UNIVERSIDAD DE LA RIOJA

TESIS DOCTORAL

Título
Discrete Harmonic Analysis related to classical orthogonal polynomials
Autor/es
Alberto Arenas Gómez
Director/es
Oscar Ciaurri Ramírez
Facultad
Facultad de Ciencia y Tecnología
Titulación
Departamento
Matemáticas y Computación
Curso Académico

publicaciones.unirioja.es
E-mail: publicaciones@unirioja.es

UNIVERSIDAD DE LA RIOJA

Discrete Harmonic Analysis related to classical orthogonal polynomials

by

Alberto Arenas Gómez

Dissertation submitted in partial fulfilment
of the requirements for the degree of
Doctor of Philosophy

Written under the supervision of
Prof. Dr. Óscar Ciaurri Ramírez

Department of Mathematics and Computer Science
Logroño, July 2019

Análisis armónico discreto asociado a polinomios ortogonales clásicos

por

Alberto Arenas Gómez

Memoria presentada en cumplimiento parcial
de los requisitos para optar al grado de
Doctor en Matemáticas

Escrita bajo la supervisión de

Dr. D. Óscar Ciaurri Ramírez

Departamento de Matemáticas y Computación

Logroño, julio de 2019

Abstract

The present dissertation belongs to the so-called non-trigonometric discrete Harmonic Analysis, specifically to the one associated with classical orthogonal polynomials. Its aim is the study of the discrete analogues of some classical operators in Harmonic Analysis. To be specific, the convergence problem of the multiplier of an interval for discrete Fourier series and the problem of the norm boundedness of the transplantation operator are studied.

Regarding the first problem, the multiplier of an interval related to Jacobi polynomials is defined and sufficient conditions are given to ensure its norm boundedness with weights. If we consider no weights, a characterization is provided. Moreover, the characterization of the convergence is also given.

Regarding the second problem, a transplantation theorem related to Jacobi coefficients is given when we consider weighted spaces. We prove that the transplantation operators are bounded in norm with weights by means of a semi-local Calderón-Zygmund theory which has been recently furnished. Moreover, some weighted weak estimates are provided. On its behalf, a transplantation theorem for Laguerre coefficients in weighted spaces is also given. In that case, we use a discrete local Calderón-Zygmund theory which is developed in the dissertation. To finish, a characterization is given when power weights are considered.

Resumen

La presente memoria se enmarca dentro del denominado análisis armónico discreto no trigonométrico, concretamente en el asociado a los polinomios ortogonales clásicos. Su objetivo es el estudio de análogos discretos de algunos operadores clásicos del análisis armónico. En concreto, se estudia el problema de la convergencia del multiplicador de un intervalo para las series de Fourier discretas y el problema de la acotación en norma del operador de transplantación.

Respecto al primer problema, se define el multiplicador de un intervalo asociado a los polinomios de Jacobi y se proporcionan condiciones suficientes para su acotación en norma con pesos, caracterizando dicha acotación en el caso en el que no se consideren pesos. Además, también se da la caracterización de la convergencia.

Respecto al segundo problema, se presenta un teorema de transplantación para coeficientes de Jacobi en espacios con pesos. Se prueba que los operadores de transplantación están acotados en norma considerando pesos, para lo cual utilizamos una teoría semi-local discreta de Calderón-Zygmund recientemente desarrollada. Además de esto, se obtienen estimaciones débiles con pesos para dichos operadores. Por su parte, también se prueba un teorema de transplantación para los coeficientes de Laguerre en espacios con pesos. En este caso se utiliza una teoría local discreta de Calderón-Zygmund que se desarrolla en la memoria. Para finalizar, se da una caracterización cuando se consideran pesos de tipo potencia.

Acknownledgements

Spleepe after toyle, port after stormie seas,
Ease after warre, death after life does greatly please.

E. Spenser, *The Faerie Queene*, Book I, Canto IX.

Once to this point, it is time to thank many people for several and fundamental reasons.

First of all, I would like to thank Prof. Dr. Óscar Ciaurri for his patience and his help throughout all past years, and for bring me the opportunity to present this dissertation under his supervision. Since the moment we started collaborating together, back in the academic year 2012-2013, it has been an incredible pleasure working with him.

I extend my thanks to Edgar Labarga, fellow-sufferer since the time we obtained the degree in Mathematics. His support and company during all these years have made all this time, without a doubt, much more pleasant.

I cannot miss this opportunity to have some words to all the people I have been the pleasure to meet. Specially to Prof. Emilio Fernández and Prof. Dr. Manuel Benito. I do not forget here of my officemates Ángela Casado and Jaime Fernández.

I am really grateful to the Department of Mathematics and Computer Science of the University of La Rioja, specially to the research group *Grupo de Teoría de la Aproximación*, which is led by Prof. Dr. Juan Luis Varona. I also thank him for his advices and his help regarding the composition of the present dissertation. I would like to mention all the people of the thematic network *Orthonet*, and in particular to Prof. Dr. Renato Álvarez-Nodarse for bring me the opportunity to work with him in my other passion: mathematical physics.

I wish to express my gratitude and affection to Prof. Dr. hab. Adam Nowak, from the Polish Academy of Science in Wrocław, for his warm hospitality during my research stay in Polland. I am really lucky to have worked with him.

I also indebted to Prof. Dr. hab. Krzysztof Stempak and Prof. Dr. Sundaram Thangavelu for their help and kindedness.

From a personal point of view, the present dissertation is dedicated to my family and, in particular, to my parents José Alberto and Consuelo. It is also dedicated to those who, unfortunately, are no longer with us.

I have to mention here each and every one of my friends, who esteem me just as I am. Carla deserves a separate mention for her endless patience and support at bad times.

Finally, I am indebted to Jorge Fernández. He was my teacher when I was at secondaty school and I firmly believe that I have submit the present dissertation is partially thanks to him.

This dissertation has been partially supported by a predoctoral research grant of the University of La Rioja funded by the Government of Comunidad Autónoma de La Rioja, and grants MTM2012-36732-C03-02, MTM2015-65888-C04-4-P, and PGC2018-096504-B-C32 of the Agencia Estatal de Investigación del Ministerio de Ciencia, Innovación y Universidades (or similar organisations and ministries with analogue competencies in the past).

Agradecimientos

El sueño tras el esfuerzo, tras la tempestad el puerto,
el reposo tras la guerra, la muerte tras la vida harto complace.

E. Spenser, *La reina hada*, Libro I, Canto IX.

Llegados a este punto, es el momento de agradecer a varias personas por muchas y muy buenas razones.

En primer lugar quiero expresar mi más sincero agradecimiento al Prof. Dr. Óscar Ciaurri Ramírez, por su paciencia y dedicación a lo largo de todos estos años, y por brindarme la oportunidad de presentar esta memoria bajo su dirección. Desde el momento en el que empezamos a colaborar juntos, allá por el curso académico 2012-2013, ha sido un enorme placer trabajar con él.

Hago extensivo este agradecimiento a Edgar Labarga, compañero de fatigas desde los tiempos en los que estudiábamos el grado en Matemáticas. Su apoyo y compañía durante todos estos años han hecho todo este tiempo, sin lugar a dudas, mucho más agradable.

No puedo dejar pasar la oportunidad de dedicar algunas palabras a los que, junto a Edgar, he tenido el placer de conocer. En particular al Prof. Emilio Fernández y al Prof. Dr. Manuel Benito. Tampoco puedo olvidarme aquí de mis compañeros de despacho Ángela Casado y Jaime Fernández.

Agradezco al Departamento de Matemáticas y Computación de la Universidad de La Rioja, en especial al *Grupo de Teoría de la Aproximación*, liderado por el Prof. Dr. Juan Luis Varona. A él le doy las gracias por sus siempre útiles consejos y su inestimable ayuda en la redacción de esta memoria. Quisiera agradecer también a los miembros de la red temática *Orthonet* y, en particular, al Prof. Dr. Renato Álvarez-Nodarse, quien de manera extremadamente generosa me ha permitido trabajar junto a él en mi otra pequeña pasión: la física matemática.

Quisiera expresar mi profunda gratitud y afecto al Prof. Dr. hab. Adam Nowak, de la Academia Polaca de Ciencias en Wrocław, por el tiempo que pude disfrutar de estancia de investigación en Polonia bajo su supervisión. He sido realmente afortunado al poder trabajar con él.

También el Prof. Dr. hab. Krzysztof Stempak y al Prof. Dr. Sundaram Thangavelu, por su ayuda y amabilidad.

Pasando a un plano más personal, esta memoria está dedicada a mi familia y, especialmente, a mis padres: José Alberto y Consuelo. También se la dedico a aquellos que, desafortunadamente, ya no están entre nosotros.

Quiero agradecer aquí a todos y cada uno de mis amigos, por quererme tal como soy. Y a Carla, por su infinita paciencia y su apoyo en los malos momentos.

Finalmente, quisiera agradecer al Prof. Jorge Fenández. Como profesor mío que fue en el instituto, creo con total firmeza que si hoy me encuentro presentando esta memoria es, en gran parte, gracias a él.

Esta memoria ha sido realizada parcialmente con un contrato predoctoral de la Universidad de La Rioja, el cual ha estado financiado por la Comunidad Autónoma de La Rioja. También ha sido financiada parcialmente por los proyectos MTM2012-36732-C03-02, MTM2015-65888-C04-4-P y PGC2018-096504-B-C32 de la Agencia Estatal de Investigación del Ministerio de Ciencia, Innovación y Universidades (u organizaciones similares y ministerios con competencias análogas en el pasado).

Contents

Introduction

The problems studied in the present dissertation are related to the discrete analogues of some classical operators in Harmonic Analysis associated with non-trigonometric orthonormal systems. Specifically, the study of the problem related to the convergence of the multiplier of an interval and the so-called transplantation theorems are treated. For the sake of clarity, we start showing both topics in the continuous context.

Let $((a,b),\zeta)$ be an appropriate metric space (although we have taken an open interval, it is possible to take it closed or half-closed). We define $L^p((a,b),d\zeta)$, with $1 \le p < \infty$, as the space of ζ-measurable real functions such that

$$\|F\|_{L^p((a,b),d\zeta)} := \left(\int_a^b |F(x)|^p \, d\zeta(x)\right)^{1/p} < \infty.$$

The application $\|\cdot\|_{L^p((a,b),d\zeta)}$ is a norm and $L^p((a,b),d\zeta)$ is a Banach space. Only the case $p=2$ is a Hilbert space with the innner product

$$\langle F, G\rangle_{d\zeta} := \int_a^b F(x)G(x)\, d\zeta(x).$$

Let us consider a family $\{\phi_n\}_{n\ge 0}$ of orthonormal functions, i.e.,

$$\langle \phi_n, \phi_m\rangle_{d\zeta} = \int_a^b \phi_n(x)\phi_m(x)\, d\zeta(x) = \delta_{n,m}, \qquad n \in \mathbb{N},$$

where $\delta_{n,m}$ denotes the Kronecker delta function given by

$$\delta_{n,m} = \begin{cases} 0, & \text{si } n = m, \\ 1, & \text{si } n \neq m. \end{cases}$$

Furthermore, we suppose that such family is complete in the space $L^2((a,b),d\zeta)$.

To each function $F \in L^2((a,b),d\zeta)$ we can associate its corresponding Fourier coefficients

$$c_n(F) := \int_a^b F(x)\phi_n(x)\, d\zeta(x)$$

with respect to the system $\{\phi_n\}_{n\ge0}$. It is well-known in this situation the fact that $c_n(F) \in \ell^2(\mathbb{N})$, the space of square summable sequences. Moreover, the application

$$\begin{array}{ccc} L^2((a,b),d\zeta) & \longrightarrow & \ell^2(\mathbb{N}) \\ F & \longmapsto & \{c_n(F)\}_{n\ge0} \end{array}$$

is an isometry. That is why the operator

$$\mathcal{F}f(x) := \sum_{n=0}^{\infty} f(n)\phi_n(x), \qquad f \in \ell^2(\mathbb{N}),$$

is called the discrete Fourier transform with respect to the system $\{\phi_n\}_{n\geq 0}$. Then, we can associate to each function $F \in L^2((a,b), d\zeta)$ the formal series

$$F(x) \sim \mathcal{F}\left(c_{(\cdot)}(F)\right)(x) = \sum_{n=0}^{\infty} c_n(F)\phi_n(x),$$

which is called the Fourier series of F with respect to $\{\phi_n\}_{n\geq 0}$. The study of its convergence (in its different meanings) initiated an enormous field of research which is still active. The problem is correctly formulated by means of the partial sum operators of the Fourier series, given by

$$\mathfrak{S}_n F(x) := \mathcal{F}\left(\chi_{[0,n]\cap\mathbb{N}}(\cdot)c_{(\cdot)}(F)\right)(x) = \sum_{k=0}^{n} c_k(F)\phi_k(x),$$

where χ_A denotes the usual characteristic function of the set A. If $1 \leq p < \infty$, with $p \neq 2$, is well-kwnon that the convergence

$$\lim_{n\to\infty} \|\mathfrak{S}_n F - F\|_{L^p((a,b),d\zeta)} = 0,$$

is equivalent to the uniform boundedness

$$\|\mathfrak{S}_n F\|_{L^p((a,b),d\zeta)} \lesssim \|F\|_{L^p((a,b),d\zeta)},$$

provided the density of the family $\{\phi_n\}_{n\geq 0}$ in the space $L^p((a,b), d\xi)$. That is a classical consequence of the Banach-Steinhaus theorem [**42**, Teorema 5.8].

In the given form, first appearance of this problem was formulated by H. Pollard in [**37**]. Besides proposing the problem, Pollard also announce an affirmative answer for $2-1/(\lambda+1) < p < 2+1/\lambda$ and negative for $1 \leq p < 2-1/(\lambda+1)$ or $p > 2+1/\lambda$, when the ultraspherical polynomials of order $\lambda > -1/2$ are considered. Pollard did not give an answer for the endpoints.

When $\lambda = 0$ the corresponding polynomials are the Chebyshov polynomials of the first kind, which are the usual trigonometrical system. The answer in this case is positive for $1 < p < \infty$ and negative in the case $p = 1$, and it was already known since the M. Riesz work regarding the conjugate function [**41**]. The case $\lambda = 1/2$ is connected with the Legendre polynomials and, according to a non-published conjecture by A. Zygmund stated in that time, the answer should be positive for $4/3 < p < 4$. Such conjecture was finally proved by Pollard in [**38**], who supplied the counterexample $F(x) = (1-x)^{-3/4}$ to answer negatively the question for $1 < p < 4/3$ or $p > 4$.

Pollard also gives a conjecture in the case of Laguerre and Hermite, stating that the answer is negative for all $1 \leq p < \infty$, except $p = 2$. Later on, Pollard himself confirms that conjeture in [**39**]. In the same work, he proves that for the Jacobi polynomials of order $\alpha,\beta \geq -1/2$ the answer is positive for

$$4\max\left\{\frac{\alpha+1}{2\alpha+3},\frac{\beta+1}{2\beta+3}\right\} < p < 4\min\left\{\frac{\alpha+1}{2\alpha+1},\frac{\beta+1}{2\beta+1}\right\},$$

and negative outside.

G. M. Wing studies in [**49**] the problem for functions associated with classical orthogonal polynomials. In particular, for Jacobi functions with $\alpha,\beta \geq -1/2$ the answer is positive for $4/3 < p < 4$ and negative for $1 \leq p \leq 4/3$ or $p \geq 4$.

Regarding the Hermite and Laguerre functions, for $\alpha \geq 0$ only, the problem was studied by R. Askey and S. Wainger in [**8**], given a positive answer for $4/3 < p < 4$ and negative outside. They proved the negative result showing that the product of the p-norm of orthonormal functions are not bounded for $1 \leq p \leq 4/3$ or $p \geq 4$ independently of its degree.

The problem was completely closed by B. Muckenhoupt in [**28**] for Jacobi functions, $\alpha,\beta > -1$; and for the Laguerre functions, $\alpha > -1$, and the Hermite ones in [**29**] and [**30**]. In all-three works Muckenhoupt includes weights. For the sake of clarity, we do not replicate here the conditions regarding the value p.

On its behalf, the weak boundedness of the Fourier series related to Jacobi polynomials has been widely studied, see for example the works [**22**] and [**23**] by J. J. Guadalupe, M. Pérez, and J. L. Varona.

Although we shall be focused on classical orthogonal polynomials and its associated functions, it is worth to mention the characterization of the convergence by G. M. Wing, in the aforementioned paper [**49**], when the Fourier expansions related to Bessel functions are considered. In this line, we also have to cite here the work by A. I. Benedek and R. Panzone in a series of papers in the seventies, e.g. [**11**].

Finally, if the orthonormal system is based on eigenfunctions of a Sturm-Liouville problem there exits solutions in particular cases. See, for instance, the works [**20**] by V. L. Generozov, [**12**] by Benedek and Panzone, and [**10**] by J. A. Barceló and A. Córdoba.

Let us consider now the transplantation problem. Let $\{\phi_n^\lambda\}_{n\geq 0}$ be a family of orthonormal functions which depends on a parameter $\lambda \in \Lambda$, in a space $L^2((a,b),d\xi)$. A transplantation operator relates two of the previous families $\{\phi_n^{\lambda_1}\}_{n\geq 0}$ and $\{\phi_n^{\lambda_2}\}_{n\geq 0}$ by means of the application

$$\Upsilon_{\lambda_1}^{\lambda_2}F(x) = \mathcal{F}_{\lambda_2}\left(c_{(\cdot)}^{\lambda_1}(F)\right)(x) = \sum_{n=0}^{\infty} c_n^{\lambda_1}(F)\phi_n^{\lambda_2}(x), \qquad F \in L^2((a,b),d\zeta),$$

where $c_{(\cdot)}^{\lambda_1}$ is the Fourier coefficient with respect to the family $\{\phi_n^{\lambda_1}\}_{n\geq 0}$ and $\mathcal{F}_{\lambda_2}$ is the discrete Fourier transform with respect to $\{\phi_n^{\lambda_2}\}_{n\geq 0}$. The transplantation operator $\Upsilon_{\lambda_1}^{\lambda_2}$ is an isometric isomorphism in the Hilbert space $L^2((a,b),d\zeta)$, and it becomes the identity operator when $\lambda_1 = \lambda_2$. That is why normally we shall suppose $\lambda_1 \neq \lambda_2$.

The question in this case is if, given $1 \leq p < \infty$, the transplantation operator is bounded in the p-norm. That is,

$$\|\Upsilon_{\lambda_1}^{\lambda_2}F\|_{L^p((a,b),d\zeta)} \lesssim \|F\|_{L^p((a,b),d\zeta)}, \qquad F \in L^2 \cap L^p((a,b),d\zeta).$$

If the previous line holds, then the transplantation operator $\mathfrak{T}_{\lambda_1}^{\lambda_2}$ extends uniquely to a bounded linear operator from $L^p((a,b),d\zeta)$ to itself.

Historically, the first transplantation theorem is stated in the paper [**24**] by D. L. Guy, who studied the Hankel transform in the positive half-line. Other transplantation theorems could be seen in the excellent survey [**45**] by K. Stempak.

The first result regarding the problem stated above is due to R. Askey and S. Wainger, who obtained a transplantation result for ultraspherical functions in [**9**]. Askey extends that work for Jacobi functions in [**7**], and again B. Muckenhoupt in [**31**] who enhanced Askey's result in several directions, such as the consideration of the complete order and the inclusion of some kind of weights.

Finally, it is worth to mention the research done by Ó. Ciaurri, A. Nowak, and K. Stempak, who revisited the problem in [**17**] using a variant of the Calderón-Zygmund theory. In that work they obtained the abovementioned Muckenhoupt's result and, moreover, weak (1,1) estimates.

On its behalf, first tranplantation theorem considering Laguerre functions is due to Y. Kanjin un [**26**], which was considerably enhanced by K. Stempak and W. Trebels in [**46**] including power weights. The latter was improved by G. Garrigós, E. Harboure, T. Signes, J. L. Torrea and B. Viviani in [**19**], given the sharpest result for that kind of weights.

Its possible to define another Laguerre functions, the so-called Laguerre functions of Hermite type. In the present disssertation they will not be considered, but there are in the literature some transplantations results involving them which could be seen in the monogrpah [**48**] by S. Thangavelu.

As it has been mentioned before, the aim of the present dissertation is the study of the discrete analogue related to the norm convergence problem and the transplantation one. Last years this line of research, in which the discrete counterpart of classical problems are considered, has been an incredible grow. For example, we cite here the excellent work [**16**] by Ó. Ciaurri, T. A. Gillespie, L. Roncal, J. L. Torrea y J. L. Varona, where a complete study of operators related to the discrete Laplacian

$$\Delta_d f(n) := f(n+1) - 2f(n) + f(n-1), \qquad n \in \mathbb{Z}.$$

is provided.

Regarding the discrete harmonic analysis related to classical orthogonal polynomials, one of the first research is due to J. J. Betancor, A. J. Castro, J. C. Fariña and L. Rodríguez-Mesa in [**13**]. In that work, besides studying classical problems such as the boundedness of the heat semigroup or the study of the so-called Littlewood-Paley-Stein functions, all of them for ultraspherical polynomials, they also give a local discrete Calderón-Zygmund theory which we shall use. That paper was enhanced in the setting of Jacobi polynomials in a series of joint works by the author with Ó. Ciaurri and E. Labarga. Specifically, in [**3**] the heat semigroup is studied, in [**4**] the same is done for the Riesz tranform, and, finally, in [**5**] the Littlewood-Paley-Stein and Laplace type multipliers are investigated. Previous three papers taking part in the doctoral dissertation [**27**] by E. Labarga, which has been recently submitted.

The discrete analogue of the partial sum operators we are interested in is given by the multiplier of an interval. To be specific, let us take

$$S_{[c,d]}f(n) = c_n\left(\chi_{[c,d]}\mathcal{F}f\right),$$

where $[c,d] \subset (a,b)$. In this manner, we want to study the hypotheses under the identity

$$\lim_{\substack{c\to a^+\\ d\to b^-}} \|S_{[c,d]}f - f\|_{\ell^p(\mathbb{N})} = 0$$

holds.

Regarding the transplantation operators, we have already seen that necessary we have to consider the functions associated with the polynomials which depend on parameters. Then, we define the discrete transplantation operator by

$$T_{\lambda_1}^{\lambda_2} f(n) = c_n^{\lambda_1}\left(\mathcal{F}_{\lambda_2} f\right), \qquad f \in \ell^2(\mathbb{N}),$$

and we would like to know the hypotheses under such operators are bounded in the p-norm, including some weights. That is,

$$\|T_{\lambda_1}^{\lambda_2} f\|_{\ell^p(\mathbb{N},w)} \lesssim \|f\|_{\ell^p(\mathbb{N},w)}, \qquad f \in \ell^2(\mathbb{N}) \cap \ell^p(\mathbb{N},w).$$

Again the transplantation operator $T_{\lambda_1}^{\lambda_2}$ extends uniquely to a bounded linear operator form $\ell^p(\mathbb{N},w)$ to itself.

That kind of discrete transplantation operators related to functions associated with classical orthogonal polynomials were studied in [**9**] by R. Askey and S. Wainger for the Fourier coefficients related to ultraspherical functions, and by Askey in [**6**] for the coefficients of Jacobi functions. In both cases only potential weights are considered. Furthermore, in [**6**], Askey gave some result concerning the Fourier coefficients of the Laguerre functions, but he does not include weights and only treats the orders α and $\alpha + 2$.

Summary of the chapters

The present dissertation contains five chapters. We give here a short summary of each one.

Chapter 1 is a short introduction to the theory of classical orthogonal polynomials. In spite of the existence of excellent monographs in the literature, we have decided to include this chapter in order to do the dissertation as selfcontained as possible. The classical orthogonal poynomials are defined from the hypergeometric differential equation. Then, we give several properties which we shall use. The functions associated with the classical orthogonal polynomials are also defined. Finally, some estimations for both the polynomials and the function are given.

Chapter 2 is devoted to show properly the discrete local Calderón-Zygmund theory in order to achieve our targets. In the first part the one developed by J. J. Betancor, A. J. Castro, J. C. Fariña and L. Rodríguez-Mesa in [**13**] is stated. We call it semi-local and we give a draft of its proof. In the second one we develop a discrete local Calderón-Zygmund analogue of the continuous one stated by A. Nowak and K. Stempak in [**34**]. In the last part, well-kwon weighted norm inequalities for some classical discrete operators are included.

Chapter 3 is the first one without an introductory nature. The convergence of the multiplier of the interval related to Jacobi polynomials is studied. Firstly we give a sufficiency result for its uniform boundedness with weights. From here, we characterize the case without weights and the convergence for such operator. As a consequence, the a.e. convergence of the multiplier follows.

Chapter 4 is devoted to the study of the transplantation operator for the Fourier coefficients associated with Jacobi functions. Using the semi-local discrete Calderón-Zygmund theory, we prove the boundedness of the transplantation operator from $\ell^p(\mathbb{N}, w)$ to itself, where w is a weight in the discrete Muckenhoupt class. Moreover, we obtain weak (1,1) estimates with weights for such operators.

Cahpter 5 is the last one. The discrete transplantation operators for the Fourier coefficients associated to Laguerre functions are studied. This time, we use the discrete local Calderón-Zygmund theory to prove the boundedness from $\ell^p(\mathbb{N}, w)$ to itself, where w is a discrete local Muckenhoupts class. Moreover, a characterization is given when power weights are considered.

Basic notation

Throughout the dissertation a standard notation will be used.

Generally speaking, we shall denote by capital letters the functions depending on a continuous variable $x \in (a, b)$, whereas the small letter are used to denote sequences in $\mathbb{N}$. Given a number p such that $1 \le p < \infty$, the space of p-summable sequences is defined by

$$\ell^p(\mathbb{N}) := \left\{ f = \{f(n)\}_{n\ge 0} : \|f\|_{\ell^p(\mathbb{N})} := \left(\sum_{n=0}^{\infty} |f(n)|^p \right)^{1/p} < \infty \right\}.$$

Occasionally we also shall use the Lorentz type space

$$\ell^{1,\infty}(\mathbb{N}) := \left\{ f = \{f(n)\}_{n\ge 0} : \|f\|_{\ell^{1,\infty}(\mathbb{N})} := \sup_{\lambda>0} \lambda \#\{n \in \mathbb{N} : |f(n)| > \lambda\} < \infty \right\}.$$

A weight sequence in $\mathbb{N}$ will be a estrictly positive sequende w. The spaces of sequences with weights p-summable are given by

$$\ell^p(\mathbb{N}, w) := \left\{ f = \{f(n)\}_{n\ge 0} : \|f\|_{\ell^p(\mathbb{N},w)} := \left(\sum_{n=0}^{\infty} |f(n)|^p w(n) \right)^{1/p} < \infty \right\},$$

and

$$\ell^{1,\infty}(\mathbb{N}, w) := \left\{ f = \{f(n)\}_{n\ge 0} : \|f\|_{\ell^{1,\infty}(\mathbb{N},w)} := \sup_{\lambda>0} \lambda \sum_{\{n\in\mathbb{N}:|f(n)|>\lambda\}} w(n) < \infty \right\}.$$

Clearly, when $w(n) = 1$ for all $n \in \mathbb{N}$, that spaces are the same as the aforementioned.

We shall use $a \lesssim b$ to denote that two quantities a and b fulfil with the relation $a \le Cb$, where C is a positive constant independent of significative quantities. On its behalf, we shall use $a \simeq b$ if there are two positive constants C_1 y C_2 independents of significative quantities such that $C_1 b \leq a \leq C_2 b$. However, there will be situations in which we do not use this convention for the sake of clarity, specially in the part regarding inequalities involving weights.

To finish this section, we usually denote by capital letters the classical orthogonal polynomials, by small letters the orthonormalized ones and by calligraphic letters the functions associated with them. Futhermore, we shall take $P_{-1}(x) = 0$, specially in the recurrence formulae.

Introducción

Los problemas estudiados en la presente memoria están relacionados con los análogos discretos de ciertos operadores clásicos del análisis armónico de sistemas ortonormales no trigonométricos. Concretamente se trata el estudio del problema de la convergencia del multiplicador para un intervalo y los conocidos como teoremas de transplantación. Para la más fácil comprensión del lector, comenzamos exponiendo ambas cuestiones en el contexto continuo.

Sea $((a,b),\zeta)$ un espacio de medida apropiado (aunque hemos tomado un intervalo abierto, es posible tomarlo cerrado o semicerrado). Definimos $L^p((a,b),d\zeta)$, donde $1 \le p < \infty$, como el espacio de las funciones ζ-medibles reales tales que

$$\|F\|_{L^p((a,b),d\zeta)} := \left(\int_a^b |F(x)|^p \, d\zeta(x)\right)^{1/p} < \infty.$$

Con esta definición, la aplicación $\|\cdot\|_{L^p((a,b),d\zeta)}$ es una norma y $L^p((a,b),d\zeta)$ es un espacio de Banach. De todos los espacios $L^p((a,b),d\zeta)$ únicamente el caso $p=2$ es un espacio de Hilbert, dotado con el producto escalar

$$\langle F, G\rangle_{d\zeta} := \int_a^b F(x)G(x)\, d\zeta(x).$$

Consideremos una familia $\{\phi_n\}_{n\ge 0}$ de funciones ortonormales, esto es,

$$\langle \phi_n, \phi_m\rangle_{d\zeta} = \int_a^b \phi_n(x)\phi_m(x)\, d\zeta(x) = \delta_{n,m}, \qquad n \in \mathbb{N},$$

donde $\delta_{n,m}$ denota la función delta de Kronecker, dada por

$$\delta_{n,m} = \begin{cases} 0, & \text{si } n = m, \\ 1, & \text{si } n \neq m. \end{cases}$$

Además, supondremos que dicha familia es completa en el espacio $L^2((a,b),d\zeta)$.

A cada función $F \in L^2((a,b),d\zeta)$ le podemos asociar sus coeficientes de Fourier

$$c_n(F) := \int_a^b F(x)\phi_n(x)\, d\zeta(x)$$

respecto al sistema $\{\phi_n\}_{n\ge0}$. Es bien conocido que en esta situación $c_n(F) \in \ell^2(\mathbb{N})$, el espacio de sucesiones cuadrado sumables. Es más, la aplicación

$$\begin{array}{ccc} L^2((a,b),d\zeta) & \longrightarrow & \ell^2(\mathbb{N}) \\ F & \longmapsto & \{c_n(F)\}_{n\ge0} \end{array}$$

es una isometría. Es por ello que al operador

$$\mathcal{F}f(x) := \sum_{n=0}^{\infty} f(n)\phi_n(x), \qquad f \in \ell^2(\mathbb{N}),$$

lo denominaremos transformada de Fourier discreta respecto al sistema $\{\phi_n\}_{n\geq 0}$. Así, podemos asociar a cada función $F \in L^2((a,b), d\zeta)$ la serie formal

$$F(x) \sim \mathcal{F}\left(c_{(\cdot)}(F)\right)(x) = \sum_{n=0}^{\infty} c_n(F)\phi_n(x),$$

denominada serie de Fourier de F respecto a $\{\phi_n\}_{n\geq 0}$. El estudio de la convergencia —en sus distintos tipos— a la función F de la serie anterior dio lugar a un amplio campo de investigación que sigue activo hoy en día. El problema se formula correctamente definiendo los operadores sumas parciales de la serie de Fourier, dados por

$$\mathfrak{S}_n F(x) := \mathcal{F}\left(\chi_{[0,n]\cap\mathbb{N}}(\cdot)c_{(\cdot)}(F)\right)(x) = \sum_{k=0}^{n} a_k(F)\phi_k(x),$$

donde χ_A denota la función característica usual de un conjunto A. Cuando $1 \leq p < \infty$, con $p \neq 2$, es conocido que la convergencia

$$\lim_{n\to\infty} \|\mathfrak{S}_n F - F\|_{L^p((a,b),d\zeta)} = 0,$$

es equivalente a la acotación uniforme

$$\|\mathfrak{S}_n F\|_{L^p((a,b),d\zeta)} \lesssim \|F\|_{L^p((a,b),d\zeta)},$$

cuando la familia $\{\phi_n\}_{n\geq 0}$ es densa en el espacio $L^p((a,b), d\xi)$. Este hecho es una consecuencia clásica del teorema de Banach-Steinhaus [**42**, Teorema 5.8].

La primera aparición de este problema en la forma dada fue propuesta por H. Pollard en [**37**]. Además de proponer el problema, Pollard anuncia, en el caso de polinomios ultraesféricos de orden $\lambda > -1/2$, una respuesta positiva para $2 - 1/(\lambda+1) < p < 2 + 1/\lambda$ y negativa para $1 \leq p < 2 - 1/(\lambda+1)$ o $p > 2 + 1/\lambda$. Para los valores extremos Pollard no da respuesta.

Cuando $\lambda = 0$ lo que se obtiene son los polinomios de Chebyshov de primera especie, que se corresponden con el sistema trigonométrico usual. La respuesta en este caso, afirmativa para $1 < p < \infty$ y negativa para $p = 1$, ya era conocida desde el célebre trabajo de M. Riesz sobre la función conjugada [**41**]. El caso $\lambda = 1/2$ se corresponde con los polinomios de Legendre y, según un conjetura no publicada de A. Zygmund de aquel tiempo, la respuesta sería positiva para $4/3 < p < 4$. Dicha conjetura fue finalmente probada por el mismo Pollard en [**38**], dando además el contraejemplo $F(x) = (1-x)^{-3/4}$ para responder negativamente a la cuestión en caso de que $1 < p < 4/3$ o $p > 4$.

