{y+1}\frac{1}{\Gamma(s)}\int_C \frac{e^{-xt}}{1-t}t^{s-1}\,dt, \end{aligned}$$

where C is a path as in Theorem 2.1 avoiding the poles at $t = 1$ and $t = -1/y$ if $y < 0$ (if $y > 0$ the path C only needs to avoid the pole $t = 1$). Then, by (3.5) and (4.5),

$$H(s,x,y) = \begin{cases} \frac{y^{1-s}}{y+1}e^{x/y}\Gamma(1-s,x/y) + \frac{1}{y+1}e^{-x}e^{\pi i s}\Gamma(1-s,-x), & y > 0; \\ \frac{|y|^{-s}y}{y+1}e^{-x/|y|}e^{\pi i s}\Gamma(1-s,-x/|y|) + \frac{1}{y+1}e^{-x}e^{\pi i s}\Gamma(1-s,-x), & -1 \neq y < 0. \end{cases}$$

Notice that the cases $y = 0$ and $y = -1$ were already studied in Section 4.

Many other cases can be studied by this technique, especially when $A(t)$ is a rational function. To conclude this section, let us also briefly consider the case $A(t) = 1$ and a given r, i.e.,

$$\frac{e^{xt}}{1 - yt^r} = \sum_{n=0}^{\infty} P_n^{[r]}(x, y) \frac{t^n}{n!}.$$

Here we have

$$P_n^{[r]}(x, y) = n! \sum_{j=0}^{\lfloor n/r \rfloor} \frac{y^j x^{n-rj}}{(n - rj)!}.$$

The function $H^{[r]}(s, x, y)$ can be easily computed. First consider each case $y > 0$ or $y < 0$ (using a denominator like $1 + |y|t^r$) and then substitute $yt^r = u$ in the integral. Having done this, we just need to compare the integral with the functions $H_1^{[r]}(s, x)$ or $H_2^{[r]}(s, x)$ of Section 5.

7. An open problem for the Appell-Dunkl case

Appell sequences of polynomials have been extended in many ways. One of them consists of changing the derivative operator in the relation $P_n'(x) = nP_{n-1}(x)$ (or a similar one) by a different operator with suitable properties. In [5] and [9], the derivative was replaced by the Dunkl operator on the real line

$$\Lambda_\alpha f(x) = \frac{d}{dx} f(x) + \frac{2\alpha + 1}{2} \left(\frac{f(x) - f(-x)}{x} \right),$$

where $\alpha > -1$ is a fixed parameter (see [10,25]). When $\alpha = -1/2$ we recover the classical case $\Lambda_{-1/2} = \frac{d}{dx}$. In that setting, an Appell-Dunkl sequence $\{P_{n,\alpha}\}_{n=0}^{\infty}$ is a sequence of polynomials that satisfies

$$\Lambda_\alpha P_{n,\alpha}(x) = \big(n + (\alpha + 1/2)(1 - (-1)^n)\big) P_{n-1,\alpha}(x).$$

This is a generalization of $\frac{d}{dx} P_n(x) = nP_{n-1}(x)$, which we recover when $\alpha = -1/2$.

In order to express Appell-Dunkl polynomials by means of a generating function, we replace the classical exponential e^z by the so-called "Dunkl exponential" $E_\alpha(z)$, which is

$$E_\alpha(z) = \sum_{n=0}^{\infty} \frac{z^n}{\gamma_{n,\alpha}}, \qquad z \in \mathbb{C},$$

with

$$\gamma_{n,\alpha} = \begin{cases} 2^{2k} k!\, (\alpha + 1)_k, & \text{if } n = 2k, \\ 2^{2k+1} k!\, (\alpha + 1)_{k+1}, & \text{if } n = 2k + 1, \end{cases}$$

and where $(a)_n = a(a + 1)(a + 2) \cdots (a + n - 1) = \Gamma(a + n)/\Gamma(a)$ denotes the Pochhammer symbol. Of course, $E_{-1/2}(z) = e^z$ and $\gamma_{n,-1/2} = n!$.

In this way, an Appell-Dunkl sequence $\{P_{n,\alpha}(x)\}_{n=0}^{\infty}$ is a sequence of polynomials defined by the generating function

$$A(t) E_\alpha(xt) = \sum_{n=0}^{\infty} P_{n,\alpha}(x) \frac{t^n}{\gamma_{n,\alpha}},$$

where $A(t)$ is analytic at $t = 0$ with $A(0) \neq 0$. The first Appell-Dunkl sequence of polynomials studied in the mathematical literature were the so-called generalized Hermite polynomials (see [25]). In recent years, the Bernoulli and the Euler polynomials (among other Appell families) have been extended to the Dunkl context; see, for instance, [5,6,11]. These polynomials have been proven to be very useful for extending some classical properties to a more general context.

Some of the Appell sequences that we studied in this paper have also been recently extended to the Dunkl context (see [20]). For instance, (3.1) and (4.4) are extended as

$$\frac{1}{1 \pm t} E_\alpha(xt) = \sum_{n=0}^{\infty} P_{n,\alpha}^{(1\pm)}(x) \frac{t^n}{n!}.$$

As expected, $P_{n,\alpha}^{(1-)}(x) = \gamma_{n,\alpha} e_{n,\alpha}(x)$ and $P_{n,\alpha}^{(1+)}(x) = (-1)^n \gamma_{n,\alpha} e_{n,\alpha}(-x)$, where now

$$e_{n,\alpha}(x) = 1 + \frac{x}{\gamma_{1,\alpha}} + \frac{x^2}{\gamma_{2,\alpha}} + \cdots + \frac{x^n}{\gamma_{n,\alpha}}$$

is the n-th truncated Dunkl exponential. Can we find special functions $H_\alpha^{(1\pm)}(s, x)$ such that $H_\alpha^{(1\pm)}(-n, x) = P_{n,\alpha}^{(1\pm)}(x)$?

In the recent paper [15], we give special functions $H(s, x)$ whose values at the negative integers yield the Bernoulli-Dunkl and Euler-Dunkl polynomials (and their generalized families) and found functions that recall the Hurwitz and Riemann zeta functions, but in a Dunkl context. We do this by taking the Mellin transform

$$\int_0^\infty A(-t) E_\alpha(-xt) t^{s-1}\, dt, \qquad \operatorname{Re}(x) > 0, \tag{7.1}$$

and, at least in part, the method is similar to the one in [21, Theorem 1], although with many difficulties. One of these is the convergence of the integral (7.1) when $t \to \infty$. In the classical case $\alpha = -1/2$, the factor $E_{-1/2}(-xt) = e^{-xt}$ decreases very quickly when $t \to +\infty$, and then the integral (7.1) converges if $A(-t)$ is continuous on $[0, +\infty)$ and has polynomial growth at $+\infty$. However, except for $\alpha = -1/2$, $E_\alpha(u)$ (for $u \in \mathbb{R}$) behaves roughly like $e^{|u|}$, so $A(-t)$ must decrease very quickly to allow the convergence of (7.1). This is the case for the Bernoulli-Dunkl and Euler-Dunkl polynomials studied in [15], for which the corresponding function $A(t)$ is able to guarantee the convergence of (7.1) for a certain range of x, but not, for instance, in the case $A(t) = 1$ or $A(t) = 1/(1 - t)$.

In particular, if we try to apply the method to $G(x, -t) = E_\alpha(-xt)/(1 + t)$ with $\alpha \neq -1/2$, we get the integral

$$\int_0^\infty \frac{E_\alpha(-xt)}{1 + t} t^{s-1}\, dt,$$

which does not converge for any $x \in \mathbb{R}$. Hence, the method fails here and we can not find in this way an s-analytic function $H_\alpha(s, x)$ such that $H_\alpha(-n, x) = \gamma_{n,\alpha} e_{n,\alpha}(x)$ (or other suitable multiplicative constants) which would be the Dunkl extension of (3.6). Is there another way to find the desired function?

Taking into account the definition (3.3) and the identity (3.7) in the classical case, this will perhaps lead to an "incomplete Gamma-Dunkl function" and/or to a "Gamma-Dunkl function" with suitable properties.

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ORIGINAL PAPER

Boole-Dunkl polynomials and generalizations

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Abstract
Appell sequences of polynomials can be extended to the Dunkl context replacing the ordinary derivative by the Dunkl operator on the real line, and the exponential function by the Dunkl kernel. In a similar way, discrete Appell sequences can be extended to the Dunkl context; here, the role of the ordinary translation is played by the Dunkl translation, which is a much more intricate operator. Some sequences as the falling factorials or the Bernoulli polynomials of the second kind have already been extended and investigated in the mathematical literature. In this paper, we study the discrete Appell version of the Euler polynomials, usually known as Euler polynomials of the second kind or Boole polynomials. We show how to define the Dunkl extension of these polynomials (and some of their generalizations), and prove some relevant properties and relations with other polynomials and with Stirling-Dunkl numbers.

