Implications of the delayed feedback effect on the stability of a SIR epidemic model

Implicaciones del efecto de retroalimentación con retardo en la estabilidad de un modelo epidémico SIR

Roxana López-Cruz

Abstract

A basic mathematical model in epidemiology is the SIR (Susceptible–Infected–Removed) model, which is commonly used to characterize and study the dynamics of the spread of some infectious diseases. In humans, the time scale of a disease can be short and not necessarily fatal, but in some animals (for example, insects) this same short time scale can make the disease fatal if we take into account their life expectancy. In this work, we will see how a positive feedback effect (decrease of the susceptible population at small densities) in a SIR model can cause a qualitative characterization of the dynamics defined by the original SIR model. Finally, we will also show with numerical simulations how a delay in the feedback effect causes very interesting qualitative changes of the system with epidemiological significance.

Keywords . Ordinary differential equation, feedback effect, Stability, Simulation.

Resumen

Un modelo matemático básico en epidemiología es el modelo SIR (Susceptible-Infectado-Removido), que se utiliza habitualmente para caracterizar y estudiar la dinámica de propagación de algunas enfermedades infecciosas. En los seres humanos, la escala temporal de una enfermedad puede ser corta y no necesariamente mortal, pero en algunos animales (por ejemplo, los insectos) esta misma escala temporal corta puede hacer que la enfermedad sea mortal si tenemos en cuenta su esperanza de vida. En este trabajo veremos cómo un efecto de retroalimentación positiva (disminución de la población susceptible a pequeñas densidades) en un modelo SIR puede provocar una caracterización cualitativa de la dinámica definida por el modelo SIR original. Por último, también mostraremos con simulaciones numéricas cómo un retraso en el efecto de retroalimentación provoca cambios cualitativos muy interesantes del sistema con trascendencia epidemiológica.

Palabras clave. Ecuación diferencial ordinaria, efecto de retroalimentación, estabilidad, simulación.

1. Introduction. The construction of mathematical models is one of the tools used today for the study of problems in a wide variety of areas of knowledge; their primary objectives are to describe, explain and predict phenomena and processes in these areas. However, their application is often limited by the lack of knowledge and information about the basic principles of mathematical modeling in these areas.

When we have an epidemiological system under study, we must ask ourselves in which context we must build it and under what conditions: temporal dynamics, spatial or other structures.

As for the parts, we must take into account the population variables and their parameters with which we must count, not in their totality (otherwise it would cease to be a model), but the most relevant ones. Finally,

---

*Departamento de Matemáticas, Universidad Nacional Mayor de San Marcos, Perú. (rlopezc@unmsm.edu.pe).

those that define the dynamics will be the interactions between the variables and the epidemiological effects they influence.

The mathematical modeling of epidemics becomes more complex depending on how more factors that characterize each disease are considered. However, there are external factors that influence its incidence or reproduction, and they are generated, for example, by prejudices about vaccination campaigns with prior information (negative feedback), as in the case of the negative concept in the population regarding the vaccine against the human papillomavirus (HPV) or the SARS-CoV-2 vaccine [1], [2], sometimes caused by misinformation. Feedback effects in epidemiological systems are classified into two types: Negative feedback effect and Positive feedback effect. Positive feedback is a type of regulation in biological systems in which the end product of a process in turn increases the stimulus of that same process. Such phenomena affect small populations by reducing the average individual fitness and, hence, the positive feedback effect is an important phenomenon to include in epidemic models, where epidemics tend to decrease population densities [3], [4], [5], [2], [6], [7].

There are some publications that analyze how feedback affects the diseases [6], [8], analyzed SIR-type models with negative feedback effects , and [3], [7] analyze Allee effect in populations in epidemiological systems. Both analyze their models with ordinary differential equations with a linear incidence rate with respect to the positive feedback effect. Delayed SIR models have been studied without feedback effects [9], [10].

In this work, first we study a modified SIR model including a positive feedback effect and then the same model but taking in account a delay in the reproduction growth. In section 2, we use a type of feedback effect that gives rise to an SIR epidemic model considering Allee effect (positive feedback) in the growth rate. In section 3, we analyze the population dynamics of the modified SIR model when the time delay in the positive feedback effect is considered. Finally, in section 4, computer simulations confirm stability results obtained in sections 2 and 3, and a sensitivity analysis of the threshold parameters is presented.

2. Modified SIR Model with a positive feedback effect. In this section, we modify the classical SIR model with a feedback effect in the reproduction rate and detail a discussion on the basic properties, existence of steady states and local stability, moreover.