Pollard conjetura que para los polinomios de Laguerre y Hermite la respuesta sea negativa para todo $1 \leq p < \infty$, excepto para $p = 2$. Es el propio Pollard quien nuevamente confirma la conjetura en [**39**]. En ese mismo trabajo demuestra que para los polinomios de Jacobi de orden $\alpha, \beta \geq -1/2$ la respuesta es positiva para

$$4 \max\left\{\frac{\alpha+1}{2\alpha+3}, \frac{\beta+1}{2\beta+3}\right\} < p < 4 \min\left\{\frac{\alpha+1}{2\alpha+1}, \frac{\beta+1}{2\beta+1}\right\},$$

y negativa fuera del intervalo anterior.

Es G. M. Wing quien estudia en [**49**] el problema para las funciones asociadas a polinomios ortonormales clásicos. En particular, para las funciones de Jacobi, también con la restricción en el orden $\alpha, \beta \geq -1/2$, la respuesta es positiva para el rango $4/3 < p < 4$ y negativa para $1 \leq p \leq 4/3$ o $p \geq 4$.

Respecto a las funciones de Hermite y Laguerre, con $\alpha \geq -1/2$, el problema es estudiado por R. Askey y S. Wainger en [**8**], dando una respuesta positiva para $4/3 < p < 4$ y negativa fuera de dicho intervalo. Askey y Wainger prueban esto último mostrando que el producto de las p-normas de las funciones ortonormales no está acotado independientemente del grado del polinomio para $1 \leq p \leq 4/3$ o $p \geq 4$.

El problema lo cierra completamente B. Muckenhoupt en [**28**] para las funciones de Jacobi, $\alpha, \beta > -1$; y para las funciones de Laguerre, $\alpha > -1$, y las de Hermite en [**29**] y [**30**]. En los tres trabajos Muckenhoupt incluye pesos. Por razones de claridad de la memoria, no reproducimos aquí las condiciones sobre el valor p y los pesos.

Por otra parte, la acotación en sentido débil de las series de Fourier respecto a polinomios de Jacobi ha sido objeto de un amplio estudio, véanse por ejemplo los artículos [**22**] y [**23**] de J. J. Guadalupe, M. Pérez y J. L. Varona.

A pesar de que nos centraremos en polinomios ortogonales clásicos y sus funciones asociadas, merece la pena comentar la caracterización de la convergencia dada por G. M. Wing, en el ya citado artículo [**49**], para series de Fourier asociadas a funciones de Bessel. En este sentido, también debemos mencionar el trabajo de A. I. Benedek y R. Panzone en una serie de artículos de los años setenta, e.g. [**11**].

Finalmente, cuando el sistema está formado por autofunciones de un problema de Sturm-Liouville existen soluciones para casos particulares. Consultar, por ejemplo, los trabajos [**20**] por V. L. Generozov, [**12**] por Benedek y Panzone, y [**10**] por J. A. Barceló y A. Córdoba.

Pasemos ahora a tratar el problema de la transplantación. Sea $\{\phi_n^\lambda\}_{n\geq 0}$ una familia de funciones ortonormales, dependientes de un parámetro $\lambda \in \Lambda$, en un espacio $L^2((a,b), d\xi)$. Un operador de transplantación relaciona dos de las anteriores familias $\{\phi_n^{\lambda_1}\}_{n\geq 0}$ y $\{\phi_n^{\lambda_2}\}_{n\geq 0}$ por medio de la aplicación

$$\Upsilon_{\lambda_1}^{\lambda_2} F(x) = \mathcal{F}_{\lambda_2}\left(c_{(\cdot)}^{\lambda_1}(F)\right)(x) = \sum_{n=0}^{\infty} c_n^{\lambda_1}(F)\phi_n^{\lambda_2}(x), \qquad F \in L^2((a,b), d\zeta),$$

donde, por supuesto, $c_{(\cdot)}^{\lambda_1}$ es el coeficiente de Fourier respecto a la familia $\{\phi_n^{\lambda_1}\}_{n\geq 0}$ y $\mathcal{F}_{\lambda_2}$ es la transformada de Fourier discreta respecto a $\{\phi_n^{\lambda_2}\}_{n\geq 0}$. El operador de transplantación $\Upsilon_{\lambda_1}^{\lambda_2}$ es un isomorfismo isométrico en $L^2((a,b), d\zeta)$, y se convierte en el operador identidad cuando $\lambda_1 = \lambda_2$. Es por esto que normalmente supondremos que $\lambda_1 \neq \lambda_2$.

La pregunta en este caso es si, dado $1 \leq p < \infty$, el operador de transplantación está acotado en la p-norma. Esto es,

$$\|\mathfrak{T}_{\lambda_1}^{\lambda_2} F\|_{L^p((a,b),d\zeta)} \lesssim \|F\|_{L^p((a,b),d\zeta)}, \qquad F \in L^2 \cap L^p((a,b),d\zeta).$$

Si esto ocurre, entonces el operador transplantación $\mathfrak{T}_{\lambda_1}^{\lambda_2}$ se extiende de manera única a un operador lineal acotado de $L^p((a,b),d\zeta)$ en sí mismo.

Desde el punto de vista histórico, el primer resultado de transplantación se debe a D. L. Guy, quien estudia la transformada de Hankel en la semirrecta positiva en [**24**]. Otros teoremas de transplantación pueden ser consultados en el excelente artículo de visión general de K. Stempak [**45**].

El primer resultado relacionado con funciones ortonormales se debe a R. Askey y S. Wainger, quienes en [**9**] obtienen un resultado de transplantación para las funciones ultraesféricas. Es Askey en solitario quien extiende el trabajo para las funciones de Jacobi en [**7**] y nuevamente B. Muckenhoupt en [**31**] quien mejora el resultado de Askey en varios sentidos, entre los cuales cabe destacar la consideración del rango completo para el orden de las funciones y la inclusión de algunos tipos de pesos.

Finalmente, merece la pena destacar la aportación de Ó. Ciaurri, A. Nowak, y K. Stempak, quienes reconsideran el problema en [**17**] utilizando una variante de la teoría de Calderón-Zygmund. En dicho trabajo obtienen el citado resultado de Muckenhoupt, además de estimaciones de tipo (1,1) débil.

Por otra parte, el primer teorema de transplantación considerando las funciones de Laguerre se debe a Y. Kanjin en [**26**], que fue mejorado considerando pesos de tipo potencia por K. Stempak y W. Trebels en [**46**]. Éste último fue refinado por G. Garrigós, E. Harboure, T. Signes, J. L. Torrea y B. Viviani en [**19**], dando además el mejor resultado para esa clase de pesos.

Es posible definir otro tipo de funciones de Laguerre por medio de los polinomios de Hermite, denominadas funciones de Laguerre de tipo Hermite. En esta memoria no las consideraremos, pero existen en la literatura resultados de transplantación para ellas que pueden ser consultados en la monografía [**48**] por S. Thangavelu.

Como ya hemos mencionado, el objetivo de esta memoria es el estudio de los análogos discretos de los problemas de la convergencia en norma y de transplantación. En los últimos años este campo de actividad en el que se consideran los análogos discretos de problemas clásicos ha tenido un florecimiento especial. Baste citar aquí el excelente trabajo [**16**] de Ó. Ciaurri, T. A. Gillespie, L. Roncal, J. L. Torrea y J. L. Varona, en donde se presenta un estudio completo de los operadores asociados al Laplaciano discreto

$$\Delta_d f(n) := f(n+1) - 2f(n) + f(n-1), \qquad n \in \mathbb{Z}.$$

En cuanto al análisis armónico discreto asociado a polinomios ortogonales clásicos, uno de los primeros estudios se debe a J. J. Betancor, A. J. Castro, J. C. Fariña y L. Rodríguez-Mesa en [**13**]. En dicho trabajo, además de estudiarse problemas clásicos como la acotación del semigrupo del calor o el estudio de las denominadas funciones de Littlewood-Paley-Stein, todo ello para polinomios ultraesféricos, también se desarrolla una teoría local discreta de Calderón-Zygmund, y que será utilizada en esta memoria. Este trabajo fue posteriormente mejorado para polinomios de Jacobi en una serie de trabajos conjuntos del autor junto con Ó. Ciaurri y E. Labarga. Concretamente en [**3**] se estudia el semigrupo del calor, en [**4**] se hace lo propio con la transformada de Riesz, y finalmente en [**5**] se investigan las funciones de Littlewood-Paley-Stein junto con multiplicadores de tipo Laplace. Cabe reseñar que los tres artículos citados forman parte de la tesis doctoral [**27**], recientemente presentada por E. Labarga.

El análogo discreto de los operadores suma parcial en que estamos interesados vendrá dado como el multiplicador de un cierto intervalo. En concreto, tomamos

$$S_{[c,d]}f(n) = c_n\left(\chi_{[c,d]}\mathcal{F}f\right),$$

donde $[c,d] \subset (a,b)$. De este modo, queremos estudiar bajo que condiciones se cumple que

$$\lim_{\substack{c\to a^+\\ d\to b^-}} \|S_{[c,d]}f - f\|_{\ell^p(\mathbb{N})} = 0.$$

Respecto a los operadores de transplantación, ya hemos visto que necesariamente hay que considerar las funciones asociadas a los polinomios dependientes de parámetros. Así, definimos el operador transplantación discreta por

$$T_{\lambda_1}^{\lambda_2}f(n) = c_n^{\lambda_1}\left(\mathcal{F}_{\lambda_2}f\right), \qquad f\in\ell^2(\mathbb{N}),$$

y nos interesará saber bajo que condiciones dichos operadores están acotados en norma p, incluyendo algunos pesos. Esto es,

$$\|T_{\lambda_1}^{\lambda_2}f\|_{\ell^p(\mathbb{N},w)} \lesssim \|f\|_{\ell^p(\mathbb{N},w)}, \qquad f\in\ell^2(\mathbb{N})\cap\ell^p(\mathbb{N},w).$$

Nuevamente el operador transplantación $T_{\lambda_1}^{\lambda_2}$ se extiende de manera única a un operador lineal acotado de $\ell^p(\mathbb{N})$ en sí mismo.

Estos operadores de transplantación discretos relacionados con los funciones asociadas a polinomios ortogonales clásicos ya fueron estudiados en [**9**] por R. Askey y S. Wainger para el caso de los coeficientes de Fourier de funciones ultraesféricas, y únicamente por Askey en [**6**] para los coeficientes de funciones de Jacobi. En ambos casos únicamente se consideran pesos potencia. Además de esto, también en [**6**], Askey da algunos resultados sobre los coeficientes de Fourier de las funciones de Laguerre, sin incluir pesos y tratando únicamente los órdenes α y $\alpha+2$.

Resumen de los capítulos

La presente memoria está dividida en cinco capítulos. Damos aquí un breve resumen de cada uno de ellos.

El CAPÍTULO 1 tiene un carácter introductorio a la teoría de polinomios ortogonales clásicos. A pesar de existir excelentes monografías dedicadas al tema, hemos decidido incluir este capítulo tratando de hacer esta memoria lo más autocontenida posible. Se construyen los polinomios ortogonales clásicos a partir de la ecuación diferencial hipergeométrica para, a continuación, dar varias propiedades que utilizaremos más adelante. También se definen las funciones asociadas a los polinomios ortogonales clásicos. Finalmente se dan varias estimaciones tanto de los polinomios como de las funciones asociadas a ellos.

El CAPÍTULO 2 está dedicado a exponer las teorías locales discretas de Calderón-Zygmund apropiadas para nuestros propósitos. En la primera parte se expone la desarrollada por J. J. Betancor, A. J. Castro, J. C. Fariña y L. Rodríguez-Mesa en [**13**]. La denominamos semi-local y damos un esbozo de su prueba. La segunda parte está dedicada al desarrollo de una teoría local discreta de Calderón-Zygmund basada en la análoga continua publicada por A. Nowak y K. Stempak en [**34**]. En la última parte se dan conocidas desigualdades en norma con pesos para algunos operadores discretos clásicos.

El CAPÍTULO 3 es el primero que no tiene un carácter preliminar. En él se estudia la convergencia del multiplicador del intervalo asociado a los polinomios de Jacobi. En primer lugar damos un resultado de suficiencia para la acotación uniforme con pesos del intervalo. A partir de él, damos una caracterización para el caso en el que no se consideran pesos, así como la caracterización de la convergencia de dicho operador. Como consecuencia de esto último, la convergencia puntual del multiplicador se sigue inmediatamente.

El CAPÍTULO 4 se dedica al estudio de los operadores de transplantación para los coeficientes de Fourier asociados a las funciones de Jacobi. Usando la teoría discreta semi-local de Calderón-Zygmund probamos la acotación para los operadores de transplantación de $\ell^p(\mathbb{N}, w)$ en sí mismo, donde w es un peso en la clase discreta de Muckenhoupt. Además de esto, obtenemos estimaciones (1,1) débiles con pesos para dichos operadores.

El CAPÍTULO 5 es el último de la memoria. En él se estudian los operadores de transplantación discretos para los coeficientes de Fourier asociados a las funciones de Laguerre. En esta ocasión utilizamos la teoría discreta local de Calderón-Zygmund para probar la acotación de $\ell^p(\mathbb{N}, w)$ en sí mismo, donde w es un peso en la clase discreta local de Muckenhoupt. Además de esto, se da una caracterización completa cuando se consideran pesos de tipo potencia.

Notación básica

A lo largo de la memoria se utilizará notación estándar bien conocida.

Con carácter general denotaremos por letras mayúsculas las funciones dependientes de una variable continua $x \in (a, b)$, mientras que las letras minúsculas se reservan para las sucesiones en $\mathbb{N}$. Dado un número p tal que $1 \le p < \infty$, el espacio de sucesiones p-sumables se define como

$$\ell^p(\mathbb{N}) := \left\{ f = \{f(n)\}_{n\ge0} : \|f\|_{\ell^p(\mathbb{N})} := \left(\sum_{n=0}^{\infty} |f(n)|^p \right)^{1/p} < \infty \right\}.$$

En ocasiones también utilizaremos el espacio de tipo Lorentz

$$\ell^{1,\infty}(\mathbb{N}) := \left\{ f = \{f(n)\}_{n\ge0} : \|f\|_{\ell^{1,\infty}(\mathbb{N})} := \sup_{\lambda>0} \lambda \#\{n \in \mathbb{N} : |f(n)| > \lambda\} < \infty \right\}.$$

Una sucesión peso en $\mathbb{N}$ será una sucesión w estrictamente positiva. Los espacios de sucesiones con pesos p-sumables vendrán dados por

$$\ell^p(\mathbb{N}, w) := \left\{ f = \{f(n)\}_{n\geq 0} : \|f\|_{\ell^p(\mathbb{N},w)} := \left(\sum_{n=0}^{\infty} |f(n)|^p w(n) \right)^{1/p} < \infty \right\},$$

y

$$\ell^{1,\infty}(\mathbb{N}, w) := \left\{ f = \{f(n)\}_{n\geq 0} : \|f\|_{\ell^{1,\infty}(\mathbb{N},w)} := \sup_{\lambda>0} \lambda \sum_{\{n\in\mathbb{N}:|f(n)|>\lambda\}} w(n) < \infty \right\}.$$

Claramente cuando $w(n) = 1$ para todo $n \in \mathbb{N}$, estos espacios coinciden con los anteriores.

A lo largo de la memoria usaremos $a \lesssim b$ para denotar que dos cantidades a y b cumplen la relación $a \leq Cb$ para una constante C independiente de cantidades significativas. Por su parte, usaremos $a \simeq b$ si existen dos constantes C_1 y C_2 independientes de cantidades significativas tales que $C_1 b \leq a \leq C_2 b$. Sin embargo, habrá situaciones en las que no utilizaremos esta convención por motivos de claridad en la exposición, especialmente en lo referente a la parte en la que se tratan desigualdades que involucran pesos.

Para finalizar, también con carácter general, denotaremos por letras mayúsculas los polinomios ortogonales clásicos, por letras minúsculas los ortonormales y por letras caligráficas mayúsculas las funciones asociadas a ellos. Además tomaremos como convención que $P_{-1}(x) = 0$, especialmente en lo referente a las fórmulas de recurrencia.

CHAPTER 1

Classical orthogonal polynomials

1. Definitions and first properties

In this section we expose briefly several results related to classical orthogonal polynomials, namely, Jacobi, Laguerre and Hermite. We do not mean to made an exhaustive study related to orthogonal polynomials because there exits in the literature excellent resources devoted to that end, such as the classical monographs by G. Szegő [**47**] and T. S. Chihara [**15**]. The interested reader are urged to consult those references.

There are several ways to introduce the classical orthogonal polynomials. In this dissertation we shall follow the way given in the book [**32**] by A. F. Nikiforov and V. B. Uvarov and, for that reason, we omit the majority of proofs. There, classical orthogonal polynomials are defined as the polynomial solutions of the second order differential equation

$$\sigma(x)\frac{d^2}{d^2x}y(x)+\tau(x)\frac{d}{dx}y(x)+\lambda y(x)=0, \qquad \lambda\in\mathbb{R}, \tag{1.1}$$

where $\sigma(x)$ and $\tau(x)$ are polynomials whose degrees are, at most, two and one, respectively. This equation is known as the hypergeometric differential equation and its solutions as functions of hypergeometric type. The reason for this nomenclature is the so-called hypergeometric property, that is, all the derivatives of functions of hypergeometric type are also functions of hypergeometric type.

Polynomial solutions $y(x)\equiv y_n(x)$ of the hypergeometric differential equation (1.1), where n denotes the degree of the polynomial, are called polynomials of hypergeometric type. It is important to note that, in this situation, the constant $\lambda\equiv\lambda_n$ also depends on the degree n by

$$\lambda_n=-\frac{n(n-1)}{2}\frac{d^2}{dx^2}\sigma(x)-n\frac{d}{dx}\tau(x), \qquad n\in\mathbb{N}.$$

Polynomials of hypergeometric type are given explicitly by the Rodrigues' formula

$$y_n(x)=\frac{B_n}{\rho(x)}\frac{d^n}{dx^n}\left(\sigma^n(x)\rho(x)\right), \tag{1.2}$$

where B_n is a normalization constant and $\rho(x)$ a function satisfying the Pearson's differential equation

$$\frac{d}{dx}\left(\sigma(x)\rho(x)\right)=\tau(x)\rho(x). \tag{1.3}$$

An immediate consequence of the previous equation is the factorization of the hypergeometric differential equation

$$\mathfrak{D} y_n(x) = \lambda_n y_n(x), \qquad \mathfrak{D} = \frac{-1}{\rho(x)} \frac{d}{dx}\left(\sigma(x)\rho(x)\frac{d}{dx}\right), \tag{1.4}$$

which shows that polynomials of hypergeometric type are eigenfunctions of the second order differential operator $\mathfrak{D}$, whose correspondent eigenvalue is λ_n.

Pearson's equation, together with the Rodrigues' formula, shows that the degree of the polynomial $\tau(x)$ is exactly one. Regarding the polynomial $\sigma(x)$, since it has degree at most two, it could be expressed by

$$\sigma(x) = (b-x)(x-a), \quad \sigma(x) = z - c, \quad \text{or} \quad \sigma(x) = 1,$$

with a, b, and c complex numbers, provided $\sigma(x)$ has not got a double zero. By means of linear change of variable, the above expressions could be transformed, up to multiplicative constants, into the three following canonical forms

$$\sigma(x) = 1 - x^2, \quad \sigma(x) = x, \quad \text{and} \quad \sigma(x) = 1.$$

Previous line and the fact that $\tau(x)$ has degree one allow us to consider the possible solutions of the Pearson's equation (1.3) and, therefore, to get, again up to constants factors, the possible expressions for the function $\rho(x)$:

$$\rho(x) = \begin{cases} (1-x)^\alpha(1+x)^\beta, & \text{for } \sigma(x) = 1-x^2, \\ x^\alpha e^{-x}, & \text{for } \sigma(x) = x, \\ e^{-x^2}, & \text{for } \sigma(x) = 1. \end{cases}$$

According to the choice of the function $\rho(x)$ among its possible forms, we obtain the following systems of polynomials.

1. Let $\sigma(x) = 1 - x^2$, $\rho(x) = (1-x)^\alpha(1+x)^\beta$. Then

$$\tau(x) = -(\alpha+\beta+2)x + \beta - \alpha.$$

The corresponding polynomials are the Jacobi polynomials of order (α, β), which are denoted by $P_n^{(\alpha,\beta)}(x)$, and whose Rodrigues' formula is given by

$$P_n^{\alpha,\beta}(x) = \frac{(-1)^n}{2^n n!(1-x)^\alpha(1+x)^\beta} \frac{d^n}{dx^n}\left((1-x)^{n+\alpha}(1+x)^{n+\beta}\right),$$

were we have taken $B_n = (-1)^n/(2^n n!)$. There are important particular cases of the Jacobi polynomials.

i. The Legendre polynomials

$$P_n(x) = P_n^{(0,0)}(x).$$

ii. The Chebyshov polynomials of the first kind T_n and second kind U_n, given by

$$T_n(x) = \frac{n!}{(1/2)_n} P_n^{(-1/2,-1/2)}(x), \text{ and}$$
$$U_n(x) = \frac{(n+1)!}{(3/2)_n} P_n^{(1/2,1/2)}(x),$$

respectively.

iii. The ultraspherical polynomials, sometimes known as Gegenbauer polynomials,

$$C_n^{\lambda}(x) = \frac{(2\lambda)_n}{(\lambda+1/2)_n} P_n^{(\lambda-1/2,\lambda-1/2)}(x).$$

2. Let $\sigma(x) = x$, $\rho(x) = x^{\alpha}e^{-x}$. Then

$$\tau(x) = -x + \alpha + 1.$$

The polynomials are the Laguerre polynomials of order α, which are denoted by $L_n^{\alpha}(x)$, and whose Rodrigues' formula is given by

$$L_n^{\alpha}(x) = \frac{1}{n!x^{\alpha}e^{-x}} \frac{d^n}{dx^n}\left(x^{n+\alpha}e^{-x}\right),$$

where we have taken $B_n = 1/n!$.

3. Let $\sigma(x) = 1$, $\rho(x) = e^{-x^2}$. Then

$$\tau(x) = -2x.$$

In this last case, the polynomials are the Hermite polynomials, which are denoted by $H_n(x)$, and whose Rodigues' formula is given by

$$H_n(x) = \frac{(-1)^n}{e^{-x^2}} \frac{d^n}{dx^n}\left(e^{-x^2}\right).$$

where we have taken $B_n = (-1)^n$.

Regarding the pathological case $\sigma(x) = (x-a)^2$, it is possible to show that the corresponding hypergeometric polynomials could be written, under some additional hypotheses, in terms of the Laguerre polynomials. However, since we shall not consider these kind of polynomials anymore we are not giving more details.

If $\rho(x)$ satisfies some additional conditions, we can obtain the orthogonality property. From now on, we shall denote by $d\mu(x) := \rho(x)\,dx$ for simplicity.

Theorem 1.1. *Let $\rho(x)$ so that for all $k \in \mathbb{N}$ satisfies*

$$\left(\sigma(x)\rho(x)x^k\right)\Big|_{x=a}^{x=b} = 0. \tag{1.5}$$

Then, the polynomials $y_n(x)$ of hypergeometric type corresponding to different constants λ_n are orthogonal on (a,b) with weight $\rho(x)$, i.e.,

$$\int_a^b y_n(x)y_m(x)\,d\mu(x) = \frac{\delta_{n,m}}{\omega_n^2}, \tag{1.6}$$

where $\delta_{n,m}$ *is the usual Kronecker's delta function and* ω_n *denotes the norm in the space* $L^2((a,b),d\mu)$ *of the polynomials* y_n*.*

PROOF. Consider the adjoint differential equation (1.4) for the polynomials y_n and y_m, and mutilply the first by y_m and the second by y_n, that is

$$y_m(x)\mathfrak{D}y_n(x) + \lambda_n y_n(x)y_m(x) = 0 \quad \text{and} \quad y_n(x)\mathfrak{D}y_m(x) + \lambda_m y_m(x)y_n(x) = 0.$$

Substracting these two identities we have

$$y_m(x)\mathfrak{D}y_n(x) - y_n(x)\mathfrak{D}y_m(x) + (\lambda_n - \lambda_m)y_n(x)y_m(x) = 0.$$

Since we can rewrite

$$y_m(x)\mathfrak{D}y_n(x) - y_n(x)\mathfrak{D}y_m(x) = \frac{1}{\rho(x)}\frac{d}{dx}\left(\sigma(x)\rho(x)W(y_m,y_n)(x)\right)$$

where $W(\cdot,\cdot)$ denotes the usual Wronskian, integrating from a to b against $\rho(x)\,dx$ we obtain

$$(\lambda_m - \lambda_n)\int_a^b y_n(x)y_m(x)\,d\mu(x) = \sigma(x)\rho(x)W(y_m,y_n)(x)\Big|_{x=a}^{x=b}$$

The right hand side is a polynomial in the variable x, which is zero due hyphotesis. Hence, the orthogonality holds provided $\lambda_m \neq \lambda_n$. □

The polynomials $y_n(x)$ for which $\rho(x)$ satisfies condition (1.5) are known as classical orthogonal polynomials. They are usually considered under the extra condition of the positivity of the functions $\rho(x)$ and $\sigma(x)$ in the interval (a,b). A possible choice is to take a and b the roots of the equation $\sigma(x) = 0$, provided they exits. These conditions are satisfied by the Jacobi polynomials $P_n^{(\alpha,\beta)}(x)$ in the interval $(-1,1)$ and $\alpha,\beta > -1$; by the Laguerre polynomials $L_n^\alpha(x)$ in the interval $(0,\infty)$ and $\alpha > -1$; and by the Hermite polynomials in the interval $(-\infty,\infty)$. Note that in all three cases the condition $\lambda_m \neq \lambda_n$ could be replaced by $n \neq m$.

In order to obtain the norm ω_n of the polynomials we could use the Rodrigues' formula in (1.6), so

$$\frac{1}{\omega_n^2} = B_n \int_a^b y_n(x)\frac{d^n}{dx^n}\left(\sigma^n(x)\rho(x)\right)\,dx.$$

Integrating by parts and using that $\frac{d^n}{dx^n}y_n(x) = n!a_n$, where a_n is the coefficient of the power x^n of the polynomial y_n, we conclude that

$$\frac{1}{\omega_n^2} = (-1)^n B_n n! a_n \int_a^b \sigma^n(x)\,d\mu(x).$$

For the Jacobi, Laguerre and Hermite polynomials the integral involved in the previous line can be evaluated in terms of Gamma functions. Then, for Jacobi polynomials we get

$$\omega_n^{(\alpha,\beta)} = \sqrt{\frac{(2n+\alpha+\beta+1)\Gamma(n+1)\Gamma(n+\alpha+\beta+1)}{2^{\alpha+\beta+1}\Gamma(n+\alpha+1)\Gamma(n+\beta+1)}},$$

where for $n = 0$ and $\alpha + \beta = -1$ the term $(2n + \alpha + \beta + 1)\Gamma(n + \alpha + \beta + 1)$ is replaced by $\Gamma(n + \alpha + \beta + 2)$. Regarding Laguerre polynomials we have

$$\omega_n^\alpha = \sqrt{\frac{\Gamma(n+1)}{\Gamma(n+\alpha+1)}},$$

and for Hermite $\omega_n = (\pi^{1/2} 2^n \Gamma(n+1))^{-1/2}$. Then, multiplying each polynomial for its norm we obtain the orthonormal classical polynomials

$$p_n(x) := \omega_n^{(\alpha,\beta)} P_n^{(\alpha,\beta)}(x), \quad \ell_n^\alpha(x) := \omega_n^\alpha L_n^\alpha(x), \quad h_n(x) := \omega_n(x) H_n(x).$$

We have already mention that all the derivatives of all order of the polynomials $y_n(x)$ of hypergeometric type are also polynomials of hypergeometric type. By the Rodrigues' formula for $y_n(x)$ and its corresponding for $y_n'(x)$ it is possible to obtain the following differentiation formulas for the Jacobi, Laguerre and Hermite polynomials:

$$\begin{aligned} \frac{d}{dx} P_n^{(\alpha,\beta)}(x) &= \frac{n+\alpha+\beta+1}{2} P_{n-1}^{(\alpha+1,\beta+1)}(x), \\ \frac{d}{dx} L_n^\alpha(x) &= -L_{n-1}^{\alpha+1}(x), \\ \frac{d}{dx} H_n(x) &= 2n H_{n-1}(x). \end{aligned} \tag{1.7}$$

Regarding the parity in the case of Jacobi and Hermite polynomials, again from the Rodrigues' formula, it is possible to prove

$$P_n^{(\alpha,\beta)}(-x) = (-1)^n P_n^{(\beta,\alpha)}(x), \qquad -1 < x < 1, \tag{1.8}$$

and, analogously,

$$H_n(-x) = (-1)^n H_n(x), \qquad -\infty < x < \infty.$$

Finally, we give here two facts involving Laguerre polynomials that we shall use in last chapter. First one is the connection formula

$$L_n^\beta(x) = \sum_{j=0}^{n} \frac{(\beta-\alpha)_{n-j}}{(n-j!)} L_j^\alpha(x), \tag{1.9}$$

which allows us to express a Laguerre polynomial of degree n and order β as a linear combination of another Laguerre polynomials of order α and degrees less or equal than n. Second one is a proper expression for the difference of two Laguerre polynomials whose degree differ in one unit, given by

$$L_n^\alpha(x) - L_{n+1}^\alpha(x) = \frac{x}{n+1} L_n^{\alpha+1}(x) - \frac{\alpha}{n+1} L_n^\alpha(x). \tag{1.10}$$

2. Functions associated with classical orthogonal polynomials

It is well known that the set $\{p_n(x)\}_{n\geq 0}$ of classical orthonormal polynomials is complete in the space $L^2((a,b),d\mu)$. Let us consider the function $\mathcal{P}_n(x) = \rho^{1/2}(x)p_n(x)$, then the family $\{\mathcal{P}_n(x)\}_{n\geq 0}$ is also a complete orthonormal system but now in the space $L^2((a,b),dx)$. Functions $\mathcal{P}_n(x)$ are called functions associated with classical orthogonal polynomials, and they fulfill analogues of the given properties for the latter. Here, we mainly use two special cases.

Let us start with Jacobi functions, which are highly commonly expressed in terms of the variable $\theta \in (0,\pi)$. This dependence is obtained by means of the change of variable $x = \cos\theta$. Then, the Jacobi functions are given by

$$\mathcal{P}_n^{(\alpha,\beta)}(\theta) := 2^{(\alpha+\beta+1)/2}(\sin\theta/2)^{\alpha+1/2}(\cos\theta/2)^{\beta+1/2}p_n^{(\alpha,\beta)}(\cos\theta). \tag{1.11}$$

As well as the Jacobi polynomials are the eigenfunctions of the second order differential operator $\mathfrak{D}^{\alpha,\beta}$ given in (1.4), Jacobi functions are also eigenfunctions of the operator

$$\mathfrak{L}^{\alpha,\beta} = -\frac{d}{d\theta^2} - \left(\frac{1-4\alpha^2}{16\sin^2(\theta/2)} + \frac{1-4\beta^2}{16\cos^2(\theta/2)}\right), \qquad \theta \in (0,\pi),$$

with eigenvalues $\nu_n^{(\alpha,\beta)} = (n + (\alpha+\beta+1)/2)^2$, that is,

$$\mathfrak{L}^{\alpha,\beta}\mathcal{P}_n^{(\alpha,\beta)}(\theta) = \nu_n^{(\alpha,\beta)}\mathcal{P}_n^{(\alpha,\beta)}(\theta). \tag{1.12}$$

The operator $\mathfrak{L}^{\alpha,\beta}$ is symmetric in the sense of

$$\int_0^\pi \mathfrak{L}^{\alpha,\beta}f(\theta)g(\theta)\,d\theta = \int_0^\pi f(\theta)\mathfrak{L}^{\alpha,\beta}g(\theta)\,d\theta,$$

at least for functions $f, g \in C_c^2(0,\pi) \subset L^2(0,\pi)$. However, for any interval $[a,b] \subset (0,\pi)$, we have

$$\int_a^b \mathfrak{L}^{\alpha,\beta}f(\theta)g(\theta)\,d\theta = W(f,g)(\theta)\Big|_{\theta=a}^{\theta=b} + \int_a^b f(\theta)\mathfrak{L}^{\alpha,\beta}g(\theta)\,d\theta, \tag{1.13}$$

where an extra term appears. Moreover, operators of different orders are related by means of the identity

$$\mathfrak{L}^{\alpha,\beta} = \mathfrak{L}^{\gamma,\delta} + R_{\alpha,\beta}^{\gamma,\delta}(\theta), \qquad R_{\alpha,\beta}^{\gamma,\delta}(\theta) = \frac{\alpha^2-\gamma^2}{4\sin^2(\theta/2)} + \frac{\beta^2-\delta^2}{4\cos^2(\theta/2)}. \tag{1.14}$$

The same situation states for the Laguerre functions

$$\mathcal{L}_n^\alpha(x) := x^{\alpha/2}e^{-x/2}\ell_n^\alpha(x). \tag{1.15}$$

They fulfil the identity $\mathfrak{L}^\alpha\mathcal{L}_n^\alpha(x) = \nu_n^\alpha\mathcal{L}_n^\alpha(x)$, where

$$\mathfrak{L}^\alpha = -x\frac{d^2}{dx^2} - (x+1)\frac{d}{dx} + \frac{\alpha^2}{4x}, \qquad \nu_n^\alpha = 4n + 2\alpha + 2.$$

In other words, they are also eigenfunctions of a second order differential operator, namely $\mathfrak{L}^\alpha$. Occasionally we shall use the so-called Laguerre functions of Hermite type

$$\varphi_n^\alpha(x) = \sqrt{2x}\mathcal{L}_n^\alpha(x^2),$$

which fulfil $\bar{\mathfrak{L}}^\alpha \varphi_n^\alpha(x) = \nu_n^\alpha \varphi_n^\alpha(x)$ for

$$\bar{\mathfrak{L}}^\alpha = -\frac{d^2}{dx^2} - \frac{1/4-\alpha^2}{x^2} + x^2.$$

3. Estimates for Jacobi polynomials and functions

Throughout the present dissertation we shall use well-known estimates for both Jacobi polynomials and Jacobi functions. It is possible to obtain similar bounds for the cases of Laguerre and Hermite, but we omit them because they will not be necessary.