Keywords Appell-Dunkl sequences · Discrete Appell-Dunkl sequences · Euler-Dunkl polynomials · Boole polynomials · Boole-Dunkl polynomials · Dunkl transform · Stirling-Dunkl numbers

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1 Introduction

An Appell sequence $\{P_k(x)\}_{k=0}^{\infty}$ is a sequence of polynomials such that

$$\frac{d}{dx} P_k(x) = k P_{k-1}(x), \quad k \geq 1, \tag{1.1}$$

and whose generating function is given by

$$A(t)e^{xt} = \sum_{k=0}^{\infty} P_k(x) \frac{t^k}{k!}, \tag{1.2}$$

where $A(t)$ is a function analytic at $t = 0$ such that $A(0) \neq 0$. Some examples of this kind of sequences are the trivial monomial case $\{x^k\}_{k=0}^{\infty}$, the Bernoulli polynomials and the Euler polynomials, whose generating functions are (1.2) with $A(t) = 1$, $A(t) = t/(e^t - 1)$ and $A(t) = 2/(e^t + 1)$, respectively.

Actually, some alternative methods can be used to introduce Appell sequences. For instance, it is not necessary to have an analytic function $A(t)$, but only a formal expansion. Also, a general framework about polynomial expansions of analytic functions (generating functions of polynomial sequences) can be used, as in [1, 2]. Moreover, more general expansions than Appell sequences are Sheffer sequences of polynomials, with generating functions of the form $A(t)e^{xB(t)}$ (see, for instance [3, Chapter 10]), and Brenke polynomials, with generating functions of the form $A(t)B(xt)$ (see [4] and [3, (24.7.2), p. 654]). Finally, it is worth mentioning that Appell and Sheffer sequences can also be studied in the framework of Roman and Rota's umbral calculus, see [5, 6]. But this is not necessary for the goals of this paper, that is focused in the Dunkl context, so we will not give more details on the above mentioned frameworks.

A discrete Appell sequence $\{p_k(x)\}_{k=0}^{\infty}$ is a sequence of polynomials such that

$$\Delta p_k(x) = p_k(x+1) - p_k(x) = k p_{k-1}(x), \quad k \geq 1.$$

Note that the differential operator in (1.1) has been changed by the forward difference operator $\Delta f(x) = f(x+1) - f(x)$. In this case, the generating function is

$$A(t)(1+t)^x = \sum_{k=0}^{\infty} p_k(x) \frac{t^k}{k!}, \tag{1.3}$$

where $A(t)$ is a function analytic at $t = 0$ such that $A(0) \neq 0$. The trivial case, that is, being $A(t) = 1$ in (1.3), is the falling factorial $\{x^{\underline{k}}\}_{k=0}^{\infty}$ defined by

$$x^{\underline{k}} = x(x-1)\cdots(x-k+1) = \prod_{j=0}^{k-1}(x-j).$$

Although other notations have been used for these polynomials, here we follow [7] and [8, §2.6, p. 47]. The corresponding discrete Bernoulli polynomials, which we will denote by $\{b_k(x)\}_{k=0}^{\infty}$, are defined taking $A(t) = t/\log(1+t)$ in (1.3), that is,

$$\frac{t}{\log(1+t)}(1+t)^x = \sum_{k=0}^{\infty} b_k(x) \frac{t^k}{k!}. \tag{1.4}$$

The above polynomials were introduced by Jordan [9] and Rey Pastor [10] in 1929 and are usually called the Bernoulli polynomials of the second kind (see also [11]).

Jordan [12, § 113, p. 317] also introduced the discrete Euler polynomials $\{e_k(x)\}_{k=0}^{\infty}$, which he called Boole polynomials. They are defined in terms of a generating function in the following way:

$$\frac{2}{2+t}(1+t)^x = \sum_{k=0}^{\infty} e_k(x)\frac{t^k}{k!}. \tag{1.5}$$

These polynomials could be called the Euler polynomials of the second kind (by analogy with the Bernoulli polynomials) as we can read in the title of [13]. In addition, we can also find them in the literature as the Changhee polynomials (see [14]). In this paper, we will refer to them as Boole polynomials, following the original work [12].

If the forward difference Δ is changed by the central difference operator

$$\Delta_c f(x) = \frac{f(x+1) - f(x-1)}{2},$$

then a new family of polynomials can be defined as follows: a central discrete Appell sequence $\{q_k(x)\}_{k=0}^{\infty}$ is a sequence of polynomials such that

$$\Delta_c q_k(x) = k q_{k-1}(x), \quad k \geq 1,$$

and they can also be defined using a Taylor generating expansion

$$A(t)\left(t + \sqrt{1+t^2}\right)^x = \sum_{k=0}^{\infty} q_k(x)\frac{t^k}{k!}, \tag{1.6}$$

where $A(t)$ is a function analytic such that $A(0) \neq 0$. The sequence obtained in the trivial case $A(t) = 1$ is the sequence of the central falling factorial polynomials that will be denoted $\{f_k(x)\}_{k=0}^{\infty}$ and satisfies

$$\left(t + \sqrt{1+t^2}\right)^x = \sum_{k=0}^{\infty} f_k(x)\frac{t^k}{k!}. \tag{1.7}$$

These polynomials have been studied in [15] and [16] (note that the so-called "of the second kind" have a different definition, see [17]). Taking the generating function (see [18, § 6])

$$\frac{t}{\log(t+\sqrt{1+t^2})}\left(t + \sqrt{1+t^2}\right)^x = \sum_{k=0}^{\infty} b_k^{II}(x)\frac{t^k}{k!}, \tag{1.8}$$

the central Bernoulli polynomials of the second kind $\{b_k^{II}(x)\}_{k=0}^{\infty}$ are defined. We have not been able to find the corresponding central Boole polynomials in the literature, so, in what follows, we are going to define the central Boole polynomials by means of the operator Δ_c.

In fact, we will consider a more general approach, working with the generalized Boole and Euler polynomials. For an integer $r \geq 0$, the generalized Boole polynomials $\{e_k^{(r)}(x)\}_{k=0}^{\infty}$ of order r are defined by means of the generating function

$$\left(\frac{2}{2+t}\right)^r (1+t)^x = \sum_{k=0}^{\infty} \frac{e_k^{(r)}(x)}{k!} t^k \tag{1.9}$$

and satisfy

$$\Delta e_k^{(r)}(x) = k e_{k-1}^{(r)}(x). \tag{1.10}$$

When $r = 1$, we obtain the classical Boole polynomials (1.5) and, when $r = 0$, we have the falling factorials $x^{\underline{k}}$. Now, we define the "mean operator"

$$Mf(x) = \frac{1}{2}\left(f(x+1) + f(x)\right).$$

In what follows, we will sometimes denote the operator M as M_x to emphasize that the involved variable is x if we apply it to a function of several variables. We will also use some abuse of notation for $M(f(\cdot,t))(x)$ such as $M_x(f(x,t))$, $M(f(x,t))(x)$, or similar.

Performing the operator M to (1.9), we see that

$$Me_k^{(r)}(x) = e_k^{(r-1)}(x), \tag{1.11}$$

since

$$M_x\left(\left(\frac{2}{2+t}\right)^r (1+t)^x\right) = \left(\frac{2}{2+t}\right)^{r-1} (1+t)^x. \tag{1.12}$$

Replacing t by $e^t - 1$ in the left part of (1.9), we obtain the function $(2/(e^t+1))^r e^{xt}$ which is the generating function of the generalized Euler polynomials $E_k^{(r)}(x)$ of order r, that is,

$$\left(\frac{2}{e^t+1}\right)^r e^{xt} = \sum_{k=0}^{\infty} E_k^{(r)}(x)\frac{t^k}{k!}. \tag{1.13}$$

Of course, when $r = 1$ we recover the classical Euler polynomials $E_k(x)$ and when $r = 0$ we have the trivial case x^k. It is well-known that these polynomials satisfy

$$ME_k^{(r)}(x) = \frac{1}{2}\left(E_k^{(r)}(x+1) + E_k^{(r)}(x)\right) = E_k^{(r-1)}(x), \tag{1.14}$$

since

$$M_x\left(\left(\frac{2}{e^t+1}\right)^r e^{xt}\right) = \left(\frac{2}{e^t+1}\right)^{r-1} e^{xt}. \tag{1.15}$$

