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Applications of the duality theory of convex analysis to the complete electrode model of electrical impedance tomography

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Abstract

The duality theory of convex analysis is applied to the complete electrode model (CEM), which is a standard model in electrical impedance tomography (EIT). This results in a dual formulation of the CEM and a general error estimate. This new formulation of the CEM is written in terms of current fields and is shown to have a unique solution. Using this formulation, the general error estimate is proved, from which two a posteriori error estimates and a well known asymptotic result on CEM solutions are obtained. The first a posteriori error estimate assesses the accuracy of solutions to approximate problems, and the second one assesses the accuracy of approximate solutions. Numerical tests to apply this second estimate are performed, employing the finite element method to obtain approximate solutions.

Keywords . Electrical impedance tomography, duality theory, complete electrode model, direct problem.

1. Introduction. Electrical impedance tomography (EIT) is an imaging modality that seeks to recover the conductivity (or conductivity and permittivity) distribution inside a physical body from electrical measurements taken on the surface of the body. The Argentinian engineer and mathematician Alberto Pedro Calderon wrote the first mathematical formulation of this problem in [1]. To perform EIT, current is sent through electrodes placed on the surface of the body and the resulting voltage on these same electrodes is measured. Then, EIT aims to recover the conductivity distribution from this knowledge of current and voltage. Due to its potential advantages over other imaging techniques, e.g. low cost, rapid response, high contrast, non-intrusiveness, portability, and absence of ionizing radiation, EIT has applications in fields such as medical imaging, geophysics, industrial process tomography, and non-destructive testing. For a recent account of the applications we refer the reader to [2].

Several EIT models have been proposed for modeling the electric potential induced within a conducting body by boundary current injection. In all of them, the electric potential u in the body Ω is governed by the elliptic partial differential equation

$$\nabla \cdot (\sigma \nabla u) = 0 \quad \text{in } \Omega,$$

where σ is the internal conductivity of Ω. This equation can be obtained from Maxwell's equations. The body Ω is considered as a domain in $\mathbb{R}^d$, with $d = 2$ or $d = 3$ if, for instance, the EIT experiment is realized with electrodes encircling the chest of a subject or around the human head, respectively. In the general case, σ is replaced by the complex admittivity $\sigma + i\omega\varepsilon$, with $i = \sqrt{-1}$, where ε is the permittivity and ω is the frequency. Starting with this equation, each EIT model proposes a different set of boundary conditions to model the electrodes attached to the body surface $\partial\Omega$ and the current application through these electrodes. For a complete description of the EIT models we refer the reader to [3, 4, 5].

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According to experiments, the *complete electrode model* (CEM) is the most accurate model. It is capable of predicting experimental measurements more accurately than 0.1 percent [5]. The CEM improves the other models by considering the voltage drops across the layer between the electrode and the body as the product of a contact impedance times the current flux. Due to the modeling of electrodes, the CEM is an elliptic problem with non-standard boundary conditions, where the conductivity distribution and contact impendances are coefficients. Typically, it is weakly formulated in an appropriate Sobolev space to analyze their existence and uniqueness of a solution. Moreover, an equivalent extremal formulation of it can be obtained.

Assuming the conductivity and contact impendaces are known, the problem of finding the electrical potential within the body and the voltages on the electrodes generated by the application of current is called the *direct problem* of EIT. It has a unique solution, and is linear and stable with respect to the current applied, that is, it is *well-posed* in the sense of Hadamard [6]. Accurate solutions to this problem serve as input to numerical methods used to recover the internal conductivity, which is the goal of EIT.

The *duality theory* of convex analysis [7] is a general and versatile framework for problems in applied mathematics that have a extremal formulation. Roughly speaking, the goal of duality theory is to formulate a "dual" problem of the original one and to explore the relations between them. This theory has been successfully applied to standard elliptic problems, providing a posteriori error estimates for approximate solutions obtained by numerical methods (see for instance [8, 9]), and solutions to approximate problems that contain coefficient, boundary condition, and domain idealizations (see for instance [10, 11]). In this paper, the duality theory of convex analysis is applied to the CEM, an elliptic problem with non-standard boundary conditions. The main contributions of this work are:

(1) A general error estimate for the difference between a solution to the CEM and any other element in the space of solutions. Its generality allows us to derive two a posteriori error estimates. The first estimate assesses the accuracy of approximate solutions, such as finite element solutions, while the second one assesses the accuracy of exact solutions to problems with idealized conductivity and contact impedances. Furthermore, the general error estimate is used to prove the convergence of the CEM solutions to the *shunt* model solutions as the contact impedances tend to zero (the shunt model is another model in EIT). A proof of this result can be found in [12]. All of these findings highlight the potential of the general error estimate.

(2) A novel formulation of the CEM in terms of current fields. It serves to prove the general error estimate. A similar formulation was proposed in [13]. There, the body domain is extended to consider electrode conductivity functions. Here, such assumption is not made.

The rest of this paper is organized as follows. In Section 2, the CEM is presented. In Section 3, we introduce the duality theory of convex analysis and summarize the main results of this theory. In Section 4, it is proved that the CEM fits into the duality theory. The extremal formulation of the CEM is interpreted as the *primal* problem. Then, the duality theory naturally provides its *dual* problem. It is shown that the dual problem has a unique solution and that there is no duality gap between the corresponding optimal values. The relation between the optimal solutions is also obtained. These results are used to prove the general error estimate. From this, the two a posteriori error estimates are derived and the asymptotic result on CEM solutions is proved. In Section 5, numerical tests are performed to determine the error of finite element solutions.

2. The complete electrode model of EIT. In order to present the CEM and its formulations, we introduce the following notations. Let Ω be an open, connected, bounded, and Lipschitz domain in $\mathbb{R}^d$ $(d = 2, 3)$ with boundary $\partial\Omega$. The outward unit normal to $\partial\Omega$ is denoted by $\mathbf{n}$. Let M be an integer and let $\mathcal{E}_1, \ldots, \mathcal{E}_M$ be open connected subsets of $\partial\Omega$ such that $\overline{\mathcal{E}_i} \cap \overline{\mathcal{E}_j} = \emptyset$ for $i \neq j$, and if $d = 3$, the boundary of each $\mathcal{E}_m$ is a smooth curve on $\partial\Omega$. The space of square integrable vector-valued functions from Ω into $\mathbb{R}^d$ is denoted by $L^2\left(\Omega, \mathbb{R}^d\right)$. The space of square integrable functions from $\partial\Omega$ into $\mathbb{R}$ is denoted by $L^2\left(\partial\Omega\right)$. The space of square integrable functions from $\mathcal{E}_m$ into $\mathbb{R}$ is denoted by $L^2\left(\mathcal{E}_m\right)$, for $m = 1, \ldots, M$. Let $H^1\left(\Omega\right)$ denote the usual Sobolev space on Ω. Let $\mathbb{R}^M_\diamond$ be the subspace of vectors with zero mean value $\left\{U \in \mathbb{R}^M \,\middle|\, \sum_{m=1}^M U_m = 0\right\}$. The space of traces on $\mathcal{E}_m$ is denoted by $H^{1/2}\left(\mathcal{E}_m\right)$ and $\gamma_m : H^1\left(\Omega\right) \to H^{1/2}\left(\mathcal{E}_m\right)$ denotes the trace operator on $\mathcal{E}_m$, for $m = 1, \ldots, M$. The space of bounded measurable functions is denoted by $L^\infty\left(\Omega\right)$. For a measurable function σ, the essential infimum of σ is denoted by $\operatorname{ess\,inf}_{\mathbf{x}\in\Omega}\sigma\left(\mathbf{x}\right)$. Finally, $\mathbf{1}$ denotes the constant function $\mathbf{1}\left(\mathbf{x}\right) = 1$ for all $\mathbf{x} \in \Omega$, $\overrightarrow{1}$ denotes the all-ones vector $\left(1_{(1)}, \ldots, 1_{(M)}\right) \in \mathbb{R}^M$, and $\overrightarrow{0}$ denotes the zero vector of $\mathbb{R}^M$.