Let us start by the estimate (c.f. [**47**, Thm. 7.32.2])

$$P_n(\cos\theta) = \begin{cases} \theta^{-\alpha-1/2} O((n+1)^{-1/2}), & C/(n+1) \le \theta \le \pi/2, \\ O((n+1)^\alpha), & 0 < \theta \le C/(n+1). \end{cases}$$

By the change of variable $x = \cos\theta$, which implies $x \simeq 1 - \theta^2/2$, we get

$$P_n^{(\alpha,\beta)}(x) = \begin{cases} (1-x)^{-\alpha/2-1/2} O((n+1)^{-1/2}), & 0 \le x \le C(1-(n+1)^{-2}) \\ O((n+1)^\alpha), & C(1-(n+1)^{-2}) \le x < 1. \end{cases}$$

From here,

$$|p_n^{(\alpha,\beta)}(x)| \lesssim \begin{cases} (1-x)^{-\alpha/2-1/2}, & 0 \le x \le C(1-(n+1)^{-2}) \\ (n+1)^{\alpha+1/2}, & C(1-(n+1)^{-2}) \le x < 1. \end{cases} \tag{1.16}$$

where we have used $\omega_n^{(\alpha,\beta)} \simeq (n+1)^{-1/2}$. Considering two cases according to the sign of the value $-\alpha/2 - 1/4$, we obtain

$$|p_n^{\alpha,\beta}(x)| \lesssim (1-x+(n+1)^{-2})^{-\alpha/2-1/4}, \qquad 0 \le x < 1.$$

The estimate for $-1 < x \le 0$ could be obtained using the parity relation (1.8). So, finally,

$$|p_n^{\alpha,\beta}(x)| \lesssim \left(1-x+\frac{1}{(n+1)^2}\right)^{-\alpha/2-1/4} \left(1+x+\frac{1}{(n+1)^2}\right)^{-\beta/2-1/4}, \quad -1 < x < 1.$$

It is worth pointing out that if $\alpha, \beta \ge -1/2$ then

$$|p_n^{(\alpha,\beta)}(x)| \lesssim (1-x)^{-\alpha/2-1/4}(1+x)^{-\beta/2-1/4}, \quad -1 < x < 1, \tag{1.17}$$

which implies the estimate for Jacobi functions

$$|\mathcal{P}_n^{(\alpha,\beta)}(\theta)| \lesssim 1, \qquad 0 < \theta < \pi, \qquad \alpha, \beta \ge -1/2. \tag{1.18}$$

The estimate (1.16) implies the following one for Jacobi functions

$$|\mathcal{P}_n^{(\alpha,\beta)}(\theta)| \lesssim \begin{cases} (n+1)^{\alpha+1/2}\theta^{\alpha+1/2}, & 0 < \theta < 1/(n+1), \\ 1, & 1/(n+1) \le \theta \le \pi - 1/(n+1), \\ (n+1)^{\beta+1/2}(\pi-\theta)^{\beta+1/2}, & \pi - 1/(n+1) < \theta < \pi. \end{cases} \tag{1.19}$$

To obtain estimates for the derivative of a Jacobi function we need to consider the operator

$$\Psi := \frac{d}{d\theta} - \frac{2\alpha+1}{4}\cot\frac{\theta}{2} + \frac{2\beta+1}{4}\tan\frac{\theta}{2}.$$

It is known, c.f. [**33**, Sec. 7.7], that it fulfils the relation

$$\Psi\mathcal{P}_n^{(\alpha,\beta)}(\theta) = -(\lambda_n^{(\alpha,\beta)})^{1/2}\mathcal{P}_{n-1}^{(\alpha+1,\beta+1)}(\theta),$$

which allows us to express the derivative of a Jacobi function by

$$\frac{d}{d\theta}\mathcal{P}_n^{(\alpha,\beta)}(\theta) = -(\lambda_n^{(\alpha,\beta)})^{1/2}\mathcal{P}_{n-1}^{(\alpha+1,\beta+1)}(\theta) + \left(\frac{2\alpha+1}{4}\cot\frac{\theta}{2} - \frac{2\beta+1}{4}\tan\frac{\theta}{2}\right)\mathcal{P}_n^{(\alpha,\beta)}(\theta). \tag{1.20}$$

Then, by (1.19), we deduce the bound

$$\left|\frac{d}{d\theta}\mathcal{P}_n^{(\alpha,\beta)}(\theta)\right| \lesssim \begin{cases} (n+1)^{\alpha+1/2}\theta^{\alpha-1/2}, & 0<\theta<1/(n+1), \\ n+1, & 1/(n+1)\le\theta\le\pi-1/(n+1), \\ (n+1)^{\beta+1/2}(\pi-\theta)^{\beta-1/2}, & \pi-1/(n+1)<\theta<\pi. \end{cases} \tag{1.21}$$

CHAPTER 2

Discrete local Calderón-Zygmund theory

1. Introduction

The present chapter is devoted to show proper local discrete Calderón-Zygmund theories. First one was developed by J. J. Betancor, A. J. Castro, J. C. Fariña, and L. Rodríguez-Mesa in [**13**]. We present it in Section 2 and we shall call it semi-local. Second one is the analogue of the continuous one developed by A. Nowak and K. Stempak in [**34**].

Firstly, we give here some preliminar definitions related to the discrete Muckenhoupt weights.

For a Banach space $\mathbb{B}$ and a weight sequence $w := \{w(n)\}_{n\geq 0}$, we consider the space

$$\ell^p_{\mathbb{B}}(\mathbb{N}, w) = \{\mathbb{B}\text{-valued sequences } f = \{f(n)\}_{n\geq 0} : \{\|f(n)\|_{\mathbb{B}}\}_{n\geq 0} \in \ell^p(\mathbb{N}, w)\}$$

for $1 \leq p < \infty$, and

$$\ell^{1,\infty}_{\mathbb{B}}(\mathbb{N}, w) = \left\{\mathbb{B}\text{-valued sequences } f = \{f(n)\}_{n\geq 0} : \{\|f(n)\|_{\mathbb{B}}\}_{n\geq 0} \in \ell^{1,\infty}(\mathbb{N}, w)\right\}.$$

As usual, we simply write $\ell^p_{\mathbb{B}}(\mathbb{N})$ and $\ell^{1,\infty}_{\mathbb{B}}(\mathbb{N})$ when $w(n) = 1$ for all $n \in \mathbb{N}$. Also, we denote by $\mathbb{B}^{\mathbb{N}}_0$ the space of $\mathbb{B}$-valued sequences such that $f(n) = 0$ for all $n > j$, for some fixed $j \in \mathbb{N}$.

A weight sequence w belongs to the discrete Muckenhoupt $A_p(\mathbb{N})$ class, where $1 < p < \infty$, if

$$[w]_{A_p} := \sup_{\substack{n,m\in\mathbb{N}\\ 0\leq n\leq m}} \frac{1}{(m-n+1)^p} \left(\sum_{l=n}^{m} w(l)\right) \left(\sum_{l=n}^{m} w(l)^{-1/(p-1)}\right)^{p-1} < \infty,$$

and to the discrete Muckenhoupt $A_1(\mathbb{N})$ class if

$$[w]_{A_1(\mathbb{N})} = \sup_{\substack{n,m\in\mathbb{N}\\ 0\leq n\leq m}} \left(\sum_{l=n}^{m} w(l)\right) \max_{n\leq l\leq m} w(l)^{-1} < \infty.$$

The value $[w]_{A_p(\mathbb{N})}$ is called the constant of the weight w.

An interesting fact, which we shall use later, regarding the discrete Muckenhoupt weights is the following.

LEMMA 2.1. *Let $1 \leq p < \infty$ and $w \in A_p(\mathbb{N})$. Then, $w(n) \simeq w(n+1)$.*

Proof. For $1 < p < \infty$ and $w \in A_p(\mathbb{N})$, it is clear that

$$[w]_{A_p(\mathbb{N})} \geq \frac{1}{2^p}(w(n)+w(n+1))(w(n)^{-1/(p-1)}+w(n+1)^{-1/(p-1)})^{p-1}, \qquad n \in \mathbb{N}.$$

Now, by means of the inequality $(a+b)^r \geq C_r(a^r+b^r)$, where $a,b,r > 0$ and $C_r = \min\{2^{r-1},1\}$, we have

$$[w]_{A_p(\mathbb{N})} > \frac{C_{p-1}}{2^p}(w(n)+w(n+1))(w(n)^{-1}+w(n+1)^{-1}) > \frac{C_{p-1}}{2^p}w(n)w(n+1)^{-1}$$

and, similarly,

$$[w]_{A_p(\mathbb{N})} > \frac{C_{p-1}}{2^p}w(n+1)w(n)^{-1}.$$

So,

$$\frac{C_{p-1}}{2^p[w]_{A_p(\mathbb{N})}}w(n) < w(n+1) < \frac{2^p[w]_{A_p(\mathbb{N})}}{C_{p-1}}w(n).$$

For $p = 1$, if we suppose first $w(n) \leq w(n+1)$, then it is clear that

$$\begin{aligned}[w]_{A_1(\mathbb{N})} &\geq \frac{1}{2}(w(n)+w(n+1))\max\{w(n)^{-1},w(n+1)^{-1}\} \\ &= \frac{1}{2}(1+w(n+1)w(n)^{-1}) > \frac{w(n+1)w(n)^{-1}}{2}\end{aligned}$$

and we obtain $w(n) \leq w(n+1) < 2[w]_{A_1(\mathbb{N})}w(n)$. On the other hand, supposing $w(n+1) < w(n)$ the procedure is exactly the same. $\square$

2. Semi-local Calderón-Zygmund theory

Let $\mathbb{B}_1$, $\mathbb{B}_2$ be Banach spaces and consider $\mathcal{L}(\mathbb{B}_1,\mathbb{B}_2)$, the space of bounded linear operators form $\mathbb{B}_1$ into $\mathbb{B}_2$. Let us suppose that the application

$$K : (\mathbb{N}\times\mathbb{N}) \setminus D \longrightarrow \mathcal{L}(\mathbb{B}_1,\mathbb{B}_2),$$

which we shall call kernel and where $D := \{(n,n) : n \in \mathbb{N}\}$, is measurable and that, for each $n,m \in \mathbb{N}$, the following conditions hold:

(a) the size condition

$$\|K(n,m)\|_{\mathcal{L}(\mathbb{B}_1,\mathbb{B}_2)} \lesssim \frac{1}{|n-m|}; \text{ and}$$

(b) the regularity properties

(b1) $$\|K(n,m)-K(n,l)\|_{\mathcal{L}(\mathbb{B}_1,\mathbb{B}_2)} \lesssim \frac{|m-l|}{|n-m|^2}, \quad |n-m| > 2|m-l|,\ n_0 \leq m,l \leq n_0^*,$$

(b2) $$\|K(n,m)-K(s,m)\|_{\mathcal{L}(\mathbb{B}_1,\mathbb{B}_2)} \lesssim \frac{|n-s|}{|n-m|^2}, \quad |n-m| > 2|n-s|,\ n_0 \leq m,s \leq n_0^*.$$

Here and from now on, we shall denote $n_0 := 2n/3$ and $n_0^* := 3n/2$ for $n \in \mathbb{N}$. A kernel $K(n,m)$ satisfying conditions (a) and (b) is called a semi-local $\mathcal{L}(\mathbb{B}_1, \mathbb{B}_2)$-standard kernel.

By a discrete semi-local Calderón-Zygmund operator we mean a linear and bounded operator T form $\ell^r_{\mathbb{B}_1}(\mathbb{N})$ into $\ell^r_{\mathbb{B}_2}(\mathbb{N})$, for some $1 < r < \infty$, and such that there exists a semi-local $\mathcal{L}(\mathbb{B}_1, \mathbb{B}_2)$-standard kernel $K(n,m)$ so that

$$Tf(n) = \sum_{m=0}^{\infty} f(m)K(n,m),$$

for every sequence $f \in (\mathbb{B}_1)_0^{\mathbb{N}}$. This representation of the operator T holds for every $n \in \mathbb{N}$ such that $f(n) = 0$. We shall use the terminology semi-local regarding this theory due to the fact that the operator is defined in the complete region of the variable $m \in \mathbb{N}$.

Next result is Theorem 2.1 in [**13**].

Theorem 2.1 (Betancor et al., 2019). *Let $\mathbb{B}_1$ and $\mathbb{B}_2$ be Banach spaces and T a semi-local Calderón-Zygmund operator. Then,*

(i) *for every $1 < p < \infty$ and $w \in A_p(\mathbb{N})$, the operator T can be extended from $\ell^r_{\mathbb{B}_1}(\mathbb{N}) \cap \ell^p_{\mathbb{B}_1}(\mathbb{N}, w)$ to $\ell^p_{\mathbb{B}_1}(\mathbb{N}, w)$ as a bounded operator form $\ell^p_{\mathbb{B}_1}(\mathbb{N}, w)$ into $\ell^p_{\mathbb{B}_2}(\mathbb{N}, w)$; and,*

(ii) *for every $w \in A_1(\mathbb{N})$, the operator T can be extended from $\ell^r_{\mathbb{B}_1}(\mathbb{N}) \cap \ell^1_{\mathbb{B}_1}(\mathbb{N}, w)$ to $\ell^1_{\mathbb{B}_1}(\mathbb{N}, w)$ as a bounded operator form $\ell^1_{\mathbb{B}_1}(\mathbb{N}, w)$ into $\ell^{1,\infty}_{\mathbb{B}_2}(\mathbb{N}, w)$.*

Proof. We split the operator T in its local and global part, given by

$$T_{\text{glob}} f(n) = T(\chi_{\mathbb{N}\setminus W_n} f)(n) \quad \text{and} \quad T_{\text{loc}} f(n) = Tf(n) - T_{\text{glob}} f(n),$$

respectively, where

$$W_n = \{j \in \mathbb{N} : n_0 \le j \le n_0^*\}.$$

Let us study first the global part. Since $\chi_{\mathbb{N}\setminus W_n}(n) = 0$ for all $n \in \mathbb{N}$, it can be rewritten by

$$T_{\text{glob} f(n)} = \sum_{m \in \mathbb{N}\setminus W_n} f(m)K(n,m).$$

Using the size condition (a), we obtain the estimate

$$\begin{aligned} \|T_{\text{glob}} f(n)\|_{\mathbb{B}_2} &\lesssim \sum_{m \in \mathbb{N}\setminus W_n} \frac{\|f(m)\|_{\mathbb{B}_1}}{|n-m|} \\ &\lesssim \frac{1}{n+1} \sum_{\substack{m \in \mathbb{N} \\ m < n_0}} \|f(m)\|_{\mathbb{B}_1} + \sum_{\substack{m \in \mathbb{N} \\ m > n_0^*}} \frac{\|f(m)\|_{\mathbb{B}_2}}{m+1} \\ &\lesssim H(\|f\|_{\mathbb{B}_1})(n) + H^*(\|f\|_{\mathbb{B}_1})(n), \end{aligned}$$

where we have denote $\|f\|_{\mathbb{B}_1} = \{\|f(m)\|_{\mathbb{B}_1}\}_{m \ge 0}$ and by H and H^* the discrete Hardy operator and its adjoint, which are defined, for any $f \in \mathbb{C}^{\mathbb{N}}$, by

$$Hf(n) = \frac{1}{n+1} \sum_{m=0}^{n} f(m) \qquad \text{and} \qquad H^*f(n) = \sum_{m=n}^{\infty} \frac{f(m)}{m+1},$$

respectively. It is well-known that both Hardy operator and its adjoint are bounded on $\ell^p(\mathbb{N}, w)$, vid. e.g. [**35**], so T_{glob} can be extended to $\ell^p_{\mathbb{B}_1}(\mathbb{N}, w)$ as a bounded operator form $\ell^p_{\mathbb{B}_1}(\mathbb{N}, w)$ into $\ell^p_{\mathbb{B}_2}(\mathbb{N}, w)$ if $1 < p < \infty$ and $w \in A_p(\mathbb{N})$. They can also be extended to $\ell^1_{\mathbb{B}_1}(\mathbb{N}, w)$ as a bounded operator from $\ell^1_{\mathbb{B}_1}(\mathbb{N}, w)$ into $\ell^{1,\infty}_{\mathbb{B}_2}(\mathbb{N}, w)$.

Let us deal now with the local part, which is expressed by

$$T_{\text{loc}} f(n) = \sum_{m \in W_n} f(m) K(n, m)$$

for all $n \in \mathbb{N}$ such that $f(n) = 0$. For $n, m \in \mathbb{N}$, $n \neq m$, we define the restricted kernel $\bar{K}(n, m) = \chi_{W_n}(m) K(n, m)$. Note that $\chi_{W_n}(m) = \chi_{W_m}(n)$. It turns out that $\bar{K}$ satisfies certain Hörmander-type conditions which are natural discrete vector-valued analogues of [**34**, Eqs. (4.4) and (4.5)]. More precisely

$$\sum_{n \in \mathbb{N} \setminus 2I} \|\bar{K}(n, m) - \bar{K}(n, l)\|_{\mathcal{L}(\mathbb{B}_1, \mathbb{B}_2)} \|f(n)\|_{\mathbb{B}_1} \lesssim \mathcal{M}(\|f\|_{\mathbb{B}_1})(m), \qquad m, l \in I, \tag{2.1}$$

and

$$\sum_{m \in \mathbb{N} \setminus 2I} \|\bar{K}(n, m) - \bar{K}(s, m)\|_{\mathcal{L}(\mathbb{B}_1, \mathbb{B}_2)} \|f(m)\|_{\mathbb{B}_1} \lesssim \mathcal{M}(\|f\|_{\mathbb{B}_1})(n), \qquad n, s \in I, \tag{2.2}$$

for all intervals I in $\mathbb{N}$, with $\mathcal{M}$ denoting the non-centered discrete Hardy-Littlewood maximal function given by

$$\mathcal{M} f(n) = \sup_{\substack{n \in I \\ I \text{ interval}}} \frac{1}{\#(I)} \sum_{m \in I} f(n), \qquad n \in \mathbb{N}.$$

Here, for $a, b \in \mathbb{N}$, an interval in $\mathbb{N}$ is given by $I = [a, b] \cap \mathbb{N}$ if $a \leq b$, setting $I = \emptyset$ in case that $b < a$. Futhermore, $2I$ denotes the interval

$$2I := \left[a - \frac{b-a}{2}, b + \frac{b-a}{2} \right].$$

The proofs of the Hörmander-type conditions essentially follow the same ideas as in [**34**]. We only show here the corresponding to (2.1) because the one for (2.2) is similar and it does add nothing different. Let $a, b \in \mathbb{N}$ with $a < b$, the interval $I = [a, b] \cap \mathbb{N}$, and $f \in (\mathbb{B}_1)^{\mathbb{N}}_0$. Suppose $m, l \in I$ and $m < l$. The case $l < m$ is similar. Along the argument, we shall apply repeatedly the chain of inequalities, vid. [**13**, Eq. (18)],

$$\frac{|n - m|}{3} \leq |n - l| \leq 3|n - m|, \tag{2.3}$$

when $n \in \mathbb{N} \setminus 2I$.

We split the left hand side of (2.1) in three parts

$$\sum_{n \in \mathbb{N} \setminus 2I} \|\bar{K}(n, m) - \bar{K}(n, l)\|_{\mathcal{L}(\mathbb{B}_1, \mathbb{B}_2)} \|f(n)\|_{\mathbb{B}_1} = S_1(m, l) + S_2(m, l) + S_3(m, l), \tag{2.4}$$

where

$$S_1(m,l) := \sum_{\substack{n\in\mathbb{N}\setminus 2I\\ n\in W_m\cap W_l}} \|K(n,m)-K(n,l)\|_{\mathcal{L}(\mathbb{B}_1,\mathbb{B}_2)}\|f(n)\|_{\mathbb{B}_1},$$

$$S_2(m,l) := \sum_{\substack{n\in\mathbb{N}\setminus 2I\\ n\in W_m\setminus W_l}} \|K(n,m)\|_{\mathcal{L}(\mathbb{B}_1,\mathbb{B}_2)}\|f(n)\|_{\mathbb{B}_1}, \text{ and}$$

$$S_3(m,l) := \sum_{\substack{n\in\mathbb{N}\setminus 2I\\ n\in W_l\setminus W_m}} \|K(n,l)\|_{\mathcal{L}(\mathbb{B}_1,\mathbb{B}_2)}\|f(n)\|_{\mathbb{B}_1}.$$

We consider two cases. If $9m<4l$ we have that $W_m\cap W_l=\emptyset$ and then $S_1(m,l)=0$; in addition, by the size condition (a) of the kernel K and (2.3) we obtain that

$$S_2(m,l)+S_3(m,l)\lesssim \sum_{\substack{n\in W_m\\ n\in\mathbb{N}\setminus 2I}} \frac{\|f\|_{\mathbb{B}_1}}{|n-m|} + \sum_{\substack{n\in W_l\\ n\in\mathbb{N}\setminus 2I}} \frac{\|f\|_{\mathbb{B}_1}}{|n-m|}.$$

Note that if $n\in\mathbb{N}\setminus 2I$, then $|n-m|>(b-a)/2\geq (l-m)/2>5l/18$ and then,

$$S_2(m,l)+S_3(m,l)\lesssim \frac{1}{l}\sum_{\substack{n\in W_m\cup W_l\\ n\in\mathbb{N}\setminus 2I}} \|f(n)\|_{\mathbb{B}_1} \lesssim \frac{1}{l}\sum_{n\in J}\|f(n)\|_{\mathbb{B}_1},$$

where $J=[m_0,l_0^*]\cap\mathbb{N}$. Since $l_0^*\geq l_0^*-m_0>65l/54$,

$$S_2(m,l)+S_3(m,l)\lesssim \mathcal{M}(\|f\|_{\mathbb{B}_1})(m). \tag{2.5}$$

On the other hand, if $9m\geq 4l$, then $m\neq 0$ because $m<l$. We have that

$$W_m\setminus W_t=[m_0,l_0)\cap\mathbb{N},\qquad W_m\cap W_t=[l_0,m_0^*]\cap\mathbb{N},\qquad W_l\setminus W_m=(m_0^*,l_0^*]\cap\mathbb{N}.$$

Clearly, $l/3<l-n$ provided $n\in W_m\setminus W_l$ and $m/2<m-n$ provided $n\in W_l\setminus W_m$. So, by the size condition (a) and (2.3), we obtain

$$\begin{aligned} S_2(m,l)+S_3(m,l) &\lesssim \sum_{\substack{m_0\leq n\leq l_0\\ n\in\mathbb{N}\setminus 2I}} \frac{\|f(n)\|_{\mathbb{B}_1}}{|l-n|} + \sum_{\substack{m_0^*\leq n\leq l_0^*\\ n\in\mathbb{N}\setminus 2I}} \frac{\|f(n)\|_{\mathbb{B}_1}}{|m-n|} \\ &\lesssim \frac{1}{l}\sum_{\substack{1\leq n\leq l\\ n\in\mathbb{N}}}\|f(n)\|_{\mathbb{B}_1} + \frac{1}{m}\sum_{\substack{m\leq n\leq 4m\\ n\in\mathbb{N}}}\|f(n)\|_{\mathbb{B}_1} \lesssim \mathcal{M}(\|f\|_{\mathbb{B}_1})(m). \end{aligned} \tag{2.6}$$

In order to study S_1 we decompose the sum into two parts

$$S_1(m,l)=S_{1,1}(m,l)+S_{1,2}(m,l), \tag{2.7}$$

where

$$S_{1,1}(m,l) := \sum_{\substack{n\in\mathbb{N}\setminus 2I\\ l_0\leq n\leq m_0^*\\ |n-m|\leq 2|m-l|}} \|f(n)\|_{\mathbb{B}_1}\|K_{n,m}-K(n,l)\|_{\mathcal{L}(\mathbb{B}_1,\mathbb{B}_2)}$$

and

$$S_{1,2}(m,l) := \sum_{\substack{n\in\mathbb{N}\setminus 2I\\ l_0\le n\le m_0^*\\ |n-m|>2|m-l|}} \|f(n)\|_{\mathbb{B}_1}\|K_{n,m}-K(n,l)\|_{\mathcal{L}(\mathbb{B}_1,\mathbb{B}_2)}.$$

To estimate $S_{1,1}$ we use the size condition (a) and (2.3) to get

$$S_{1,1}(m,l) \lesssim \sum_{n\in\mathbb{N}\setminus 2I} \|f(n)\|_{\mathbb{B}_1}\frac{|m-l|}{|n-m|^2}$$

and the same estimate holds for $S_{1,2}$ using (b1). Therefore,

$$\begin{aligned} S_{1,1}(m,l)+S_{1,2}(m,l) &\lesssim \sum_{k=1}^{\infty}\sum_{n\in 2^{k+1}I\setminus 2^kI}\frac{|m-l|}{|n-m|^2}\|f(n)\|_{\mathbb{B}_1} \\ &\lesssim \sum_{k=1}^{\infty}\frac{\#(I)}{2^{2k}(\#(I))^2}\sum_{n\in 2^{k+1}I}\|f(n)\|_{\mathbb{B}_1}\lesssim \mathcal{M}(\|f\|_{\mathbb{B}_1})(m) \end{aligned} \tag{2.8}$$

The Hörmander type condition (2.1) follows from (2.4), (2.5), (2.6), (2.7), and (2.8).

Now it is clear that T_{loc} is bounded from $\ell^r_{\mathbb{B}_1}(\mathbb{N})$ into $\ell^r_{\mathbb{B}_1}(\mathbb{N})$ because so are T and T_{glob}. By [**21**, Thm. 1.1] and the Hörmander-type conditions (2.1) and (2.2), for $1<p<\infty$, the operator T_{loc} can be extended from $\ell^p_{\mathbb{B}_1}(\mathbb{N})\cap\ell^r_{\mathbb{B}_1}(\mathbb{N})$ to $\ell^p_{\mathbb{B}_1}(\mathbb{N})$ as a bounded operator from $\ell^p_{\mathbb{B}_1}(\mathbb{N})$ into $\ell^p_{\mathbb{B}_2}(\mathbb{N})$, and T_{loc} can be extended from $\ell^1_{\mathbb{B}_1}(\mathbb{N})\cap\ell^r_{\mathbb{B}_1}(\mathbb{N})$ to $\ell^1_{\mathbb{B}_1}(\mathbb{N})$ as a bounded operator from $\ell^1_{\mathbb{B}_1}(\mathbb{N})$ into $\ell^{1,\infty}_{\mathbb{B}_2}(\mathbb{N})$. Moreover, this properties also hold for T because T_{glob} also verifies them.

Finally, by adapting the arguments in Lemmas 5.15, 7.9, and 7.10, and Theorems 7.11 and 7.12 in [**18**] to vector-valued homogeneous settings, we conclude that T_{loc}, and therefore T, can be extended from $\ell^p_{\mathbb{B}_1}(\mathbb{N},w)\cap\ell^r_{\mathbb{B}_1}(\mathbb{N})$ to $\ell^p_{\mathbb{B}_1}(\mathbb{N},w)$ as bounded operators from $\ell^p_{\mathbb{B}_1}(\mathbb{N},w)$ into $\ell^p_{\mathbb{B}_2}(\mathbb{N},w)$, for $1<p<\infty$ and $w\in A_p(\mathbb{N})$, and from $\ell^1_{\mathbb{B}_1}(\mathbb{N},w)\cap\ell^r_{\mathbb{B}_1}(\mathbb{N})$ to $\ell^1_{\mathbb{B}_1}(\mathbb{N},w)$ as bounded operators from $\ell^1_{\mathbb{B}_1}(\mathbb{N},w)$ into $\ell^{1,\infty}_{\mathbb{B}_2}(\mathbb{N},w)$, for every $w\in A_1(\mathbb{N})$. □

Applications of this theory in the present dissertation will be focused in the special case $\mathbb{B}_1=\mathbb{B}_2=\mathbb{C}$ and $r=2$. In this situation, conditions (a) and (b) are given by

(a') the size condition

$$|K(n,m)|\lesssim\frac{1}{|n-m|};\ \text{and}$$

(b') the regularity properties

$$|K(n,m)-K(n,l)|\lesssim\frac{|m-l|}{|n-m|^2},\quad |n-m|>2|m-l|,\ n_0\le m,l\le n_0^*, \tag{b1'}$$

$$|K(n,m)-K(s,m)|\lesssim\frac{|n-s|}{|n-m|^2},\quad |n-m|>2|n-s|,\ n_0\le m,s\le n_0^*. \tag{b2'}$$

Also in our case we shall denote by c_{00} instead of $\mathbb{C}_0^{\mathbb{N}}$ the space of sequences f such that $f(n) = 0$, with $n > j$, for some $j \in \mathbb{N}$.

Futhermore, for the sake of clarity, we state here the appropriate version of Theorem 2.1 which we shall use along next chapters.

THEOREM 2.2. *Supppose that T is a linear and bounded operator from $\ell^2(\mathbb{N})$ into itself and such that there exists a semi-local standard kernel K such that, for every sequence $f \in c_{00}$,*

$$Tf(n) = \sum_{m=0}^{\infty} f(m)K(n,m),$$

for every $n \in \mathbb{N}$ such that $f(n) = 0$. Then

(i) *for every $1 < p < \infty$ and $w \in A_p(\mathbb{N})$ the operator T can be extended from $\ell^2(\mathbb{N}) \cap \ell^p(\mathbb{N}, w)$ to $\ell^p(\mathbb{N}, w)$ as a bounded operator from $\ell^p(\mathbb{N}, w)$ into itself;*

(ii) *for every $w \in A_1(\mathbb{N})$ the operator T can be extended from $\ell^2(\mathbb{N}) \cap \ell^1(\mathbb{N}, w)$ to $\ell^{1,\infty}(\mathbb{N}, w)$ as a bounded operator from $\ell^1(\mathbb{N}, w)$ into $\ell^{1,\infty}(\mathbb{N}, w)$.*

3. Local Calderón-Zygmund theory

A. Nowak and K. Stempak developed in [**34**] a local Calderón-Zygmund theory in the continuous setting. In this section, we present the discrete counterpart of their work.

We call local standard kernel a semi-local standard kernel $K(n,m)$ supported in the region $n_0 \le m \le n_0^*$. By a discrete local Calderón-Zygmund operator we mean a linear and bounded operator T from $\ell^r(\mathbb{N})$ into $\ell^r(\mathbb{N})$ for some $1 < r < \infty$, and such that there exists a local standard kernel so that, for every sequence $f \in c_{00}$,

$$Tf(n) = \sum_{\substack{m \in \mathbb{N} \\ n_0 \le m \le n_0^*}} f(m)K(n,m)$$

for every $n \in \mathbb{N}$ such that $f(n) = 0$.

The following result, which is the discrete counterpart of [**34**, Thm. 4.3.] could be proved in the same way given for the proof of Theorem 2.1.

THEOREM 2.3. *Assume that T is a local Calderón-Zygmund operator. Then,*

(i) *for every $1 < p < \infty$ and $w \in A_p(\mathbb{N})$, the operator T can be extended from $\ell^r(\mathbb{N}) \cap \ell^p(\mathbb{N}, w)$ to $\ell^p(\mathbb{N}, w)$ as a bounded operator form $\ell^p(\mathbb{N}, w)$ into $\ell^p(\mathbb{N}, w)$;*

(ii) *for every $w \in A_1(\mathbb{N})$, the operator T can be extended from $\ell^r(\mathbb{N}) \cap \ell^1(\mathbb{N}, w)$ to $\ell^1(\mathbb{N}, w)$ as a bounded operator form $\ell^1(\mathbb{N}, w)$ into $\ell^{1,\infty}(\mathbb{N}, w)$.*

Previous theorem can be strengthened by considering a more general class of weights that we introduce next. For any $k \in \mathbb{N}$, $k > 1$, consider the set

$$\mathcal{I}_k = \{[a,b] : a, b \in \mathbb{N},\ 0 \le a \le b \le k(a+1)\}.$$

For some fixed $k \in \mathbb{N}$, we say that any weight sequence w is a local $A_p(\mathbb{N})$ weight, for $1 < p < \infty$, if

$$[w]_{A^{\mathrm{loc}}_{p,k}(\mathbb{N})} := \sup_{I \in \mathcal{I}_k} \frac{1}{|I|} \left(\sum_{\ell \in I} w(\ell) \right)^{1/p} \left(\sum_{\ell \in I} w(\ell)^{-q/p} \right)^{1/q} < \infty, \tag{2.9}$$

and a local $A_1(\mathbb{N})$ weight if

$$[w]_{A^{\mathrm{loc}}_{1,k}(\mathbb{N})} := \sup_{I \in \mathcal{I}_k} \frac{1}{|I|} \left(\sum_{\ell \in I} w(\ell) \right) \max_{\ell \in I} w(\ell)^{-1} < \infty, \tag{2.10}$$

The value $[w]_{A^{\mathrm{loc}}_{p,k}(\mathbb{N})}$ is called the local constant of the weight w. We shall denote by $A^{\mathrm{loc}}_{p,k}(\mathbb{N})$ the class of all local $A_p(\mathbb{N})$ weights.