With the aim of also adapting the operator M, let M_{c} be the "central mean operator" defined by

$$M_{\mathrm{c}} f(x) = \frac{1}{2}(f(x+1) + f(x-1))$$

(we will also use the same kind of above-mentioned abuses of notation). We would like M_{c} to satisfy a property analogous to (1.12) with the new generating function. As $\left(t+\sqrt{1+t^2}\right)^x$ plays the role of $(1+t)^x$ in the central case, we are looking for a function $A(t)$ such that

$$M_{\mathrm{c},x}\left(A(t)^r\left(t+\sqrt{1+t^2}\right)^x\right) = A(t)^{r-1}\left(t+\sqrt{1+t^2}\right)^x, \tag{1.16}$$

by analogy to (1.15). Since

$$\begin{aligned} M_{\mathrm{c},x}\left(\left(t+\sqrt{1+t^2}\right)^x\right) &= \frac{1}{2}\left(\left(t+\sqrt{1+t^2}\right)^{x+1} + \left(t+\sqrt{1+t^2}\right)^{x-1}\right) \\ &= \frac{1}{2}\left(t+\sqrt{1+t^2}\right)^x\left(t+\sqrt{1+t^2}+\frac{1}{t+\sqrt{1+t^2}}\right) \\ &= \frac{t^2+1+t\sqrt{1+t^2}}{t+\sqrt{1+t^2}}\left(t+\sqrt{1+t^2}\right)^x, \end{aligned} \tag{1.17}$$

we define the generalized central Boole polynomials of order r, $\{e_{k,c}^{(r)}(x)\}_{k=0}^{\infty}$, by means of the generating function

$$\left(\frac{t+\sqrt{1+t^2}}{t^2+1+t\sqrt{1+t^2}}\right)^r \left(t+\sqrt{1+t^2}\right)^x = \sum_{k=0}^{\infty} e_{k,c}^{(r)}(x)\frac{t^k}{k!}. \tag{1.18}$$

It is clear that this function satisfies (1.16). Note that, when $r=0$, the polynomials $e_{k,c}^{(0)}(x)$ are the central falling factorial polynomials $f_k(x)$ defined in (1.7). Moreover, it is easy to see that

$$\Delta_c e_{k,c}^{(r)}(x) = k e_{k-1,c}^{(r)}(x),$$

and

$$M_c(e_{k,c}^{(r)})(x) = e_{k,c}^{(r-1)}(x), \tag{1.19}$$

obtaining the analogous properties to (1.10) and (1.11), respectively.

Replacing $t+\sqrt{1+t^2}$ by e^t in (1.18) we obtain $(2e^t/(e^{2t}+1))^r e^{xt}$, which is the generating function of a sequence of polynomials $\{E_{k,c}^{(r)}(x)\}_{k=0}^{\infty}$ that we call generalized central Euler polynomials of order r, that is,

$$\left(\frac{2e^t}{e^{2t}+1}\right)^r e^{xt} = \sum_{k=0}^{\infty} E_{k,c}^{(r)}(x)\frac{t^k}{k!}. \tag{1.20}$$

Performing the operator M_c to (1.20) we obtain

$$M_c E_{k,c}^{(r)}(x) = E_{k,c}^{(r-1)}(x), \tag{1.21}$$

which is the central version of (1.14).

The main goals of this paper are to extend the Boole polynomials to the Dunkl context, and to prove some relevant properties of them. The paper is structured as follows. Section 2 is devoted to define Appell-Dunkl polynomials and discrete Appell-Dunkl polynomials (Sect. 2.1), and all the necessary concepts and notation concerning the Dunkl context are provided (Sect. 2.2). Sections 3 and 4 are dedicated to the Boole-Dunkl polynomials and their properties.

2 The Dunkl framework

2.1 Appell-Dunkl polynomials and discrete Appell-Dunkl polynomials

In the mathematical literature, there are many generalizations of Appell polynomials by means of parameters in the function $A(t)$. Other kind of extensions can be obtained by replacing the derivative operator as shown in [19]. In this regard, an interesting generalization of Appell polynomials is given in [20] by replacing the operator $\frac{d}{dx}$ in (1.1) by the Dunkl operator Λ_α on the real line (for the group $\mathbb{Z}_2$)

$$\Lambda_\alpha f(x) = \frac{d}{dx} f(x) + \frac{2\alpha+1}{2}\left(\frac{f(x)-f(-x)}{x}\right), \tag{2.1}$$

where $\alpha > -1$ is a fixed parameter (see [21, 22]); many typical problems of the classical world have been extended to the Dunkl context (see, for instance, [23–36]). In that setting, an Appell-Dunkl sequence $\{P_{k,\alpha}\}_{k=0}^{\infty}$ is a sequence of polynomials that satisfies

$$\Lambda_\alpha P_{k,\alpha}(x) = \theta_{k,\alpha} P_{k-1,\alpha}(x), \qquad \text{where } \theta_{k,\alpha} = k + (\alpha + 1/2)(1 - (-1)^k)$$

(instead of $\Lambda_\alpha P_{k,\alpha} = k P_{k-1,\alpha}$, the previous definition uses another multiplicative constant $\theta_{k,\alpha}$ for convenience with the notation). Of course, in the case $\alpha = -1/2$, the operator Λ_α is the ordinary derivative and Appell-Dunkl sequences become classical Appell sequences (and $\theta_{k,\alpha} = k$).

Appell-Dunkl polynomials can be also defined by means of the generating function

$$A(t)E_\alpha(xt) = \sum_{k=0}^{\infty} P_{k,\alpha}(x)\frac{t^k}{\gamma_{k,\alpha}}, \tag{2.2}$$

where $A(t)$ is an analytic function defined in a neighborhood of 0 such that $A(0) \neq 0$. The Dunkl kernel $E_\alpha(xt)$ and the sequence $\{\gamma_{k,\alpha}\}_{k=0}^{\infty}$ will be explained in detail in Subsection 2.2. They play the roles of the exponential function and the factorial, respectively, in (1.2). Indeed, $E_{-1/2}(z) = e^z$ and $\gamma_{k,-1/2} = k!$ when $\alpha = -1/2$. For any $\lambda \in \mathbb{C}$, it holds

$$\Lambda_\alpha E_\alpha(\lambda x) = \lambda E_\alpha(\lambda x),$$

which generalizes the property $\frac{d}{dx}e^{\lambda x} = \lambda e^{\lambda x}$ of the exponential function. Note that $P_{k,\alpha}(x)$ in (2.2) are particular cases of Brenke polynomials (and their discrete versions $p_{k,\alpha}(x)$ that we will see later in (2.5) can be studied with the ideas of [3, Theorem 10.1.4] for Sheffer A-type zero polynomials relative to an operator T). Although these approaches are no doubt of interest by themselves, their generality does not allow a deeper study of the polynomials in terms of the Dunkl theory. In addition, a specific umbral calculus in the Dunkl context has not been explored yet; we plan to develop this idea in a forthcoming paper.

The first Appell polynomials extended to the Dunkl context were the Hermite polynomials. These new polynomials are called the generalized Hermite polynomials (see, for example, [22, 37]). The extensions of the Bernoulli and Euler polynomials to the Dunkl context were introduced in [20] and [38], respectively. In [39, 40] and [41], many properties of the Bernoulli-Dunkl and Euler-Dunkl polynomials are presented. Some other Appell-Dunkl polynomials have been studied in [42] (see also the open problem proposed in [43, § 7]).

To extend discrete Appell polynomials to the Dunkl context is necessary to define a suitable difference operator. From (2.1), we see that the role of 0 and 1 in the classical case is played by -1 and 1 in the Dunkl context. Hence, it seems more natural to generalize the central difference operator Δ_c instead of the forward difference operator Δ.

The generalization of Δ_c in the Dunkl context is given in [44] in the following way:

$$\Delta_\alpha f(x) = (\alpha + 1)(\tau_1 - \tau_{-1}) f(x).$$

Here, τ_y is the Dunkl translation operator (see [22]) that, for a function f, is

$$\tau_y f(x) = \sum_{k=0}^{\infty} \Lambda_\alpha^k f(x)\frac{y^k}{\gamma_{k,\alpha}}, \qquad \alpha > -1, \tag{2.3}$$

where Λ_α^0 is the identity operator and $\Lambda_\alpha^{n+1} = \Lambda_\alpha(\Lambda_\alpha^n)$. Note that, for $\alpha = -1/2$, the translation $\tau_y f$ is just the Taylor expansion of a function f around a fixed point x, that is,

$$f(x+y) = \sum_{k=0}^{\infty} f^{(k)}(x)\frac{y^k}{k!},$$

and $\Delta_{-1/2} = \Delta_c$. In order to define the Dunkl translation (2.3), it is assumed that the function f is in C^∞ and also that the series on the right-hand side is convergent. In particular, this is true when f is a polynomial, because the series (2.3) has only a finite number of nonzero terms. Some other properties of the Dunkl translation, including an integral expression that is more general than (2.3), can be found in [22, 45], and [35].