In what follows, the domain Ω represents a body with an internal conductivity and the subsets $\mathcal{E}_1, \ldots, \mathcal{E}_M$ represent M electrodes attached on the surface $\partial\Omega$. Moreover, there is a contact impedance associated to each electrode.

The CEM reads as: given the conductivity $\sigma \in L^\infty\left(\Omega\right)$ satisfying $\sigma_- = \operatorname{ess\,inf}_{\mathbf{x}\in\Omega}\sigma\left(\mathbf{x}\right) > 0$, positive contact impedances $z_1, \ldots, z_M$, and current pattern $I = \left(I_1, \ldots, I_M\right) \in \mathbb{R}^M_\diamond$ applied through the electrodes, find the electric potential $(u, U) \in H^1(\Omega) \times \mathbb{R}^M$ such that

$$\begin{aligned} \nabla \cdot (\sigma \nabla u) = 0 \quad & \text{in } \Omega, \\ \sigma \nabla u \cdot \mathbf{n} = 0 \quad & \text{on } \partial\Omega \setminus \cup_{m=1}^{M} \mathcal{E}_m, \\ u + z_m \sigma \nabla u \cdot \mathbf{n} = U_m \quad & \text{on } \mathcal{E}_m, \text{ for } m = 1, \ldots, M, \\ \int_{\mathcal{E}_m} \sigma \nabla u \cdot \mathbf{n} \, \mathrm{ds} = I_m \quad & \text{for } m = 1, \ldots, M. \end{aligned} \tag{2.1}$$

The vector $U = (U_1, \ldots, U_M)$ contains the voltages generated on the electrodes by the application of the current I. Observe that from the second and third equations it can be deduced that $\sigma \nabla u \cdot \mathbf{n} \in L^2(\partial\Omega)$. In consequence, the integral in the fourth equation can be interpreted in the classical sense.

It is well known that the weak formulation of the CEM reads as: the electric potential $(u, U) \in H^1(\Omega) \times \mathbb{R}^M$ satisfies (recall that $\gamma_1, \ldots, \gamma_M$ are trace functions)

$$\int_\Omega \sigma \nabla u \cdot \nabla v \, \mathrm{d}\mathbf{x} + \sum_{m=1}^{M} \int_{\mathcal{E}_m} \frac{(\gamma_m u - U_m)(\gamma_m v - V_m)}{z_m} \, \mathrm{ds} = \sum_{m=1}^{M} I_m V_m, \tag{2.2}$$

for all $(v, V) \in H^1(\Omega) \times \mathbb{R}^M$. In fact, this weak formulation has an associated minimization problem: $(\bar{u}, \bar{U})$ is a solution to (2.2) if and only if $(\bar{u}, \bar{U})$ is a minimizer of

$$\min_{(u,U) \in H^1(\Omega) \times \mathbb{R}^M} \frac{1}{2} \left(\int_\Omega \sigma |\nabla u|^2 \, \mathrm{d}\mathbf{x} + \sum_{m=1}^{M} \int_{\mathcal{E}_m} \frac{(\gamma_m u - U_m)^2}{z_m} \, \mathrm{ds} \right) - \sum_{m=1}^{M} I_m U_m, \tag{2.3}$$

It is easy to check that the minimum value is exactly $-(1/2) \sum_{m=1}^{M} I_m \bar{U}_m$. The quantity $\sum_{m=1}^{M} I_m \bar{U}_m$ is the power dissipated during current injection. Using the *Lax-Milgram* theorem one can prove that there exists $(\bar{u}, \bar{U}) \in H^1(\Omega) \times \mathbb{R}^M$ such that $\left\{ \left(\bar{u} + \lambda \mathbf{1}, \bar{U} + \lambda \overrightarrow{1} \right) \mid \lambda \in \mathbb{R} \right\}$ is the set of solutions to (2.2). From this, it is easy to verify that there exists a unique solution to (2.2) in the closed subspace $H^1(\Omega) \times \mathbb{R}^M_\diamond$. See [5] for more details.

3. Duality theory. In this section, we review some standard results of the duality theory of convex analysis. Our references in this subject are [14, 7, 11].

Let us begin by defining the primal problem and its dual problem. Let X and Y be two normed spaces and let $X^\star$ and $Y^\star$ be their duals. The duality pairing between X and $X^\star$ is denoted by $\langle \cdot, \cdot \rangle_{X^\star \times X}$. The duality pairing between Y and $Y^\star$ is denoted by $\langle \cdot, \cdot \rangle_{Y^\star \times Y}$. Let $\Lambda : X \to Y$ be a linear continuous operator and let $\Lambda^\star : Y^\star \to X^\star$ its adjoint, that is, $\langle \Lambda^\star y^\star, x \rangle_{X^\star \times X} = \langle y^\star, \Lambda x \rangle_{Y^\star \times Y}$ for all $x \in X$ and all $y^\star \in Y^\star$. Let $J : X \times Y \to \overline{\mathbb{R}}$ be a functional and let $J^\star : X^\star \times Y^\star \to \overline{\mathbb{R}}$ be its *conjugate*, that is,

$$J^\star(x^\star, y^\star) = \sup_{x \in X, y \in Y} \left\{ \langle x^\star, x \rangle_{X^\star \times X} + \langle y^\star, y \rangle_{Y^\star \times Y} - J(x, y) \right\}.$$

Here $\overline{\mathbb{R}}$ denotes the extended real line. Consider the optimization problems

$$\inf_{x \in X} J(x, \Lambda x) \qquad \text{and} \qquad \sup_{y^\star \in Y^\star} -J^\star(\Lambda^\star y^\star, -y^\star).$$

The first is the *primal* problem and the second is its *dual* problem.

The following theorem has a special interest when the primal problem has no solution.

Theorem 3.1. *Suppose that J is convex, that $\inf_{x \in X} J(x, \Lambda x)$ is finite, and that there exists $x_0 \in X$ such that $J(x_0, \Lambda x_0) < \infty$ and $y \mapsto J(x_0, y)$ is continuous at $y = \Lambda x_0$. Then*

$$\inf_{x \in X} J(x, \Lambda x) = \sup_{y^\star \in Y^\star} -J^\star(\Lambda^\star y^\star, -y^\star) \tag{3.1}$$

and the dual problem has at least one solution $\bar{y}^\star$. If, moreover, $\bar{x}$ is a solution to the primal problem, then $\bar{x}$ and $\bar{y}^\star$ satisfy the extremality relation

$$J(\bar{x}, \Lambda \bar{x}) + J^\star(\Lambda^\star \bar{y}^\star, -\bar{y}^\star) = 0.$$

The proof of this result can be found in [14, Chap. 9] [7, Chap. 3].

This theorem provides a general framework for a posteriori error estimates.