Following proposition shows that in fact the class $A^{\mathrm{loc}}_{p,k}(\mathbb{N})$ is independent of the choice of the value $k > 1$.

PROPOSITION 2.1. *Let $1 \leq p < \infty$. Then, $A^{loc}_{p,k}(\mathbb{N}) = A^{loc}_{p,2}(\mathbb{N})$ for any $k \in \mathbb{N}$ with $k > 1$.*

PROOF. Let us take $k_1, k_2 \in \mathbb{N}$ with $1 < k_1 < k_2$. On the one hand, the inclusion $A^{\mathrm{loc}}_{p,k_2}(\mathbb{N}) \subseteq A^{\mathrm{loc}}_{p,k_1}(\mathbb{N})$ is immediate. In order to show the converse inclusion, we use the same ideas given in [**2**, Lem. 1] and [**34**, Prop. 6.1]. In what follows, we shall use $\lfloor \cdot \rfloor$ to denote the usual floor function.

First, let us suppose the case $1 < p < \infty$. By the Hölder's inequality

$$\left\lfloor \sqrt{k_1}(a+1) \right\rfloor - a + 1 \leq \left(\sum_{\ell=a}^{\lfloor \sqrt{k_1}(a+1) \rfloor} w(\ell) \right)^{1/p} \left(\sum_{\ell=a}^{\lfloor \sqrt{k_1}(a+1) \rfloor} w(\ell)^{1-q} \right)^{1/q},$$

so

$$\left(\sum_{\ell=a}^{\lfloor \sqrt{k_1}(a+1) \rfloor} w(\ell)^{-1/(p-1)} \right)^{1-p} \leq \left(\left\lfloor \sqrt{k_1}(a+1) \right\rfloor - a + 1 \right)^{-p} \sum_{\ell=a}^{\lfloor \sqrt{k_1}(a+1) \rfloor} w(\ell). \tag{2.11}$$

Since the local Muckenhoupt-type condition (2.9) could be reformulated by

$$\left(\sum_{\ell=a}^{b} w(\ell) \right) \left(\sum_{\ell=a}^{b} w(\ell)^{-1/(p-1)} \right)^{p-1} \lesssim (b-a+1)^p, \quad a, b \in \mathbb{N}, \quad 0 \leq a \leq b \leq k_1(a+1), \tag{2.12}$$

we obtain

$$\sum_{\ell=a}^{k_1(a+1)} w(\ell) \lesssim (k_1(a+1) - a + 1)^p \left(\sum_{\ell=a}^{k_1(a+1)} w(\ell)^{-1/(p-1)} \right)^{1-p}.$$

Then, using inequality (2.11) and the previous one, we get

$$\begin{aligned}\sum_{\ell=\lfloor\sqrt{k_1}(a+1)\rfloor}^{k_1(a+1)} w(\ell) &\lesssim \sum_{\ell=a}^{k_1(a+1)} w(\ell) \lesssim (k_1(a+1)-a+1)\left(\sum_{\ell=a}^{k_1(a+1)} w(\ell)^{-1/(p-1)}\right)^{1-p}\\ &\lesssim (k_1(a+1)-a+1)^p\left(\sum_{\ell=a}^{\lfloor\sqrt{k_1}(a+1)\rfloor} w(\ell)^{-1/(p-1)}\right)^{1-p}\\ &\lesssim \left(\frac{k_1(a+1)-a+1}{\lfloor\sqrt{k_1}(a+1)\rfloor-a+1}\right)^p \sum_{\ell=a}^{\lfloor\sqrt{k_1}(a+1)\rfloor} w(\ell)\\ &\le C_{p,k_1}\sum_{\ell=a+1}^{\lfloor\sqrt{k_1}(a+1)\rfloor} w(\ell),\end{aligned}$$

where the constant in the last line depends on both p and k_1. Iterating this process

$$\sum_{\ell=a}^{b} w(\ell) \le C_{p,k_1,k_2}\sum_{\ell=a}^{\lfloor\sqrt{k_1}(a+1)\rfloor} w(\ell)$$

whenever $k_1(a+1) \le b \le k_2(a+1)$. One could prove a similar inequality for $\sum_{l=a}^{b} w(l)^{-1/(p-1)}$. Therefore, condition (2.12) holds with a new constant depending on k_1 and k_2 whenever $a, b \in \mathbb{N}$ and $0 \le a \le b \le k_2(a+1)$.

We do not give the details for $p = 1$ because the reasoning in that case is essentially the same. ☐

Since the classes $A^{\text{loc}}_{p,k}(\mathbb{N})$ are independent of k and all the constants of the weight $[w]_{A^{\text{loc}}_{p,k}(\mathbb{N})}$ are comparable to each other, by the local $A_p(\mathbb{N})$ condition we shall mean conditions (2.9) and (2.10) for $k = 2$, and for the local $A_p(\mathbb{N})$ constant of the weight the value $[w]_{A^{\text{loc}}_{p,2}(\mathbb{N})}$, which will be simply denote by $[w]_{A^{\text{loc}}_{p}(\mathbb{N})}$. Moreover, we shall denote by $A^{\text{loc}}_p(\mathbb{N})$ the class of local $A_p(\mathbb{N})$ weights.

THEOREM 2.4. *Assume that T is a local Calderón-Zygmund operator. Let w be a weight sequence such that $w \in A^{loc}_p(\mathbb{N})$. Then,*

(i) *if $1 < p < \infty$, the operator T can be extended from $\ell^r(\mathbb{N}) \cap \ell^p(\mathbb{N}, w)$ to $\ell^p(\mathbb{N}, w)$ as a bounded operator from $\ell^p(\mathbb{N}, w)$ into $\ell^p(\mathbb{N}, w)$; and*

(ii) *if $p = 1$, the operator T can be extended from $\ell^r(\mathbb{N}) \cap \ell^1(\mathbb{N}, w)$ to $\ell^1(\mathbb{N}, w)$ as a bounded operator from $\ell^1(\mathbb{N}, w)$ into $\ell^{1,\infty}(\mathbb{N}, w)$.*

PROOF. We shall use the discrete counterpart of a standard argument in the continuous case, given e.g. in [**2**, Section 5], together with Theorem 2.3.

Suppose that $1 < p < \infty$, $w \in A^p_{\text{loc}}(\mathbb{N})$, and $f \in \ell^r(\mathbb{N}) \cap \ell^p(\mathbb{N}, w)$. Let us consider the intervals $I_{-2} = \{0\}$ and $I_k = [2^k, 2^{k+3}) \cap \mathbb{N}$ in case that $k \ge -1$. For each $k \ge -1$, define the weight $\overline{w}_k$ on $\mathbb{Z}$ given by

$$\overline{w}_k(n) = \begin{cases} w(n), & \text{if } n \in I_k,\\ w(2^{k+1}-n), & \text{if } n \in [-3\times 2^{k+1}, 2^k),\end{cases}$$

and we extend it to $\mathbb{Z}$ so that it is $2|I_k|$-periodical. The restriction of the weight $\overline{w}_k$ to the set $\mathbb{N}$ will be denoted w_k. In the case $k = -2$, we simply take $w_{-2}(n) = w(0)$ for all $n \in \mathbb{N}$. By Proposition 2.1, $[w_k]_{A_p(\mathbb{N})} \leq 2[w]_{A^{\mathrm{loc}}_{p,8}(\mathbb{N})} \simeq [w]_{A^{\mathrm{loc}}_p(\mathbb{N})}$ holds, so $w_k \in A_p(\mathbb{N})$.

Let us define now the restricted sequence $f_k(n) = \chi_{I_k}(n)f(n)$ and the new intervals $J_k = [2^{k+1}, 2^{k+2}) \cap \mathbb{N}$ for $k \geq -1$, with $J_{-2} = \{0\}$. Then, $Tf(n) = Tf_k(n)$ for $n \in J_k$. This can be proved for $k \geq -1$ considering the decomposition $Tf(n) = T(f_k)(n) + T(\chi_{(I_k)^c}f)(n)$ and, since $n_0 \leq m \leq n_0^*$ implies $m \in I_k$, we have

$$T(\chi_{(I_k)^c}f)(n) = \sum_{\substack{m\in\mathbb{N}\\ n_0\leq m\leq n_0^*}} \chi_{(I_k)^c}(m)f(m)K(n,m) = 0.$$

The case $k = -2$ could be treated in a similar way. Then,

$$\sum_{n=0}^{\infty} |Tf(n)|^p w(n) = \sum_{k=-2}^{\infty} \sum_{n\in J_k} |Tf_k(n)|^p w_k(n)$$

and, by Theorem 2.3,

$$\|Tf\|_{\ell^p(\mathbb{N},w)} \lesssim [w]_{A^{\mathrm{loc}}_p(\mathbb{N})} \sum_{k=-2}^{\infty} \sum_{n=0}^{\infty} |f_k(n)|^p w_k(n) \lesssim 3[w]_{A^{\mathrm{loc}}_p(\mathbb{N})} \|f\|_{\ell^p(\mathbb{N},w)}.$$

In the weak case essentially the same argument works, so we do not provide the details. □

4. Weighted norm inequalities for some classical discrete operators

This section is devoted to give some boundedness results of the discrete Hardy operator and its adjoint, and the discrete Hilbert transform. All of them are well-known, so we shall not present their proofs.

First, recall that the discrete Hardy operator and its adjoint are defined by

$$Hf(n) = \frac{1}{n+1}\sum_{m=0}^{n} f(m) \qquad \text{and} \qquad H^*f(n) = \sum_{m=n}^{\infty} \frac{f(m)}{m+1}.$$

Given the parameters $\alpha, \beta > -1$ fixed and power weights $w_a(n) = (n+1)^a$, with $a \in \mathbb{R}$, for a weight sequence w we consider the following couple of conditions when $1 < p < \infty$ and $1/p + 1/q = 1$:

$$(2.13) \quad [w]^{\alpha}_{H_p} := \sup_{N\geq 0} \left(\sum_{n=N}^{\infty} w(n) w_{-(\alpha/2+1)p}(n)\right)^{1/p} \left(\sum_{n=0}^{N} w(n)^{-q/p} w_{\alpha q/2}(n)\right)^{1/q} < \infty,$$

$$(2.14) \quad [w]^{\beta}_{H^*_p} := \sup_{N\geq 0} \left(\sum_{n=N}^{\infty} w(n)^{-q/p} w_{-(\beta/2+1)q}(n)\right)^{1/q} \left(\sum_{n=0}^{N} w(n) w_{\beta p/2}(n)\right)^{1/p} < \infty.$$

The usual maximum interpretation could be considered in the case $p = 1$, but we shall skip it.

Although it is a bit misleading, the values $[w]^{\alpha}_{H_p}$ and $[w]^{\beta}_{H^*_p}$ are also called the constant of the weight w. The two previous conditions are adjoints in the sense that

$[w]^{\alpha}_{H_p} < \infty$ if and only if $[w^{-q/p}]^{\alpha}_{H^*_p} < \infty$. Moreover, note that for any non-negative value $\delta \geq 0$, the inequalities $[w]^{\alpha}_{H_p} \geq [w]^{\alpha+\delta}_{H_p}$ and $[w^{-q/p}]^{\alpha}_{H^*_p} \geq [w]^{\beta+\delta}_{H_{p^*}}$ hold. Finally, if a power weight is considered, it is easy to obtain the characterizations

$$[w_a]^{\alpha}_{H_p} < \infty \iff \frac{a+1}{p} < \frac{\alpha}{2} + 1 \quad \text{and} \quad [w_a]^{\beta}_{H^*_p} < \infty \iff -\frac{\beta}{2} < \frac{a+1}{p}. \tag{2.15}$$

It is well-known (c.f. [**35**] for instance) that condition (2.13) is necessary and sufficient for the weighted inequality

$$\|w_{-\alpha/2} H(w_{\alpha/2} f)\|_{\ell^p(\mathbb{N},w)} \lesssim \|f\|_{\ell^p(\mathbb{N},w)}, \tag{2.16}$$

whereas (2.14) is necessary and sufficient for

$$\|w_{\beta/2} H^*(w_{-\beta/2} f)\|_{\ell^p(\mathbb{N},w)} \lesssim \|f\|_{\ell^p(\mathbb{N},w)}. \tag{2.17}$$

Therefore, by using (2.15), we have the characterizations

$$\|w_{-\alpha/2} H(w_{\alpha/2} f)\|_{\ell^p(\mathbb{N},w_a)} \lesssim \|f\|_{\ell^p(\mathbb{N},w_a)} \iff \frac{a+1}{p} < \frac{\alpha}{2} + 1 \tag{2.18}$$

and

$$\|w_{-\alpha/2} H^*(w_{\alpha/2} f)\|_{\ell^p(\mathbb{N},w_a)} \lesssim \|f\|_{\ell^p(\mathbb{N},w_a)} \iff -\frac{\beta}{2} < \frac{a+1}{p}. \tag{2.19}$$

REMARK 1. In spite of the previous characterization related to the weights, we should note that since the estimation

$$Hf(n) \lesssim \mathcal{M}f(n), \qquad f(n) \geq 0,$$

holds, where $\mathcal{M}$ is the non-centered discrete Hardy-Littlewood maximal function (vid. p. 12), the bounds

$$\|Hf\|_{\ell^p(\mathbb{N},w)} \lesssim \|f\|_{\ell^p(\mathbb{N},w)} \quad \text{and} \quad \|H^* f\|_{\ell^p(\mathbb{N},w)} \lesssim \|f\|_{\ell^p(\mathbb{N},w)} \tag{2.20}$$

for $1 < p < \infty$, and is analogues for the case $p = 1$, hold for weights so that $w \in A_p(\mathbb{N})$. This fact has been already used in the proof of Theorem 2.1, and it will be used again later.

Other operator we shall use is the discrete Hilbert transform

$$\mathfrak{H}f(n) := \sum_{\substack{m=0 \\ m \neq n}}^{\infty} \frac{f(m)}{n-m}.$$

Its boundedness with weights was treated in [**25**, Thm. 10] by R. Hunt, B. Muckenhoupt, and R. Wheeden. There, it was proved that

$$\|\mathfrak{H}f\|_{\ell^p(\mathbb{N},w)} \lesssim \|f\|_{\ell^p(\mathbb{N},w)} \quad \text{if and only if} \quad w \in A_p(\mathbb{N}) \tag{2.21}$$

for $1 < p < \infty$, and

$$\|\mathfrak{H}f\|_{\ell^{1,\infty}(\mathbb{N},w)} \lesssim \|f\|_{\ell^1(\mathbb{N},w)} \quad \text{if and only if} \quad w \in A_1(\mathbb{N}). \tag{2.22}$$

CHAPTER 3

The convergence of discrete Fourier-Jacobi series

1. Introduction and main results

Given a function $F \in L^2((-1,1), d\mu_{\alpha,\beta})$, its Fourier-Jacobi coefficients related to Jacobi polynomials are defined by

$$a_n^{(\alpha,\beta)}(F) = \int_{-1}^{1} F(x) p_n^{(\alpha,\beta)}(x)\, d\mu_{\alpha,\beta}(x).$$

The application

$$\begin{array}{ccc} L^2((-1,1), d\mu_{\alpha,\beta}) & \longrightarrow & \ell^2(\mathbb{N}) \\ F & \longmapsto & \{a_n^{(\alpha,\beta)}(F)\}_{n\geq 0} \end{array}$$

is an isometry and the Parseval's type identity

$$\|F\|_{L^2((-1,1),d\mu_{\alpha,\beta})} = \|a_n^{(\alpha,\beta)}(F)\|_{\ell^2(\mathbb{N})}$$

holds. On its behalf, given an appropriate sequence $\{f(n)\}_{n\geq 0}$, its discrete Fourier-Jacobi transform $\mathcal{F}_{\alpha,\beta}$ is defined by

$$\mathcal{F}_{\alpha,\beta} f(x) = \sum_{m=0}^{\infty} f(m) p_m^{(\alpha,\beta)}(x).$$

For functions $F \in L^p((-1,1), d\mu_{\alpha,\beta})$, we define the n-th partial sum operator by

$$\mathfrak{S}_n^{(\alpha,\beta)} F(x) = \mathcal{F}_{\alpha,\beta}\left(\chi_{[0,n]} a_{(\cdot)}^{(\alpha,\beta)}(F)\right)(x).$$

As it has been already mentioned in the introduction, the problem of the mean convergence of the partial sum operator is equivalent to its uniform boundedness in the p-norm. The study of the mean convergence of the Fouier-Jacobi expansions in a continuous setting was initiated by H. Pollard in the paper [**40**].

Theorem 3.1 (Pollard, 1949). *Let $\alpha, \beta > -1/2$, $1 < p < \infty$, and*

$$\|\mathfrak{S}_n^{(\alpha,\beta)} F\|_{L^p((-1,1),d\mu_{\alpha,\beta})} \lesssim \|F\|_{L^p((-1,1),d\mu_{\alpha,\beta})}$$

if and only if

$$4 \max\left\{\frac{\alpha+1}{2\alpha+3}, \frac{\beta+1}{2\beta+3}\right\} < p < 4 \min\left\{\frac{\alpha+1}{2\alpha+1}, \frac{\beta+1}{2\beta+1}\right\}.$$

In the present chapter we are mainly interested in, given an appropriate sequence $f := \{f(n)\}_{n\geq 0}$, recover it by means of the multiplier of an interval for $\mathcal{F}_{\alpha,\beta}$. The multiplier of an interval $[a,b]\subset(-1,1)$, denoted by $S_{[a,b]}^{(\alpha,\beta)}$ and simply by $S_r^{(\alpha,\beta)}$ when $[a,b]=[-r,r]$ for $0<r<1$, is defined by the relation

$$S_{[a,b]}^{(\alpha,\beta)}f(n)=a_n^{(\alpha,\beta)}(\chi_{[a,b]}\mathcal{F}_{\alpha,\beta}f).$$

In a more concrete way, we want to obtain necessary and sufficient conditions under the limit

$$\lim_{r\to1^-}\|S_r^{(\alpha,\beta)}f-f\|_{\ell^p(\mathbb{N})}=0$$

holds. In order to do this task, we shall give a complete characterization of the uniform boundedness of the multiplier $S_{[a,b]}^{(\alpha,\beta)}$ on the spaces $\ell^p(\mathbb{N})$. Here, the term uniform must be understood respect to the interval $[a,b]$. That characterization will be a consequence of the following general theorem in which discrete weights are considered.

Theorem 3.2. *Let $\alpha,\beta\geq-1/2$, $[a,b]\subset(-1,1)$, $1\leq p<\infty$, and $w\in A_p(\mathbb{N})$. Then,*

$$\|S_{[a,b]}^{(\alpha,\beta)}f\|_{\ell^p(\mathbb{N},w)}\lesssim\|f\|_{\ell^p(\mathbb{N},w)}$$

for $1<p<\infty$, and

$$\|S_{[a,b]}^{(\alpha,\beta)}f\|_{\ell^{1,\infty}(\mathbb{N},w)}\lesssim\|f\|_{\ell^1(\mathbb{N},w)}$$

for $p=1$.

As a consequence of the previous theorem, it is possible to characterize the uniform boundedness of the mutiplier of the interval.

Theorem 3.3. *Let $\alpha,\beta\geq-1/2$, $[a,b]\subset(-1,1)$, and $1\leq p<\infty$. Then*

$$\|S_{[a,b]}^{(\alpha,\beta)}f\|_{\ell^p(\mathbb{N})}\lesssim\|f\|_{\ell^p(\mathbb{N})}$$

if and only if $1<p<\infty$.

Finally, to end this section, we characterize the norm convergence of the multiplier.

Theorem 3.4. *Let $\alpha,\beta\geq-1/2$ and $1\leq p<\infty$. Then*

$$\lim_{r\to1^-}\|S_r^{(\alpha,\beta)}f-f\|_{\ell^p(\mathbb{N})}=0$$

if and only if $1<p<\infty$.

Since the measure in this setting is the counting one, it is clear that from the previous characterization the pointwise convergence

$$\lim_{r\to1^-}S_r^{(\alpha,\beta)}f(n)=f(n),\qquad n\in\mathbb{N},$$

follows for $f\in\ell^p(\mathbb{N})$.

The proof of Theorem 3.2 will be showed in next section, whereas the corresponding for both characterizations in Theorem 3.3 and Theorem 3.4 will be given in last section.

2. Proof of the main theorem

To prove Theorem 3.2 we shall analyse the related operator $\mathcal{S}_b^{(\alpha,\beta)} := S_{(-1,b]}^{(\alpha,\beta)}$, restricted to sequences $f \in \ell^2(\mathbb{N}) \cap \ell^p(\mathbb{N}, w)$. Thus, the result follows by the identity $\chi_{[a,b]}(x) = \chi_{(-1,b]}(x) - \chi_{(-1,a)}(x)$ and a standard density argument.

The operator $\mathcal{S}_b^{(\alpha,\beta)}$ can by rewritten by the series

$$\mathcal{S}_b^{(\alpha,\beta)} f(n) = \sum_{m=0}^{\infty} f(m) K_b^{(\alpha,\beta)}(m,n),$$

where

$$K_b^{(\alpha,\beta)}(m,n) = \int_{-1}^{b} p_m^{(\alpha,\beta)}(x) p_n^{(\alpha,\beta)}(x)\, d\mu_{\alpha,\beta}(x)$$

is the kernel of the operator $\mathcal{S}_b^{(\alpha,\beta)}$. It is possible to obtain an explicit expression for the kernel, which is contained in the following lemma. Recall that for Jacobi polynomials we have

$$\sigma(x) \equiv \sigma_{\alpha,\beta}(x) = 1 - x^2, \qquad \rho(x) \equiv \rho_{\alpha,\beta}(x) = (1-x)^{\alpha}(1+x)^{\beta},$$

and $\lambda_n^{(\alpha,\beta)} = n(n+\alpha+\beta+1)$.

Lemma 3.1. *Let $\alpha, \beta > -1$ and $-1 < b < 1$. Then,*

$$K_b^{(\alpha,\beta)}(n,m) = \frac{A_{\alpha,\beta}(b)}{\lambda_n^{(\alpha,\beta)} - \lambda_m^{(\alpha,\beta)}} W(p_n^{(\alpha,\beta)}, p_m^{(\alpha,\beta)})(b), \qquad n \neq m,$$

where $A_{\alpha,\beta}(x) := \sigma_{\alpha,\beta}(x)\rho_{\alpha,\beta}(x)$.

Proof. It is almost immediate having in mind the identity (1.4) and integrating by parts twice. In fact,

$$\begin{aligned}
\lambda_n^{(\alpha,\beta)} K_b^{(\alpha,\beta)}(n,m) &= \int_{-1}^{b} p_m^{(\alpha,\beta)}(x) \mathfrak{D}^{\alpha,\beta} p_n^{(\alpha,\beta)}(x)\, d\mu_{\alpha,\beta}(x) \\
&= \Big(A_{\alpha,\beta}(x) W(p_n^{(\alpha,\beta)}, p_m^{(\alpha,\beta)})(x)\Big)\Big|_{x=-1}^{x=b} \\
&\qquad\qquad + \int_{-1}^{b} \mathfrak{D}^{\alpha,\beta} p_m^{(\alpha,\beta)}(x) p_n^{(\alpha,\beta)}(x)\, d\mu_{\alpha,\beta}(x) \\
&= \Big(A_{\alpha,\beta}(x) W(p_n^{(\alpha,\beta)}, p_m^{(\alpha,\beta)})(x)\Big)\Big|_{x=-1}^{x=b} + \lambda_m^{(\alpha,\beta)} K_b^{(\alpha,\beta)}(n,m),
\end{aligned}$$

which leads us to the statement of the lemma. □

The explicit expression of the kernel allows us to write $\mathcal{S}_r^{(\alpha,\beta)}$ in terms of the discrete Hilbert transform $\mathfrak{H}$ (vid. p. 19) and the operator

$$\mathcal{Q}_a f(n) = \sum_{\substack{m=0 \\ m \neq n}}^{\infty} \frac{f(m)}{n+m+a},$$

where a is a non-negative constant. In the definition of $\mathcal{Q}_a$ we have required $m \neq n$ because it is more convenient for our purposes, but that value could be included without problem. Then, we shall need weighted norm inequalities for both operators $\mathfrak{H}$ and $\mathcal{Q}_a$.

The characterization of weighted norm boundedness of the discrete Hilbert transform has been shown in Chap. 2, Sec. 4, specifically in Eq. (2.21) and (2.22). Regarding the operator $\mathcal{Q}_a$, it can be bounded by the discrete Hardy operator H and its adjoint H^*. Indeed,

$$|\mathcal{Q}_a f(n)| \lesssim H(|f|)(n) + H^*(|f|)(n).$$

We have already shown that both H and H^* fulfil weighted norm inequalities (vid. Eq. (2.20) in Remark 1, p. 19), so

$$\|\mathcal{Q}_a f\|_{\ell^p(\mathbb{N},w)} \lesssim \|f\|_{\ell^p(\mathbb{N},w)} \tag{3.1}$$

for $1 < p < \infty$ and $w \in A_p(\mathbb{N})$. The weak-type

$$\|\mathcal{Q}_a f\|_{\ell^{1,\infty}(\mathbb{N},w)} \lesssim \|f\|_{\ell^1(\mathbb{N},w)} \tag{3.2}$$

also holds when $w \in A_1(\mathbb{N})$.

Proof of Theorem 3.2. As it has been mentioned, it is enough to give the proof for the operator $\mathcal{S}_b^{(\alpha,\beta)}$. First, let us fix the notations

$$r_b(n) := (1-b)^{\alpha/2+1/4}(1+b)^{\beta/2+1/4} p_n^{(\alpha,\beta)}(b)$$

and

$$R_b(n) := \frac{(1-b)^{\alpha/2+3/4}(1+b)^{\beta/2+3/4}}{2n+\alpha+\beta+1}(p_n^{(\alpha,\beta)})'(b).$$

Note that previous quantities fulfil the bounds

$$|r_b(n)| \lesssim 1 \qquad \text{and} \qquad |R_b(n)| \lesssim 1, \tag{3.3}$$

which could be easily proved by the expression for the derivative of a Jacobi polynomial (1.7) and the uniform estimate (1.17).

By Lemma 3.1, it is clear that

$$\begin{aligned}\mathcal{S}_b^{(\alpha,\beta)} f(n) = A_{\alpha,\beta}(b)\Bigg(p_n^{(\alpha,\beta)}(b) \sum_{\substack{m=0\\ m\neq n}}^{\infty} \frac{f(m)(p_m^{(\alpha,\beta)})'(b)}{\lambda_n^{(\alpha,\beta)} - \lambda_m^{(\alpha,\beta)}} \\ - (p_n^{(\alpha,\beta)})'(b) \sum_{\substack{m=0\\ m\neq n}}^{\infty} \frac{f(m) p_m^{(\alpha,\beta)}(b)}{\lambda_n^{(\alpha,\beta)} - \lambda_m^{(\alpha,\beta)}}\Bigg) + f(n) K_b^{(\alpha,\beta)}(n,n).\end{aligned}$$

The elementary identities

$$\frac{1}{\lambda_n^{(\alpha,\beta)} - \lambda_m^{(\alpha,\beta)}} = \frac{1}{2m+\alpha+\beta+1}\left(\frac{1}{n-m} - \frac{1}{m+n+\alpha+\beta+1}\right) \tag{3.4}$$

$$= \frac{1}{2n+\alpha+\beta+1}\left(\frac{1}{n-m} + \frac{1}{m+n+\alpha+\beta+1}\right), \tag{3.5}$$

together with an appropriate factorization of the function $A_{\alpha,\beta}(x)$, give us the relations

$$A_{\alpha,\beta}(b)p_n^{(\alpha,\beta)}(b)\sum_{\substack{m=0\\m\neq n}}^{\infty}\frac{f(m)(p_m^{(\alpha,\beta)})'(b)}{\lambda_n^{(\alpha,\beta)}-\lambda_m^{(\alpha,\beta)}}=r_b(n)\left(\mathcal{H}\left(R_bf\right)(n)-\mathcal{Q}_{\alpha+\beta+1}(R_bf)(n)\right),$$

and

$$A_{\alpha,\beta}(b)(p_n^{(\alpha,\beta)})'(b)\sum_{\substack{m=0\\m\neq n}}^{\infty}\frac{f(m)p_m^{(\alpha,\beta)}(b)}{\lambda_n^{(\alpha,\beta)}-\lambda_m^{(\alpha,\beta)}}=R_b(n)\left(\mathcal{H}\left(r_bf\right)(n)-\mathcal{Q}_{\alpha+\beta+1}(r_bf)(n)\right).$$

Using them, we get the fundamental decomposition

$$\begin{aligned}\mathcal{S}_b^{(\alpha,\beta)}f(n)=r_b(n)\mathcal{H}(R_bf)(n)-R_b(n)\mathcal{H}(r_bf)(n)-r_b(n)\mathcal{Q}_{\alpha+\beta+1}(R_bf)(n)\\-R_b(n)\mathcal{Q}_{\alpha+\beta+1}(r_bf)(n)+f(n)K_b^{(\alpha,\beta)}(n,n).\end{aligned}$$

So, taking into account the bounds (3.3) and the trivial estimate $K_b(n,n)\le 1$, we obtain

$$\begin{aligned}\|\mathcal{S}_b^{(\alpha,\beta)}f\|_{\ell^p(\mathbb{N},w)}\lesssim\|\mathcal{H}(R_bf)\|_{\ell^p(\mathbb{N},w)}+\|\mathcal{H}(r_bf)\|_{\ell^p(\mathbb{N},w)}+\|\mathcal{Q}_{\alpha+\beta+1}(R_bf)\|_{\ell^p(\mathbb{N},w)}\\+\|\mathcal{Q}_{\alpha+\beta+1}(r_bf)\|_{\ell^p(\mathbb{N},w)}+\|f\|_{\ell^p(\mathbb{N},w)}.\end{aligned}$$

From here, the first part of the theorem follows by the weighted norm inequality (3.1) and the one for the discrete Hilbert transform (2.21). The reasoning is the same to prove the weak type but using the corresponding inequalities (3.2) and (2.22). □

3. Proofs of the characterizations

In order to prove both Theorem 3.3 and Theorem 3.4 we need to study first some estimates and the belonging of the sequence $\{K_b^{(\alpha,\beta)}(m,n)\}_{n\ge0}$ of kernels to the space $\ell^p(\mathbb{N})$.

LEMMA 3.2. *Let $\alpha,\beta\ge-1/2$ and $m\in\mathbb{N}$. Then,*

$$|K_b^{(\alpha,\beta)}(m,n)|\lesssim\frac{1}{|n-m|},\qquad n\neq m,\tag{3.6}$$

and $K_b^{(\alpha,\beta)}(m,\cdot)\in\ell^p(\mathbb{N})$ for $1<p<\infty$. Moreover,

$$\sum_{n=m+1}^{2m}\left|\int_0^{1-1/m^2}p_m^{(\alpha,\beta)}(x)p_n^{(\alpha,\beta)}(x)\,d\mu_{\alpha,\beta}(x)\right|\simeq\log m.\tag{3.7}$$

PROOF. By Lemma 3.1 and the identities (3.4) and (3.5), we get the relation

$$K_b^{(\alpha,\beta)}(m,n)=\frac{r_b(n)R_b(m)+r_b(m)R_b(n)}{n-m}-\frac{r_b(n)R_b(m)-r_b(m)R_b(n)}{m+n+\alpha+\beta+1}.$$

Using the estimates (3.3) for r_b and R_b, the bound for the kernel $K_b^{(\alpha,\beta)}(m,n)$ is easily obtained. Furthermore, for $1 < p < \infty$, we have

$$\sum_{n=0}^{\infty} |K_b^{(\alpha,\beta)}(m,n)|^p \lesssim \sum_{\substack{n=0\\ n\neq m}}^{\infty} \frac{1}{|n-m|^p} + |K_b^{(\alpha,\beta)}(m,m)| \lesssim 1,$$

so $K_b^{(\alpha,\beta)}(m,\cdot) \in \ell^p(\mathbb{N})$ really.

The proof of the second part of the lemma is based on the reasoning given in [**6**], whose details appear in [**9**]. For this reason, we shall skip some points that could be found there. First, let us denote

$$I(m,n) := \int_0^{1-1/m^2} p_m^{(\alpha,\beta)}(x) p_n^{(\alpha,\beta)}(x)\, d\mu_{\alpha,\beta}(x),$$

which can be rewritten, applying the change of variable $x = \cos\theta$ and denoting $\varphi_m := \arccos(1-1/m^2)$, in terms of the Jacobi functions by

$$I(m,n) = \int_{\varphi_m}^{\pi/2} \mathcal{P}_m^{(\alpha,\beta)}(\theta)\mathcal{P}_n^{(\alpha,\beta)}(\theta)\, d\theta.$$

Note that, since $m < n \le 2m$, the equivalences $\varphi_m \simeq 1/m \simeq 1/n$ hold.