Using the operator Δ_α, discrete Appell-Dunkl polynomials are defined in [44] as a sequence of polynomials $\{p_{k,\alpha}(x)\}_{k=0}^\infty$ such that

$$\Delta_\alpha p_{k,\alpha}(x) = \theta_{k,\alpha} p_{k-1,\alpha}(x), \qquad \theta_{k,\alpha} = k + (\alpha + 1/2)(1 - (-1)^k) \tag{2.4}$$

and they can also be defined by the Taylor generating expansion

$$A(t)E_\alpha(xG_\alpha^{-1}(t)) = \sum_{k=0}^\infty p_{k,\alpha}(x)\frac{t^k}{\gamma_{k,\alpha}}, \tag{2.5}$$

where $A(t)$ is an analytic function such that $A(0) \neq 0$ and $G_\alpha^{-1}(t)$ is the inverse of a function $G_\alpha(t)$ that will be defined in Sect. 2.2; this $G_\alpha^{-1}(t)$ plays the role of $\log(t + \sqrt{1 + t^2})$ in (1.6), where we can write

$$\left(t + \sqrt{1 + t^2}\right)^x = \exp\left(x \log(t + \sqrt{1 + t^2})\right).$$

Also in [44], the analogous sequence in the Dunkl context of the (central) falling factorial is introduced and denoted by $\{f_{k,\alpha}\}_{k=0}^\infty$. That is,

$$E_\alpha(xG_\alpha^{-1}(t)) = \sum_{k=0}^\infty f_{k,\alpha}(x)\frac{t^k}{\gamma_{k,\alpha}}. \tag{2.6}$$

These polynomials have been used in [46] to define the Stirling numbers in the Dunkl context. Moreover, the Bernoulli-Dunkl polynomials of the second kind $\{b_{k,\alpha}(x)\}_{k=0}^\infty$ are also introduced in [44, § 6] by taking the function $A(t) = t/G_\alpha^{-1}(t)$ in (2.5):

$$\frac{t}{G_\alpha^{-1}(t)} E_\alpha(xG_\alpha^{-1}(t)) = \sum_{k=0}^\infty b_{k,\alpha}(x)\frac{t^k}{\gamma_{k,\alpha}}. \tag{2.7}$$

Of course, (2.7) becomes (1.8) when $\alpha = -1/2$.

2.2 Details of the notation for the Dunkl context

To define the Dunkl exponential function or Dunkl kernel $E_\alpha(z)$ and some related functions, we need to introduce some previous notation.

Let $\gamma_{n,\alpha}$ denote the numbers

$$\gamma_{n,\alpha} = \begin{cases} 2^{2k}k!\,(\alpha+1)_k, & \text{if } n = 2k, \\ 2^{2k+1}k!\,(\alpha+1)_{k+1}, & \text{if } n = 2k+1, \end{cases}$$

where $(a)_n$ denotes the Pochhammer symbol

$$(a)_n = a(a+1)(a+2)\cdots(a+n-1) = \frac{\Gamma(a+n)}{\Gamma(a)}, \quad n \geq 0;$$

they satisfy $\gamma_{n,\alpha}/\gamma_{n-1,\alpha} = \theta_{n,\alpha}$ for $n \geq 1$. Let $\binom{n}{k}_\alpha$ be the numbers

$$\binom{n}{k}_\alpha = \frac{\gamma_{n,\alpha}}{\gamma_{k,\alpha}\gamma_{n-k,\alpha}}.$$

If $\alpha = -1/2$, then $\gamma_{n,-1/2} = n!$ and $\binom{n}{k}_{-1/2}$ is the classical combinatorial number $\binom{n}{k}$.

Now, let $J_\alpha(z)$ be the Bessel function of order $\alpha > -1$ and consider the entire function

$$\mathcal{I}_\alpha(z) = 2^\alpha \Gamma(\alpha+1)\,\frac{J_\alpha(iz)}{(iz)^\alpha} = \Gamma(\alpha+1)\sum_{n=0}^{\infty}\frac{(z/2)^{2n}}{n!\,\Gamma(n+\alpha+1)} = \sum_{n=0}^{\infty}\frac{z^{2n}}{\gamma_{2n,\alpha}}$$

(the function $\mathcal{I}_\alpha$ is a small variation of the so-called modified Bessel function of the first kind and order α, usually denoted by I_α, see [47] or [48]). We denote $G_\alpha(z) := z\mathcal{I}_{\alpha+1}(z)$ and then, it is easy to see that

$$\frac{1}{2(\alpha+1)}G_\alpha(z) = \sum_{n=0}^{\infty}\frac{z^{2n+1}}{\gamma_{2n+1,\alpha}}.$$

The Dunkl kernel $E_\alpha(z)$ is defined in terms of these functions as

$$E_\alpha(z) = \mathcal{I}_\alpha(z) + \frac{1}{2(\alpha+1)}\,G_\alpha(z) = \sum_{n=0}^{\infty}\frac{z^n}{\gamma_{n,\alpha}}, \qquad z \in \mathbb{C}.$$

In the case $\alpha = -1/2$, this is the decomposition $E_{-1/2}(z) = \cosh(z) + \sinh(z) = e^z$. Recall that E_α is invariant under the Dunkl operator (2.1) in the same way that the exponential function is invariant under the ordinary derivative. Hence, it is easy to check that (2.2) is the characterization of Appell-Dunkl sequences by means of a generating function.

We also need to give the notation for some families of polynomials (and numbers) that have been defined in other works, and explain some of their properties that will be used later in this paper.

In [39], the generalized Bernoulli-Dunkl polynomials of order r ($r \geq 0$ integer), $\{\mathfrak{B}^{(r)}_{k,\alpha}(x)\}_{k=0}^{\infty}$, are introduced as

$$\frac{E_\alpha(xt)}{\mathcal{I}_{\alpha+1}(t)^r} = \sum_{k=0}^{\infty}\mathfrak{B}^{(r)}_{k,\alpha}(x)\,\frac{t^k}{\gamma_{k,\alpha}}. \tag{2.8}$$

In the same way, the generalized Euler-Dunkl polynomials of order r ($r \geq 0$ integer), $\{\mathcal{E}^{(r)}_{k,\alpha}(x)\}_{k=0}^{\infty}$, are given by

$$\frac{E_\alpha(xt)}{\mathcal{I}_\alpha(t)^r} = \sum_{k=0}^{\infty}\mathcal{E}^{(r)}_{k,\alpha}(x)\,\frac{t^k}{\gamma_{k,\alpha}}. \tag{2.9}$$

When $r = 1$, $\{\mathfrak{B}^{(1)}_{k,\alpha}(x)\}_{k=0}^{\infty}$ and $\{\mathcal{E}^{(1)}_{k,\alpha}(x)\}_{k=0}^{\infty}$ correspond to the Bernoulli-Dunkl and Euler-Dunkl polynomials introduced in [20] and [38], respectively.

In order to define the discrete Appell-Dunkl polynomials, we need to see that (2.5) is well-defined. This is the case because the function

$$G_\alpha(z) = z\mathcal{I}_{\alpha+1}(z) = z\,{}_0F_1(\alpha+2, z^2/4) = 2(\alpha+1)\sum_{n=0}^{\infty}\frac{z^{2n+1}}{\gamma_{2n+1,\alpha}}, \quad z \in \mathbb{C},$$

is odd, non-negative for $z > 0$ and increasing (for $z > 0$, the derivative term by term of the series is positive), so there exists the inverse function $G_\alpha^{-1}(z)$. Therefore, we say that a discrete Appell-Dunkl sequence $\{p_{k,\alpha}(x)\}_{k=0}^{\infty}$ is a sequence that satisfies (2.5) (moreover, recall (2.4)).

Then, the Dunkl (central) falling factorial polynomials $\{f_{k,\alpha}(x)\}_{k=0}^{\infty}$ are defined in [44, § 3] as the polynomials obtained in the trivial case; i.e., taking $A(t)=1$ in (2.5), as we have seen in (2.6).