Theorem 3.2. *Let $\bar{x}$ be a solution to the primal problem. Assume that the directional derivative of J at $(\bar{x}, \Lambda\bar{x})$ in the direction $(x, \Lambda x) - (\bar{x}, \Lambda\bar{x})$ exists and denote it by $J(\bar{x}, \Lambda\bar{x})(x - \bar{x}, \Lambda x - \Lambda\bar{x})$. Let $x \in X$ be any element satisfying $J(x, \Lambda x) < \infty$. Define*

$$D(x, \bar{x}) = J(x, \Lambda x) - J(\bar{x}, \Lambda\bar{x}) - J'(\bar{x}, \Lambda\bar{x})(x - \bar{x}, \Lambda x - \Lambda\bar{x}).$$

Under the same assumptions as in Theorem 3.1, we have the estimate

$$D(x, \bar{x}) \leq J(x, \Lambda x) + J^\star(\Lambda^\star y^\star, -y^\star),$$

for all $y^\star \in Y^\star$. The proof of this result can be found in [11, Chap. 2].

Remark 3.1. *Observe that $D(x, \bar{x})$ is the Bregman distance between $(x, \Lambda x)$ and $(\bar{x}, \Lambda\bar{x})$ with respect to J [15].*

Remark 3.2. *In the case in which J can be written in the form*

$$J(x, y) = F(x) + G(y), \tag{3.2}$$

with the functionals $F : X \to \overline{\mathbb{R}}$ and $G : Y \to \overline{\mathbb{R}}$, the primal and dual problems read as

$$\inf_{x \in X} F(x) + G(\Lambda x) \qquad \text{and} \qquad \sup_{y^\star \in Y^\star} -F^\star(\Lambda^\star y^\star) - G^\star(-y^\star),$$

respectively. The functionals $F^\star : X^\star \to \overline{\mathbb{R}}$ and $G^\star : Y^\star \to \overline{\mathbb{R}}$ are the conjugates of F and G, and are defined as

$$F^\star(x^\star) = \sup_{x \in X} \left\{ \langle x^\star, x \rangle_{X^\star \times X} - F(x) \right\} \quad \text{and} \quad G^\star(y^\star) = \sup_{y \in Y} \left\{ \langle y^\star, y \rangle_{Y^\star \times Y} - G(y) \right\}.$$

The extremality relation given in Theorem 3.1 decouples into the equations

$$F(\bar{x}) + F^\star(\Lambda^\star \bar{y}^\star) - \langle \Lambda^\star \bar{y}^\star, \bar{x} \rangle_{X^\star \times X} = 0, \tag{3.3}$$

$$G(\Lambda\bar{x}) + G^\star(-\bar{y}^\star) + \langle \bar{y}^\star, \Lambda\bar{x} \rangle_{Y^\star \times Y} = 0. \tag{3.4}$$

Moreover, if F is linear over $\operatorname{dom} F = \{x \in X \mid F(x) < \infty\}$ and G is real-valued over Y and Gâteaux-differentiable at $y = \Lambda\bar{x}$, then $\operatorname{dom} F = \{x \in X \mid J(x, \Lambda x) < \infty\}$ and the Bregman distance between $(x, \Lambda x)$ and $(\bar{x}, \Lambda\bar{x})$ given in Theorem 3.2 becomes

$$D(x, \bar{x}) = G(\Lambda x) - G(\Lambda\bar{x}) - \langle G'(\Lambda\bar{x}), (\Lambda x - \Lambda\bar{x}) \rangle_{Y^\star \times Y}, \tag{3.5}$$

where $G'(\Lambda\bar{x})$ is the Gâteaux derivative of G at $y = \Lambda\bar{x}$. For more details see [14, Thm. 9.8.1][7, Rem. 4.2].

Remark 3.3. *When X and Y are Hilbert spaces, one can consider the Riesz-Fréchet isomorphism between these spaces and their dual spaces, and identify $X^\star, Y^\star$ with X, Y, respectively. All of the above remains true. For instance, the adjoint $\Lambda^\star$ reads as the operator $\Lambda^\star : Y \to X$ defined by*

$$\langle \Lambda^\star y, x \rangle_X = \langle y, \Lambda x \rangle_Y \quad \text{for all } x \in X \text{ and all } y \in Y,$$

where $\langle \cdot, \cdot \rangle_X$ and $\langle \cdot, \cdot \rangle_Y$ are the inner products of X and Y.

4. Results. In this section, we fit the CEM into the framework presented in the previous section. A dual version of the CEM in terms of current fields is obtained. This dual formulation will allow to obtain a general error estimate, from which a posteriori error estimates for approximate solutions and for solutions to approximate problems are derived.

We begin by deriving the dual version of the extremal formulation of the CEM.

Lemma 4.1. *The dual problem of (2.3) is*

$$\max_{(p,P) \in L^2(\Omega, \mathbb{R}^d) \times (L^2(\mathcal{E}_1) \times \ldots \times L^2(\mathcal{E}_M))} -\frac{1}{2}\left(\int_\Omega \frac{1}{\sigma} |p|^2 \, \mathrm{d}\mathbf{x} + \sum_{m=1}^{M} \int_{\mathcal{E}_m} z_m P_m^2 \, \mathrm{d}\mathbf{s} \right)$$

subject to

$$\begin{aligned} \nabla \cdot p &= 0 && \text{in } \Omega, \\ p \cdot \mathbf{n} &= 0 && \text{on } \partial\Omega \setminus \cup_{m=1}^{M} \mathcal{E}_m, \\ p \cdot \mathbf{n} &= -P_m && \text{on } \mathcal{E}_m, \text{ for } m = 1, \ldots, M, \\ \int_{\mathcal{E}_m} P_m \, \mathrm{d}\mathbf{s} &= I_m && \text{for } m = 1, \ldots, M. \end{aligned} \tag{4.1}$$

Moreover, the unique solution to (4.1) is

$$\left(\bar{p},\bar{P}\right)=-\left(\sigma\nabla\bar{u},\frac{\gamma_1\bar{u}-\bar{U}_1}{z_1},\dots,\frac{\gamma_M\bar{u}-\bar{U}_M}{z_M}\right),\tag{4.2}$$

with $\left(\bar{u},\bar{U}\right)$ *being a solution to the primal problem (2.3).*

Proof: First note that the minimization problem (2.3) can be written in the form

$$\inf_{(u,U)\in X}F\left(u,U\right)+G\left(\Lambda\left(u,U\right)\right),$$

where

$$\begin{aligned}
&X=H^1\left(\Omega\right)\times\mathbb{R}^M,\ Y=L^2\left(\Omega,\mathbb{R}^d\right)\times\left(L^2\left(\mathcal{E}_1\right)\times\ldots\times L^2\left(\mathcal{E}_M\right)\right),\\
&\Lambda:X\to Y\text{ is defined by }\Lambda\left(u,U\right)=\left(\nabla u,\gamma_1u-U_1,\dots,\gamma_Mu-U_M\right),\\
&F:X\to\mathbb{R}\text{ is defined by }F\left(u,U\right)=-\textstyle\sum_{m=1}^M I_mU_m,\ \text{ and}\\
&G:Z\to\mathbb{R}\text{ is defined by }G\left(p,P\right)=\tfrac{1}{2}\left(\int_\Omega\sigma\left|p\right|^2\,\mathrm{d}\mathbf{x}+\textstyle\sum_{m=1}^M\int_{\mathcal{E}_m}\frac{P_m^2}{z_m}\,\mathrm{ds}\right).
\end{aligned}$$