Positive feedback effects (Allee type) into population growth rates are defined by:
Logistic Positive Feedback Effect

$$r(x) = x(x-1).$$

Strong Positive Feedback Effect

$$r(x) = x(x-1)(x-a), \ o < a < 1.$$

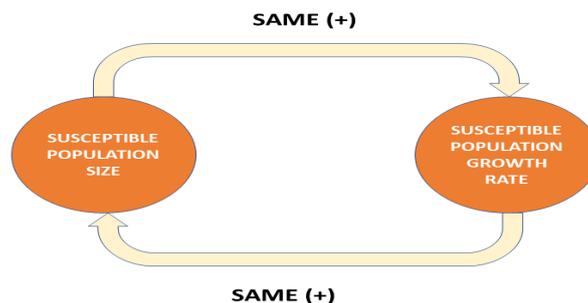Weak Positive Feedback Effect

$$r(x) = x(x-1)(x+b), \ b > 0.$$

Figure 2.1: Strong Positive Feedback effect.

For the construction of this model, we consider three subpopulations of the total population, suceptible individuals $S(t)$, infected individuals $I(t)$ and removed individuals $R(t)$. The parameters are defined in the table (2.1) Let's also consider some assumptions:

- incidence rate is linear
- there is no disease related mortality

Parameters	Meaning
r	birth rate per capita
K	carrying capacity
M	minimum viable population size (positive feedback threshold)
β	incidence rate
α	removing rate

Table 2.1: System Parameter Meanings.

- the infected I is removed by recovery or death at rate α
- the equation for S considers a strong positive feedback effect (Fig. 2.1), where we distinguished two different thresholds in the susceptible population: the per capita susceptible population growth is negative below M and when $0 < M < S < K$, the per capita growth rate is positive.

The transitions between these three subpopulations occur as follows in the flow diagram (2.2), with all these considerations.
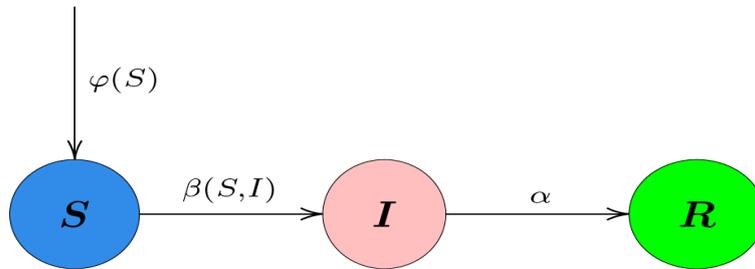

Figure 2.2: Compartamental flow diagram, where the population growth rate is $\varphi(S) = r\left(1 - \dfrac{S(t)}{K}\right)(S - M)$ and the incidence rate is $\beta(S, I) = \beta I$.

The modified SIR model with positive feedback effect is given by the system (2.1)

$$
\begin{cases}
\dfrac{dS}{dT} = \left(r\left(1 - \dfrac{S}{K}\right)(S - M) - \beta I\right)S, \\[3mm]
\dfrac{dI}{dT} = \beta IS - \alpha I, \\[3mm]
\dfrac{dR}{dT} = \alpha I.
\end{cases}
\tag{2.1}
$$

where $0 < M < K$ and initial conditions $S(0) \geq 0$, $I(0) \geq 0$, and $R(0) \geq 0$.

2.1. System Reduction. From the above system (2.1) we can infer that S and I are free from the effect of R, in consequence we reduce the model to

$$
\begin{cases}
\dfrac{dS}{dT} = \left(r\left(1 - \dfrac{S}{K}\right)(S - M) - \beta I\right)S, \\[3mm]
\dfrac{dI}{dT} = \beta IS - \alpha I.
\end{cases}
\tag{2.2}
$$

with initial conditions $S(0) \geq 0$, and $I(0) \geq 0$. Now, to reduce the number of parameters and to determine

which parameter combinations control the behavior of the system, we choose the following transformation

$$(S, I, T) = \left(Ks, \frac{rK}{\beta}i, \frac{1}{rK}t \right). \tag{2.3}$$

and change of parameters

$$(m, b, a) = \left(\frac{M}{K}, \frac{\beta}{r}, \frac{\alpha}{\beta K} \right), \tag{2.4}$$

where $m < 1$. Consequently, the system (2.1) is reduced to the new system

$$\begin{cases} \dfrac{ds}{dt} &= \; ((1 - s)(s - m) - i)\; s, \\[4mm] \dfrac{di}{dt} &= \; b(s - a)\; i, \end{cases} \tag{2.5}$$

with the initial conditions $s(0) \geq 0$, and $i(0) \geq 0$.

The change of coordinates (2.3) and reparametrization (2.4) define a biunique orientation-preserving diffeomorphism between trajectories of system (2.2) and trajectories of system (2.5), it means both systems are topological equivalent (solutions have same qualitative structure).

2.2. Stability Analysis.

Steady states. The steady states of the system 2.5 are given by the following Lemma.