To deal with $I(m,n)$ we use the following asymptotic expression for Jacobi functions (in what follows, C will be denote a positive constant independent of significative quantities and possibly different in each line)

$$\mathcal{P}_n^{(\alpha,\beta)}(\theta) = C\cos(N\theta - \phi_\alpha) + C\frac{\sin(N\theta-\phi_\alpha)}{N\theta} + O\left(N^{-1}\right) + O\left((N\theta)^{-2}\right),$$

where $\phi_\alpha := (2\alpha+1)\pi/4$, $N := n + (\alpha+\beta+1)/2$, and $\delta/n < \theta \le \pi/2$ for $\delta > 0$. Thus

$$I(m,n) = I_1(m,n) + I_2(m,n) + I_3(m,n) + O(M^{-1}),$$

where $M := m + (\alpha+\beta+1)/2$ and

$$I_1(m,n) := C\int_{1/m}^{\pi/2} \cos(N\theta-\phi_\alpha)\cos(M\theta-\phi_\alpha)\, d\theta,$$
$$I_2(m,n) := \frac{C}{N}\int_{1/m}^{\pi/2} \sin(N\theta-\phi_\alpha)\cos(M\theta-\phi_\alpha)\frac{d\theta}{\theta},$$
$$I_3(m,n) := \frac{C}{M}\int_{1/m}^{\pi/2} \cos(N\theta-\phi_\alpha)\sin(M\theta-\phi_\alpha)\frac{d\theta}{\theta}.$$

First integral is

$$I_1(m,n) = \frac{C}{n-m} + O(M^{-1}),$$

while second and third are

$$I_2(m,n) = \frac{C}{N}\log\left(\frac{N}{n-m}\right) + O(M^{-1}),$$

and

$$I_3(m,n) = \frac{C}{M}\log\left(\frac{M}{n-m}\right) + O(M^{-1})$$

respectively. Therefore,

$$I(m,n) \simeq \frac{1}{n-m} + \frac{1}{N}\log\left(\frac{N}{n-m}\right) + \frac{1}{M}\log\left(\frac{M}{n-m}\right) + O(M^{-1}).$$

From this, the equivalence (3.7) follows because

$$\sum_{n=m+1}^{2m} \frac{1}{n-m} \simeq \log m$$

and

$$\sum_{n=m+1}^{2m} \left(\frac{1}{N}\log\left(\frac{N}{n-m}\right) + \frac{1}{M}\log\left(\frac{M}{n-m}\right)\right) \lesssim 1. \qquad \square$$

Proof of Theorem 3.3. Let us suppose first that $1 < p < \infty$. Due to the fact that $w(n) = 1$ is a weight in the $A_p(\mathbb{N})$ class, we get the uniform boundedness directly by Theorem 3.2.

For the converse let us prove that exists a sequence $f \in \ell^1(\mathbb{N})$ such that the norm inequality

$$\|S_{[a,b]}^{(\alpha,\beta)} f\|_{\ell^1(\mathbb{N})} \lesssim \|f\|_{\ell^1(\mathbb{N})}$$

does not hold for some interval $[a,b]$. To this end, let us take $m \geq 1$, the interval $[0, 1-1/m^2]$, and consider the sequence $f_m(n) = \delta_{n,m}$. Note that in this situation the identity $\mathcal{F}_{\alpha,\beta} f_m(x) = p_m^{(\alpha,\beta)}(x)$ holds, so

$$\|S_{[0,1-1/m^2]}^{(\alpha,\beta)} f_m\|_{\ell^1(\mathbb{N})} = \sum_{n=0}^{\infty} \left| \int_0^{1-1/m^2} p_m^{(\alpha,\beta)}(x) p_n^{(\alpha,\beta)}(x)\, d\mu_{\alpha,\beta}(x) \right|.$$

Since $\|f_m\|_{\ell^1(\mathbb{N})} = 1$, if the uniform boundedness for $p = 1$ was true, it would imply

$$\sum_{n=m+1}^{2m} \left| \int_0^{1-1/m^2} p_m^{(\alpha,\beta)}(x) p_n^{(\alpha,\beta)}(x)\, d\mu_{\alpha,\beta}(x) \right| \lesssim 1.$$

However, this inequality is not possible because the left hand side is greater than $\log m$ by the equivalence (3.7). $\square$

Remark 2. It is worth pointing out that the same reasoning works for the operator $S_r^{(\alpha,\beta)}$. By means of the identity (1.8) and proceeding as in the proof of Lemma 3.2, it is possible to show

$$\int_{-1+1/m^2}^{1-1/m^2} p_m^{(\alpha,\beta)}(x) p_n^{(\alpha,\beta)}(x)\, d\mu_{\alpha,\beta}(x) \simeq \frac{1}{n-m} + \frac{1}{N}\log\left(\frac{N}{n-m}\right) + \frac{1}{M}\log\left(\frac{M}{n-m}\right) + O(M^{-1}),$$

which implies that $S_r^{(\alpha,\beta)}$ are not bounded from $\ell^1(\mathbb{N})$ into itself either.

To end this chapter, let us prove Theorem 3.4. The strategy we shall follow is to prove it for sequences in the space c_{00} first, and then approximate any sequence $f \in \ell^p(\mathbb{N})$ by means of them.

LEMMA 3.3. *Let* $\alpha, \beta \geq -1/2$, $1 < p < \infty$, *and* $f \in c_{00}$. *Then*

$$\lim_{r \to 1^-} \|S_r^{(\alpha,\beta)} f - f\|_{\ell^p(\mathbb{N})} = 0.$$

PROOF. First, note that each sequence $f \in c_{00}$ could be expressed by a linear combination of Dirac deltas $f_m(n) = \delta_{n,m}$, so it is enough to prove the statement of the lemma for the sequence $f_m(n)$.

Due to the orthonormality of the normalized Jacobi polynomials, the integral representation

$$f_m(n) = \int_{-1}^{1} p_m^{(\alpha,\beta)}(x) p_n^{(\alpha,\beta)}(x)\, d\mu_{\alpha,\beta}(x)$$

is trivial. By means of it, we have

$$\begin{aligned} S_r^{(\alpha,\beta)} f_m(n) - f_m(n) &= \int_{-1}^{1} \Big(\chi_{[-r,r]}(x) - 1\Big)\, p_n^{(\alpha,\beta)}(x) p_m^{(\alpha,\beta)}(x)\, d\mu_{\alpha,\beta}(x) \\ &= -\int_{-1}^{-r} p_n^{(\alpha,\beta)}(x) p_m^{(\alpha,\beta)}(x)\, d\mu_{\alpha,\beta}(x) \\ &\qquad - \int_{r}^{1} p_n^{(\alpha,\beta)}(x) p_m^{(\alpha,\beta)}(x)\, d\mu_{\alpha,\beta}(x), \end{aligned} \tag{3.8}$$

and using the identity

$$\int_{r}^{1} p_n^{(\alpha,\beta)}(x) p_m^{(\alpha,\beta)}(x)\, d\mu_{\alpha,\beta}(x) = -\int_{-1}^{r} p_n^{(\alpha,\beta)}(x) p_m^{(\alpha,\beta)}(x)\, d\mu_{\alpha,\beta}(x), \qquad n \neq m,$$

we deduce that

$$S_r^{(\alpha,\beta)} f_m(n) - f_m(n) = K_r^{(\alpha,\beta)}(m,n) - K_{-r}^{(\alpha,\beta)}(m,n), \qquad n \neq m.$$

Regarding the case $n = m$, applying the estimate (1.17) in the expression (3.8) we obtain that the bound

$$|S_r^{(\alpha,\beta)} f_m(m) - f_m(m)| \lesssim (1-r)^{1/2}$$

is attained. Then,

$$\lim_{r \to 1^-} \|S_r^{(\alpha,\beta)} f_m - f_m\|_{\ell^p(\mathbb{N})}^p = \lim_{r \to 1^-} \sum_{\substack{n=0 \\ n \neq m}}^{\infty} |K_r^{(\alpha,\beta)}(m,n) - K_{-r}^{(\alpha,\beta)}(m,n)|^p. \tag{3.9}$$

From (3.6) we have the estimate $|K_r^{(\alpha,\beta)}(m,n) - K_{-r}^{(\alpha,\beta)}(m,n)| \lesssim |n-m|^{-1}$ when $n \neq m$. Then applying the dominated convergence theorem [**42**, Thm. 1.34] the result follows (note that $|n-m|^{-1}$ for $n \neq m$ is p-summable for $1 < p < \infty$) from (3.9) because $\lim_{r \to 1^-}(K_{-r}(m,n) - K_r(m,n)) = 0$. □

PROOF OF THEOREM 3.4. Let us take a sequence $f \in \ell^p(\mathbb{N})$ and $1 < p < \infty$. Given $\varepsilon > 0$, exists $m = m(\varepsilon) \in \mathbb{N}$ such that $\|f - g_m\|_{\ell^p(\mathbb{N})} < \varepsilon$, with $g_m(n) = \chi_{[0,m] \cap \mathbb{N}}(n) f(n) \in c_{00}$. Therefore, by Theorem 3.3 and Lemma 3.3, we get

$$\begin{aligned} \|S_r^{(\alpha,\beta)} f - f\|_{\ell^p(\mathbb{N})} &\leq \|S_r^{(\alpha,\beta)}(f - g_m)\|_{\ell^p(\mathbb{N})} + \|S_r^{(\alpha,\beta)} g_m - g_m\|_{\ell^p(\mathbb{N})} + \|g_m - f\|_{\ell^p(\mathbb{N})} \\ &\lesssim \|f - g_m\|_{\ell^p(\mathbb{N})} + \varepsilon \lesssim \varepsilon \end{aligned}$$

for r close to 1.

To finish, let us prove by means of an argument to absurdity that the convergence does not hold for the case $p = 1$. Let us suppose that it holds, then there exits a constant $C(f)$, which actually depends on the sequence f, so that $\|S_r^{(\alpha,\beta)} f\|_{\ell^p(\mathbb{N})} \leq C(f)$. The Banach-Steinhaus theorem [**42**, Thm. 5.8] implies the uniform boundedness of the multiplier $S_r^{(\alpha,\beta)}$ in $\ell^1(\mathbb{N})$, but we have already proof that it is impossible (vid. Remark 2, p. 27). □

CHAPTER 4

A weighted transplantation theorem for Jacobi coefficients

1. Introduction and main result

The family $\{\mathcal{P}_n^{(\alpha,\beta)}\}_{n\geq 0}$, where $\mathcal{P}_n^{(\alpha,\beta)}$ is the Jacobi function associated with the Jacobi polynomial of degree n and order (α,β), $\alpha,\beta>-1$, is a complete orthonormal system in the space $L^2(0,\pi)$. Given a function $F\in L^2(0,\pi)$, its Fourier-Jacobi coefficients are defined by

$$b_n^{(\alpha,\beta)}(F)=\int_0^\pi F(\theta)\mathcal{P}_n^{(\alpha,\beta)}(\theta)\,d\theta.$$

It turns out that under that assumption $b_n^{(\alpha,\beta)}(F)$ is a sequence in $\ell^2(\mathbb{N})$ and it is possible to recover the original function F by means of the discrete Fourier-Jacobi transform, which is given by

$$\mathcal{G}_{\alpha,\beta}f(\theta)=\sum_{n=0}^{\infty}f(n)\mathcal{P}_n^{(\alpha,\beta)}(\theta),\qquad f\in\ell^2(\mathbb{N}).$$

Then, taking $F\in L^2(0,\pi)$, we define the transplantation operator for Jacobi expansions by

$$\mathfrak{T}_{\alpha,\beta}^{\gamma,\delta}F(\theta)=\mathcal{G}_{\gamma,\delta}\left(b_{(\cdot)}^{(\alpha,\beta)}(F)\right)(\theta),$$

where $\alpha,\beta,\gamma,\delta>-1$. Naturally, it becomes the identity operator in the case $(\alpha,\beta)=(\gamma,\delta)$.

First transplantation result considering Jacobi expansions in terms of the functions $\mathcal{P}_n^{(\alpha,\beta)}(\theta)$ is due to R. Askey in [**7**]. He considered functions in the space $L^p_{\sigma,\tau}(0,\pi)$, the set of measurable functions such that the norm

$$\|F\|_{L^p_{\sigma,\tau}(0,\pi)}:=\int_0^\pi |F(\theta)|^p(\sin\theta/2)^\sigma(\cos\theta/2)^\tau\,d\theta$$

is finite. Moreover, to state his result he defined the series

$$\mathfrak{G}_rF(\theta):=\mathcal{G}_{\gamma,\delta}\left(r^{(\cdot)}b_{(\cdot)}^{(\alpha,\beta)}(F)\right)(\theta),\qquad 0<r<1.$$

Theorem 4.1 (Askey, 1969). *Let $1<p<\infty$, $\alpha,\beta,\gamma,\delta\geq -1/2$, $-1<\sigma<p-1$, and $-1<\tau<p-1$. Let $F\in L^p_{\sigma,\tau}(0,\pi)$. Then*

$$\|\mathfrak{G}_rF\|_{L^p_{\sigma,\tau}(0,\pi)}\lesssim\|F\|_{L^p_{\sigma,\tau}(0,\pi)}.$$

Also there exists $G(\theta)$ so that $\mathfrak{G}_rF(\theta)\to G(\theta)$ a.e.,

$$\|\mathfrak{G}_rF-G\|_{L^p_{\sigma,\tau}(0,\pi)}\to 0,\qquad \text{and}\qquad \|G\|_{L^p_{\sigma,\tau}(0,\pi)}\lesssim\|F\|_{L^p_{\sigma,\tau}(0,\pi)}.$$

It is worth to mention here the paper [**17**] by Ó. Ciaurri, A. Nowak, and K. Stempak, extending a previous analysis by B. Muckenhoupt in [**31**], where that transplantation problem was revisited and a variant of a Calderón-Zygmund theory was used.

From the discrete point of view, the application $\mathcal{G}_{\alpha,\beta}$ is an isometry from the space $\ell^2(\mathbb{N})$ into $L^2(0,\pi)$ and, of course, its inverse is the Fourier-Jacobi coefficient $\mathcal{G}_{\alpha,\beta}^{-1}F(n) = b_n^{(\alpha,\beta)}(F)$. Thus it is possible to recover the original sequence by means of it, i.e., $f(n) = b_n^{(\alpha,\beta)}(\mathcal{G}_{\alpha,\beta}f)$. In view of the above, for a sequence $f \in \ell^2(\mathbb{N})$, we define the discrete transplantation operator for Jacobi expansions by

$$T_{\alpha,\beta}^{\gamma,\delta}f(n) = b_n^{(\gamma,\delta)}(\mathcal{G}_{\alpha,\beta}f),$$

for any $\alpha,\beta,\gamma,\delta > -1$, which becomes the identity operator when $(\alpha,\beta) = (\gamma,\delta)$.

A remarkable result in this context, which shows that the size of the Fourier coefficients (measured in the p-norm) remain the same whatever its order is, is again due to Askey in [**6**].

Theorem 4.2 (Askey, 1967). *Let $\alpha,\beta,\gamma,\delta \geq -1/2$ and $F \in L^1(0,\pi)$. Then, if $1 < p < \infty$, $-1 < a < p-1$ and if either $\|b_n^{(\alpha,\beta)}(F)\|_{\ell^p(\mathbb{N},w_a)}$ or $\|b_n^{(\gamma,\delta)}(F)\|_{\ell^p(\mathbb{N},w_a)}$ is finite so is the other and*

$$\frac{1}{C}\|b_n^{(\alpha,\beta)}(F)\|_{\ell^p(\mathbb{N},w_a)} \leq \|b_n^{(\gamma,\delta)}(F)\|_{\ell^p(\mathbb{N},w_a)} \leq C\|b_n^{(\alpha,\beta)}(F)\|_{\ell^p(\mathbb{N},w_a)},$$

where C is a positive constant independent of F and thus of $b_n^{(\alpha,\beta)}(F)$ and $b_n^{(\gamma,\delta)}(F)$.

The main result of this chapter reads as follows.

Theorem 4.3. *Let $\alpha,\beta,\gamma,\delta \geq -1/2$ so that $(\alpha,\beta) \neq (\gamma,\delta)$.*

(i) *If $1 < p < \infty$ and $w \in A_p(\mathbb{N})$, then*

$$\|T_{\alpha,\beta}^{\gamma,\delta}f\|_{\ell^p(\mathbb{N},w)} \lesssim \|f\|_{\ell^p(\mathbb{N},w)}, \qquad f \in \ell^2(\mathbb{N}) \cap \ell^p(\mathbb{N},w).$$

Consequently, the transplantation operator $T_{\alpha,\beta}^{\gamma,\delta}$ extends uniquely to a bounded linear operator from $\ell^p(\mathbb{N},w)$ into itself.

(ii) *If $w \in A_1(\mathbb{N})$, then*

$$\|T_{\alpha,\beta}^{\gamma,\delta}f\|_{\ell^{1,\infty}(\mathbb{N},w)} \lesssim \|f\|_{\ell^1(\mathbb{N},w)}, \qquad f \in \ell^2(\mathbb{N}) \cap \ell^1(\mathbb{N},w).$$

Consequently, the transplantation operator $T_{\alpha,\beta}^{\gamma,\delta}$ extends uniquely to a bounded linear operator from $\ell^1(\mathbb{N},w)$ into $\ell^{1,\infty}(\mathbb{N},w)$.

An immediate consequence of this theorem is the following result.

Corollary 4.1. *Let $1 < p < \infty$, $w \in A_p(\mathbb{N})$, and $\alpha,\beta,\gamma,\delta \geq -1/2$ so that $(\alpha,\beta) \neq (\gamma,\delta)$. Then, there exits a positive constant C independent of f such that*

$$\frac{1}{C}\|f\|_{\ell^p(\mathbb{N},w)} \leq \|T_{\alpha,\beta}^{\gamma,\delta}f\|_{\ell^p(\mathbb{N},w)} \leq C\|f\|_{\ell^p(\mathbb{N},w)}$$

for any sequence $f \in \ell^p(\mathbb{N},w)$.

Out main result is actually a transplantation theorem. It is a version of Theorem 4.2 by Askey including general weights for functions $F \in L^q_v(0,\pi)$, where $v \in A_q(0,\infty)$ and $L^q_v(0,\pi)$ is the set of measurable functions such that the norm

$$\|F\|_{L^q_v(0,\pi)} \left(\int_0^\infty |F(x)|^q v(x)\,dx\right)^{1/q}, \qquad 1 < q < \infty,$$

is finite. To see that, we define the partial sum operator associated with Jacobi functions

$$\mathfrak{S}_n^{(\alpha,\beta)} F(m) := \mathcal{G}_{\alpha,\beta}\left(\chi_{[0,n]\cap\mathbb{N}} b_{(\cdot)}^{(\alpha,\beta)}(F)\right)(m).$$

V. Almeida, J. J. Betancor, A. J. Castro, A. Sanabria, and R. Scotto proved in [**1**, Prop. 2.2] that

$$\lim_{n\to\infty} \|\mathfrak{S}_n^{(\alpha,\beta)} F - F\|_{L^q_v(0,\pi)} = 0.$$

Then, by the Hölder's inequality and the bound (1.18) for Jacobi functions, we deduce

$$b_m^{(\alpha,\beta)}(F) = \lim_{n\to\infty} \int_0^\pi \mathfrak{S}_n^{(\gamma,\delta)} F(\theta) \mathcal{P}_m^{(\alpha,\beta)}(\theta)\,d\theta.$$

In this way,

$$b_m^{(\alpha,\beta)} = \lim_{n\to\infty} \sum_{k=0}^n b_k^{(\alpha,\beta)}(F) \int_0^\infty \mathcal{P}_k^{(\gamma,\delta)}(\theta) \mathcal{P}_m^{(\alpha,\beta)}(\theta)\,d\theta = T_{\alpha,\beta}^{\gamma,\delta} f(m)$$

where we have taken $f(n) = b_m^{(\alpha,\beta)}(F)$ and in last step we have used the expression (4.1), which is further down the line. Previous argument proves the following corollary.

COROLLARY 4.2. *Let $\alpha,\beta,\gamma,\delta \geq -1/2$ so that that $(\alpha,\beta) \neq (\gamma,\delta)$, $1 < p, q < \infty$, $v \in A_q(0,\pi)$, and $w \in A_p(\mathbb{N})$. Then, for $F \in L^q_v(0,\pi)$, there exists a positive constant C independent of F such that*

$$\frac{1}{C}\|b_n^{(\alpha,\beta)}(F)\|_{\ell^p(\mathbb{N},w)} \leq \|b_n^{(\gamma,\delta)}(F)\|_{\ell^p(\mathbb{N},w)} \leq C\|b_n^{(\alpha,\beta)}(F)\|_{\ell^p(\mathbb{N},w)}.$$

The present chapter is organised in the following way: in Section 2 a couple of preliminary results which we shall use later are presented. Section 3 is devoted to prove that actually the transplantation operator is a semi-local Calderón-Zygmund operator and, finally, in last section we use that fact in order to give the proof of the main theorem.

2. Preparatory results

The blueprint to prove Theorem 4.3 is to show that actually the transplantation operator $T_{\alpha,\beta}^{\gamma,\delta}$ is a semi-local Calderón-Zygmund operator, so we can apply the discrete semi-local Calderón-Zygmund theory (vid. Chap. 2, Sec. 2). To this end, note that the transplantation operator can be expressed by the series

$$(4.1)\quad T_{\alpha,\beta}^{\gamma,\delta} f(n) = \sum_{m=0}^\infty f(m) K_{\alpha,\beta}^{\gamma,\delta}(n,m), \qquad K_{\alpha,\beta}^{\gamma,\delta}(n,m) := \int_0^\pi \mathcal{P}_n^{(\gamma,\delta)}(x) \mathcal{P}_m^{(\alpha,\beta)}(x)\,dx,$$

at least for functions $f \in c_{00}$. The function $K_{\alpha,\beta}^{\gamma,\delta}(n,m)$ is the kernel of the transplantation operator $T_{\alpha,\beta}^{\gamma,\delta}$.

Throughout the present chapter we shall use profusely the estimate (1.19) for the Jacobi functions and (1.21) for its derivative. We shall also need proper bounds of the difference of two Jacobi functions whose degrees differ in two units. That is the statement of the following lemma.

Lemma 4.1. *Let $n \in \mathbb{N}$ and $\alpha, \beta > -1$. Then,*

$$|\mathcal{P}_{n+2}^{(\alpha,\beta)}(\theta)-\mathcal{P}_n^{(\alpha,\beta)}(\theta)| \lesssim \begin{cases} (n+1)^{\alpha-1/2}\theta^{\alpha+1/2}, & 0<\theta<1/(n+1), \\ \theta(\pi-\theta), & 1/(n+1)\le\theta\le\pi-1/(n+1), \\ (n+1)^{\beta-1/2}(\pi-\theta)^{\beta+1/2}, & \pi-1/(n+1)<\theta<\pi. \end{cases}$$

Proof. We give the result for values $0<\theta\le\pi/2$ firstly. To be specific, we are going to prove that the estimate

$$|\mathcal{P}_{n+2}^{(\alpha,\beta)}(\theta)-\mathcal{P}_n^{(\alpha,\beta)}(\theta)| \lesssim \begin{cases} (n+1)^{\alpha-1/2}\theta^{\alpha+1/2}, & 0<\theta<1/(n+1), \\ \theta, & 1/(n+1)\le\theta\le\pi/2, \end{cases} \tag{4.2}$$

holds. From the definition (1.11) of the Jacobi functions, we can put the difference we are interested in by

$$\mathcal{P}_{n+2}^{(\alpha,\beta)}(\theta)-\mathcal{P}_n^{(\alpha,\beta)}(\theta) = \left(\frac{\omega_{n+2}^{(\alpha,\beta)}}{\omega_n^{(\alpha,\beta)}}-1\right)\mathcal{P}_n^{(\alpha,\beta)}(\theta)+A^{\alpha,\beta}(n,\theta) \tag{4.3}$$

where we have denoted

$$A^{\alpha,\beta}(n,\theta) := \left(\sin\frac{\theta}{2}\right)^{\alpha+1/2}\left(\cos\frac{\theta}{2}\right)^{\beta+1/2}\omega_{n+2}^{(\alpha,\beta)}\left(P_{n+2}^{(\alpha,\beta)}(\cos\theta)-P_n^{(\alpha,\beta)}(\cos\theta)\right).$$

Using that

$$\left|\frac{\omega_{n+2}^{(\alpha,\beta)}}{\omega_n^{(\alpha,\beta)}}-1\right| \lesssim \frac{1}{n+1},$$

and the estimate (1.19), we have that the first addend fulfils the bound

$$\left|\frac{\omega_{n+2}^{(\alpha,\beta)}}{\omega_n^{(\alpha,\beta)}}-1\right| |\mathcal{P}_n^{(\alpha,\beta)}(\theta)| \lesssim \begin{cases} (n+1)^{\alpha-1/2}\theta^{\alpha+1/2}, & 0<\theta<1/(n+1), \\ \theta, & 1/(n+1)\le\theta\le\pi/2, \end{cases} \tag{4.4}$$

For the remaining part $A^{\alpha,\beta}(n,\theta)$ of the right and side of (4.3), we shall need an appropriate expression for the difference of two Jacobi polynomials whose degrees differ in one unit. We use the identity

$$-\frac{2n+\alpha+\beta+2}{2}(1-x)P_n^{(\alpha+1,\beta)}(x)+\alpha P_n^{(\alpha,\beta)}(x) = (n+1)\left(P_{n+1}^{(\alpha,\beta)}(x)-P_n^{(\alpha,\beta)}(x)\right)$$

for $x \in (-1,1)$, which could be consulted in [**36**, Eq. 18.9.6]. By means of it, we have

$$\left(\sin\frac{\theta}{2}\right)^{\alpha+1/2}\left(\cos\frac{\theta}{2}\right)^{\beta+1/2}\omega_{n+2}^{(\alpha,\beta)}|P_{n+1}^{(\alpha,\beta)}(\cos\theta)-P_n^{\alpha,\beta}(\cos\theta)|$$
$$\leq \frac{2n+\alpha+\beta+2}{n+1}\frac{\omega_{n+2}^{(\alpha,\beta)}}{\omega_n^{(\alpha+1,b)}}\left(\sin\frac{\theta}{2}\right)|\mathcal{P}_n^{(\alpha+1,\beta)}(\theta)|+\frac{|\alpha|}{n+1}\frac{\omega_{n+2}^{(\alpha,\beta)}}{\omega_n^{(\alpha,\beta)}}|\mathcal{P}_n^{(\alpha,\beta)}(\theta)|.$$

In this way, (1.19) implies that the latter is bounded by

$$\begin{cases}(n+1)^{\alpha-1/2}\theta^{\alpha+1/2}, & 0<\theta<1/(n+1),\\ \theta, & 1/(n+1)\leq\theta\leq\pi/2,\end{cases}\tag{4.5}$$

times a multiplicative constant. The same bound hold for the term

$$\left(\sin\frac{\theta}{2}\right)^{\alpha+1/2}\left(\cos\frac{\theta}{2}\right)^{\beta+1/2}\omega_{n+2}^{(\alpha,\beta)}|P_{n+2}^{(\alpha,\beta)}(\cos\theta)-P_{n+1}^{\alpha,\beta}(\cos\theta)|.$$

Then (4.2) follows from (4.3), (4.4) and (4.5).

The result for the remaining values of the variable θ could be obtained having in mind the parity property (1.8). Indeed,

$$\mathcal{P}_{n+2}^{(\alpha,\beta)}(\pi-\theta)-\mathcal{P}_n^{(\alpha,\beta)}(\pi-\theta)=(-1)^n\left(\mathcal{P}_{n+2}^{(\beta,\alpha)}(\theta)-\mathcal{P}_n^{(\beta,\alpha)}(\theta)\right)$$

for $0<\theta<\pi$. Then, by an analogous argument as before, we get

$$|\mathcal{P}_{n+2}^{(\alpha,\beta)}(\theta)-\mathcal{P}_n^{(\alpha,\beta)}(\theta)|\lesssim\begin{cases}\pi-\theta & \pi/2<\theta\leq\pi-1/(n+1),\\ (n+1)^{\beta-1/2}(\pi-\theta)^{\beta+1/2}, & \pi-1/(n+1)<\theta<\pi.\end{cases}$$

The union of the previous estimates with the ones in (4.2) gives the statement of the lemma. □

Next lemma provides us of estimates for the derivative of the difference of two Jacobi functions whose degrees differ in two units.

LEMMA 4.2. *Let $n\in\mathbb{N}$ and $\alpha,\beta>-1$. Then,*

$$\left|\frac{d}{d\theta}(\mathcal{P}_{n+2}^{(\alpha,\beta)}-\mathcal{P}_n^{(\alpha,\beta)})(\theta)\right|$$
$$\lesssim\begin{cases}(n+1)^{\alpha-1/2}\theta^{\alpha-1/2}, & 0<\theta<1/(n+1),\\ (n+1)\theta(\pi-\theta), & 1/(n+1)\leq\theta\leq\pi-1/(n+1),\\ (n+1)^{\beta-1/2}(\pi-\theta)^{\beta-1/2}, & \pi-1/(n+1)<\theta<\pi.\end{cases}$$

PROOF. In the same way as in the proof of Lemma 4.1, we suppose firstly that $0<\theta\leq\pi/2$ and we are going to prove that

$$\left|\frac{d}{d\theta}(\mathcal{P}_{n+2}^{(\alpha,\beta)}-\mathcal{P}_n^{(\alpha,\beta)})(\theta)\right|\lesssim\begin{cases}(n+1)^{\alpha-1/2}\theta^{\alpha-1/2}, & 0<\theta<1/(n+1),\\ (n+1)\theta, & 1/(n+1)\leq\theta\leq\pi/2.\end{cases}\tag{4.6}$$

The expression (1.20) for the derivative of a Jacobi function is used to put

$$\frac{d}{d\theta}(\mathcal{P}_{n+2}^{(\alpha,\beta)} - \mathcal{P}_{n}^{(\alpha,\beta)})(\theta) = B_1^{\alpha,\beta}(n,\theta) - B_2^{\alpha,\beta}(n,\theta) - B_3^{\alpha,\beta}(n,\theta),$$

where

$$\begin{aligned} B_1^{\alpha,\beta}(n,\theta) &:= \left(\frac{2\alpha+1}{4}\cot\frac{\theta}{2} - \frac{2\beta+1}{4}\tan\frac{\theta}{2}\right)\left(\mathcal{P}_{n+2}^{(\alpha,\beta)}(\theta) - \mathcal{P}_{n}^{(\alpha,\beta)}(\theta)\right), \\ B_2^{\alpha,\beta}(n,\theta) &:= (\lambda_{n+2}^{(\alpha,\beta)})^{1/2}\left(\mathcal{P}_{n+1}^{(\alpha+1,\beta+1)}(\theta) - \mathcal{P}_{n-1}^{(\alpha+1,\beta+1)}(\theta)\right), \\ B_3^{\alpha,\beta}(n,\theta) &:= \left((\lambda_{n+2}^{(\alpha,\beta)})^{1/2} - (\lambda_{n}^{(\alpha,\beta)})^{1/2}\right)\mathcal{P}_{n-1}^{(\alpha+1,\beta+1)}(\theta). \end{aligned}$$

As a consequence of the estimate in Lemma 4.1, first two parts are respectively bounded by

$$|B_1^{\alpha,\beta}(n,\theta)| \lesssim \begin{cases} (n+1)^{\alpha-1/2}\theta^{\alpha-1/2}, & 0<\theta<1/(n+1), \\ 1, & 1/(n+1)\le\theta\le\pi-1/(n+1), \end{cases}$$

and

$$|B_2^{\alpha,\beta}(n,\theta)| \lesssim \begin{cases} (n+1)^{\alpha+3/2}\theta^{\alpha+3/2}, & 0<\theta<1/(n+1), \\ (n+1)\theta, & 1/(n+1)\le\theta\le\pi-1/(n+1), \end{cases}$$

Regarding the last one, using the estimate (1.19), we obtain

$$|B_3^{\alpha,\beta}(n,\theta)| \lesssim \begin{cases} (n+1)^{\alpha+3/2}\theta^{\alpha+3/2}, & 0<\theta<1/(n+1), \\ 1, & 1/(n+1)\le\theta\le\pi-1/(n+1), \end{cases}$$

and (4.6) follows.

In the case $\pi/2<\theta<\pi$ we use again (1.8) to deduce the corresponding bounds. The statement of the lemma is obtained putting all the previous estimates together. □

3. Calderón-Zygmund properties for the kernel

An important tool to analyse the kernel $K_{\alpha,\beta}^{\gamma,\delta}(n,m)$ is the identity (1.13) involving the second order differential operator $\mathfrak{L}^{\alpha,\beta}$, and the relation (1.14) between two operators of different order.

Next proposition gives the size condition (a') for the kernel.

PROPOSITION 4.1. *Let $n,m\in\mathbb{N}$ with $n\neq m$, and $\alpha,\beta,\gamma,\delta\ge -1/2$. Then,*

$$|K_{\alpha,\beta}^{\gamma,\delta}(n,m)| \lesssim \frac{1}{|n-m|}.$$

PROOF. Firstly, note that we may suppose $n>m$ due to the symmetry. The idea is to decompose the kernel according the intervals suggested by the estimate (1.19) and apply it in each part. Then, considering the intervals $I_1 := (0, 1/(n+1))$, $I_2 := [1/(n+1), \pi - 1/(n+1)]$, and $I_3 := (\pi - 1/(n+1), \pi)$, we put

$$K_{\alpha,\beta}^{\gamma,\delta}(n,m) = K_1(n,m) + K_2(n,m) + K_3(n,m)$$

where we have denoted

$$K_\ell(n,m) := \int_{I_\ell} \mathcal{P}_n^{(\gamma,\delta)}(\theta)\mathcal{P}_m^{(\alpha,\beta)}(\theta)\,d\theta, \qquad \ell = 1,2,3.$$

First and third integrals are bounded by

$$|K_1(n,m)| \lesssim (n+1)^{\gamma+1/2}(m+1)^{\alpha+1/2}\int_{I_1}\theta^{\alpha+\gamma+1}\,d\theta \lesssim \frac{1}{n+1},$$

and

$$|K_3(n,m)| \lesssim (n+1)^{\delta+1/2}(m+1)^{\beta+1/2}\int_{I_3}(\pi-\theta)^{\beta+\delta+1}\,d\theta \lesssim \frac{1}{n+1},$$

respectively.