It is easy to check that F is linear and that G is convex. With these choices, the minimization problem (2.3) fits into the assumptions of Section 3 and can be interpreted as the primal problem. We consider X and Y as Hilbert spaces equipped with the inner products induced by the direct sum operation, that is,

$$\begin{aligned}
&\left\langle\left(u,U\right),\left(v,V\right)\right\rangle_X=\left\langle u,v\right\rangle_{H^1(\Omega)}+\left\langle U,V\right\rangle_{\mathbb{R}^M}\quad\text{and}\\
&\left\langle\left(p,P\right),\left(q,Q\right)\right\rangle_Y=\left\langle p,q\right\rangle_{L^2(\Omega,\mathbb{R}^d)}+\sum_{m=1}^M\left\langle P_m,Q_m\right\rangle_{L^2(\mathcal{E}_m)}.
\end{aligned}$$

The Hilbert spaces $H^1\left(\Omega\right)$, $\mathbb{R}^M$, $L^2\left(\Omega,\mathbb{R}^d\right)$, $L^2\left(\mathcal{E}_m\right)$ are equipped with their usual inner products. With this, it follows that F, G, and Λ are continuous. One can easily prove that

$$\begin{aligned}
F^\star\left(u,U\right)&=\sup_{(v,V)\in X}\left\{\left\langle\left(u,U\right),\left(v,V\right)\right\rangle_X+\sum_{m=1}^M I_mV_m\right\}\\
&=\begin{cases}0 & \text{if }\left\langle\left(u,U\right),\left(v,V\right)\right\rangle_X=-\sum_{m=1}^M I_mV_m\text{ for all }\left(v,V\right)\in X\\ \infty & \text{otherwise}\end{cases}.
\end{aligned}$$

On the other hand, a direct application of the Lax-Milgram theorem yields

$$\begin{aligned}
G^\star\left(p,P\right)&=\sup_{(q,Q)\in Y}\left\{\left\langle\left(p,P\right),\left(q,Q\right)\right\rangle_Y-\frac{1}{2}\left(\int_\Omega\sigma\left|q\right|^2\,\mathrm{d}\mathbf{x}+\sum_{m=1}^M\int_{\mathcal{E}_m}\frac{Q_m^2}{z_m}\,\mathrm{ds}\right)\right\}\\
&=\sup_{(q,Q)\in Y}\left\{\left\langle\left(p,P\right),\left(q,Q\right)\right\rangle_Y-\frac{1}{2}\left\langle\left(\sigma q,\left(\frac{Q_m}{z_m}\right)_{m=1}^M\right),\left(q,Q\right)\right\rangle_Y\right\}\\
&=\frac{1}{2}\left\langle\left(p,P\right),\left(\bar{q},\bar{Q}\right)\right\rangle_Y\quad\text{with}\ \left(\sigma\bar{q},\left(\frac{\bar{Q}_m}{z_m}\right)_{m=1}^M\right)=\left(p,P\right)\\
&=\frac{1}{2}\left(\int_\Omega\frac{1}{\sigma}\left|p\right|^2\,\mathrm{d}\mathbf{x}+\sum_{m=1}^M\int_{\mathcal{E}_m}z_mP_m^2\,\mathrm{ds}\right).
\end{aligned}$$

Combining the above results, we obtain

$$-F^\star\left(\Lambda^\star\left(p,P\right)\right)-G^\star\left(-\left(p,P\right)\right)=$$

$$\begin{cases}-\frac{1}{2}\left(\displaystyle\int_\Omega\frac{1}{\sigma}\left|p\right|^2\,\mathrm{d}\mathbf{x}+\sum_{m=1}^M\int_{\mathcal{E}_m}z_mP_m^2\,\mathrm{ds}\right) & \text{if }\left\langle\left(p,P\right),\Lambda\left(v,V\right)\right\rangle_Y=-\displaystyle\sum_{m=1}^M I_mV_m\text{ for all }\left(v,V\right)\in X\\ -\infty & \text{otherwise}\end{cases}.$$

Therefore, the dual problem is given by

$$\begin{aligned}
&\sup_{(p,P)\in Y}-\frac{1}{2}\left(\int_\Omega\frac{1}{\sigma}\left|p\right|^2\,\mathrm{d}\mathbf{x}+\sum_{m=1}^M\int_{\mathcal{E}_m}z_mP_m^2\,\mathrm{ds}\right)\\
&\text{subject to }\int_\Omega p\cdot\nabla v\,\mathrm{d}\mathbf{x}+\sum_{m=1}^M\int_{\mathcal{E}_m}P_m\left(\gamma_mv-V_m\right)\,\mathrm{ds}=-\sum_{m=1}^M I_mV_m\text{ for all }\left(v,V\right)\in X\quad.
\end{aligned}$$

It suffices to rewrite the constraint to obtain (4.1). Since the primal problem has solution, the dual problem has at least one solution by Theorem 3.1. Moreover, the extremality conditions (3.3) and (3.4) hold. Note that the extremality condition (3.4) can be written as

$$E\left(\left(\bar{u},\bar{U}\right),\left(\bar{p},\bar{P}\right)\right)=0,$$

where the functional $E : X \times Y \to [0,\infty[$ is defined as

$$E\left(\left(u,U\right),\left(p,P\right)\right)=\frac{1}{2}\int_{\Omega}\left|\frac{p}{\sqrt{\sigma}}+\sqrt{\sigma}\nabla u\right|^{2}\mathrm{d}\mathbf{x}+\frac{1}{2}\sum_{m=1}^{M}\int_{\mathcal{E}_m}\left(\sqrt{z_m}P_m+\frac{\gamma_m u-U_m}{\sqrt{z_m}}\right)^{2}\mathrm{d}\mathbf{s}.$$

Hence, the relation (4.2) follows, and the dual problem has a unique solution. □

Remark 4.1. *Let us make some quick comments about the dual problem. First, observe that the solution* $\left(\bar{p},\bar{P}\right)$ *does not depend on the choice of the solution* $\left(\bar{u},\bar{U}\right)$ *since the set of solution to the primal problem is of the form* $\left\{\left(\bar{u},\bar{U}\right)+\lambda(\mathbf{1},\overrightarrow{1})\mid\lambda\in\mathbb{R}\right\}$. *Second, it is easy to check that the variational problem associated to (4.1) is the following: find* $\left(\bar{p},\bar{P}\right)\in L$ *satisfying*

$$\begin{aligned}\int_{\Omega}\frac{1}{\sigma}\bar{p}\cdot p\,\mathrm{d}\mathbf{x}+\sum_{m=1}^{M}\int_{\mathcal{E}_m}z_m\bar{P}_mP_m\,\mathrm{d}\mathbf{s}&=0\qquad\textit{for all }\ (p,P)\in L_0,\\ \int_{\mathcal{E}_m}\bar{P}_m\,\mathrm{d}\mathbf{s}&=I_m\quad\textit{for } m=1,\ldots,M,\end{aligned}$$

where

$$L=\left\{(p,P)\,\middle|\,\int_{\Omega}p\cdot\nabla v\,\mathrm{d}\mathbf{x}+\sum_{m=1}^{M}\int_{\mathcal{E}_m}P_m\gamma_m v\,\mathrm{d}\mathbf{s}=0\ \textit{ for all } v\in H^1\left(\Omega\right)\right\}$$

and

$$L_0=\left\{(p,P)\in L\,\middle|\,\int_{\mathcal{E}_m}P_m\,\mathrm{d}\mathbf{s}=0\ \textit{ for } m=1,\ldots,M\right\}$$

are closed subspaces of $L^2\left(\Omega,\mathbb{R}^d\right)\times\left(L^2\left(\mathcal{E}_1\right)\times\ldots\times L^2\left(\mathcal{E}_M\right)\right)$. *Third, from (3.1) we obtain upper and lower bounds for the power dissipated in an EIT experiment, namely*

$$\begin{cases}-F\left(u,U\right)-G\left(\Lambda\left(u,U\right)\right)\leq\displaystyle\sum_{m=1}^{M}I_m\bar{U}_m\leq F^{\star}\left(\Lambda^{\star}\left(p,P\right)\right)+G^{\star}\left(-\left(p,P\right)\right)\\ \textit{for all }\ (u,U)\in H^1\left(\Omega\right)\times\mathbb{R}^M\ \textit{and all }\ (p,P)\in L^2\left(\Omega,\mathbb{R}^d\right)\times\left(L^2\left(\mathcal{E}_1\right)\times\ldots\times L^2\left(\mathcal{E}_M\right)\right)\end{cases}$$

holds. Finally, we point out that (4.1) is not the unique dual problem of (2.3) since it depends on the choices of X, Y, Λ, F, *and* G.