In the classical case, if $\{p_k(x)\}_{k=0}^{\infty}$ is a discrete Appell sequence (see (1.3)), it satisfies

$$p_k(x+y)=\sum_{j=0}^{k}\binom{k}{j}p_j(x)y^{\underline{k-j}}.$$

In [44, Theorem 3.1], the analogous formula for the Appell-Dunkl polynomials is proved, where $f_{k,\alpha}(y)$ plays the role of the falling factorial $y^{\underline{k}}$. More precisely, if $\{p_{k,\alpha}(x)\}_{k=0}^{\infty}$, $\alpha>-1$, is a discrete Appell-Dunkl sequence of polynomials defined by (2.5), then

$$\tau_y(p_{k,\alpha}(\cdot))(x)=\sum_{j=0}^{k}\binom{k}{j}_{\alpha}p_{j,\alpha}(x)\,f_{k-j,\alpha}(y). \tag{2.10}$$

From [44, Theorem 5.1] it is easy to see that the Dunkl (central) falling factorials $\{f_{k,\alpha}(x)\}_{k=0}^{\infty}$ are expressed in terms of the generalized Bernoulli-Dunkl polynomials $\{\mathfrak{B}_{k,\alpha}^{(k)}(x)\}_{k=0}^{\infty}$ as

$$f_{k,\alpha}(x)=\sum_{j=0}^{k}\frac{j}{k}\binom{k}{j}_{\alpha}\mathfrak{B}_{k-j,\alpha}^{(k)}(0)x^j,\quad k=1,2,\dots. \tag{2.11}$$

Then, relation (2.11) is used in [46] to define the Stirling-Dunkl numbers of the first kind of order $\alpha>-1$ as

$$s^{\alpha}(k,j)=\frac{j}{k}\binom{k}{j}_{\alpha}\mathfrak{B}_{k-j,\alpha}^{(k)}(0), \tag{2.12}$$

which, replaced in (2.11), gives

$$f_{k,\alpha}(x)=\sum_{j=0}^{k}s^{\alpha}(k,j)x^j,\quad k=1,2,\dots.$$

As we can see in [46, Theorem 3.4], the generating function of the Stirling-Dunkl numbers of the first kind of order α is

$$\frac{G_{\alpha}^{-1}(t)^r}{\gamma_{r,\alpha}}=\sum_{k=r}^{\infty}s^{\alpha}(k,r)\frac{t^k}{\gamma_{k,\alpha}}=\sum_{k=r}^{\infty}\frac{r}{k}\binom{k}{r}_{\alpha}\mathfrak{B}_{k-r}^{(k)}(0)\frac{t^k}{\gamma_{k,\alpha}}. \tag{2.13}$$

Finally, the Stirling-Dunkl numbers of the second kind of order $\alpha>-1$, $S_{\alpha}(k,j)$, are also defined in [46] as

$$x^k=\sum_{j=0}^{k}S_{\alpha}(k,j)\,f_{j,\alpha}(x),$$

and their generating function is

$$\frac{(\alpha+1)^r}{\gamma_{r,\alpha}}(E_{\alpha}(t)-E_{\alpha}(-t))^r=\frac{G_{\alpha}(t)^r}{\gamma_{r,\alpha}}=\sum_{k=r}^{\infty}S_{\alpha}(k,r)\frac{t^k}{\gamma_{k,\alpha}}. \tag{2.14}$$

3 Boole-Dunkl and generalized Boole-Dunkl polynomials

In order to extend the Boole polynomials to the Dunkl context, we focus on the operator M_{c} and define the Dunkl (central) mean operator M_α as

$$M_\alpha f(x) = \frac{1}{2}(\tau_{1,x} + \tau_{-1,x}) f(x).$$

It was proved in [39, Theorem 8.5] that the generalized Euler-Dunkl polynomials of order r ($r \geq 0$ integer) given by (2.9) satisfy

$$M_\alpha \mathcal{E}_{k,\alpha}^{(r)}(x) = \mathcal{E}_{k,\alpha}^{(r-1)}(x),$$

which is the extension of property (1.21). Note that

$$\begin{aligned} M_{\alpha,x}(E_\alpha(xG_\alpha^{-1}(t))) &= \frac{1}{2}\sum_{j=0}^{\infty}\frac{1}{\gamma_{j,\alpha}}G_\alpha^{-1}(t)^j E_\alpha(xG_\alpha^{-1}(t)) \\ &\quad + \frac{1}{2}\sum_{j=0}^{\infty}\frac{(-1)^j}{\gamma_{j,\alpha}}G_\alpha^{-1}(t)^j E_\alpha(xG_\alpha^{-1}(t)) \\ &= \frac{E_\alpha(xG_\alpha^{-1}(t))}{2}\sum_{j=0}^{\infty}\frac{2}{\gamma_{2j,\alpha}}G_\alpha^{-1}(t)^{2j} \\ &= E_\alpha(xG_\alpha^{-1}(t))\mathcal{I}_\alpha(G_\alpha^{-1}(t)). \end{aligned}$$

So, taking (1.17) into account, we choose the analytic function $1/\mathcal{I}_\alpha(G_\alpha^{-1}(t))$ to be the Dunkl version of $(t + \sqrt{1+t^2})/(t^2 + 1 + t\sqrt{1+t^2})$.

Now, we can define the generalized Boole-Dunkl polynomials of order r, $\{e_{k,\alpha}^{(r)}(x)\}_{k=0}^{\infty}$, by means of the generating function

$$\frac{E_\alpha(xG_\alpha^{-1}(t))}{\mathcal{I}_\alpha(G_\alpha^{-1}(t))^r} = \sum_{k=0}^{\infty} e_{k,\alpha}^{(r)}(x)\frac{t^k}{\gamma_{k,\alpha}}. \tag{3.1}$$

The function $\mathcal{I}_\alpha(G_\alpha^{-1}(t))^r$ is even, and this easily implies that $e_{2k,\alpha}^{(r)}(x)$ is an even polynomial for $k \geq 0$ and $e_{2k+1,\alpha}^{(r)}(x)$ is an odd polynomial for $k \geq 0$ (and hence, it vanishes at $x = 0$).

The first Boole-Dunkl polynomials of order r are

$$\begin{aligned} &e_{0,\alpha}^{(r)}(x) = 1, \qquad e_{1,\alpha}^{(r)}(x) = x, \qquad e_{2,\alpha}^{(r)}(x) = x^2 - r, \\ &e_{3,\alpha}^{(r)}(x) = x^3 - \frac{1+\alpha+r(2+\alpha)}{1+\alpha}x, \\ &e_{4,\alpha}^{(r)}(x) = x^4 - \frac{2(2+2\alpha+r(2+\alpha))}{1+\alpha}x^2 + \frac{r(5+4\alpha+r(2+\alpha))}{1+\alpha}, \\ &e_{5,\alpha}^{(r)}(x) = x^5 - \frac{2(3+\alpha)(3+3\alpha+r(2+\alpha))}{(1+\alpha)(2+\alpha)}x^3 \\ &\qquad + \left(5 + \frac{6}{2+\alpha} + \frac{r(3+\alpha)(7+6\alpha+r(2+\alpha))}{(1+\alpha)^2}\right)x. \end{aligned}$$

In the following result, we extend formula (1.19) to the Dunkl context:

Theorem 3.1 *Let $\{e_{k,\alpha}^{(r)}(x)\}_{k=0}^{\infty}$ be the sequence of generalized Boole-Dunkl polynomials of order r with $r \geq 0$ integer. Then,*

$$\Delta_\alpha e_{k,\alpha}^{(r)}(x) = \theta_{k,\alpha} e_{k-1,\alpha}^{(r)}(x), \quad k \geq 1, \tag{3.2}$$

and

$$M_\alpha e_{k,\alpha}^{(r)}(x) = e_{k,\alpha}^{(r-1)}(x). \tag{3.3}$$

Proof As $\Delta_\alpha(E_\alpha(xG_\alpha^{-1}(t))) = tE_\alpha(xG_\alpha^{-1}(t))$, it is clear that (3.2) holds. By the choice of the generating function, we know that

$$M_{\alpha,x}\left(\frac{E_\alpha(xG_\alpha^{-1}(t))}{\mathcal{I}_\alpha(G_\alpha^{-1}(t))^r}\right) = \frac{E_\alpha(xG_\alpha^{-1}(t))}{\mathcal{I}_\alpha(G_\alpha^{-1}(t))^{r-1}}. \tag{3.4}$$

Writing each function of (3.4) as power series using (3.1), we get

$$\sum_{k=0}^{\infty} M_\alpha(e_{k,\alpha}^{(r)}(x))\frac{t^k}{\gamma_{k,\alpha}} = \sum_{k=0}^{\infty} e_{k,\alpha}^{(r-1)}(x)\frac{t^k}{\gamma_{k,\alpha}},$$

and then we have proved (3.3). □

A nice property of the generalized Boole polynomials is (see [13, (3.15)])

$$e_k^{(r+s)}(x+y) = \sum_{l=0}^{k}\binom{k}{l} e_l^{(r)}(x) e_{k-l}^{(s)}(y),$$

which taking $s = 0$ gives

$$e_k^{(r)}(x+y) = \sum_{l=0}^{k}\binom{k}{l} e_l^{(r)}(x) y^{\underline{k-l}}.$$