To estimate $K_2(n,m)$ we consider two cases: $n - m \le |\alpha+\beta-\gamma-\delta|$ and $n - m > |\alpha+\beta-\gamma-\delta|$. In the first case we can use that

$$|K_2(n,m)| \le \|\mathcal{P}_n^{(\gamma,\delta)}\|_{L^2(0,\pi)}\|\mathcal{P}_m^{(\alpha,\beta)}\|_{L^2(0,\pi)} = 1,$$

which is enough to obtain the result in this situation.

Now, we focus on the case $n - m > |\alpha+\beta-\gamma-\delta|$, which implies $\nu_n^{(\gamma,\delta)} > \nu_m^{(\alpha,\beta)}$. To analyse $K_2(n,m)$ in this situation let us set the notations

$$U(n,m) := W(\mathcal{P}_n^{(\gamma,\delta)}, \mathcal{P}_m^{(\alpha,\beta)})(\theta)\Big|_{\theta=1/(n+1)}^{\theta=\pi-1/(n+1)}$$

and

$$V(n,m) := \int_{I_2} R_{\alpha,\beta}^{\gamma,\delta}(\theta)\mathcal{P}_n^{(\gamma,\delta)}(\theta)\mathcal{P}_m^{(\alpha,\beta)}(\theta)\,d\theta.$$

By the identity (1.12) we have

$$\begin{aligned}\nu_n^{(\gamma,\delta)}K_2(m,n) &= \int_{I_2}\mathfrak{L}^{\gamma,\delta}\mathcal{P}_n^{(\gamma,\delta)}(\theta)\mathcal{P}_m^{(\alpha,\beta)}(\theta)\,d\theta\\ &= U(n,m) + \int_{I_2}\mathcal{P}_n^{(\gamma,\delta)}(\theta)\mathfrak{L}^{\gamma,\delta}\mathcal{P}_m^{(\alpha,\beta)}(\theta)\,d\theta\\ &= U(n,m) + \nu_m^{(\alpha,\beta)}K_2(n,m) + V(m,n),\end{aligned}$$

where in the last line we have used the relation (1.14) first and then again (1.12). Hence,

$$K_2(n,m) = \frac{U(n,m)+V(n,m)}{\nu_n^{(\gamma,\delta)} - \nu_m^{(\alpha,\beta)}}. \tag{4.7}$$

Let us study first the part $U(n,m)$ related to the Wronskian. Using the estimates (1.19) and (1.21) for Jacobi functions, we obtain, by a tedious but simple computation, $|U(n,m)| \lesssim n+m$.

To estimate $V(n,m)$ split I_2 into the intervals $J_1 := [1/(n+1), 1/(m+1))$, $J_2 := [1/(m+1), \pi/2]$, $J_3 := (\pi/2, \pi - 1/(m+1)]$, and $J_4 := (\pi - 1/(m+1), \pi - 1/(n+1)]$. Then,

$$V(n,m) = \sum_{\ell=1}^{4} V_\ell(n,m), \qquad V_\ell(n,m) := \int_{J_\ell} R^{\gamma,\delta}_{\alpha,\beta}(\theta)\mathcal{P}^{(\gamma,\delta)}_n(\theta)\mathcal{P}^{(\alpha,\beta)}_m(\theta)\,d\theta,$$

Again by the bounds (1.19) and having in mind the constraint $\alpha \geq -1/2$ we obtain

$$|V_1(n,m)| \lesssim (m+1)^{\alpha+1/2}\int_{J_1} \theta^{\alpha-3/2}\,d\theta \lesssim \int_{1/(n+1)}^{\pi/2} \frac{d\theta}{\theta^2} \lesssim n+1$$

and

$$|V_2(n,m)| \lesssim \int_{J_2} \frac{d\theta}{\theta^2} \lesssim m+1.$$

The treatment of the parts $V_3(m,n)$ and $V_4(m,n)$ is analogous and we omit it. So, $|V(m,n)| \lesssim n+m$. Therefore, we conclude that

$$|K_2(m,n)| \lesssim \frac{1}{|n-m|}$$

and the proof is complete. □

Following proposition implies that some auxiliary kernels, which are related to $K^{\gamma,\delta}_{\alpha,\beta}(n,m)$ and will be defined later, fulfil regularity conditions (b1') and (b2').

PROPOSITION 4.2. *Let $n,m \in \mathbb{N}$ such that $m/4 \leq n \leq 4m$ and $n \neq m$, and let $\alpha,\beta,\gamma,\delta \geq -1/2$. Then,*

$$|K^{\gamma,\delta}_{\alpha,\beta}(n+2,m) - K^{\gamma,\delta}_{\alpha,\beta}(n,m)| \lesssim \frac{1}{|n-m|^2} \tag{4.8}$$

and

$$|K^{\gamma,\delta}_{\alpha,\beta}(n,m) - K^{\gamma,\delta}_{\alpha,\beta}(n,m+2)| \lesssim \frac{1}{|n-m|^2}. \tag{4.9}$$

PROOF. We focus on the proof of the estimation (4.8) because (4.9) could be deduced in a similar way. The proof follows the steps of the one for Proposition 4.1 and we shall use several notations taken from there. However, we shall study separately the situations $n > m$ and $m > n$.

Let us suppose first $n > m$. We decompose the difference

$$K^{\gamma,\delta}_{\alpha,\beta}(n+2,m) - K^{\gamma,\delta}_{\alpha,\beta}(n,m) = \mathcal{K}_1(n,m) + \mathcal{K}_2(n,m) + \mathcal{K}_3(n,m)$$

where

$$\mathcal{K}_\ell(n,m) := \int_{I_\ell} \left(\mathcal{P}^{(\gamma,\delta)}_{n+2}(\theta) - \mathcal{P}^{(\gamma,\delta)}_n(\theta)\right)\mathcal{P}^{(\alpha,\beta)}_m(\theta)\,d\theta, \qquad \ell = 1,2,3.$$

Here, the intervals I_ℓ are the same as the ones defined in the proof of Proposition 4.1. For the first and third parts we use the estimate (1.19) and Lemma 4.1, together with the constraint $\alpha, \beta, \gamma, \delta \geq -1/2$, to get

$$|\mathcal{K}_1(n,m)| \lesssim (n+1)^{\gamma-1/2}(m+1)^{\alpha+1/2}\int_{I_1}\theta^{\alpha+\gamma+1}\,d\theta \lesssim \frac{1}{(n+1)^2}$$

and

$$|\mathcal{K}_3(n,m)| \lesssim (n+1)^{\delta-1/2}(m+1)^{\beta+1/2}\int_{I_3}(\pi-\theta)^{\beta+\delta+1}\,d\theta \lesssim \frac{1}{(n+1)^2},$$

which are enough for our purpose.

In order to estimate $\mathcal{K}_2(n,m)$ we consider the cases $|n-m| \leq |\alpha+\beta-\gamma-\delta|$ and $|n-m| > |\alpha+\beta-\gamma-\delta|$. First one is easy to handle with, so we shall focus on the second one.

Note that $|n-m| > |\alpha+\beta-\gamma-\delta|$ implies again $\nu_n^{(\gamma,\delta)} > \nu_m^{(\alpha,\beta)}$ and also $\nu_{n+2}^{(\gamma,\delta)} > \nu_n^{(\alpha,\beta)}$. Then, using (4.7) we have

$$\mathcal{K}_2(n,m) = \frac{U(n+2,m)+V(n+2,m)}{\nu_{n+2}^{(\gamma,\delta)}-\nu_m^{(\alpha,\beta)}} - \frac{U(n,m)+V(n,m)}{\nu_n^{(\gamma,\delta)}-\nu_m^{(\alpha,\beta)}}, \tag{4.10}$$

where the notation given in the proof of Proposition 4.1 is followed. To avoid cumbersome notations we denote analogously as there

$$\mathcal{U}(n,m) := W(\mathcal{P}_{n+2}^{(\gamma,\delta)}-\mathcal{P}_n^{(\gamma,\delta)}, \mathcal{P}_m^{(\alpha,\beta)})(\theta)\Big|_{\theta=1/(n+1)}^{\theta=\pi-1/(n+1)}$$

and

$$\mathcal{V}(n,m) := \int_{I_2} R_{\gamma,\delta}^{\alpha,\beta}(\theta)\left(\mathcal{P}_{n+2}^{(\gamma,\delta)}(\theta)-\mathcal{P}_n^{(\gamma,\delta)}(\theta)\right)\mathcal{P}_m^{(\alpha,\beta)}(\theta)\,d\theta.$$

Then,

$$\begin{aligned}
\mathcal{K}_2(n,m) &= \frac{\mathcal{U}(n,m)+\mathcal{V}(n,m)}{\nu_{n+2}^{(\gamma,\delta)}-\nu_m^{(\alpha,\beta)}} - \frac{(\nu_{n+2}^{(\gamma,\delta)}-\nu_n^{(\gamma,\delta)})(U(n,m)+V(n,m))}{(\nu_{n+2}^{(\gamma,\delta)}-\nu_m^{(\alpha,\beta)})(\nu_n^{(\gamma,\delta)}-\nu_m^{(\alpha,\beta)})} \\
&= \frac{\mathcal{U}(n,m)+\mathcal{V}(n,m)}{\nu_{n+2}^{(\gamma,\delta)}-\nu_m^{(\alpha,\beta)}} - \frac{2(2n+\gamma+\delta+3)(U(n,m)+V(n,m))}{(\nu_{n+2}^{(\gamma,\delta)}-\nu_m^{(\alpha,\beta)})(\nu_n^{(\gamma,\delta)}-\nu_m^{(\alpha,\beta)})}.
\end{aligned}$$

The terms $U(n,m)$ and $V(n,m)$ could be treated proceeding as in the proof of Proposition 4.1, so we obtain

$$\left|\frac{2(2n+\gamma+\delta+3)(U(n,m)+V(n,m))}{(\nu_{n+2}^{(\gamma,\delta)}-\nu_m^{(\alpha,\beta)})(\nu_n^{(\gamma,\delta)}-\nu_m^{(\alpha,\beta)})}\right| \lesssim \frac{1}{|n-m|^2}. \tag{4.11}$$

On the other hand, from the estimates for Jacobi functions (1.19) and for its derivative (1.21), together with Lemma 4.1 and Lemma 4.2, we deduce that $|\mathcal{U}(n,m)| \lesssim 1$, which implies

$$\left|\frac{\mathcal{U}(n,m)}{\nu_{n+2}^{(\gamma,\delta)}-\nu_m^{(\alpha,\beta)}}\right| \lesssim \frac{1}{|n-m|^2}. \tag{4.12}$$

Finally, in order to analyse $\mathcal{V}(n,m)$, we denote

$$\overline{\mathcal{U}}(n,m) := W\left(\mathcal{P}_m^{(\alpha,\beta)}, R_{\gamma,\delta}^{\alpha,\beta}(\mathcal{P}_{n+2}^{(\gamma,\delta)} - \mathcal{P}_n^{(\gamma,\delta)})\right)(\theta)\Big|_{\theta=1/(n+1)}^{\theta=\pi-1/(n+1)},$$

$$\mathcal{W}_1(n,m) := \int_{I_2}\left((R_{\gamma,\delta}^{\alpha,\beta})^2(\theta) + \frac{d^2}{d\theta^2}R_{\gamma,\delta}^{\alpha,\beta}(\theta)\right)\left(\mathcal{P}_{n+2}^{(\gamma,\delta)}(\theta) - \mathcal{P}_n^{(\gamma,\delta)}(\theta)\right)\mathcal{P}_m^{(\alpha,\beta)}(\theta)\,d\theta,$$

and

$$\mathcal{W}_2(n,m) := \int_{I_2}\frac{d}{d\theta}R_{\gamma,\delta}^{\alpha,\beta}(\theta)\frac{d}{d\theta}\left(\mathcal{P}_{n+2}^{(\gamma,\delta)} - \mathcal{P}_n^{(\gamma,\delta)}\right)(\theta)\mathcal{P}_m^{(\alpha,\beta)}(\theta)\,d\theta.$$

Then, using (1.19) and the identity

$$\mathfrak{L}^{\alpha,\beta}(R_{\gamma,\delta}^{\alpha,\beta}\cdot f)(\theta) = R_{\gamma,\delta}^{\alpha,\beta}(\theta)\mathfrak{L}^{\gamma,\delta}f(\theta) - \left((R_{\gamma,\delta}^{\alpha,\beta})^2(\theta) - \frac{d^2}{d\theta^2}R_{\gamma,\delta}^{\alpha,\beta}(\theta)\right)f(\theta) - 2\frac{d}{d\theta}R_{\gamma,\delta}^{\alpha,\beta}(\theta)\frac{d}{d\theta}f(\theta),$$

we can deduce that

$$\begin{aligned}\nu_m^{(\alpha,\beta)}\mathcal{V}(n,m) &= \int_{I_2}R_{\gamma,\delta}^{\alpha,\beta}(\theta)\left(\mathcal{P}_{n+2}^{(\gamma,\delta)}(\theta) - \mathcal{P}_n^{(\gamma,\delta)}(\theta)\right)\mathfrak{L}^{\alpha,\beta}\mathcal{P}_m^{(\alpha,\beta)}(\theta)\,d\theta\\ &= \overline{\mathcal{U}}(n,m) + \int_{I_2}\mathfrak{L}^{\alpha,\beta}\left(R_{\gamma,\delta}^{\alpha,\beta}(\mathcal{P}_{n+2}^{(\gamma,\delta)} - \mathcal{P}_n^{(\gamma,\delta)})\right)(\theta)\mathcal{P}_m^{(\alpha,\beta)}(\theta)\,d\theta\\ &= \overline{\mathcal{U}}(n,m) + \int_{I_2}R_{\gamma,\delta}^{\alpha,\beta}(\theta)\mathfrak{L}^{\gamma,\delta}(\mathcal{P}_{n+2}^{(\gamma,\delta)} - \mathcal{P}_n^{(\gamma,\delta)})(\theta)\mathcal{P}_m^{(\alpha,\beta)}(\theta)\,d\theta\\ &\qquad - \mathcal{W}_1(n,m) - 2\mathcal{W}_2(n,m).\end{aligned}$$

Now, note that

$$\mathfrak{L}^{\gamma,\delta}(\mathcal{P}_{n+2}^{(\gamma,\delta)} - \mathcal{P}_n^{(\gamma,\delta)})(\theta) = \nu_{n+2}^{(\gamma,\delta)}(\mathcal{P}_{n+2}^{\gamma,\delta}(\theta) - \mathcal{P}_n^{(\gamma,\delta)}(\theta)) + (\nu_{n+2}^{(\gamma,\delta)} - \nu_n^{(\gamma,\delta)})\mathcal{P}_n^{(\gamma,\delta)}(\theta)$$

and hence

$$\begin{aligned}\nu_m^{(\alpha,\beta)}\mathcal{V}(n,m) = \overline{\mathcal{U}}(n,m) + \nu_{n+2}^{(\gamma,\delta)}\mathcal{V}(n,m) + (\nu_{n+2}^{(\gamma,\delta)} - \nu_n^{(\gamma,\delta)})V(n,m)\\ - \mathcal{W}_1(n,m) - 2\mathcal{W}_2(n,m).\end{aligned}$$

Thus,

$$\frac{\mathcal{V}(n,m)}{\nu_{n+2}^{(\gamma,\delta)} - \nu_m^{(\alpha,\beta)}} = \frac{\mathcal{W}_1(n,m) + 2\mathcal{W}_2(n,m) - \overline{\mathcal{U}}(n,m) - 2(2n+\gamma+\delta+3)V(n,m)}{(\nu_{n+2}^{(\gamma,\delta)} - \nu_m^{(\alpha,\beta)})^2}. \tag{4.13}$$

It is again straightforward from Proposition 4.1 to deduce that

$$\left|\frac{2(2n+\gamma+\delta+3)}{(\nu_{n+2}^{(\gamma,\delta)} - \nu_m^{(\alpha,\beta)})^2}V(n,m)\right| \lesssim \frac{1}{|n-m|^2}.$$

In this way, by (1.19), (1.21), together with Lemma 4.1 and Lemma 4.2, showing that

$$|\overline{\mathcal{U}}(n,m)| + |\mathcal{W}_1(n,m)| + |\mathcal{W}_2(n,m)| \lesssim (n+1)^2$$

the proof of (4.8) will be completed.

For the first part we use again the estimates (1.19) and (1.21), together with Lemma 4.1 and Lemma 4.2, and we get $|\overline{\mathcal{U}}(n,m)| \lesssim (n+1)^2$.

In order to estimate $\mathcal{W}_1(n,m)$ and $\mathcal{W}_2(n,m)$ we consider I_2 as the union of the interval $[1/(n+1), \pi/2]$ and its complementary $(\pi/2, \pi - 1/(n+1)]$. We shall only be focused on the first one interval because the second one could be studied in an analogous way. The corresponding integrals will be denoted by

$$\overline{\mathcal{W}}_1(n,m) := \int_{1/(n+1)}^{\pi/2} \left((R_{\gamma,\delta}^{\alpha,\beta})^2(\theta) + \frac{d^2}{d\theta^2} R_{\gamma,\delta}^{\alpha,\beta}(\theta) \right) \left(\mathcal{P}_{n+2}^{(\gamma,\delta)}(\theta) - \mathcal{P}_n^{(\gamma,\delta)}(\theta) \right) \mathcal{P}_m^{(\alpha,\beta)}(\theta)\, d\theta$$

and

$$\overline{\mathcal{W}}_1(n,m) := \int_{1/(n+1)}^{\pi/2} \frac{d}{d\theta} R_{\gamma,\delta}^{\alpha,\beta}(\theta) \frac{d}{d\theta} \left(\mathcal{P}_{n+2}^{(\gamma,\delta)} - \mathcal{P}_n^{(\gamma,\delta)} \right)(\theta) \mathcal{P}_m^{(\alpha,\beta)}(\theta)\, d\theta.$$

First, using the bound

$$(R_{\gamma,\delta}^{\alpha,\beta})^2(\theta) + \left| \frac{d^2}{d\theta^2} R_{\gamma,\delta}^{\alpha,\beta}(\theta) \right| \lesssim \frac{1}{\theta^4}, \qquad 1/(n+1) \le \theta \le \pi/2,$$

we have

$$|\overline{\mathcal{W}}_2(n,m)| \lesssim \int_{1/(n+1)}^{\pi/2} |\mathcal{P}_{n+2}^{(\gamma,\delta)}(\theta) - \mathcal{P}_n^{(\gamma,\delta)}(\theta)||\mathcal{P}_m^{(\alpha,\beta)}(\theta)| \frac{d\theta}{\theta^4}.$$

Then, using the estimate (1.19) and Lemma 4.1, we obtain

$$\begin{aligned} |\overline{\mathcal{W}}_1(n,m)| &\lesssim \left((m+1)^{\alpha+1/2} \int_{1/(n+1)}^{1/(m+1)} \theta^{\alpha-5/2}\, d\theta + \int_{1/(m+1)}^{\pi/2} \frac{d\theta}{\theta^3} \right) \\ &\lesssim \int_{1/(n+1)}^{\pi/2} \frac{d\theta}{\theta^3} \lesssim (n+1)^2. \end{aligned}$$

Finally, using the bounds

$$\left| \frac{d}{d\theta} R_{\gamma,\delta}^{\alpha,\beta}(\theta) \right| \lesssim \frac{1}{\theta^3}, \qquad 1/(n+1) \le \theta \le \pi/2,$$

and (1.19), together with Lemma 4.2, we deduce that

$$\begin{aligned} |\overline{\mathcal{W}}_2(n,m)| &\lesssim (n+1) \left((m+1)^{\alpha+1/2} \int_{1/(n+1)}^{1/(m+1)} \theta^{\alpha-3/2}\, d\theta + \int_{1/(m+1)}^{\pi/2} \frac{dx}{\theta^2} \right) \\ &\lesssim (n+1) \int_{1/(n+1)}^{\pi/2} \frac{d\theta}{\theta^2} \lesssim (n+1)^2. \end{aligned}$$

Let us deal now with the case $n < m$. At this point, we find convenient to reset the notation fixed for the case $n > m$. Again, we consider the intervals $I_1 := (0, 1/(n+1))$, $I_2 := [1/(n+1), \pi - 1/(n+1)]$, and $I_3 := (\pi - 1/(n+1), \pi)$ respectively, and we decompose the difference

$$K_{\alpha,\beta}^{\gamma,\delta}(n+2,m) - K_{\alpha,\beta}^{\gamma,\delta}(n,m) = \mathcal{K}_1(n,m) + \mathcal{K}_2(n,m) + \mathcal{K}_3(n,m)$$

where

$$\mathcal{K}_\ell(n,m) := \int_{I_\ell} \left(\mathcal{P}_{n+2}^{(\gamma,\delta)}(\theta) - \mathcal{P}_n^{(\gamma,\delta)}(\theta) \right) \mathcal{P}_m^{(\alpha,\beta)}(\theta)\, d\theta, \qquad \ell = 1,2,3.$$

Using the estimates (1.19) for Jacobi functions, Lemma 4.1, and the fact that $m/4 \le n \le 4m$, we have

$$|\mathcal{K}_1(n,m)| \lesssim (n+1)^{\gamma-1/2}\left((m+1)^{\alpha+1/2}\int_0^{1/(m+1)} \theta^{\alpha+\gamma+1}\,d\theta + \int_{1/(m+1)}^{1/(n+1)} \theta^{\gamma+1/2}\,d\theta\right) \lesssim \frac{1}{(m+1)^2}$$

and

$$|\mathcal{K}_3(n,m)| \lesssim (n+1)^{\delta-1/2}\left(\int_{\pi-1/(n+1)}^{\pi-1/(m+1)} (\pi-\theta)^{\delta+1/2}\,d\theta + (m+1)^{\beta+1/2}\int_{\pi-1/(m+1)}^{\pi} (\pi-\theta)^{\beta+\delta+1}\,d\theta\right) \lesssim \frac{1}{(m+1)^2}.$$

We analyse now the part corresponding to $\mathcal{K}_2(n,m)$. Again the case $m-n \le |\alpha+\beta-\gamma-\delta-4|$ is elementary. For $m-n > |\alpha+\beta-\gamma-\delta-4|$, note first that (4.10) still holds in the present situation. Following the same procedure as in the proof of Proposition 4.1 we have that $|U(n,m)| \lesssim n+m$. On its behalf, the estimates (1.19) leads to

$$|V(n,m)| \lesssim \int_{1/(n+1)}^{\pi/2} \frac{d\theta}{\theta^2} \lesssim n+1.$$

Furthermore, the use again of the estimates (1.19) for Jacobi functions and (1.21) for its derivative, together with the bounds in Lemma 4.1 and Lemma 4.2, and the fact $m/4 \le n \le 4m$, implies $|\mathcal{U}(n,m)| \lesssim 1$.

Now, to analyse $\mathcal{V}(n,m)$, note firstly that (4.13) remains true. We have already studied the term $V(n,m)$ so we shall focus on the remaining ones. One more time, by the estimates (1.19) and (1.21), and the bounds in Lemma 4.1 and Lemma 4.2, we have that $|\overline{U}(n,m)| \lesssim (m+1)^2$.

Finally, treating $\mathcal{W}_1$ and $\mathcal{W}_2$ as in the case corresponding to $n > m$, we obtain that $|\mathcal{W}_1(n,m)| \lesssim (m+1)^2$ and $|\mathcal{W}_2(n,m)| \lesssim (m+1)^2$.

From all of these bounds, the condition (4.8) follows for $m > n$ and then the proof is complete. □

4. Proof of the main theorem

We give now the proof of Theorem 4.3. Since we have obtained estimates for the differences

$$K^{\gamma,\delta}_{\alpha,\beta}(n+2,m) - K^{\gamma,\delta}_{\alpha,\beta}(n,m) \quad \text{and} \quad K^{\gamma,\delta}_{\alpha,\beta}(n,m) - K^{\gamma,\delta}_{\alpha,\beta}(n,m+2) \tag{4.14}$$

instead of

$$K^{\gamma,\delta}_{\alpha,\beta}(n+1,m) - K^{\gamma,\delta}_{\alpha,\beta}(n,m) \quad \text{and} \quad K^{\gamma,\delta}_{\alpha,\beta}(n,m) - K^{\gamma,\delta}_{\alpha,\beta}(n,m+1),$$

we should decompose the transplantation operator into four parts according the parity of the variables n and m. The differences (4.14) are more appropriate due to the appearance of $\mathcal{P}^{(\alpha,\beta)}_{n+2}(\theta) - \mathcal{P}^{(\alpha,\beta)}_{n}(\theta)$ which, as it is well-known, behaves better than $\mathcal{P}_{n+1}^{(\alpha,\beta)}(\theta) - \mathcal{P}_{n}^{(\alpha,\beta)}(\theta)$. This fact was observed in the forties of past century by H. Pollard in [**38**].

Having in mind the above argument, we split the transplantation operator according the parity of the variable m to get

$$T_{\alpha,\beta}^{\gamma,\delta}f(n) = \sum_{n=0}^{\infty} f(2m)K_{\alpha,\beta}^{\gamma,\delta}(n,2m) + \sum_{n=0}^{\infty} f(2m+1)K_{\alpha,\beta}^{\gamma,\delta}(n,2m+1).$$

On its behalf, the parity of the variable n motivates the definition of the following kernels

$$\begin{aligned} {}_e^eK_{\alpha,\beta}^{\gamma,\delta}(n,m) &:= K_{\alpha,\beta}^{\gamma,\delta}(2n,2m), & {}_o^eK_{\alpha,\beta}^{\gamma,\delta}(n,m) &:= K_{\alpha,\beta}^{\gamma,\delta}(2n,2m+1),\\ {}_e^oK_{\alpha,\beta}^{\gamma,\delta}(n,m) &:= K_{\alpha,\beta}^{\gamma,\delta}(2n+1,2m), & {}_o^oK_{\alpha,\beta}^{\gamma,\delta}(n,m) &:= K_{\alpha,\beta}^{\gamma,\delta}(2n+1,2m+1). \end{aligned}$$

Each one of these kernels produces an auxiliary operator

$$\begin{aligned} {}_e^eT_{\alpha,\beta}^{\gamma,\delta}f(n) &:= \sum_{m=0}^{\infty} f(m)\,{}_e^eK_{\alpha,\beta}^{\gamma,\delta}(n,m), & {}_o^eT_{\alpha,\beta}^{\gamma,\delta}f(n) &:= \sum_{m=0}^{\infty} f(m)\,{}_o^eK_{\alpha,\beta}^{\gamma,\delta}(n,m),\\ {}_e^oT_{\alpha,\beta}^{\gamma,\delta}f(n) &:= \sum_{m=0}^{\infty} f(m)\,{}_e^oK_{\alpha,\beta}^{\gamma,\delta}(n,m), & {}_o^oT_{\alpha,\beta}^{\gamma,\delta}f(n) &:= \sum_{m=0}^{\infty} f(m)\,{}_o^oK_{\alpha,\beta}^{\gamma,\delta}(n,m). \end{aligned}$$

By means of them, we obtain that

$$T_{\alpha,\beta}^{\gamma,\delta}f(2n) = {}_e^eT_{\alpha,\beta}^{\gamma,\delta}\tilde{f}(n) + {}_o^eT_{\alpha,\beta}^{\gamma,\delta}\hat{f}(n) \quad \text{and} \quad T_{\alpha,\beta}^{\gamma,\delta}f(2n+1) = {}_e^oT_{\alpha,\beta}^{\gamma,\delta}\tilde{f}(n) + {}_o^oT_{\alpha,\beta}^{\gamma,\delta}\hat{f}(n)$$

where $\tilde{f}(n) = f(2n)$ and $\hat{f}(n) = f(2n+1)$, $n \in \mathbb{N}$. In addition, note that all the operators ${}_e^eT_{\alpha,\beta}^{\gamma,\delta}$, ${}_o^eT_{\alpha,\beta}^{\gamma,\delta}$, ${}_e^oT_{\alpha,\beta}^{\gamma,\delta}$, and ${}_o^oT_{\alpha,\beta}^{\gamma,\delta}$, are bounded in the space $\ell^2(\mathbb{N})$ because so it the transplantation one $T_{\alpha,\beta}^{\gamma,\delta}$. Indeed, let us define the functions

$$g(n) := \chi_{\mathcal{E}}(n)f(n/2) \qquad \text{and} \qquad h(n) := \chi_{\mathcal{O}}(n)f((n-1)/2),$$

with $\mathcal{E}$ and $\mathcal{O}$ denoting the sets of even and odd numbers respectively. Then, it is verified that

$$\begin{aligned} {}_e^eT_{\alpha,\beta}^{\gamma,\delta}f(n) &= T_{\alpha,\beta}^{\gamma,\delta}g(2n), & {}_o^eT_{\alpha,\beta}^{\gamma,\delta}f(n) &= T_{\alpha,\beta}^{\gamma,\delta}h(2n),\\ {}_e^oT_{\alpha,\beta}^{\gamma,\delta}f(n) &= T_{\alpha,\beta}^{\gamma,\delta}g(2n+1), & {}_o^oT_{\alpha,\beta}^{\gamma,\delta}f(n) &= T_{\alpha,\beta}^{\gamma,\delta}h(2n+1), \end{aligned}$$

so the boundedness in $\ell^2(\mathbb{N})$ of each operator follows immediately.

By the triangle inequality, it is easy to check that for the kernels ${}_e^eK_{\alpha,\beta}^{\gamma,\delta}(n,m)$, ${}_o^eK_{\alpha,\beta}^{\gamma,\delta}(n,m)$, ${}_e^oK_{\alpha,\beta}^{\gamma,\delta}(n,m)$, and ${}_o^oK_{\alpha,\beta}^{\gamma,\delta}(n,m)$, the properties (a') and (b') are implied by Proposition 4.1 and Proposition 4.2, so they are semi-local discrete kernels. Let us denote the sequence weights $w_e(n) := w(2n)$ and $w_o := w(2n+1)$, which belongs also to the $A_p(\mathbb{N})$ class because $w \in A_p(\mathbb{N})$. Then, by Theorem 2.2, we have

$$\begin{aligned} \|{}_e^eT_{\alpha,\beta}^{\gamma,\delta}\tilde{f}\|_{\ell^p(\mathbb{N},w_e)} &\lesssim \|\bar{f}\|_{\ell^p(\mathbb{N},w_e)}, & \|{}_o^eT_{\alpha,\beta}^{\gamma,\delta}\tilde{f}\|_{\ell^p(\mathbb{N},w_e)} &\lesssim \|\hat{f}\|_{\ell^p(\mathbb{N},w_e)},\\ \|{}_e^oT_{\alpha,\beta}^{\gamma,\delta}\tilde{f}\|_{\ell^p(\mathbb{N},w_o)} &\lesssim \|\tilde{f}\|_{\ell^p(\mathbb{N},w_o)}, & \|{}_e^oT_{\alpha,\beta}^{\gamma,\delta}\hat{f}\|_{\ell^p(\mathbb{N},w_o)} &\lesssim \|\hat{f}\|_{\ell^p(\mathbb{N},w_o)}, \end{aligned}$$

for $1 < p < \infty$, and the corresponding weak inequalities for $p = 1$.

To complete the proof, it is enough to observe that

$$\|\hat{f}\|_{\ell^p(\mathbb{N},w_e)} \lesssim \|\hat{f}\|_{\ell^p(\mathbb{N},w_o)} \lesssim \|f\|_{\ell^p(\mathbb{N},w)} \quad \text{and} \quad \|\tilde{f}\|_{\ell^p(\mathbb{N},w_o)} \lesssim \|\tilde{f}\|_{\ell^p(\mathbb{N},w_e)} \lesssim \|f\|_{\ell^p(\mathbb{N},w)}$$

by Lemma 2.1.

CHAPTER 5

Weighted transplantation for Laguerre coefficients

1. Introduction

The family $\{\mathcal{L}_n^\alpha\}_{n\geq 0}$, where $\mathcal{L}_n^\alpha$ is the Laguerre function associated with the Laguerre polynomial of degree n and order $\alpha > -1$, is a complete orthonormal system in the space $L^2(0,\infty)$. Given a function $F \in L^2(0,\infty)$, its Fourier-Laguerre coefficients are defined by

$$b_n^\alpha(F) = \int_0^\infty F(x)\mathcal{L}_n^\alpha(x)\,dx.$$

It turns out that under that assumption $b_n^\alpha(F)$ is a sequence in $\ell^2(\mathbb{N})$ and it is possible to recover the original function F by means of the discrete Fourier-Laguerre transform, which is given by

$$\mathcal{G}_\alpha f(x) = \sum_{n=0}^\infty f(x)\mathcal{L}_n^\alpha(x), \qquad f \in \ell^2(\mathbb{N}).$$

Then, taking a function $F \in L^2(0,\infty)$, we define the transplantation operator for Laguerre expansions by

$$\Upsilon_\alpha^\beta F(x) = \mathcal{G}_\alpha\left(b_{(\cdot)}^\beta(F)\right)(x),$$

where $\alpha,\beta > -1$. Naturally, it becomes the identity operator in the case $\alpha = \beta$. From now on we shall use the notation $\gamma = \min\{\alpha,\beta\}$.

Historically, the first transplantation result considering Laguerre expansions in terms of the functions $\mathcal{L}_n^\alpha$ is due to Y. Kanjin in [**26**].

Theorem 5.1 (Kanjin, 1991). *Let $\alpha,\beta > -1$ and $1 < p < \infty$. If $\gamma \geq 0$, then*

$$\|\Upsilon_\alpha^\beta F\|_{L^p(0,\infty)} \lesssim \|F\|_{L^p(0,\infty)}, \qquad F \in L^2 \cap L^p(0,\infty).$$

If $\gamma < 0$ the above inequality holds in the restricted range $2/(\gamma+2) < p < -2/\gamma$.