The theorem below is our main result. There an estimate that measures the energy norm difference between $\left(\bar{u},\bar{U}\right)$ (solution to the CEM) and any other element of $H^1\left(\Omega\right)\times\mathbb{R}^M$ is provided. Later we will use this result to get a posteriori error estimates.

Theorem 4.1. *Let* $\left(\bar{u},\bar{U}\right)$ *be a solution to the primal problem (2.3). Then we have the estimate*

$$\begin{aligned}&\frac{1}{2}\left(\int_{\Omega}\sigma\left|\nabla\left(\bar{u}-u\right)\right|^{2}\mathrm{d}\mathbf{x}+\sum_{m=1}^{M}\int_{\mathcal{E}_m}\frac{\left(\gamma_m\left(\bar{u}-u\right)-\left(\bar{U}_m-U_m\right)\right)^{2}}{z_m}\mathrm{d}\mathbf{s}\right)\\&\leq\frac{1}{2}\left(\int_{\Omega}\sigma\left|\nabla u\right|^{2}\mathrm{d}\mathbf{x}+\sum_{m=1}^{M}\int_{\mathcal{E}_m}\frac{\left(\gamma_m u-U_m\right)^{2}}{z_m}\mathrm{d}\mathbf{s}\right)-\sum_{m=1}^{M}I_mU_m\\&\quad+\frac{1}{2}\left(\int_{\Omega}\frac{1}{\sigma}\left|p\right|^{2}\mathrm{d}\mathbf{x}+\sum_{m=1}^{M}\int_{\mathcal{E}_m}z_mP_m^{2}\,\mathrm{d}\mathbf{s}\right)\end{aligned}\tag{4.3}$$

for all $(u,U)\in H^1\left(\Omega\right)\times\mathbb{R}^M$ *and all* $(p,P)\in L^2\left(\Omega,\mathbb{R}^d\right)\times\left(L^2\left(\mathcal{E}_1\right)\times\ldots\times L^2\left(\mathcal{E}_M\right)\right)$ *such that* $\nabla\cdot p=0$ *in* Ω, $p\cdot\mathbf{n}=0$ *on* $\partial\Omega\backslash\cup_{m=1}^{M}\mathcal{E}_m$, $p\cdot\mathbf{n}=-P_m$ *on* $\mathcal{E}_m$, *and* $\int_{\mathcal{E}_m}P_m\,\mathrm{d}\mathbf{s}=I_m$, *for* $m=1,\ldots,M$.

Proof: Here we use the same notations as in the proof of Lemma 4.1. First note the following properties of the functional G:

$$G(p,P) = \tfrac{1}{2}\left\langle \left(\sigma p, \tfrac{P_1}{z_1}, \dots, \tfrac{P_M}{z_M}\right), (p,P)\right\rangle_Y,$$
$$\langle G'(p,P),(q,Q)\rangle_{Y^\star\times Y} = \left\langle \left(\sigma p, \tfrac{P_1}{z_1}, \dots, \tfrac{P_M}{z_M}\right), (q,Q)\right\rangle_Y,$$
$$\text{and } \langle G'(p,P),(p,P)\rangle_{Y^\star\times Y} = 2\times G(p,P)$$

for all $(p,P),(q,Q)\in Y$. Also, observe that $F(u,U)+G(\Lambda(u,U))<\infty$ for all $(u,U)\in X$. Let $(\bar u,\bar U)$ be a solution to the primal problem and $(u,U)\in X$. Denote $(p,P)=\Lambda(u,U)$ and $(\bar p,\bar P)=\Lambda(\bar u,\bar U)$. By Remark 3.2 (eq. (3.5)) and applying the above properties of G it follows that

$$\begin{aligned} D\left((u,U),(\bar u,\bar U)\right) &= G(p,P)+G(\bar p,\bar P)-\langle G'(\bar p,\bar P),(p,P)\rangle_{Y^\star\times Y} \\ &= \frac{1}{2}\left\langle \left(\sigma(\bar p-p), \left(\frac{\bar P_m - P_m}{z_m}\right)_{m=1}^M\right), (\bar p - p, \bar P - P)\right\rangle_Y. \end{aligned}$$

This gives the left-hand side of (4.3). On the other hand, from the computation in the proof of Lemma 4.1,

$$F^\star(\Lambda^\star(p,P)) + G^\star(-(p,P)) = \frac{1}{2}\left(\int_\Omega \frac{1}{\sigma}|p|^2\,\mathrm{d}\mathbf{x} + \sum_{m=1}^M \int_{\mathcal{E}_m} z_m P_m^2\,\mathrm{ds}\right)$$

for all $(p,P)\in Y$ such that $\langle (p,P),\Lambda(v,V)\rangle_Y = -\sum_{m=1}^M I_m V_m$ for all $(v,V)\in X$. Since, by Theorem 3.2 and Remark 3.2,

$$D\left((u,U),(\bar u,\bar U)\right) \le F(u,U)+G(\Lambda(u,U))+F^\star(\Lambda^\star(p,P))+G^\star(-(p,P))$$

for all $(u,U)\in X$ and all $(p,P)\in Y$, the right-hand side of (4.3) follows. □

Remark 4.2. *Observe that the estimate (4.3) can be rewritten as*

$$\frac{1}{2}\left(\int_\Omega \sigma|\nabla(\bar u - u)|^2\,\mathrm{d}\mathbf{x} + \sum_{m=1}^M \int_{\mathcal{E}_m} \frac{\left(\gamma_m(\bar u - u) - (\bar U_m - U_m)\right)^2}{z_m}\,\mathrm{ds}\right) \le E\left((u,U),(p,P)\right).$$

Thus, E can be interpreted as an error functional. Moreover, it can be proved that equality holds in (4.3) if and only if (p,P) is chosen to be the solution $(\bar p,\bar P)$ of the dual problem (4.1) [10].

We use the estimate (4.3) to derive a posteriori error estimates for the CEM. These results will be expressed using the following norm. Let $\|\cdot\|: H^1(\Omega)\times\mathbb{R}^M\to[0,\infty[$ be defined by

$$\|(u,U)\| = \left(\int_\Omega |\nabla u|^2\,\mathrm{d}\mathbf{x} + \sum_{m=1}^M \int_{\mathcal{E}_m} (\gamma_m u - U_m)^2\,\mathrm{ds}\right)^{1/2}.$$

It can be proved that $\|\cdot\|$ is a semi-norm on $H^1(\Omega)\times\mathbb{R}^M$ and a norm on $H^1(\Omega)\times\mathbb{R}^M_\diamond$ equivalent to the norm induced by the direct sum operation of $H^1(\Omega)$ and $\mathbb{R}^M$ (considering both spaces with their usual norms) [16]. Also, the inequality

$$C\|(u,U)\|^2 \le \int_\Omega \sigma|\nabla u|^2\,\mathrm{d}\mathbf{x} + \sum_{m=1}^M \int_{\mathcal{E}_m} \frac{(\gamma_m u - U_m)^2}{z_m}\,\mathrm{ds} \tag{4.4}$$

holds, where $C=\min\left\{\sigma_-, z_1^{-1},\dots,z_M^{-1}\right\}$.