In the Dunkl context, we have the following:

Theorem 3.2 *For $\alpha > -1$, the generalized Boole-Dunkl polynomials satisfy*

$$\tau_y(e_{k,\alpha}^{(r+s)})(x) = \sum_{j=0}^{k}\binom{k}{j}_\alpha e_{j,\alpha}^{(r)}(x) e_{k-j,\alpha}^{(s)}(y), \tag{3.5}$$

with $r \geq 0$, $s \geq 0$ integers. When we take $s = 0$, we have

$$\tau_y(e_{k,\alpha}^{(r)})(x) = \sum_{j=0}^{k}\binom{k}{j}_\alpha e_{j,\alpha}^{(r)}(x) f_{k-j,\alpha}(y). \tag{3.6}$$

Proof From (3.1) for r and s, we have

$$\begin{aligned}
\frac{E_\alpha(xG_\alpha^{-1}(t))}{\mathcal{I}_\alpha(G_\alpha^{-1}(t))^r}\frac{E_\alpha(yG_\alpha^{-1}(t))}{\mathcal{I}_\alpha(G_\alpha^{-1}(t))^s} &= \left(\sum_{k=0}^{\infty} e_{k,\alpha}^{(r)}(x)\frac{t^k}{\gamma_{k,\alpha}}\right)\left(\sum_{k=0}^{\infty} e_{k,\alpha}^{(s)}(y)\frac{t^k}{\gamma_{k,\alpha}}\right) \\
&= \sum_{k=0}^{\infty}\sum_{j=0}^{k}\frac{e_{j,\alpha}^{(r)}(x)}{\gamma_{j,\alpha}}\frac{e_{k-j,\alpha}^{(s)}(y)}{\gamma_{k-j,\alpha}} t^k \\
&= \sum_{k=0}^{\infty}\left(\sum_{j=0}^{k}\binom{k}{j}_\alpha e_{j,\alpha}^{(r)}(x) e_{k-j,\alpha}^{(s)}(y)\right)\frac{t^k}{\gamma_{k,\alpha}}. \qquad (3.7)
\end{aligned}$$

On the other hand, using (3.1) for $r+s$ and (2.6),

$$\frac{E_\alpha(xG_\alpha^{-1}(t))}{\mathcal{I}_\alpha(G_\alpha^{-1}(t))^{r+s}}E_\alpha(yG_\alpha^{-1}(t)) = \left(\sum_{k=0}^{\infty} e_{k,\alpha}^{(r+s)}(x)\frac{t^k}{\gamma_{k,\alpha}}\right)\left(\sum_{k=0}^{\infty} f_{k,\alpha}(y)\frac{t^k}{\gamma_{k,\alpha}}\right)$$
$$= \sum_{k=0}^{\infty}\sum_{j=0}^{k}\binom{k}{j}_\alpha e_{j,\alpha}^{(r+s)}(x)\, f_{k-j,\alpha}(y)\frac{t^k}{\gamma_{k,\alpha}}. \quad (3.8)$$

Now, equating the coefficients of the series in (3.7) and (3.8) and using (2.10),

$$\sum_{j=0}^{k}\binom{k}{j}_\alpha e_{j,\alpha}^{(r)}(x)e_{k-j,\alpha}^{(s)}(y) = \sum_{j=0}^{k}\binom{k}{j}_\alpha e_{j,\alpha}^{(r+s)}(x)\, f_{k-j,\alpha}(y) = \tau_y(e_{k,\alpha}^{(r+s)})(x).$$

Equation (3.6) is obtained taking $s=0$ in (3.5) since $e_{j,\alpha}^{(0)}(y) = f_{j,\alpha}(y)$. □

Remark 3.3 Note that an expression as (3.5) can be proved for any generalized discrete Appell-Dunkl sequence of order r. That is, if for an analytic function $A(t)$ we take $A(t)^r E_\alpha(xG_\alpha^{-1}(t)) = \sum_{k=0}^{\infty} p_k^{(r)}(x)t^k/\gamma_{k,\alpha}$, the polynomials $p_k^{(r)}(x)$ will satisfy a relation as (3.5), reproducing the same proof.

In [13, (3.18)], it is proved a formula that expresses the generalized Boole polynomials in terms of the generalized Euler and Bernoulli polynomials,

$$e_k^{(r)}(x) = \sum_{j=0}^{k}\binom{k}{j}E_j^{(r)}(x)B_{k-j}^{(k+1)}(1),$$

for $r \geq 0$ integer. Recall that the generalized Euler polynomials of order r that appear in this formula have been mentioned in the introduction (see (1.13)). In a similar way, the generalized Bernoulli polynomials of order r are defined by means of the generating function $\big(t/(e^t-1)\big)^r e^{xt}$. We extend this result below to the Dunkl context.

Theorem 3.4 *For $\alpha > -1$ and $r \geq 0$ integer, the generalized Boole-Dunkl polynomials of order r satisfy*

$$e_{k,\alpha}^{(r)}(x) = \sum_{l=0}^{k}\frac{l}{k}\binom{k}{l}_\alpha \mathcal{E}_{l,\alpha}^{(r)}(x)\mathfrak{B}_{k-l,\alpha}^{(k)}(0), \quad (3.9)$$

and

$$\Lambda_\alpha e_{k,\alpha}^{(r)}(x) = \frac{\theta_{k,\alpha}}{k}\sum_{l=0}^{k-1}(l+1)\binom{k-1}{l}_\alpha \mathcal{E}_{l,\alpha}^{(r)}(x)\mathfrak{B}_{k-l-1,\alpha}^{(k)}(0), \quad (3.10)$$

where $\mathcal{E}_{l,\alpha}^{(r)}(x)$ are the generalized Euler-Dunkl polynomials of order r (2.9) and $\mathfrak{B}_{l,\alpha}^{(k)}(x)$ are the generalized Bernoulli-Dunkl polynomials of order k (2.8).

Proof Put $t = G_\alpha^{-1}(u)$ in (2.9). Then, we have

$$\frac{E_\alpha(xG_\alpha^{-1}(u))}{\mathcal{I}_\alpha(G_\alpha^{-1}(u))^r} = \sum_{k=0}^{\infty}\mathcal{E}_{k,\alpha}^{(r)}(x)\frac{(G_\alpha^{-1}(u))^k}{\gamma_{k,\alpha}}.$$

From (2.13),

$$\frac{E_\alpha(xG_\alpha^{-1}(u))}{\mathcal{I}_\alpha(G_\alpha^{-1}(u))^r} = \sum_{k=0}^{\infty} \mathcal{E}_{k,\alpha}^{(r)}(x) \sum_{l=k}^{\infty} \frac{k}{l} \binom{l}{k}_\alpha \mathfrak{B}_{l-k}^{(l)}(0) \frac{u^l}{\gamma_{l,\alpha}}$$
$$= \sum_{k=0}^{\infty} \left(\sum_{l=0}^{k} \frac{l}{k} \binom{k}{l}_\alpha \mathcal{E}_{l,\alpha}^{(r)}(x) \mathfrak{B}_{k-l}^{(k)}(0) \right) \frac{u^k}{\gamma_{k,\alpha}}.$$

Equating coefficients of this series and (3.1), we get (3.9). Formula (3.10) is obtained applying the operator Λ_α to (3.9). □

Another property of the generalized Boole polynomials is

$$E_k^{(r)}(x) = \sum_{n=0}^{k} \frac{\Delta^n((\cdot)^k)(0)}{n!} e_n^{(r)}(x)$$

(see, for instance, [13, (3.21)]). In the Dunkl case, we have the following:

Theorem 3.5 *For $\alpha > -1$ and $r \geq 0$ integer, the generalized Euler-Dunkl polynomials of order r satisfy*

$$\mathcal{E}_{k,\alpha}^{(r)}(x) = \sum_{n=0}^{k} \frac{\Delta_\alpha^n((\cdot)^k)(0)}{\gamma_{n,\alpha}} e_{n,\alpha}^{(r)}(x).$$

Proof First, we make the substitution $t = G_\alpha(u)$ in (3.1), and then, we use (2.9). We get

$$\sum_{n=0}^{\infty} e_{n,\alpha}^{(r)}(x) \frac{G_\alpha(u)^n}{\gamma_{n,\alpha}} = \frac{E_\alpha(xu)}{(\mathcal{I}_\alpha(u))^r} = \sum_{n=0}^{\infty} \mathcal{E}_{n,\alpha}^{(r)}(x) \frac{u^n}{\gamma_{n,\alpha}}. \tag{3.11}$$