Later on, Kanjin's theorem was enhanced in a weighted setting by K. Stempak and W. Trebels in [**46**].

Theorem 5.2 (Stempak and Trebels, 1994). *Let $\alpha,\beta > -1$ and $1 < p < \infty$. Then*

$$\|\Upsilon_\alpha^\beta F\|_{L^p((0,\infty),x^a)} \lesssim \|F\|_{L^p((0,\infty),x^a)}, \qquad F \in L^2(0,\infty) \cap L^p((0,\infty),x^a),$$

where $-1/p < a < 1-1/p$ if $\gamma \geq 0$ and $-\gamma/2 - 1/p < a < 1 - 1/p + \gamma/2$ if $\gamma < 0$.

Finally, this result was refined in an optimal way by G. Garrigós, E. Harboure, T. Signes, J. L. Torrea, and B. Viviani in [**19**].

THEOREM 5.3 (Garrigós et al., 2007). *Let $\alpha,\beta > -1$ and $1 < p < \infty$. Then,*

$$\|\Upsilon_\alpha^\beta F\|_{L^p((0,\infty),x^a)} \lesssim \|F\|_{L^p((0,\infty),x^a)}, \qquad F \in L^2(0,\infty) \cap L^p((0,\infty),x^a),$$

if and only if $-\gamma/2 - 1/p < a < 1 - 1/p + \gamma/2$.

Other transplantations theorems for Laguerre expansions defined in terms of another different functions than $\mathcal{L}_n^\alpha$, such as the Laguerre functions of Hermite type, could be looked up in the monograph [**48**] by S. Thangavelu.

From the discrete point of view, the application $\mathcal{G}_\alpha$ is an isometry from the space $\ell^2(\mathbb{N})$ into $L^2(0,\infty)$. Furthermore, the Parseval's type identity

$$\int_0^\infty |\mathcal{G}_\alpha f(x)|^2\,dx = \sum_{n=0}^\infty |f(n)|^2$$

holds, as well as its generalization

$$\int_0^\infty \mathcal{G}_\alpha f_1(x)\mathcal{G}_\alpha f_2(x)\,dx = \sum_{n=0}^\infty f_1(n)f_2(n), \qquad f_1,f_2 \in \ell^2(\mathbb{N}). \tag{5.1}$$

The inverse of the $\mathcal{G}_\alpha$ is $b_n^\alpha(F)$, which implies that it is possible to recover the original sequence by means of the composition $b_n^\alpha(\mathcal{G}_\alpha f) = f$. In view of the above, we define the discrete transplantation operator for Laguerre expansions by

$$T_\alpha^\beta f(n) = b_n^\alpha(\mathcal{G}_\beta f), \qquad f \in \ell^2(\mathbb{N}),$$

for any $\alpha,\beta > -1$, which of course becomes the identity operator when $\alpha = \beta$.

This operator has been already studied in the special case $\beta = \alpha + 2$ by R. Askey in [**6**]. To be precise, he stated that the size of the coefficients $b_n^\alpha(F)$ and $b_n^{\alpha+2}(F)$, measured in the $\ell^p(\mathbb{N})$ norm, remain equivalent.

THEOREM 5.4 (Askey, 1967). *Let $F \in L^1(0,\infty)$. Then, there is a positive constant C independent of F such that, if $\alpha > 0$,*

$$\frac{1}{C}\|b_n^{\alpha+2}(F)\|_{\ell^p(\mathbb{N})} \le \|b_n^\alpha(F)\|_{\ell^p(\mathbb{N})} \le C\|b_n^{\alpha+2}(F)\|_{\ell^p(\mathbb{N})}$$

for $1 \le p < \infty$. If $-1 < \alpha < 0$, then the above holds in the restricted range $2/(\alpha+2) < p < -2/\alpha$.

Our aim in the present chapter is to prove the boundedness of the transplantation operator T_α^β with some weights and, as a corollary, improve Askey's result for a natural range of the parameters. Moreover, we give a characterization of such boundedness when power weights are considered, which is the dual theorem of the previously mentioned one by Garrigós et al.

In order to state our results we use the special case of condition (2.9):

$$[w]_{A_p^{\mathrm{loc}}} = \sup_{0\le m\le n\le 2(m+1)} \frac{1}{n-m+1}\left(\sum_{k=m}^n w(k)\right)^{1/p}\left(\sum_{k=m}^n w(k)^{-q/p}\right)^{1/q} < \infty. \tag{5.2}$$

The main theorem of this chapter is the following one.

THEOREM 5.5. *Let $\alpha, \beta > -1$ with $\alpha \neq \beta$, $1 < p < \infty$ and w a weight sequence that satisfies: $w(n) \simeq w(n+1)$ and condition* (2.13) *if $\beta = \alpha + 2k$ for some $k \in \mathbb{N}$; $w(n) \simeq w(n+1)$ and condition* (2.14) *if $\alpha = \beta + 2k$ for some $k \in \mathbb{N}$; and conditions* (2.13), (2.14), *and* (5.2) *if $|\alpha - \beta| \neq 2k$ for every $k \in \mathbb{N}$. Then*

$$\|T_\alpha^\beta f\|_{\ell^p(\mathbb{N},w)} \lesssim \|f\|_{\ell^p(\mathbb{N},w)}, \qquad f \in \ell^2(\mathbb{N}) \cap \ell^p(\mathbb{N}, w).$$

Consequently, the transplantation operator T_α^β extends uniquely to a bounded linear operator from $\ell^p(\mathbb{N}, w)$ into itself.

The reason to split in three different cases the hypotheses of the theorem according to $\alpha = \beta + 2k$, $\beta = \alpha + 2k$, and $|\beta - \alpha| \neq 2k$ is because in the first and second cases the transplantation operator is essentially equivalent to the discrete Hardy operator and its adjoint (vid. p. 18). This phenomenon is not strange and, for example, it is the same as the one occurring for the Hankel transform in the work [**34**] by A. Nowak and K. Stempak. On its behalf, in the last case $|\beta - \alpha| \neq 2k$, the transplantation operator is bounded again by the Hardy operator and is adjoint in the global part, whereas it is bounded by a Calderón-Zygmund operator in the local part.

It is worth to pointing out that the condition $w(m) \simeq w(m+1)$, which we only consider in the cases $|\beta-\alpha| = 2k$, is required to get the boundedness in Theorem 5.5 because an extra factor appears when we write the transplantation operator in terms of the Hardy operator and its adjoint. When $|\beta-\alpha| \neq 2k$ condition (5.2) is required for the weight sequence w to deduce the boundedness and, using Lemma 2.1 (note that its proof could be replicate for local $A_p(\mathbb{N})$ weights), it is possible to prove that (5.2) implies $w(m) \simeq w(m+1)$. Then, this condition does not appear explicitly in that case.

An immediate consequence of the main theorem is the following result.

COROLLARY 5.1. *Let $\alpha, \beta > -1$ with $\alpha \neq \beta$, $1 < p < \infty$, and w a weight sequence that satisfies conditions* (2.13), (2.14), *and* (5.2). *Then*

$$\|T_\alpha^\beta f\|_{\ell^p(\mathbb{N},w)} \simeq \|f\|_{\ell^p(\mathbb{N},w)}, \qquad f \in \ell^p(\mathbb{N}, w).$$

When power weights w_a are considered, the following full characterization of the boundedness of the transplantation operator is possible. Remind here that $\gamma = \min\{\alpha, \beta\}$.

THEOREM 5.6. *Let $\alpha, \beta > -1$ with $\alpha \neq \beta$, $1 < p < \infty$ and w_a a power weight sequence with $a \in \mathbb{R}$. Then,*

$$\|T_\alpha^\beta f\|_{\ell^p(\mathbb{N},w_a)} \lesssim \|f\|_{\ell^p(\mathbb{N},w_a)} \iff -\frac{\gamma}{2} < \frac{a+1}{p} < \frac{\gamma}{2} + 1.$$

Previous theorem gives the characterization when all possible values of the parameters α and β are considered. However, there are two special situations in which it is improved. Indeed, for $\alpha, \beta > -1$ and $k \in \mathbb{N} \setminus \{0\}$,

$$\|T_\alpha^{\alpha+2k} f\|_{\ell^p(\mathbb{N},w_a)} \lesssim \|f\|_{\ell^p(\mathbb{N},w_a)} \iff \frac{a+1}{p} < \frac{\alpha}{2} + 1. \tag{5.3}$$

and

$$\|T^{\beta}_{\beta+2k} f\|_{\ell^p(\mathbb{N},w_a)} \lesssim \|f\|_{\ell^p(\mathbb{N},w_a)} \iff -\frac{\beta}{2} < \frac{a+1}{p}. \tag{5.4}$$

Theorem 5.6 is actually a transplantation theorem with powers weights which extends Theorem 5.4 for functions $F \in L^1_\sigma(0,\infty)$, space defined as the set of measurable functions on the positive half-line such that the norm

$$\|F\|_{L^1_\sigma(0,\infty)} := \int_0^\infty |F(x)| x^\sigma \, dx$$

is finite. To see that, we define firstly the heat-diffusion semigroup associated with Laguerre functions by

$$W^\alpha_t F(x) := \sum_{n=0}^\infty e^{-\left(n+\frac{\alpha+1}{2}\right)t} b^\alpha_n(F) \mathcal{L}^\alpha_n(x), \qquad t > 0.$$

The convergence of this operator is well-known in the literature since the pioneer work [**43**] by K. Stempak. One of the latest result of this kind is due to A. Chicco-Ruiz and E. Harboure, who proved in [**14**] that

$$W^\alpha_t F(x) \longrightarrow F(x) \quad \text{a.e.}$$

for functions in $L^1_\sigma(0,\infty)$ provided $-\alpha/2 \le \sigma+1 \le \alpha/2+1$ if $\alpha \neq 0$ and $0 < \sigma+1 \le 1$ if $\alpha = 0$. Although details could be consulted in the adoresaid reference, it is worth to comment that the upper bounds of the hypotheses ensure that the semigroup is finite for each function in the space $L^1_\sigma(0,\infty)$, whereas the lower ones are necessary and sufficient to obtain the boundedness of the corresponding maximal operator from $L^1_\sigma(0,\infty)$ into $L^{1,\infty}_\sigma(0,\infty)$.

Using the procedure given by Askey and Wainger in [**9**, p. 394] in the setting of ultraspherical expansions, we deduce that

$$b^\alpha_m(F) = \lim_{t\to 0^+} \int_0^\infty W^\alpha_r F(x) \mathcal{L}^\beta_m(x) \, dx.$$

In this way, by the expression (5.5) (which we shall see further down the line),

$$b^\beta_m(F) = \lim_{t\to 0^+} \sum_{n=0}^\infty e^{(n+\frac{\alpha+1}{2})t} b^\alpha_n(F) K^\beta_\alpha(n,m) = T^\beta_\alpha f(m),$$

where we have taken $f(n) = b^\alpha_n(F)$. Previous argument proves the following corollary, for which we shall need to define the sets U and V given by

$$U = \{(\alpha,0) : \alpha \ge 0\} \cup \{(0,\beta) : \beta \ge 0\} \quad \text{and} \quad V = \{(\alpha,\beta) : \alpha,\beta > -1\} \setminus U.$$

Of course, the motivation to define the preceding sets is the aforementioned hypotheses for the convergence of the heat-diffusion semigroup W^α_t.

Corollary 5.2. *Let $\alpha,\beta > -1$ with $\alpha \neq \beta$, $1 < p < \infty$, w_a a power weight sequence with $a \in \mathbb{R}$, and $\sigma \in \mathbb{R}$ so that $-\gamma/2 \le \sigma+1 \le \gamma/2+1$ if $(\alpha,\beta) \in V$ and*

$0 < \delta + 1 \leq 1$ *if* $(\alpha, \beta) \in U$*. Then, there exists a positive constant* C *independent of* F *such that*

$$\frac{1}{C}\|b_n^\alpha(F)\|_{\ell^p(\mathbb{N},w_a)} \leq \|b_n^\beta(F)\|_{\ell^p(\mathbb{N},w_a)} \leq C\|b_n^\alpha(F)\|_{\ell^p(\mathbb{N},w_a)}, \qquad F \in L^1_\sigma(0,\infty),$$

if and only if

$$-\frac{\gamma}{2} < \frac{a+1}{p} < \frac{\gamma}{2} + 1.$$

On the other hand, it is possible to repeat the showed procedure for functions in $L^p_\sigma(0,\infty)$ having in mind the corresponding modifications in the convergence of the heat-disffusion semigroup, which are also treated in the same paper [**14**].

2. Preparatory results

In order to prove Theorem 5.5 we shall study separately the transplantation operator T_α^β according the global and local parts. In the global part, a fundamental tool to prove our results is the boundedness with weights of the discrete Hardy operator H and its adjoint H^* given in (2.16) and (2.17), p. 19. On its behalf, in the local part the proof relies on the discrete local Calderón-Zygmund theory exposed in Chap. 2, Sec. 3.

First of all we note that the transplantation operator T_α^β can be expressed for sequences $f \in c_{00}$ by the series

$$T_\alpha^\beta f(m) = \sum_{n=0}^{\infty} f(n) K_\alpha^\beta(n,m), \qquad K_\alpha^\beta(n,m) := \int_0^\infty \mathcal{L}_n^\alpha(x)\mathcal{L}_m^\beta(x)\,dx. \tag{5.5}$$

The function $K_\alpha^\beta(n,m)$ is the kernel of the transplantation operator, and it fulfils the trivial identity $K_\alpha^\beta(n,m) = K_\beta^\alpha(m,n)$. Moreover, the fundamental property

$$K_\alpha^\beta(n,m) = \sum_{k=0}^{\infty} K_\alpha^\delta(n,k) K_\delta^\beta(k,m),$$

which can be proved by means of the generalised Parseval's type identity (5.1), implies the decomposition

$$T_\alpha^\beta f(m) = T_\delta^\beta \circ T_\alpha^\delta f(m), \qquad f \in \ell^2(\mathbb{N}).$$

The proof of Theorem 5.5 will be based on particular cases of the transplantation operator. One of them will be Askey's case $\beta = \alpha + 2$, with $\alpha > -1$, for which the expression of the kernel is closed. Recall here that

$$\omega_n^\alpha = \sqrt{\frac{\Gamma(n+1)}{\Gamma(n+\alpha+1)}}.$$

Lemma 5.1. *Let* $n, m \in \mathbb{N}$ *and* $\alpha > -1$*. Then,*

$$K_\alpha^{\alpha+2}(n,m) = (\alpha+1)\frac{\omega_m^{\alpha+2}}{\omega_n^\alpha}, \qquad 0 \leq n \leq m,$$

$$K_\alpha^{\alpha+2}(m+1,m) = \left(\frac{m+1}{m+\alpha+2}\right)^{1/2},$$

and $K_\alpha^{\alpha+2}(n,m) = 0$ *for* $0 \le m \le n-1$.

Proof. For the case $\beta = \alpha+2$ the kernel is given by

$$K_\alpha^{\alpha+2}(n,m) = \omega_n^\alpha \omega_m^{\alpha+2} \int_0^\infty L_n^\alpha(x) L_m^{\alpha+2}(x) x^{\alpha+1} e^{-x}\, dx.$$

Then, one can try to express each Laguerre polynomial in terms of another Laguerre polynomial of order $\alpha+1$ and take advantage of the orthogonality. To this end, we use the connection formula (1.9) to obtain

$$L_n^\alpha(x) = \sum_{j=0}^{n} \frac{(-1)_{n-j}}{(n-j)!} L_j^{\alpha+1}(x) \quad \text{and} \quad L_m^{\alpha+2}(x) = \sum_{j=0}^{m} \frac{(-1)_{m-j}}{(m-j)!} L_j^{\alpha+1}(x).$$

Note that in these particular cases we have the well-known identities

$$L_n^\alpha(x) = L_n^{\alpha+1}(x) - L_{n-1}^{\alpha+1}(x), \qquad n \ge 1,$$

and

$$L_m^{\alpha+2}(x) = \sum_{j=0}^{m} L_j^{\alpha+1}(x).$$

Therefore, due to the orthogonality, the kernel is given by

$$K_\alpha^{\alpha+2}(n,m) = \omega_n^\alpha \omega_m^{\alpha+2} \sum_{j=0}^{m} \left(\delta_{j,n}(\omega_j^{\alpha+1})^{-2} - \delta_{j,n-1}(\omega_j^{\alpha+1})^{-2}\right) \qquad n \ge 1,$$

and, using that $L_0^\alpha(x) = L_0^{\alpha+1}(x) = 1$,

$$K_\alpha^{\alpha+2}(0,m) = (\alpha+1)\frac{\omega_m^{\alpha+2}}{\omega_0^{\alpha+1}}.$$

From these identities, the statement of the lemma is obtained in a straightforward way by checking the cases $m+1<n$, $n=m+1$, and $1 \le n \le m$. □

Except in the cases related to the Askey's one, we are no able to give a closed expression for the kernel. That is why we shall need to obtain estimates for the kernel in the remaining situations. First step is rewriting the kernel by a proper expression in which the difference of eigenvalues of the Laguerre functions appears explicitly.

Lemma 5.2. *Let* $\alpha, \beta > -1$ *such that* $\alpha+\beta > 0$ *and* $|\alpha-\beta| \ne 2k$ *for every* $k \in \mathbb{N}$, *and* $n, m \in \mathbb{N}$. *Then*

$$K_\alpha^\beta(n,m) = \frac{\beta^2-\alpha^2}{\nu_m^\beta - \nu_n^\alpha}\mathcal{K}_\alpha^\beta(n,m), \qquad \mathcal{K}_\alpha^\beta(n,m) := \int_0^\infty \mathcal{L}_n^\alpha(x)\mathcal{L}_m^\beta(x)\frac{dx}{x}.$$

Proof. The proof relies on the direct application of the integrating by parts formula over the kernel $K_\alpha^\beta(n,m)$, but it is more straightforward if the Laguerre functions of Hermite type $\varphi_n^\alpha(x)$ are considered. By means of them, the kernel can be rewritten as

$$K_\alpha^\beta(n,m) = \int_0^\infty \varphi_n^\alpha(x)\varphi_m^\beta(x)\,dx.$$

Recall that $\varphi_n^\alpha(x)$ are eigenfunctions of the operator $\bar{\mathfrak{L}}^\alpha$ whose corresponding eigenvalue is ν_n^α (vid. p. 7). Then, since $|\beta-\alpha| \neq 2k$ for every $k \in \mathbb{N}$ implies $\nu_n^\alpha \neq \nu_m^\beta$, the statement is obtained directly from the identity

$$\int_0^\infty \bar{\mathfrak{L}}^\alpha \varphi_n^\alpha(x)\varphi_m^\beta(x)\,dx = \int_0^\infty \varphi_n^\alpha(x)\bar{\mathfrak{L}}^\beta\varphi_m^\beta(x)\,dx + (\beta^2-\alpha^2)\int_0^\infty \varphi_n^\alpha(x)\varphi_m^\beta(x)\frac{dx}{x^2}.$$

Note that last term in the previous line appears naturally due to the difference

$$\bar{\mathfrak{L}}^\alpha - \bar{\mathfrak{L}}^\beta = \frac{\beta^2-\alpha^2}{x^2}$$

of the second order differential operators of different order. □

Throughout the present chapter we shall use frequently, without an explicit mention to it, the equivalence

$$|\nu_m^\beta - \nu_n^\alpha| \simeq \begin{cases} m+1, & 0 \le n < m_0, \\ |m-n|, & m_0 \le n \le m_0^*,\ n \neq m, \\ n+1, & m_0^* < n. \end{cases}$$

It holds for $\alpha, \beta > -1$ such that $\alpha < \beta < \alpha+2$ and its proof is elementary.

We need to estimate the function $\mathcal{K}_\alpha^\beta(n,m)$.

Lemma 5.3. *Let $n, m \in \mathbb{N}$ and $\alpha, \beta > -1$ such that $\alpha+\beta > 0$ and $\alpha < \beta < \alpha+2$. Then*

$$\mathcal{K}_\alpha^\beta(n,m) \lesssim \left(\frac{n+1}{m+1}\right)^{\alpha/2}, \qquad 0 \le n \le m, \tag{5.6}$$

and

$$\mathcal{K}_\alpha^\beta(n,m) \lesssim \left(\frac{m+1}{n+1}\right)^{\beta/2}, \qquad m \le n < \infty. \tag{5.7}$$

Moreover,

$$\mathcal{K}_\alpha^\beta(n,m) \lesssim 1, \qquad m_0 \le n \le m_0^*. \tag{5.8}$$

Proof. Firstly, we have to observe that $\mathcal{K}_\alpha^\beta(n,m)$ is a convergent integral due to the hypothesis $\alpha+\beta > 0$. In the spirit of the proof of Lemma 5.1, we use the connection formula (1.9) to obtain

$$L_n^\alpha(x) = \sum_{j=0}^n \frac{(\alpha/2-\beta/2+1)_{n-j}}{(n-j)!} L_j^{\alpha/2+\beta/2+1}(x)$$

and, similarly,

$$L_m^\beta(x) = \sum_{j=0}^m \frac{(\beta/2-\alpha/2+1)_{m-j}}{(m-j)!} L_j^{\alpha/2+\beta/2+1}(x).$$

Therefore, due to the orthogonality

$$\mathcal{K}_\alpha^\beta(n,m) = \sum_{j=0}^{\min\{n,m\}} \frac{(\alpha/2-\beta/2+1)_{n-j}}{(n-j)!} \frac{(\beta/2-\alpha/2+1)_{m-j}}{(m-j)!} \frac{\omega_n^\alpha \omega_m^\beta}{(\omega_j^{\alpha/2+\beta/2-1})^2}$$

(note that by this identity we can deduce the positivity of $\mathcal{K}_\alpha^\beta(n,m)$ directly). Then, we have to estimate the previous sum. We can put the Pochhammer symbols in terms of Gamma functions and, by the well-known equivalence [**36**, Eq. 5.11.12]

$$\frac{\Gamma(x+a)}{\Gamma(x+b)} \simeq x^{a-b}, \qquad x \to \infty,$$

we get the estimates

$$\frac{(\alpha/2-\beta/2+1)_{n-j}}{(n-j)!} = \frac{\Gamma(n-j+\alpha/2-\beta/2+1)}{\Gamma(n-j+1)\Gamma(\alpha/2-\beta/2+1)} \lesssim (n+1-j)^{\alpha/2+\beta/2}$$

and the analogous for

$$\frac{(\beta/2-\alpha/2+1)_{m-j}}{(m-j)!} = \frac{\Gamma(m-j+\beta/2-\alpha/2+1)}{\Gamma(m-j+1)\Gamma(\beta/2-\alpha/2+1)} \lesssim (m+1-j)^{\beta/2-\alpha/2}.$$

So, having in mind the equivalence $\omega_n^\alpha \simeq (n+1)^{-\alpha/2}$, we obtain that

$$\mathcal{K}_\alpha^\beta(n,m) \lesssim (n+1)^{-\alpha/2}(m+1)^{-\beta/2} R_\alpha^\beta(n,m), \tag{5.9}$$

with

$$R_\alpha^\beta(n,m) = \sum_{j=0}^{\min\{n,m\}} (n+1-j)^{\alpha/2-\beta/2}(m+1-j)^{\beta/2-\alpha/2}(j+1)^{\alpha/2+\beta/2-1}.$$

For $0 \le n \le m$, we have

$$\begin{aligned} R_\alpha^\beta(n,m) &\lesssim (m+1)^{\beta/2-\alpha/2} \sum_{j=0}^{n} (n+1-j)^{\alpha/2-\beta/2}(j+1)^{\alpha/2+\beta/2-1} \\ &\simeq (m+1)^{\beta/2-\alpha/2}(n+1)^{\alpha}. \end{aligned}$$

Then, by the above bound (5.9),

$$\mathcal{K}_\alpha^\beta(n,m) \lesssim \left(\frac{n+1}{m+1}\right)^{\alpha/2}, \qquad 0 \le n \le m,$$

and the proof of (5.6) is completed.

To prove (5.7), we estimate $R_\alpha^\beta(n,m)$ distinguishing the cases $m \le n \le m_0^*$ and $m_0^* < n$. When $m \le n \le m_0^*$ we have

$$\begin{aligned} R_\alpha^\beta(n,m) &\lesssim (m+1)^{\beta/2-\alpha/2}(n+1)^{\alpha/2-\beta/2}\sum_{j=0}^{[m/2]}(j+1)^{\alpha/2+\beta/2-1} \\ &\qquad + (m+1)^{\beta-1}\sum_{j=[m/2]+1}^{m}(n+1-j)^{\alpha/2-\beta/2} \\ &\lesssim (m+1)^{\beta}(n+1)^{\alpha/2-\beta/2} + (m+1)^{\beta-1}\sum_{j=0}^{n}(n+1-j)^{\alpha/2-\beta/2} \\ &\lesssim (m+1)^{\beta}(n+1)^{\alpha/2-\beta/2} + (m+1)^{\beta-1}(n+1)^{\alpha/2-\beta/2+1} \\ &\lesssim (m+1)^{\beta}(n+1)^{\alpha/2-\beta/2}, \end{aligned}$$

where in the last step we have used that $n \simeq m$ in this case. Now, for $m_0^* < n$ we have

$$\begin{aligned} R_\alpha^\beta(n,m) &\lesssim (m+1)^{\beta/2-\alpha/2}(n+1-m)^{\alpha/2-\beta/2}\sum_{j=0}^{m}(j+1)^{\alpha/2+\beta/2-1} \\ &\lesssim (m+1)^{\beta}(n+1)^{\alpha/2-\beta/2}. \end{aligned}$$

Then, using (5.9),

$$\mathcal{K}_\alpha^\beta(n,m) \lesssim \left(\frac{m+1}{n+1}\right)^{\beta/2}$$

and we finish the proof of (5.7).

Obviously, the estimate (5.8) is an immediate consequence of the previous ones (5.6) and (5.7), and the restriction $m_0 \le n \le m_0^*$. □

To end this section, we give here an extra lemma which we shall use in the next one.

Lemma 5.4. *Let $n, m \in \mathbb{N}$ and $\alpha, \beta > -1$ such that $\alpha+\beta > 0$ and $\alpha < \beta < \alpha+2$. Then*

$$\int_0^\infty \mathcal{L}_n^{\alpha+1}(x)\mathcal{L}_m^\beta(x)\frac{dx}{x^{1/2}} \lesssim \frac{(n+1)^{1/2}}{m+1-n}\left(\frac{n+1}{m+1}\right)^{\alpha/2}, \qquad 0 \le n \le m,$$

and

$$\int_0^\infty \mathcal{L}_n^{\alpha+1}(x)\mathcal{L}_m^\beta(x)\frac{dx}{x^{1/2}} \lesssim \frac{(n+1)^{1/2}}{n+1-m}\left(\frac{m+1}{n+1}\right)^{\beta/2}, \qquad 0 \le m \le n.$$

Proof. Let us denote

$$\bar{\mathcal{K}}_\alpha^\beta(n,m) = \omega_n^{\alpha+1}\omega_m^\beta\int_0^\infty L_n^{\alpha+1}(x)L_m^\beta(x)x^{\alpha/2+\beta/2}e^{-x}\,dx.$$

Proceeding with the connection formula (1.9) in the same way as in the proof of Lemma 5.3 we obtain the estimate

$$\bar{\mathcal{K}}_\alpha^\beta(n,m) \lesssim (n+1)^{-\alpha/2-1/2}(m+1)^{-\beta/2}\bar{R}_\alpha^\beta(n,m),$$

where

$$\bar{R}_\alpha^\beta(n,m) = \sum_{j=0}^{\min\{n,m\}} (n+1-j)^{\alpha/2-\beta/2}(m+1-j)^{\beta/2-\alpha/2-1}(j+1)^{\alpha/2+\beta/2}.$$

In case that $n \le m$ we have (see proof of Lemma 5.3)

$$\begin{aligned}\bar{R}_\alpha^\beta(n,m) &\lesssim \frac{n+1}{m+1-n}\sum_{j=0}^{n}(n+1-j)^{\alpha/2-\beta/2}(m+1-j)^{\beta/2-\alpha/2}(j+1)^{\alpha/2+\beta/2-1} \\ &\lesssim \frac{(n+1)^{\alpha+1}(m+1)^{\beta/2-\alpha/2}}{m+1-n},\end{aligned}$$

whereas if $m \le n$,

$$\begin{aligned}\bar{R}_\alpha^\beta(n,m) &\lesssim (n+1-m)^{\alpha/2-\beta/2}\sum_{j=0}^{m}(m+1-j)^{\beta/2-\alpha/2-1}(j+1)^{\alpha/2+\beta/2} \\ &\lesssim (n+1-m)^{\alpha/2-\beta/2}(m+1)^{\alpha/2+\beta/2}\sum_{j=0}^{m}(m+1-j)^{\beta/2-\alpha/2-1} \\ &\lesssim (n+1-m)^{\alpha/2-\beta/2}(m+1)^{\beta} \lesssim \frac{(n+1)^{\alpha/2-\beta/2+1}(m+1)^{\beta}}{n+1-m}.\end{aligned}$$

From these two estimates, the lemma follows immediately. □

3. Calderón-Zygmund properties for the kernel

Let us prove in this section that the kernel $K_\alpha^\beta(n,m)$ fulfils the estimates (a') and (b').

PROPOSITION 5.1. *Let $n,m \in \mathbb{N}$ such that $n \neq m$ and $m_0 \le n \le m_0^*$; and let $\alpha,\beta > -1$ so that $\alpha+\beta > 0$ and $\alpha < \beta < \alpha+2$. Then*

$$|K_\alpha^\beta(n,m)| \lesssim \frac{1}{|n-m|}.$$

PROOF. The proof is immediate by means of Lemma 5.2 and the estimate (5.8) given in Lemma 5.3. □

PROPOSITION 5.2. *Let $n,m \in \mathbb{N}$ such that $n \neq m$ and $m_0 \le n \le m_0^*$; and let $\alpha,\beta > -1$ so that $\alpha+\beta > 0$ and $\alpha < \beta < \alpha+2$. Then*

$$|K_\alpha^\beta(n,m) - K_\alpha^\beta(n+1,m)| \lesssim \frac{1}{|n-m|^2} \tag{5.10}$$

and

$$|K_\alpha^\beta(n,m) - K_\alpha^\beta(n,m+1)| \lesssim \frac{1}{|n-m|^2}. \tag{5.11}$$

PROOF. We focus on the proof for the bound (5.10). The one corresponding to (5.11) could be deduced in a similar way.

The key point in the proof is an appropriate decomposition of the difference of the involved kernels. By Lemma 5.2, we rewrite that difference by the expression

$$|K_\alpha^\beta(n,m) - K_\alpha^\beta(n+1,m)| = \frac{\beta^2-\alpha^2}{|\nu_m^\beta - \nu_n^\alpha||\nu_m^\beta - \nu_{n+1}^\alpha|}\,|J_1(n,m) + J_2(n,m)|\,,$$

where we have denoted

$$J_1(n,m) := (\nu_n^\alpha - \nu_{n+1}^\alpha)\mathcal{K}_\alpha^\beta(n,m)$$

and

$$J_2(n,m) := (\nu_m^\beta - \nu_n^\alpha)\int_0^\infty \left(\mathcal{L}_n^\alpha(x) - \mathcal{L}_{n+1}^\alpha(x)\right)\mathcal{L}_m^\beta(x)\frac{dx}{x}.$$

Since $|\nu_m^\beta - \nu_n^\alpha| \simeq |\nu_m^\beta - \nu_{n+1}^\alpha| \simeq |n-m|$, provided $n \neq m$ and $n \neq m-1$, we have to check the bound

$$|J_1(n,m) + J_2(n,m)| \lesssim 1 \tag{5.12}$$

for $m_0 \le n \le m-2$ with $m \ge 4$, and $m+1 \le n \le m_0^*$ with $m \ge 2$.

In the special case $n = m-1$ we can obtain condition (5.10) by showing that

$$|K_\alpha^\beta(n,n+1) - K_\alpha^\beta(n+1,n+1)| \lesssim 1,$$

but this is immediate by the Cauchy-Schwarz's inequality and the orthonormality of the Laguerre functions.

Let us prove (5.12). On the one hand, it is easy to obtain the bound

$$|J_1(n,m)| \lesssim \mathcal{K}_\alpha^\beta(n,m) \lesssim 1$$

using the estimate (5.8). However, J_2 is more difficult to deal with and we have to split it in two parts, namely

$$J_2(n,m) = I_1(n,m) + I_2(n,m),$$

with

$$I_1(n,m) := (\nu_m^\beta - \nu_n^\alpha)\omega_n^{\alpha+1}\sqrt{n+\alpha+1}\int_0^\infty \left(L_n^\alpha(x) - L_{n+1}^\alpha(x)\right)\mathcal{L}_m^\beta(x)x^{\alpha/2-1}e^{-x/2}\,dx$$

and

$$\begin{aligned} I_2(n,m) &:= \frac{\nu_m^\beta - \nu_n^\alpha}{\sqrt{n+1}}\left(\sqrt{n+\alpha+1} - \sqrt{n+1}\right)\mathcal{K}_\alpha^\beta(n,m) \\ &= \frac{\alpha(\nu_m^\beta - \nu_n^\alpha)}{\sqrt{n+1}(\sqrt{n+\alpha+1} - \sqrt{n+1})}\mathcal{K}_\alpha^\beta(n,m). \end{aligned}$$

Last expression can be bounded using the estimates in Lemma 5.3. Indeed,

$$|I_2(n,m)| \lesssim \frac{|n-m|}{m+1} \lesssim 1.$$

Regarding $I_1(n,m)$, we use the identity (1.10). By means of it, we obtain the decomposition

$$|I_1(n,m)| = |I_{1,1}(n,m) - I_{1,2}(n,m)|,$$

where

$$I_{1,1}(n,m) := (\nu_m^\beta - \nu_n^\alpha)\frac{\sqrt{n+\alpha+1}}{n+1}\int_0^\infty \mathcal{L}_n^{\alpha+1}(x)\mathcal{L}_m^\beta(x)\frac{dx}{x^{1/2}}$$

and

$$I_{1,2}(n,m) := (\nu_m^\beta - \nu_n^\alpha)\frac{\alpha}{n+1}\mathcal{K}_\alpha^\beta(n,m).$$

One more time, by the estimate (5.8), we get $|I_{1,2}(n,m)| \lesssim 1$.