The following result provides an a posteriori error estimate for exact solutions of approximate problems (also called *idealizations* [11]). This result can be applied when quite accurate solutions to model approximations of the CEM are available.

Corollary 4.1. *Suppose that σ_0 and $z_{0,1},\dots,z_{0,M}$ are approximations of σ and $z_1,\dots,z_M$, respectively. Let $(\bar u_0,\bar U_0)$ be a solution to the primal problem (2.3) formulated with σ_0 and $z_{0,1},\dots,z_{0,M}$. Then the estimate*

$$\begin{aligned} &C\left\|(\bar u,\bar U)-(\bar u_0,\bar U_0)\right\|^2 \\ &\le \int_\Omega \frac{(\sigma-\sigma_0)^2}{\sigma}|\nabla \bar u_0|^2\,\mathrm{d}\mathbf{x} + \sum_{m=1}^M \int_{\mathcal{E}_m} z_m\left(\frac{1}{z_m}-\frac{1}{z_{0,m}}\right)^2(\gamma_m \bar u_0 - \bar U_{0,m})^2\,\mathrm{ds}. \end{aligned} \tag{4.5}$$

holds.

Proof: By Lemma 4.1,

$$\left(\bar{p}_0, \bar{P}_0\right) = -\left(\sigma \nabla \bar{u}_0, \frac{\gamma_1 \bar{u}_0 - \bar{U}_{0,1}}{z_{0,1}}, \dots \frac{, \gamma_M \bar{u}_0 - \bar{U}_{0,M}}{z_{0,M}}\right)$$

is the unique solution to the dual problem (4.1) formulated with σ_0 and $z_{0,1}, \dots, z_{0,M}$. In particular, $\left(\bar{p}_0, \bar{P}_0\right)$ satisfies the constraints of (4.1). The estimate (4.5) follows by setting $(u, U) = \left(\bar{u}_0, \bar{U}_0\right)$ and $(p, P) = \left(\bar{p}_0, \bar{P}_0\right)$ in the estimate (4.3) of Theorem 4.1 and by considering the norm inequality (4.4). $\square$

Here an a posteriori error estimate for approximate solutions of the CEM. For example, approximate solutions obtained by numerical methods can be evaluated with the following estimate.

Corollary 4.2. *Let $(u, U) \in H^1(\Omega) \times \mathbb{R}^M$ be an approximation of $\left(\bar{u}, \bar{U}\right)$. Then*

$$C\left\|\left(\bar{u}, \bar{U}\right) - (u, U)\right\|^2 \leq \int_\Omega \sigma \left|\nabla u\right|^2 \mathrm{d}\mathbf{x} + \sum_{m=1}^{M} \int_{\mathcal{E}_m} \frac{\left(\gamma_m u - U_m\right)^2}{z_m} \mathrm{d}\mathbf{s} - 2\sum_{m=1}^{M} I_m U_m + \sum_{m=1}^{M} I_m \bar{U}_m, \quad (4.6)$$

where the quantity $\sum_{m=1}^{M} I_m \bar{U}_m$ is the power dissipated in the EIT experiment associated to $\left(\bar{u}, \bar{U}\right)$.

Proof: It suffices to set $(p, P) = -\left(\sigma \nabla \bar{u}, \left(\frac{\gamma_m \bar{u} - \bar{U}_m}{z_m}\right)_{m=1}^{M}\right)$ in the estimate (4.3) of Theorem 4.1 and to consider the norm inequality (4.4). $\square$

Setting the contact impedances $z_1, \dots, z_M = 0$ in the CEM, we obtain the *shunt model* [4, 5]. Namely, the shunt model reads as: given the conductivity $\sigma \in L^\infty(\Omega)$ satisfying $\operatorname{ess\,inf}_{\mathbf{x} \in \Omega} \sigma(\mathbf{x}) > 0$ and current pattern $I = (I_1, \dots, I_M) \in \mathbb{R}^M_\diamond$ applied through the electrodes, find the electric potential $(u, U) \in H^1(\Omega) \times \mathbb{R}^M$ such that

$$\begin{aligned} \nabla \cdot (\sigma \nabla u) &= 0 && \text{in } \Omega, \\ \sigma \nabla u \cdot \mathbf{n} &= 0 && \text{on } \partial\Omega \setminus \cup_{m=1}^{M} \mathcal{E}_m, \\ u &= U_m && \text{on } \mathcal{E}_m, \text{ for } m = 1, \dots, M, \\ \int_{\mathcal{E}_m} \sigma \nabla u \cdot \mathbf{n}\, \mathrm{d}\mathbf{s} &= I_m && \text{for } m = 1, \dots, M. \end{aligned} \quad (4.7)$$

With the help of Green's formula it is easy to obtain the weak formulation of the shunt model: the electric potential $(u, U) \in H^1(\Omega) \times \mathbb{R}^M$ satisfies $\gamma_m u = U_m$ for $m = 1, \dots, M$ and

$$\int_\Omega \sigma \nabla u \cdot \nabla v \, \mathrm{d}\mathbf{x} = \sum_{m=1}^{M} I_m V_m \quad (4.8)$$

for all $(v, V) \in H^1(\Omega) \times \mathbb{R}^M$ such that $\gamma_m v = V_m$ for $m = 1, \dots, M$. One can use the Lax-Milgram theorem to prove that there exists $(\tilde{u}, \tilde{U}) \in H^1(\Omega) \times \mathbb{R}^M$, satisfying $\gamma_m \tilde{u} = \tilde{U}_m$ for $m = 1, \dots, M$, such that $\left\{\left(\tilde{u} + \lambda \mathbf{1}, \tilde{U} + \lambda \overrightarrow{1}\right) \mid \lambda \in \mathbb{R}\right\}$ is the set of solutions to (4.8). Thus, the shunt model has a unique solution in $H^1(\Omega) \times \mathbb{R}^M_\diamond$, just like the CEM.

Next, the estimate (4.3) is used to prove that the solution of the CEM converges to the solution of the shunt model as the contact impedances tend to zero on all electrodes. For this, it is assumed that the normal component of the vector field $\sigma \nabla \tilde{u}$, with $\tilde{u}$ being a solution to the shunt model, belongs to $L^2(\partial\Omega)$ (recall that, in general, the normal component of a vector field is viewed as a functional in $H^{-1/2}(\partial\Omega)$). Although for the CEM such an assumption is true, the same cannot be deduced from the equations of the shunt model. This asymptotic results can be found in [12], where such an assumption is not made. There, it was established that the error between the solutions of the CEM and the shunt model is of order $O\left(\max\left\{z_1, \dots, z_M\right\}^s\right)$, with $s < 1/2$.