Now, using (2.14) it follows that

$$\sum_{n=0}^{\infty} e_{n,\alpha}^{(r)}(x) \frac{G_\alpha(u)^n}{\gamma_{n,\alpha}} = \sum_{n=0}^{\infty} e_{n,\alpha}^{(r)}(x) \left(\sum_{k=n}^{\infty} S_\alpha(k,n) \frac{u^k}{\gamma_{k,\alpha}} \right)$$
$$= \sum_{k=0}^{\infty} \left(\sum_{n=0}^{k} e_{n,\alpha}^{(r)}(x) S_\alpha(k,n) \right) \frac{u^k}{\gamma_{k,\alpha}}. \tag{3.12}$$

Equating coefficients of the series (3.11) and (3.12),

$$\mathcal{E}_{k,\alpha}^{(r)}(x) = \sum_{n=0}^{k} S_\alpha(k,n) e_{n,\alpha}^{(r)}(x),$$

and, since (see [44, Theorem 5.3])

$$S_\alpha(k,n) = \frac{\Delta_\alpha^n((\cdot)^k)(0)}{\gamma_{n,\alpha}},$$

the result is proved. □

Remark 3.6 Tacitly, in Theorem 3.4 and Theorem 3.5 we have found some beautiful formulas relating the Euler-Dunkl polynomials with the Boole-Dunkl polynomials of order r through the Stirling-Dunkl numbers of the first and of the second kind. These formulas, of independent interest by themselves, are

$$e_{k,\alpha}^{(r)}(x) = \sum_{n=0}^{k} s^{\alpha}(k,n)\mathcal{E}_{n,\alpha}^{(r)}(x), \qquad \mathcal{E}_{k,\alpha}^{(r)}(x) = \sum_{n=0}^{k} S_{\alpha}(k,n)e_{n,\alpha}^{(r)}(x);$$

these kind of expressions are known in the literature as connection formulas. The first one is (3.9) viewed under (2.12); the second one is included in the proof of Theorem 3.5. These formulas are dual, and any of them can be deduced from the other one if we use that the matrices whose elements are the Stirling-Dunkl numbers of the first and of the second kind are inverse (see [46, Theorem 4.1]).

Let us conclude this section by mentioning that not every classical property can be extended to the Dunkl case with a suitable description. From (2.10) or (3.6) (and using that $\tau_y f(x) = \tau_x f(y)$), we readly get

$$e_{k,\alpha}^{(r)}(x) = \sum_{j=0}^{k} \binom{k}{j}_{\alpha} e_{j,\alpha}^{(r)}(0)\, f_{k-j,\alpha}(x). \tag{3.13}$$

In the classical case, this is

$$e_{k}^{(r)}(x) = \sum_{j=0}^{k} \binom{k}{j} e_{j}^{(r)}(0) x^{\underline{k-j}}.$$

In addition, we can easily identify the coefficients $e_j^{(r)}(0)$. Taking $x = 0$ in (1.9) we have

$$\sum_{j=0}^{\infty} \frac{e_j^{(r)}(0)}{j!} t^j = (1+t/2)^{-r} = \sum_{j=0}^{\infty} \binom{-r}{j} \frac{t^j}{2^j} = \sum_{j=0}^{\infty} (-1)^j \binom{r+j-1}{j} \frac{t^j}{2^j},$$

so $e_j^{(r)}(0) = (-1)^j 2^{-j} r(r+1)\cdots(r+j-1)$. In the particular case $r = 1$ we have $e_j(0) = e_j^{(1)}(0) = (-1)^j 2^{-j} j!$ and then

$$e_k(x) = \sum_{j=0}^{k} \frac{(-1)^j}{2^j} k(k-1)\cdots(k-j+1) x^{\underline{k-j}} = k! \sum_{j=0}^{k} \frac{(-1)^j}{2^j} \binom{x}{k-j},$$

a formula that can be found in [12, § 113, (4), p. 318].

However, in the Dunkl case, instead of $A(t) = (2/(2+t))^r$ that appears in (1.9), in (3.1) we have $A(t) = 1/\mathcal{I}_{\alpha}(G_{\alpha}^{-1}(t))^r$. To identify the coefficients $e_{j,\alpha}^{(r)}(0)$ (and then to substitute them in (3.13)) is equivalent to find the expansion of this function $A(t)$ in powers of t, and this does not seem to be an easy task, neither for the case $r = 1$.

4 Some additional properties

In the classical case, each polynomial, $P_n(x)$, of degree n can be expressed as linear combination of the Boole polynomials in the following way:

$$P_n(x) = c_{n,0} + c_{n,1}e_1(x) + c_{n,2}e_2(x) + \cdots + c_{n,n}e_n(x),$$

where

$$c_{n,k} = \frac{M\Delta^k P_n(0)}{k!}, \qquad k = 0, 1, \ldots, n.$$

The analogous result in the Dunkl case is the following:

Theorem 4.1 *Let $P_n(x)$ be a polynomial of degree n. Then,*

$$P_n(x) = c_{n,0} + c_{n,1}e_{1,\alpha}(x) + c_{n,2}e_{2,\alpha}(x) + \cdots + c_{n,n}e_{n,\alpha}(x),$$

where

$$c_{n,k} = \frac{M_\alpha \Delta_\alpha^k P_n(0)}{\gamma_{k,\alpha}}, \quad k = 0, 1, \ldots, n.$$

Proof It is clear that $P_n(x)$ can be expressed as a linear combination of the Boole-Dunkl polynomials, that is,

$$P_n(x) = c_{n,0}e_{0,\alpha}(x) + c_{n,1}e_{1,\alpha}(x) + c_{n,2}e_{2,\alpha}(x) + \cdots + c_{n,n}e_{n,\alpha}(x).$$

From Theorem 3.1, $M_\alpha e_{k,\alpha}(x) = f_{k,\alpha}(x)$ and then

$$M_\alpha P_n(x) = c_{n,0}\, f_{0,\alpha}(x) + c_{n,1}\, f_{1,\alpha}(x) + c_{n,2}\, f_{2,\alpha}(x) + \cdots + c_{n,n}\, f_{n,\alpha}(x).$$

Evaluating at $x = 0$ and, since $f_{0,\alpha}(x) = 1$ and $f_{k,\alpha}(0) = 0$ for $k = 1, \ldots, n$ (this follows from (2.6) taking $x = 0$), we obtain that

$$c_{n,0} = M_\alpha P_n(0).$$

Performing the operator Δ_α to the function $M_\alpha P_n(x)$ we have

$$M_\alpha \Delta_\alpha P_n(x) = c_{n,1}\theta_{1,\alpha}\, f_{0,\alpha}(x) + c_{n,2}\theta_{2,\alpha}\, f_{1,\alpha}(x) + \cdots + c_{n,n}\theta_{n,\alpha}\, f_{n-1,\alpha}(x),$$

so evaluating at $x = 0$,

$$M_\alpha \Delta_\alpha P_n(0) = c_{n,1}\theta_{1,\alpha}.$$

Performing k times, with $k = 0, \ldots, n$, the operator Δ_α to the function $M_\alpha P_n(x)$, we obtain the result. □

To finish the paper, let us see how to extend to the Dunkl context some expressions that in the classical case have the form of integrals.

In [12, Sect. 114], we can find these formulas for the primitive and the integral of $e_n(x)$ in [0, 1]:

$$\int e_n(x)\, dx = \sum_{k=0}^{n+1} \frac{(-1)^{n-k}}{2^{n-k}} \frac{b_{k+1}(x)}{(k+1)!} + c_n,$$

where $b_k(x)$ is the Bernoulli polynomial of the second kind of degree k defined in (1.4) and c_n is a real number. Moreover, the definite integral in [0, 1] is

$$\int_0^1 e_n(x)\, dx = \sum_{k=0}^{n+1} \frac{(-1)^{n-k}}{2^{n-k}} \frac{b_{k+1}(1) - b_{k+1}(0)}{(k+1)!},$$

and, as $b_{k+1}(1) - b_{k+1}(0) = (k+1)b_k(0)$, then

$$\int_0^1 e_n(x)\, dx = \sum_{k=0}^{n+1} \frac{(-1)^{n-k}}{2^{n-k}} \frac{b_k(0)}{k!}.$$

In order to consider this problem in the Dunkl context, we need to take the inverse of the Dunkl operator, defined in [39]: a function F is a Dunkl primitive of f if $\Lambda_\alpha F = f$. Then, for $\alpha > -1$, the Dunkl integral of a function f is

$$\oint f(x)\, d_\alpha x = F(x) + c,$$

where $c \in \mathbb{R}$ is a constant. Then, we can denote

$$\oint_a^b f(x)\, d_\alpha x = F(b) - F(a),$$

where $F(x)$ is a Dunkl primitive of $f(x)$.