Finally, by Lemma 5.4, it is easy to obtain

$$|I_{1,1}(n,m)| \lesssim \begin{cases} \dfrac{m-n}{m+1-n}\left(\dfrac{n+1}{m+1}\right)^{\alpha/2}, & m_0 \le n \le m-2, \\ \dfrac{n-m}{n+1-m}\left(\dfrac{m+1}{n+1}\right)^{\beta/2}, & m+1 \le n \le m_0^*, \end{cases}$$

so $|I_{1,1}(n,m)| \lesssim 1$, which concludes the proof. □

The estimates (5.10) and (5.11) ensure the regularity properties (b1') and (b2') respectively. Let us see for instance that (5.10) implies (b1') (the proof for (5.11) implies (b2') is analogous). Let us suppose that $n < \ell$, so, by the triangle inequality,

$$|K_\alpha^\beta(n,m) - K_\alpha^\beta(\ell,m)| \lesssim \sum_{j=0}^{\ell-n-1} |K_\alpha^\beta(n+j,m) - K_\alpha^\beta(n+1+j,m)|.$$

If $n > m$ we apply (5.10) to get the desired estimate. When $n < m$ we apply again (5.10) and, then, use the fact $|n-m| > 2|n-\ell|$, so the result follows. The case $\ell < n$ is similar and we omit the detalis.

Therefore the kernel $K_\alpha^\beta(n,m)$ is a local standard kernel under the hypotheses of Proposition 5.1 and Proposition 5.2.

4. Boundedness of some particular cases

As it has been previously mentioned, the proof of Theorem 5.5 is based on the boundedness of the transplantation operator in several special situations.

Let us start with Askey's case. Lemma 5.1 actually shows how the transplantation operator $T_\alpha^{\alpha+2}$ is decomposed in the difference

$$T_\alpha^{\alpha+2}f(m) = (\alpha+1)\omega_m^{\alpha+2}\sum_{n=0}^{m}\frac{f(y)}{\omega_n^\alpha} - \left(\frac{m+1}{m+\alpha+2}\right)^{1/2} f(m+1). \tag{5.13}$$

Last identity, in conjunction with the characterization (2.18) of the discrete Hardy operator, is the crucial point to prove the following proposition.

Proposition 5.3. *Let $\alpha > -1$, $k \in \mathbb{N}$, and $1 < p < \infty$. Let w a weight sequence that satisfies $w(m) \simeq w(m+1)$. Then, the weighted inequality*

$$\|T_\alpha^{\alpha+2k}f\|_{\ell^p(\mathbb{N},w)} \lesssim \|f\|_{\ell^p(\mathbb{N},w)}, \qquad k \ge 1, \tag{5.14}$$

holds if and only if the weight w satisfies condition (2.13).

PROOF. First of all, we note that the condition $w(m) \simeq w(m+1)$ implies the equivalence

$$\left(\sum_{m=0}^{\infty} |f(m+1)|^p w(m) \right)^{1/p} \simeq \|f\|_{\ell^p(\mathbb{N},w)}.$$

Then, from the expression (5.13) and using the equivalence $\omega_n^\alpha \simeq (n+1)^{-\alpha/2}$, it is clear that the boundedness (5.14) for $k=1$ is equivalent to

$$\|w_{-\alpha/2} H(w_{\alpha/2} f)\|_{\ell^p(\mathbb{N},w)} \lesssim \|f\|_{\ell^p(\mathbb{N},w)}.$$

So the result follows from the characterization (2.18) of the Hardy operator.

Now, to finish the proof we only have to prove that (5.14) also holds for $k>1$ when the weight w satisfies condition (2.13). In this situation, the transplantation operator can be written by the composition

$$T_\alpha^{\alpha+2k} = T_{\alpha+2(k-1)}^{\alpha+2k} \circ \cdots \circ T_\alpha^{\alpha+2}.$$

Since condition (2.13) holds by hypothesis, it is also verified that the constants $[w]_{H_p}^{\alpha+2\ell}$, where $\ell = 1, \ldots, k-1$, are finite (remind the inequality $[w]_{H_p}^{\alpha+\delta} \leq [w]_{H_p}^{\alpha}$ is satisfied for any $\delta \geq 0$). Then, every operator $T_{\alpha+2\ell}^{\alpha+2(\ell+1)}$ is bounded with the weight sequence w. □

The transplantation operator $T_{\beta+2}^{\beta}$ is closely related to the previous one. Since $K_{\beta+2}^{\beta}(n,m) = K_\beta^{\beta+2}(m,n)$, by Lemma 5.1 we have

$$T_{\beta+2}^{\beta} f(m) = -\left(\frac{m}{m+\beta+1} \right)^{1/2} f(m-1) + \frac{\beta+1}{\omega_m^\beta} \sum_{n=m}^{\infty} \omega_n^{\beta+2} f(n).$$

PROPOSITION 5.4. *Let $\beta > -1$, $k \in \mathbb{N}$, and $1 < p < \infty$. Let w a weight sequence that satisfies $w(m) \simeq w(m+1)$. Then, the weighted inequality*

$$\|T_{\beta+2}^{\beta} f\|_{\ell^p(\mathbb{N},w)} \lesssim \|f\|_{\ell^p(\mathbb{N},w)}, \qquad k \geq 1, \tag{5.15}$$

holds if and only if the weight w satisfies condition (2.14).

The proof is analogue to the one given for Proposition 5.3 but using the characterization (2.19) for the adjoint H^* of the Hardy operator, so we omit it.

Note that the characterization (5.3) and (5.4) are consequences of Proposition 5.3 and Proposition 5.4, together with the characterization for the power weight given in (2.15).

Next particular case in which we shall base the proof of the main theorem corresponds with the restrictions $\alpha < \beta < \alpha + 2$ and $\alpha + \beta > 0$, with $\alpha, \beta > -1$.

PROPOSITION 5.5. *Let $\alpha, \beta > -1$ such that $\alpha + \beta > 0$ and $\alpha < \beta < \alpha + 2$, and $1 < p < \infty$. If w is a weight sequence that satisfies conditions* (2.13), (2.14), *and* (5.2), *then*

$$\|T_\alpha^\beta f\|_{\ell^p(\mathbb{N},w)} \lesssim \|f\|_{\ell^p(\mathbb{N},w)}.$$

Proof. From Lemma 5.2 it is clear that

$$T_\alpha^\beta f(m) = (\beta^2 - \alpha^2) \sum_{\substack{n\in\mathbb{N}\\ n\neq m}} \frac{f(n)}{\nu_m^\beta - \nu_n^\alpha} \mathcal{K}_\alpha^\beta(n,m) + f(m) K_\alpha^\beta(m,m).$$

Splitting the transplantation operator in four different operators according to the global and local regions we obtain

$$T_\alpha^\beta f(m) = O_\alpha^\beta f(m) + Q_\alpha^\beta f(m) + (O^*)_\alpha^\beta f(m) + N_\alpha^\beta f(m),$$

where last operators are given explicitly by

$$O_\alpha^\beta f(m) = (\beta^2 - \alpha^2) \sum_{\substack{n\in\mathbb{N}\\ n< m_0}} \frac{f(n)}{\nu_m^\beta - \nu_n^\alpha} \mathcal{K}_\alpha^\beta(n,m),$$

$$(O^*)_\alpha^\beta f(m) = (\beta^2 - \alpha^2) \sum_{\substack{n\in\mathbb{N}\\ m_0^*< n}} \frac{f(n)}{\nu_m^\beta - \nu_n^\alpha} \mathcal{K}_\alpha^\beta(n,m)$$

in the global part,

$$Q_\alpha^\beta f(m) = (\beta^2 - \alpha^2) \sum_{\substack{n\in\mathbb{N},\ n\neq m\\ m_0\le n\le m_0^*}} \frac{f(n)}{\nu_m^\beta - \nu_n^\alpha} \mathcal{K}_\alpha^\beta(n,m),$$

in the local part, and $N_\alpha^\beta f(m) = f(m) K_\alpha^\beta(n,m)$.

Applying the Cauchy-Schwarz's inequality, it is immediate that $|K_\alpha^\beta(m,m)| \le 1$ and then

$$\|N_\alpha^\beta f\|_{\ell^p(\mathbb{N},w)} \lesssim \|f\|_{\ell^p(\mathbb{N},w)}, \qquad 1<p<\infty, \tag{5.16}$$

holds. Regarding the operator O_α^β, by (5.6), it is clear the estimate

$$|O_\alpha^\beta f(m)| \lesssim w_{-(\alpha/2+1)}(m) \sum_{\substack{n\in\mathbb{N}\\ n<m_0}} w_{\alpha/2}(n)|f|(n) \lesssim w_{-\alpha/2}(m) H(w_{\alpha/2}|f|)(m).$$

So, O_α^β is bounded by a Hardy operator and then by (2.16), the weighted norm inequality

$$\|O_\alpha^\beta f\|_{\ell^p(\mathbb{N},w)} \lesssim \|f\|_{\ell^p(\mathbb{N},w)}, \qquad 1<p<\infty, \tag{5.17}$$

holds provided w satisfies condition (2.13). On its behalf, the treatment of the operator $(O^*)_\alpha^\beta$ is analogous. Indeed, by using now (5.7), we have

$$|(O^*)_\alpha^\beta f(m)| \lesssim w_{\beta/2}(m) H(w_{-\beta/2}|f|)(m).$$

Then, $(O^*)_\alpha^\beta$ is bounded by the adjoint of the Hardy operator and, by (2.19), the weighted norm inequality

$$\|(O^*)_\alpha^\beta f\|_{\ell^p(\mathbb{N},w)} \lesssim \|f\|_{\ell^p(\mathbb{N},w)}, \qquad 1<p<\infty, \tag{5.18}$$

holds provided w satisfies condition (2.14).

To study the local part Q_α^β we use the local Calderón-Zygmund theory. Note that the transplantation operator T_α^β is obviously a bounded operator from $\ell^2(\mathbb{N})$

into itself. By taking $p = 2$ and $w(n) = 1$ for all $n \in \mathbb{N}$ in the previous inequalities (5.16), (5.17) and (5.18), it follows that both global operators O_α^β and $(O^*)_\alpha^\beta$, as well as N_α^β, are bounded in the space $\ell^2(\mathbb{N})$. Since

$$Q_\alpha^\beta f(m) = T_\alpha^\beta f(m) - O_\alpha^\beta f(m) - (O^*)_\alpha^\beta f(m) - N_\alpha^\beta f(m),$$

the operator Q_α^β is also bounded in $\ell^2(\mathbb{N})$.

Therefore, as a consequence of Proposition 5.4 and Proposition 5.5, the kernel $K_\alpha^\beta(n,m)$ is a local standard kernel and the operator Q_α^β is a local Calderón-Zygmund operator. So, by Theorem 2.4, the weighted norm inequality

$$\|Q_\alpha^\beta f\|_{\ell^p(\mathbb{N},w)} \lesssim \|f\|_{\ell^p(\mathbb{N},w)}$$

holds for $1 < p < \infty$, provided w satisfies condition (5.2). This fact finishes the proof. ☐

From previous proposition we can deduce the boundedness of the transplantation operator for $\beta < \alpha < \beta + 2$ and $\alpha + \beta > 0$, with $\alpha, \beta > -1$.

Proposition 5.6. *Let $\alpha, \beta > -1$ such that $\alpha + \beta > 0$ and $\beta < \alpha < \beta + 2$, and $1 < p < \infty$. If w is a weight sequence that satisfies conditions* (2.13)*,* (2.14)*, and* (5.2)*, then*

$$\|T_\alpha^\beta f\|_{\ell^p(\mathbb{N},w)} \lesssim \|f\|_{\ell^p(\mathbb{N},w)}.$$

Proof. This result follows from Proposition 5.5 by a duality argument. Indeed,

$$\begin{aligned}
\|T_\alpha^\beta f\|_{\ell^p(\mathbb{N},w)} &= \sup_{\substack{g\in\ell^q(\mathbb{N},w)\\ \|g\|_{\ell^q(\mathbb{N},w)=1}}} \sum_{m=0}^{\infty} g(m) T_\alpha^\beta f(m) w(m) \\
&= \sup_{\substack{g\in\ell^q(\mathbb{N},w)\\ \|g\|_{\ell^q(\mathbb{N},w)=1}}} \sum_{m=0}^{\infty} T_\beta^\alpha(wg)(m) f(m) \\
&= \|f\|_{\ell^p(\mathbb{N},w)} \sup_{\substack{g\in\ell^q(\mathbb{N},w)\\ \|g\|_{\ell^q(\mathbb{N},w)=1}}} \|T_\beta^\alpha(wg)\|_{\ell^q(\mathbb{N},w^{-q/p})}.
\end{aligned}$$

Now, using the identities $[w^{-q/p}]_{H_q}^\beta = [w]_{H_p^*}^\beta$, $[w^{-q/p}]_{H_q}^\alpha = [w]_{H_p^*}^\alpha$, and $[w^{-q/p}]_{A_q^{\mathrm{loc}}} = [w]_{A_p^{\mathrm{loc}}}$, and the conditions (2.13), (2.14), and (5.2), by Proposition 5.5 we have

$$\|T_\beta^\alpha(wg)\|_{\ell^q(\mathbb{N},w)} \lesssim \|wg\|_{\ell^q(\mathbb{N},w^{-q/p})} = \|g\|_{\ell^q(\mathbb{N},w)},$$

which finishes the proof. ☐

5. Proof of the main theorem and the characterization

Let us prove Theorem 5.5.

Proof of Theorem 5.5. We only need to prove the cases not covered by Proposition 5.3, Proposition 5.4, Proposition 5.5, and Proposition 5.6.

(i) Case $\alpha > -1$ and $\alpha + 2k < \beta < \alpha + 2(k+1)$ for $k \in \{1, 2, \dots\}$. In this situation the transplantation operator can be written as the composition

$$T_\alpha^\beta = T_{\alpha+2k}^\beta \circ T_\alpha^{\alpha+2k}.$$

Then, in this case the result follows from Proposition 5.5 and Proposition 5.3.

(ii) Case $\beta > -1$ and $\beta + 2k < \alpha < \beta + 2(k+1)$ for $k \in \{1, 2, \dots\}$. This case can be deduced from (i) by a duality argument as we did to prove Proposition 5.6 from Proposition 5.5, so we omit the details.

(iii) Case $-1 < \alpha \leq 0$, $-1 < \beta \leq 1$, $\alpha + \beta \leq 0$, and $\alpha < \beta$. Here we put the transplantation operator as the composition

$$T_\alpha^\beta = T_{\alpha+2}^\beta \circ T_\alpha^{\alpha+2}.$$

The operators involved in this case can be controlled by applying Proposition 5.6 and Proposition 5.3.

(iv) Case $-1 < \alpha \leq 1$, $-1 < \beta \leq 0$, $\alpha + \beta \leq 0$, and $\beta < \alpha$. In this last case, we can use duality and, again, we omit the details. ☐

To finish the present chapter, we give here the proof of Theorem 5.6. Before that, we note that clearly $w_a(m) \simeq w_a(m+1)$ for any $a \in \mathbb{R}$.

Proof of Theorem 5.6. Under the hypothesis $-\gamma/2 < (a+1)/p < \gamma/2 + 1$, the weight w_a satisfies conditions (2.13), (2.14), and (5.2). Then, the boundedness of the transplantation operator with the weight w_a is an immediate consequence of Theorem 5.5.

The necessity of the condition $-\gamma/2 < (a+1)/p < \gamma/2 + 1$ is a consequence of Proposition 5.3 and Proposition 5.3 with power weights w_a. ☐

Conclusions and further work

The work furnished in the present dissertation was done with the intention of researching the discrete analogues of the mean convergence problem of the partial sum operator and the problem of the boundedness of some transplantation operators, where both operators are associated with classical ortogonal polynomials.

In Chapter 3 we have given a sufficiency result to ensure the norm convergence with weights of the multiplier of an interval related to Jacobi polynomials, with $\alpha, \beta \geq -1/2$. There, the considered weights belong to the discrete Muckenhoupt class. We have also characterized the case without weights and thus the convergence for such operator. Moreover, as a consequence of the latter, we have obtained its a.e. convergence.

There is an open line of research to continue in that direction, such as the study of the norm convergence problem for the multiplier of the interval related to Laguerre and Hermite polynomials. In that way, we have already proved some partial results by means of similar estimates to the used ones in the case of Jacobi polynomials. To be concrete, we have obtained sufficient conditions to ensure that convergence, and where we have also considered its boundedness with weights in the discrete Muckenhoupt class. Nowadays we are working on the proof to show that the sufficient conditions are actually necessary when weights are not considered.

On its behalf, Chapters 4 and 5 were devoted to the problem of transplantation associated to functions of classical orthogonal polinomials. In the former we have obtained the boundedness of the transplantation operator related to Jacobi functions, with $\alpha, \beta \geq 1/2$, where weights in the discrete Muckenhoupt class were included. Furthermore, we have also get weak (1,1) estimates with weights for such operator. In the latter we have considered the transplantation operator related to Laguerre functions, and its boundedness with weights in the discrete Muckenhoupt class was obtained. We also proved a characterization of the boundedness when power weights are addressed.

It would be desirable to enhance our results regarding transplantation operators associated with Jacobi functions in the complete range of the order, i.e., for $\alpha, \beta > -1$. The main difficult to achieve this task is that the uniform estimates of the Jacobi functions depend on the degree of its associated polynomials in such situation. On the other hand, the way used in the case of Laguerre functions does not seem appropriate at all due to the expression of the connection formula for Jacobi polynomials.

In addition to the above-mentioned, in our future work we also consider the study of transplantation operators respect to the Bessel system $\{j_n^\alpha(x)\}_{n\geq 1}$, where

$$j_n^\alpha(x) = t_n^\alpha \sqrt{x} J_\alpha(\sigma_n^\alpha x), \qquad x \in (0,1).$$

Here $J_\alpha(x)$ denotes the Bessel function of the first kind of order $\alpha > -1$, t_n^α is a constant which provides the orthonormalization in the space $L^2(0,1)$, and $\{\sigma_n^\alpha\}_{n\geq 1}$ are de zeros of the function J_α. Our aim is to enhance the work [**44**] by K. Stempak, where the case $\alpha \geq -1/2$ is studied, in two folds: including general weights and consider the maximum range for the value of the order α.

Conclusiones y trabajo futuro

El trabajo desarrollado en la presente memoria se realizó con la intención de investigar los análogos discretos correspondientes al problema de la convergencia en media del operador suma parcial y el problema de la acotación en norma de ciertos operadores de transplantación, en ambos casos definidos respecto a polinomios ortogonales clásicos.

En el Capítulo 3 hemos dado un resultado de suficiencia que asegura la convergencia en norma con pesos del multiplicador del intervalo asociado a polinomios de Jacobi, con $\alpha, \beta \geq -1/2$. Los pesos que fueron considerados en ese caso pertenecían a la clase de Muckenhoupt discreta. También hemos caracterizado el caso en el que no se incluyen pesos y la convergencia para dicho operador. Además, como consecuencia de esto último, hemos obtenido su convergencia en casi todo punto.

Existe una puerta abierta para continuar por esta línea de investigación, como por ejemplo el estudio de la convergencia en norma para el multiplicador del intervalo respecto a los polinomios de Laguerre y Hermite. En este sentido ya hemos probado resultados parciales utilizando estimaciones similares a las usadas en el caso de los polinomios de Jacobi. En concreto, hemos obtenido condiciones suficientes para dicho tipo de convergencia, considerando también la acotación con pesos en la clase de Muckenhoupt discreta. Actualmente estamos trabajando para demostrar que dichas condiciones son además necesarias cuando no se consideran pesos.

Por otra parte, en los Capítulos 4 y 5 se ha estudiado el problema de la transplantación asociado a funciones de polinomios ortogonales clásicos. En el primero hemos obtenido la acotación del operador de transplantación asociado a polinomios de Jacobi, con $\alpha, \beta \geq -1/2$, y donde se han incluido pesos en la clase de Muckenhoupt discreta. Además, también hemos conseguido estimaciones (1,1) débiles para el citado operador. En el segundo hemos considerado el operador de transplantación asociado a las funciones de Laguerre, y hemos obtenido su acotación con pesos en la clase local de Muckenhoupt discreta. Además de esto, hemos dado una caracterización de la acotación cuando se consideran pesos de tipo potencia.

Sería deseable poder extender nuestros resultados referentes a la transplantación respecto a las funciones de Jacobi en el rango completo del orden, es decir, extenderlas para valores $\alpha, \beta > -1$. La mayor dificultad para obtener esta extensión es la dependencia, en esa situación, del grado de las estimaciones uniformes para funciones de Jacobi. Por otra parte, la técnica que hemos utilizado para el caso de las funciones de Laguerre no parece viable en este caso debido a que la fórmula de conexión para los polinomios de Jacobi es difícil de tratar.

En línea con lo dicho, en nuestro trabajo futuro también consideraremos el estudio de operadores de transplantación asociados al sistema de Bessel $\{j_n^\alpha\}_{n\geq 1}$, donde

$$j_n^\alpha(x) = t_n^\alpha \sqrt{x} J_\alpha(\sigma_n^\alpha x) \qquad x \in (0,1).$$

Aquí $J_\alpha(x)$ denota la función de Bessel de primera especie de orden $\alpha > -1$, t_n^α es una constante de normalización en el espacio $L^2(0,1)$, y $\{\sigma_n^\alpha\}_{n\geq 1}$ son los ceros de la función J_α. Nuestro objetivo es extender el trabajo de K. Stempak en [**44**], donde se estudia el caso $\alpha \geq -1/2$, en dos sentidos: incluir pesos generales y considerar el máximo rango posible de valores del orden α.

Publications

We include here the abstracts of the publications and pre-publications obtained from the study given in the present dissertation.

- A. Arenas, Ó. Ciaurri, and E. Labarga,
  The convergence of the discrete Fourier-Jacobi series,
  submitted. `arXiv:1906.08004`

  The discrete counterpart of the problem related to the convergence of the Fourier-Jacobi series is studied. To this end, given a sequence, we construct the analogue of the partial sum operator related to Jacobi polynomials and characterize its convergence in the $\ell^p(\mathbb{N})$ norm.

- A. Arenas, Ó. Ciaurri, and E. Labarga,
  A weighted transplantation theorem for Jacobi coefficients,
  submitted. `arXiv:1812.08422`

  We present a transplantation theorem for Jacobi coefficients in weighted spaces. In fact, by using a discrete vector-valued local Calderón- Zygmund theory, which has recently been furnished, we prove the boundedness of transplantation operators from $\ell^p(\mathbb{N}, w)$ into itself, where w is a weight in the discrete Muckenhoupt class $A_p(\mathbb{N})$. Moreover, we obtain weighted weak $(1, 1)$ estimates for those operators.

- A. Arenas, Ó. Ciaurri, and E. Labarga,
  Weighted transplantation for Laguerre coefficients,
  submitted.

  We present a transplantation theorem for Laguerre coefficients in weighted spaces by means of a discrete local Calderón-Zygmund theory. Moreover, a full characterization is given when power weights are addressed.

Publicaciones

Incluimos aquí los resúmenes de las publicaciones y prepublicaciones a las que han dado lugar el estudio presentado en la presente memoria.

- A. Arenas, Ó. Ciaurri y E. Labarga,
  The convergence of the discrete Fourier-Jacobi series,
  enviado para su publicación. `arXiv:1906.08004`

  Se estudia el análogo discreto del problema de la convergencia de la serie de Fourier-Jacobi. Para ello, dada una sucesión, construimos el análogo del operador suma parcial asociado a los polinomios de Jacobi y caracterizamos su convergencia en la norma $\ell^p(\mathbb{N})$.

- A. Arenas, Ó. Ciaurri y E. Labarga,
  A weighted transplantation theorem for Jacobi coefficients,
  enviado para su publicación. `arXiv:1812.08422`

  Presentamos un teorema de transplantación para los coeficientes de Jacobi involucrando pesos. De hecho, usando una teoría local discreta vectorvaluada de Calderón-Zygmund, que ha sido recientemente desarrollada, probamos la acotación de los operadores de transplantación de $\ell^p(\mathbb{N}, w)$ en sí mismo, donde w es un peso en la clase discreta de Muckenhoupt $A_p(\mathbb{N})$. Además de esto, obtenemos estimaciones $(1, 1)$ débiles con pesos para dichos operadores.

- A. Arenas, Ó. Ciaurri y E. Labarga,
  Weighted transplantation for Laguerre coefficients,
  enviado para su publicación.

  Presentamos un teorema de transplantación para los coeficientes de Laguerre en espacios con pesos por medio de una teoría local discreta de Calderón-Zygmund. Además de esto, se da una caracterización total cuando se consideran pesos de tipo potencia.

Bibliography

[1] V. Almeida, J. J. Betancor, A. J. Castro, A. Sanabria, and R. Scotto, Variable exponent Sobolev spaces associated with Jacobi expansions, *J. Math. Inequal.* **11** (2017), 901–946.

[2] K. F. Andersen and B. Muckenhoupt, Weighted weak type Hardy inequalities with applications to Hilbert transforms and maximal functions, *Studia Math.* **72** (1982), 9–26.

[3] A. Arenas, Ó. Ciaurri, and E. Labarga, Discrete Harmonic Analysis associated with Jacobi expansions I: the Heat semigroup, preprint, arXiv: 1806.00056 (2018).

[4] A. Arenas, Ó. Ciaurri, and E. Labarga, Discrete Harmonic Analysis associated with Jacobi expansions II: the Riesz trasnform, preprint, arXiv: 1902.01761 (2019).

[5] A. Arenas, Ó. Ciaurri, and E. Labarga, Discrete Harmonic Analysis associated with Jacobi expansions III: the Littlewood-Paley-Stein g_k-functions and the Laplace type multipliers, preprint, arXiv: 1906.07999 (2019).

[6] R. Askey, A transplantation theorem for Jacobi coefficients, *Pacific J. Math.* **21** (1967), 393–404.

[7] R. Askey, A transplantation theorem for Jacobi series, *Ill. J. Math.* **13** (1969), 583–590.

[8] R. Askey and S. Wainger, Mean convergence of expansions in Laguerre and Hermite series, *Amer. J. Math.* **87** (1965), 695–708.

[9] R. Askey and S. Wainger, A transplantation theorem for ultraspherical coefficients, *Pacific J. Math.* **16** (1966), 393–405.

[10] J. A. Barceló and A. Córdoba, Band-limited functions: L^p-convergence, *Trans. Amer. Math. Soc.* **313** (1989), 655–669.

[11] A. I. Benedek and R. Panzone, Mean convergence of series of Bessel function, *Rev. Un. Mat. Arg.* **26** (1972), 42–61.

[12] A. I. Benedek and R. Panzone, Bessel expansions arising from anomalous Sturm-Liouville differential systems, *Rev. Un. Mat. Arg.* **28** (1972), 161–185.

[13] J. J. Betancor, A. J. Castro, J. C. Fariña, and L. Rodríguez-Mesa, Discrete harmonic analysis associated with ultraspherical expansions, to appear in *Potential Analysis.* arXiv: 1512.01379 (2015).

[14] A. Chicco-Ruiz and E. Harboure,
Weighted norm inequalities for heat-diffusion Laguerre's semigroups,
Math. Z. **257** (2007), 329–354.

[15] T. S. Chihara,
An Introduction to Orthogonal Polynomials,
Gordon and Breach, New York, 1978.

[16] Ó. Ciaurri, T. A. Gillespie, L. Roncal, J. L. Torrea, and J. L. Varona,
Harmonic analysis associated with discrete Laplacian,
J. Anal. Math. **132** (2017), 109–131.

[17] Ó. Ciaurri, A. Nowak, and K. Stempak,
Jacobi transplantation revisited,
Math. Z. **257** (2007), 355–380.

[18] J. Duoandikoetxea,
Fourier Analysis,
Graduate Studies in Mathematics, 29, Amer. Math. Soc., Providence, R. I., 2001.

[19] G. Garrigós, E. Harboure, T. Signes, J. L. Torrea, and B. Viviani,
A sharp weighted transplantation theorem for Laguerre function expansions,
J. Funct. Anal. **244** (2007), 247–276.

[20] V. L. Generozov,
L_p-convergence for expansions in terms of the eigenfunctions of a Sturm-Liouville,
Math. Notes **3** (1968), 436–441.

[21] L. Grafakos, L. Liu, and D. Yang
Vector-valued singular integrals and maximal functions on spaces of homogeneous type,
Math. Scand. **104** (2009), 296–310.

[22] J. J. Guadalupe, M. Pérez, and J. L. Varona
Weighted L^p-boundedness of Fourier series with respect to generalized Jacobi weights,
Publicacions Matemàtiques **35** (1991), 449–459.

[23] J. J. Guadalupe, M. Pérez, and J. L. Varona
Weighted weak behaviour of Fourier-Jacobi series,
Math. Nachr. **158** (1992), 162-174.

[24] D. L. Guy,
Hankel multiplier transformations and weighted p-norms,
Trans. Amer. Math. Soc. **95** (1960), 137–189.

[25] R. Hunt, B. Muckenhoupt, and R. Wheeden,
Weighted norm inequalities for the conjugate function and the Hilbert transform,
Trans. Amer. Math. Soc. **176** (1973), 227–251.

[26] Y. Kanjin,
A transplantation theorem for Laguerre series,
Tohoku Math. J. **43** (1991), 537–555.

[27] E. Labarga,
Discrete Harmonic Analysis associated with Jacobi expansions,
PhD dissertation, to appear.

[28] B. Muckenhoupt,
Mean convergence of Jacobi series,
Proc. Amer. Math. Soc. **23** (1969), 306–310.

[29] B. Muckenhoupt,
Mean convergence of Hermite and Laguerre series I,
Trans. Amer. Math. Soc. **147** (1970), 419–431.

[30] B. Muckenhoupt,
Mean convergence of Hermite and Laguerre series II,
Trans. Amer. Math. Soc. **147** (1970), 433–460.

[31] B. Muckenhoupt,
Transplantation theorems and mutipliers for Jacobi series,
Mem. Amer. Math. Soc. **64** (1986).

[32] A. F. Nikiforov and V. B. Uvarov,
Special Functions of Mathematical Physics,
Birkhäuser Verlag, Basel, 1988.
Originally published as *Specjal'nye funkcii matematičeskoj fiziki*, Moscow, 1978.

[33] A. Nowak and K. Stempak,
L^2-theory of Riesz transforms for orthogonal expansions,
J. Fourier Anal. Appl. **12** (2006), 675–711.

[34] A. Nowak and K. Stempak,
Weighted estimates for the Hankel transform transplantation operator,
Tohoku Math. J. **58** (2006), 277–301.

[35] C. A. Okpoti, L.-E. Persson, and A. Wedestig,
Weight characterizations for the discrete Hardy inequality with kernel,
J. Inequal. Appl. **2006**, 1–14.

[36] F. W. J. Olver (editor-in-chief),
NIST Handbook of Mathematical Functions,
Cambridge University Press, New York, 2010.

[37] H. Pollard,
The mean convergence of orthogonal series of polynomials,
Proc. Nat. Acad. Sci. U.S.A. **32** (1946), 8–10.

[38] H. Pollard,
The mean convergence of orthogonal series I,
Trans. Amer. Math. Soc. **62** (1947), 387–403.

[39] H. Pollard,
The mean convergence of orthogonal series II,
Trans. Amer. Math. Soc. **63** (1948), 355–367.

[40] H. Pollard,
The mean convergence of orthogonal series III,
Duke Math. J. **16** (1949), 189–191.

[41] M. Riesz,
Sur les fonctions conguguées,
Math. Zeit. **27** (1927), 218–244.

[42] W. Rudin,
Real and Complex Analysis,
$3^{\text{th.}}$ ed., McGraw-Hill International Editions, Singapore, 1987.

[43] K. Stempak,
Heat-diffusion and poisson integrals for Laguerre expansions,
Tohoku Math. J. **46** (1994), 83–104.

[44] K. Stempak,
A transplantation theorem for Fourier-Bessel coefficients,
Anal. Math. **24** (1998), 311–318.

[45] K. Stempak,
Transplantation theorems - A survey,
J. Fourier Anal. Appl. **17** (2011), 408–430.

[46] K. Stempak and W. Trebels,
On weighted transplantation and multipliers for Laguerre expansions,
Math. Ann. **300** (1994), 203–219.

[47] G. Szegő,
Orthogonal polynomials,
$3^{\text{th.}}$ ed., Amer. Math. Soc. Colloq. Publ. vol. 23, Amer. Math. Soc., Providence, R. I., 1967.

[48] S. Thangavelu,
Lectures on Hermite and Laguerre expansions,
Mathematical Notes, 42, Princeton University Press, Princeton, NJ, 1993.

[49] G. M. Wing,
The mean convergence of orthogonal series,
Amer. J. Math. **72** (1950), 792–808.