Corollary 4.3. *Let $(\bar{u}, \bar{U})$ be a solution to the weak formulation of the CEM. Let $(\tilde{u}, \tilde{U})$ be a solution to the weak formulation of the shunt model and assume that $\sigma \nabla \tilde{u} \cdot \mathbf{n} \in L^2(\partial\Omega)$. Then the estimate*

$$\left\|\left(\bar{u}, \bar{U}\right) - \left(\tilde{u}, \tilde{U}\right)\right\|^2 \leq \left(\frac{\max\left\{z_1, \dots, z_M\right\}}{\min\left\{\sigma_-, z_1^{-1}, \dots, z_M^{-1}\right\}}\right) \left\|\sigma \nabla \tilde{u} \cdot \mathbf{n}\right\|^2_{L^2(\partial\Omega)}. \quad (4.9)$$

holds. In particular, if $H^1(\Omega) \times \mathbb{R}^M_\diamond$ is equipped with the topology induced by the norm $\|\cdot\|$ and the solutions are chosen in this space, then we can assert that $\left(\bar{u}, \bar{U}\right)$ converges to $(\tilde{u}, \tilde{U})$ in $H^1(\Omega) \times \mathbb{R}^M_\diamond$ as $z_1, \dots, z_M$ tend to zero and that the error between these solutions is of order $O\left(\max\left\{z_1, \dots, z_M\right\}^{1/2}\right)$.

Proof: Note that, since $(\tilde{u}, \tilde{U})$ is a solution to the shunt model and $\sigma \nabla \tilde{u} \cdot \mathbf{n} \in L^2(\partial\Omega)$,

$$\begin{aligned} &(u, U) = (\tilde{u}, \tilde{U}) \quad \text{and} \\ &(p, P) = \left(-\sigma \nabla \tilde{u} \cdot \mathbf{n}, \left(\sigma \nabla \tilde{u} \cdot \mathbf{n}|_{\mathcal{E}_1}, \ldots, \sigma \nabla \tilde{u} \cdot \mathbf{n}|_{\mathcal{E}_M}\right)\right) \end{aligned}$$

satisfy the constraints of the estimate (4.3) of Theorem 4.1. These choices yield

$$E\left((u, U), (p, P)\right) = \frac{1}{2} \sum_{m=1}^{M} z_m \int_{\mathcal{E}_m} P_m^2 \, \mathrm{d}s \leq \frac{\max\{z_1, \ldots, z_M\}}{2} \sum_{m=1}^{M} \int_{\mathcal{E}_m} \left(\sigma \nabla \tilde{u} \cdot \mathbf{n}|_{\mathcal{E}_m}\right)^2 \mathrm{d}s.$$

Hence, (4.9) follows by considering the constant C and the fact that $\sigma \nabla \tilde{u} \cdot \mathbf{n} = 0$ on $\partial\Omega \setminus \cup_{m=1}^{M} \mathcal{E}_m$. □

Remark 4.3. *An extension of the estimate (4.3) can be obtained where the auxiliary variable (p, P) is unconstrained. Consider $Y = L^2(\Omega, \mathbb{R}^d) \times \left(L^2(\mathcal{E}_1) \times \ldots \times L^2(\mathcal{E}_M)\right)$ as in the proof of Lemma 4.1 and let $\|\cdot\|_Y$ be its norm. Using Young's inequality it can be shown that*

$$\begin{aligned} &\frac{1}{2}\left(\int_{\Omega} \sigma \left|\nabla(\bar{u} - u)\right|^2 \mathrm{d}\mathbf{x} + \sum_{m=1}^{M} \int_{\mathcal{E}_m} \frac{\left(\gamma_m(\bar{u} - u) - (\bar{U}_m - U_m)\right)^2}{z_m} \mathrm{d}s\right) \\ &\leq (1+\gamma) E\left((u, U), (p, P)\right) + \frac{1}{2}\left(1 + \frac{1}{\gamma}\right) \frac{1}{C} \left\|(p, P) + \left(\nabla w, (\gamma_m w - W_m)_{m=1}^{M}\right)\right\|_Y^2 \end{aligned}$$

for all $\gamma > 0$, all $(u, U) \in H^1(\Omega) \times \mathbb{R}^M$, and all $(p, P) \in L^2(\Omega, \mathbb{R}^d) \times \left(L^2(\mathcal{E}_1) \times \ldots \times L^2(\mathcal{E}_M)\right)$, where $(w, W) \in H^1(\Omega) \times \mathbb{R}^M$ is a solution to the CEM with $\sigma = \mathbf{1}$ and $z_1, \ldots, z_M = 1$, that is

$$\begin{aligned} \Delta w &= 0 && \text{in } \Omega, \\ \nabla w \cdot \mathbf{n} &= 0 && \text{on } \partial\Omega \setminus \cup_{m=1}^{M} \mathcal{E}_m, \\ u + \nabla w \cdot \mathbf{n} &= U_m && \text{on } \mathcal{E}_m, \text{ for } m = 1, \ldots, M, \\ \int_{\mathcal{E}_m} \nabla w \cdot \mathbf{n} \, \mathrm{d}s &= I_m && \text{for } m = 1, \ldots, M. \end{aligned}$$

This technique is also used in [8, 9].

5. Numerical examples. The a posteriori error estimate obtained in Corollary 4.2 is used to compute the error of approximate solutions. Consider the domain $\Omega =]0,1[\times]0,1[\subset \mathbb{R}^2$ with $M = 12$ electrodes, contact impedances $z_1, \ldots, z_M = 0.5$, and a conductivity $\sigma : \Omega \to]0, \infty[$ defined as

$$\sigma(\mathbf{x}) = 1 + 4 \times \mathbb{1}_{\text{square}}(\mathbf{x}) + 2 \times \mathbb{1}_{\text{rectangle}}(\mathbf{x}) \quad \mathbf{x} = (x_1, x_2) \in \Omega,$$

where $\mathbb{1}_{\text{square}}$ and $\mathbb{1}_{\text{rectangle}}$ are indicator functions. See Figure 5.1. Here, an approximate solution consists of a piecewise linear function defined over an admissible triangulation of Ω and of a second component represented by its coordinates in a basis of $\mathbb{R}_\diamond^M$. Numerical tests with triangulations of $T_k = 2 \times (7 \times k)^2$ triangles and $N_k = (1 + 7 \times k)^2$ nodes, for $k = 1, \ldots, 13$, were performed. In the $k-$th test the area of each triangle is $h_k = 1/T_k$. The randomly generated current

$$\begin{aligned} I = (&0.15452372, -0.52951106, 0.32024663, 0.30371975, 0.0185459, 0.17877045 \\ &0.08249105, -0.47137138, -0.16242009, 0.2910693, 0.1427927, -0.32885696) \in \mathbb{R}_\diamond^M \end{aligned}$$

of norm 1 is applied to generate the approximate solutions $(u_k, U_k) \in C^0(\bar{\Omega}) \times \mathbb{R}_\diamond^M$. Then, according to the right-hand side of the estimate (4.6), the errors

$$e_k^2 = \int_{\Omega} \sigma \left|\nabla u_k\right|^2 \mathrm{d}\mathbf{x} + \sum_{m=1}^{M} \int_{\mathcal{E}_m} \frac{\left(\gamma_m u_k - U_{k,m}\right)^2}{z_m} \mathrm{d}s - 2 \sum_{m=1}^{M} I_m U_{k,m} + \sum_{m=1}^{M} I_m \bar{U}_m$$

are calculated. Note that $e_k^2 = 2 \times J(u_k, U_k) + \sum_{m=1}^{M} I_m \bar{U}_m$, where J is the objective functional of the primal problem (2.3). The power dissipated into heat $\sum_{m=1}^{M} I_m \bar{U}_m$ is calculated from an reference solution $(\bar{u}, \bar{U}) \in H^1(\Omega) \times \mathbb{R}_\diamond^M$ generated by choosing $k = 25$. Since $1 \leq \sigma, 1/z_1, \ldots, 1/z_M, C = 1$ in (4.4), and therefore the relative errors in norm $\|\cdot\|$ are bounded as follows

$$\frac{\left\|(\bar{u}, \bar{U}) - (u_k, U_k)\right\|}{\left\|(\bar{u}, \bar{U})\right\|} \leq \frac{e_k}{\left\|(\bar{u}, \bar{U})\right\|}.$$