Now, we are going to prove the analogous formulas in the Dunkl case. Actually, we do it for the generalized Boole-Dunkl polynomials $e^{(r)}_{k,\alpha}(x)$, that can be expressed in terms of the falling factorial Dunkl polynomials using (3.13). On the other hand, from [44, Theorem 6.2], we know that if $\{b_{k,\alpha}(x)\}_{k=0}^n$ is the sequence of Bernoulli-Dunkl polynomials of the second kind defined in (2.7), we have

$$\Lambda_\alpha b_{k,\alpha}(x) = \theta_{k,\alpha} f_{k-1,\alpha}(x), \qquad k \geq 1.$$

Therefore, if we take the inverse of the Dunkl operator,

$$\oint f_{k,\alpha}(x)\, d_\alpha x = \frac{b_{k+1,\alpha}(x)}{\theta_{k+1,\alpha}} + c_k, \tag{4.1}$$

where c_k is a real number, and we denote

$$\oint_{-1}^{1} f_{k,\alpha}(x)\, d_\alpha x = \frac{b_{k+1,\alpha}(1) - b_{k+1,\alpha}(-1)}{\theta_{k+1,\alpha}}, \tag{4.2}$$

we can prove the following result:

Theorem 4.2 *Let $\{e^{(r)}_{n,\alpha}(x)\}_{n=0}^{\infty}$ be the sequence of generalized Boole-Dunkl polynomials. Then,*

$$\oint e^{(r)}_{n,\alpha}(x)\, d_\alpha x = \sum_{k=0}^{n} \binom{n}{k}_\alpha e^{(r)}_{n-k,\alpha}(0) \frac{b_{k+1,\alpha}(x)}{\theta_{k+1,\alpha}} + c_n,$$

where c_n is a real number, and

$$\oint_{-1}^{1} e^{(r)}_{n,\alpha}(x)\, d_\alpha x = \frac{1}{\alpha+1} \sum_{k=0}^{n} e^{(r)}_{n-k,\alpha}(0) b_{k,\alpha}(0).$$

Proof From (3.13) and (4.1),

$$\oint e^{(r)}_{n,\alpha}(x)\, d_\alpha x = \sum_{k=0}^{n} \binom{n}{k}_\alpha e^{(r)}_{n-k,\alpha}(0) \oint f_{k,\alpha}(x)\, d_\alpha x = \sum_{k=0}^{n} \binom{n}{k}_\alpha e^{(r)}_{n-k,\alpha}(0) \frac{b_{k+1,\alpha}(x)}{\theta_{k+1,\alpha}} + c_n,$$

and we have proved the first equality of the theorem. Therefore,

$$\oint_{-1}^{1} e^{(r)}_{n,\alpha}(x)\, d_\alpha x = \sum_{k=0}^{n} \binom{n}{k}_\alpha e^{(r)}_{n-k,\alpha}(0) \oint_{-1}^{1} f_{k,\alpha}(x)\, d_\alpha x.$$

From (4.2), we have

$$\oint_{-1}^{1} e^{(r)}_{n,\alpha}(x)\, d_\alpha x = \sum_{k=0}^{n} \binom{n}{k}_\alpha e^{(r)}_{n-k,\alpha}(0) \frac{b_{k+1,\alpha}(1) - b_{k+1,\alpha}(-1)}{\theta_{k+1,\alpha}}, \tag{4.3}$$

and in [44, Theorem 6.3] it is shown that

$$\frac{b_{k+1,\alpha}(1) - b_{k+1,\alpha}(-1)}{\theta_{k+1,\alpha}} = \frac{1}{\alpha+1} b_{k,\alpha}(0). \tag{4.4}$$

Then, joining (4.3) and (4.4), the result is proved. □

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Capítulo 5

Memoria sobre los resultados obtenidos

La tesis que aquí se ha presentado está dedicada a profundizar en el conocimiento de las sucesiones de Appell y Appell-Dunkl. Hemos definido nuevos tipos de polinomios y funciones especiales, y demostrando diversas propiedades novedosas.

Los resultados más importantes están detallados en el capítulo 3. Allí no sólo se explica cuáles eran los objetivos de cada artículo —qué se conocía hasta ahora y qué se pretendía estudiar— sino qué se logró en cada caso, pues nos ha parecido más claro explicarlo de manera conjunta para poder mostrar el avance conseguido sin necesidad de ser demasiado repetitivo.

A modo de resumen, podemos decir que los principales resultados del primer artículo son los teoremas 3.1 y 3.2, que aquí mostramos en las páginas 19 y 21, respectivamente. De hecho, estos son posiblemente los principales resultados de la tesis.

El principal resultado del segundo artículo es, sin duda, el teorema 3.3 (página 23), y también queremos mencionar los ejemplos en los que se muestra cómo aplicar el teorema.

En el tercer artículo, queremos destacar cómo hemos logrado definir unos polinomios que legítimamente merecen llamarse polinomios de Boole-Dunkl puesto que son una clara extensión de los polinomios de Boole clásicos, ya que conservan sus propiedades esenciales, en particular su relación con los números de Stirling.

En nuestro trabajo investigador conjunto es imposible especificar cuáles han sido las aportaciones concretas de cada persona. Nuestro objetivo es siempre encontrar nuevos resultados matemáticos y todos compartimos ese objetivo, y realizamos la misma labor: pensar, explorar caminos para llegar a lo que se pretende, experimentar con programas de ordenador —fundamentalmente de cálculo simbólico— para intentar conjeturar cuál puede ser el resultado correcto, intentar demostraciones que a veces funcionan y a veces no, y así sucesivamente. Todo esto, con continuas comunicaciones informales, tanto en reuniones presenciales como en mensajes electrónicos, y trasladando con premura a los demás las nuevas ideas y los resultados exitosos, a menudo con gran alegría cuando se consigue un objetivo importante.

Por supuesto, en una tesis doctoral, como en este caso, las ideas originales de qué intentar explorar y demostrar provienen del director, sobre todo en los primeros estadios del desarrollo de la tesis, y es el doctorando el que dedica más tiempo de trabajo. Además, una vez que la tesis ya estaba muy encaminada, con dos artículos ya elaborados, el trabajo investigador dejamos de hacerlo exclusivamente entre el doctorando y el director, sino que comenzamos a colaborar con dos compañeros de nuestro grupo de investigación —Edgar Labarga y Judit Mínguez— que ya tenían experiencia en el contexto de las sucesiones de Appell-Dunkl (ver [13, 25, 42, 43, 44]), y fruto de esa colaboración es el tercer artículo recogido en esta tesis doctoral.

Capítulo 6

Conclusiones

En la tesis hemos profundizado en el conocimiento de las sucesiones de Appell y Appell-Dunkl, definiendo nuevos tipos de polinomios y funciones especiales, y demostrando diversas propiedades que, tal como se hace en matemáticas, han sido plasmadas en forma de teoremas.

En concreto, y teniendo en cuenta que esta tesis es un compendio de tres publicaciones, estos son los resultados más importantes que se han abordado y que se corresponden, respectivamente, con el contenido de las tres publicaciones:

- Desarrollo de un método para encontrar funciones especiales $F(s, x)$, con s en el plano complejo, tal que $F(-n, x)$ son polinomios de Appell-Dunkl prefijados, e interpretación como funciones que se asemejan a las funciones zeta de Hurwitz.

- Extensión de un método previo para interpolar polinomios de Appell debilitando las hipótesis de manera considerable, y planteamiento de un problema abierto para el caso de polinomios de Appell-Dunkl.

- Definición de los polinomios de Boole-Dunkl (y algunas generalizaciones suyas) y estudio de sus propiedades.

Por supuesto, esto no es un punto final en el camino, y hay muchos temas relacionados con el operador de Dunkl en la recta real, y los polinomios de Appell-Dunkl que, bien no hemos sabido resolverlos, bien están iniciados y pendientes de publicación, bien permanecen sin explorar. En particular, podemos mencionar lo siguiente:

- Desarrollo de fórmulas de linealización para productos de polinomios de Appell-Dunkl.

- Definición y estudio de nuevos tipos de polinomios de Appell-Dunkl que extiendan lo conocido para polinomios de Appell clásicos.

- Análisis de la viabilidad de extender algún tipo de funciones clásicas al contexto de Dunkl; en particular, ¿existe un función que merezca llamarse función gamma de Dunkl?

- Estudio de problemas de momentos relacionados con sucesiones vinculadas con los operadores y las funciones típicas del contexto de Dunkl.
- Definición y estudio de polinomios de Sheffer-Dunkl, como extensión de los polinomios de Sheffer clásicos.
- Estudio de una teoría umbral vinculada al mundo de Dunkl.

Estos temas serán el objetivo de nuestra trabajo investigador futuro —de hecho, en algunos casos ya está empezado—, posiblemente con diversos colaboradores.

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