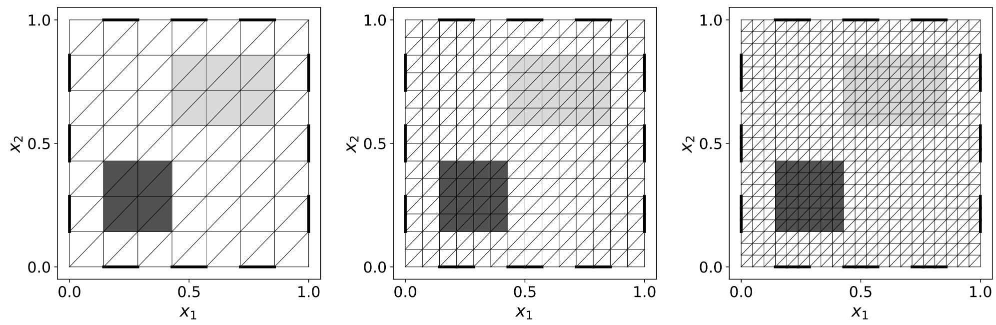

Figure 5.1: Triangulations of Ω corresponding to $k = 1$ (left), $k = 2$ (center), and $k = 3$ (right). The test conductivity has value 4 in the square, 2 in the rectangle, and 1 in the background. The thick lines on the boundary represent the positions of the $M = 12$ electrodes. The components of the current I are applied counterclockwise around the domain, beginning from the first electrode in the x_1-axis.

k	T_k	e_k^2	$\frac{e_k}{\|(\bar{u},\bar{U})\|} \times 100\%$	$\frac{J(u_k,U_k)}{J(u_{k-1},U_{k-1})}$
1	98	0.2151	29.33%	-
2	392	0.0834	18.26%	1.0324
3	882	0.0443	13.31%	1.0093
4	1568	0.0275	10.49%	1.0040
5	2450	0.0187	08.64%	1.0021
6	3528	0.0135	07.34%	1.0012
7	4802	0.0101	06.36%	1.0008
8	6272	0.0078	05.59%	1.0005
9	7938	0.0062	04.96%	1.0004
10	9800	0.0049	04.44%	1.0003
11	11858	0.0040	04.00%	1.0002
12	14112	0.0033	03.61%	1.0002
13	16562	0.0027	03.28%	1.0001

Table 5.1: *Errors of the approximate solutions* (u_k, U_k) *of the CEM*. The third and fourth columns show the a posteriori errors and the bounds of the relative errors in norm $\|\cdot\|$. Observe that the sequence $J(u_k, U_k)/J(u_{k-1}, U_{k-1})$ is decreasing and converges to 1.

The numerical results are presented in Table 5.1. There, the quotients $J(u_k, U_k)/J(u_{k-1}, U_{k-1})$ are also presented. It is worth pointing out that it is not necessary to discretize the error estimates since the conductivity is a piecewise constant function. In Figures 5.2 and 5.3 are plotted, respectively, the three dimensional surface and contour lines of the approximate solution u_k, for $k = 1, 2, 3$. Finally, note that, since the power $\sum_{m=1}^{M} I_m \bar{U}_m$ is a constant value in the error formula, to compare the errors of two approximate solutions it suffices to compute the values of the objective functional J at these approximations.

6. Conclusions. In this work, we have applied the duality theory of convex analysis to the complete electrode model (CEM). A new dual formulation of this model in terms of current fields has been derived. Using this formulation, a general error estimate has been proved. Particular choices of the auxiliary variables in the general estimate have lead to two a posteriori error estimates and to an estimate of the error between the CEM and shunt model solutions. One a posteriori error estimate is for solutions to approximate problems, and the other one is for approximate solutions. In particular, the error of approximate solutions obtained by numerical methods can be assessed. We emphasize that the fact that there is no gap between the optimal values (strong duality) allows precise upper and lower bounds for the power dissipated during current injection.

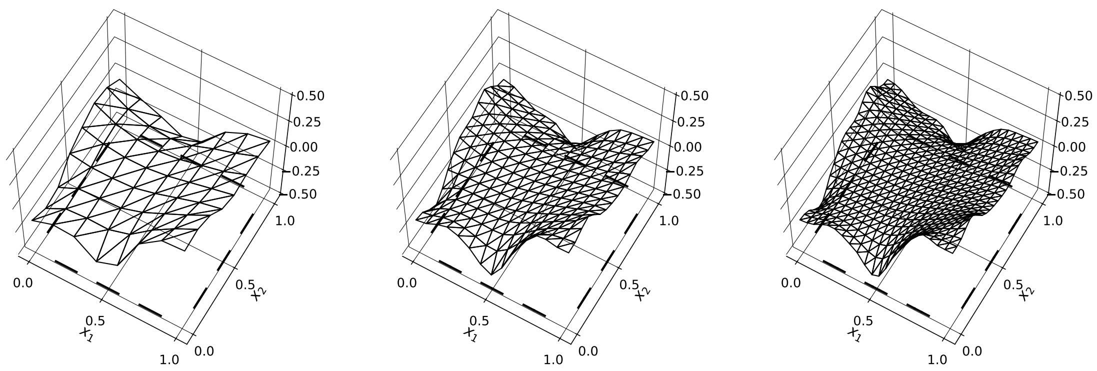

Figure 5.2: Surface of u_k, for $k = 1$ (left), $k = 2$ (center), and $k = 3$ (right). In the x_1-x_2 plane are drawn the positions of the $M = 12$ electrodes.

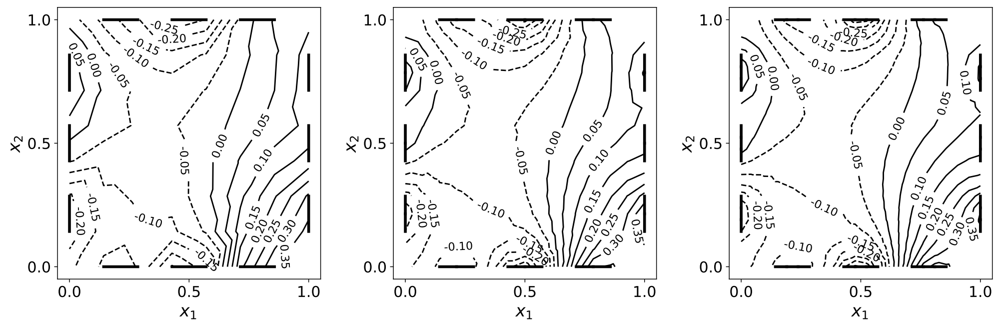

Figure 5.3: Contour lines of u_k, for $k = 1$ (left), $k = 2$ (center), and $k = 3$ (right). Observe that in the gap between electrodes, the contour lines tend to be orthogonal to the boundary. This is in concordance with the zero flux equation of the CEM.

It is worth noting that other EIT models can also fit into the duality theory of convex analysis, even when a conductivity matrix is considered (anisotropic case). Moreover, using the same ideas, it is possible to obtain a dual formulation and a posteriori error estimates for the voltage excitation case.

Future work might be concerned with solving numerically the dual formulation of the CEM and adaptive methods based on the general error estimate.

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