Lemma 2.1. *The reduced model has a minimum of three equilibrium points and a maximum of four.*
  a) *Without conditions on the parameters, there exists three equilibrium points*
      – $P_1(0, 0)$, *total extinction equilibrium point.*
      – $P_2(1, 0)$, *disease free equilibrium point (eradication boundary equilibrium).*
      – $P_3(m, 0)$, *disease free equilibrium point (saturation boundary equilibrium)*
  b) *If $m < a < 1$ in equation (2.2), then it has one singularity in the interior of the first quadrant given by*
      – $P_4(a, (1 - a)(a - m))$, *endemic equilibrium point.*

The system (2.5) evaluated at the bifurcation parameter $a = 1$ have a simple zero eigenvalue, and other eigenvalue is real and negative $\lambda_2 = m - 1 < 0$.

Basic reproduction number. The basic reproduction number is defined by

$$\mathscr{R}_0 = \frac{b}{ba} = \frac{1}{a},$$

it means, the expected number of secondary cases produced in a completely susceptible population, by a typical infective individual.

Local stability. If $\mathscr{R}_0 > 1$, the infectives grows and if $\mathscr{R}_0 \leq 1$ the infectives decreases to zero.

Theorem 2.1. *For the system (2.5), we have*
  a) *if $\mathscr{R}_0 > 1$, $P_2(1, 0)$ is locally unstable,*
  b) *if $\mathscr{R}_0 < 1$, $P_2(1, 0)$ is locally stable,*
  c) *if $\mathscr{R}_0 = 1$, $P_2(1, 0)$ is linearly neutrally stable.*
Proof: The characteristic equation of the system (2.5) evaluated at $P_1(1, 0)$ is

$$(\lambda - b(1 - a))(\lambda - (m - 1)) = 0.$$

It has one negative real root $\lambda_1 = m - 1 < 0$ and the other root is $\lambda_2 = b(1 - a)$. In consequence,
  a) if $\mathscr{R}_0 > 1$, $\lambda_2 = b(1 - a) > 0$, then $P_2(1, 0)$ is locally unstable,
  b) if $\mathscr{R}_0 < 1$, $\lambda_2 = b(1 - a) < 0$, then $P_2(1, 0)$ is locally stable,
  c) if $\mathscr{R}_0 = 1$, the system (2.5), has a simple zero eigenvalue, then $P_2(1, 0)$ is linearly neutrally stable. Hence, to investigate the stability of P_2, have to be study with the application of the bifurcation theory (center manifold theory).

$\square$

Theorem 2.2. *For the system (2.5), the disease free steady state $P_3(m, 0)$ is unstable.*

a) If $\mathscr{R}_0 < \dfrac{1}{m}$, P_3 is a saddle point.

b) If $\mathscr{R}_0 > \dfrac{1}{m}$, P_3 is a source node.

Proof: The characteristic equation of the system (2.5) evaluated at $P_3\,(m,0)$ is as follows:

$$\lambda^2 - [m(1-m) + b(m-a)]\lambda + bm(1-m)(m-a) = 0$$

or

$$(\lambda - (1-m)m)(\lambda - b(m-a)) = 0$$

As $0 < m < 1$, we have $\lambda_1 = (1-m)m > 0$, in consequence, $P_3(m,0)$ is unstable. Moreover, the second eigenvalue is $\lambda_2 = b(m-a)$, then

a) If $\mathscr{R}_0 < \dfrac{1}{m} \Rightarrow m < a$, the eigenvalue $\lambda_2 = b(m-a) < 0$, implies P_3 is a saddle point.

b) If $\mathscr{R}_0 > \dfrac{1}{m} \Rightarrow m > a$, the eigenvalue $\lambda_2 = b(m-a) > 0$, implies P_3 is a source node.

$\square$

Theorem 2.3. *For the system (2.5), the endemic steady state $P_4(a, (1-a)(a-m))$ is*
a) *locally asymptotically stable, if $(2a - m - 1) > 0$,*
b) *unstable if $(2a - m - 1) < 0$.*

Proof: The characteristic equation of the system (2.5) evaluated at $P_4 = (a, (1-a)(a-m))$ is as follows:

$$\mathcal{P}(\lambda) = \lambda^2 + a(2a - m - 1)\lambda + ab(1-a)(a-m) = 0.$$

As $(1-a)(a-m) > 0$, using Routh-Hurwitz method
- if $(2a - m - 1) > 0$, $\mathcal{P}(\lambda)$ has both roots with negative real part, in consequence P_4 is locally asymptotically stable and
- if $(2a - m - 1) < 0$, $\mathcal{P}(\lambda)$ has one root with positive real part, it means P_4 is unstable.

$\square$

3. Modified SIR Model with delay. The SIR model modified by the positive feedback effect and a lag in the growth rate where τ represents the age of maximum reproductive capacity of an individual in the population, is given by the following system of three-dimensional differential equations:

$$\begin{cases} \dfrac{dS}{dT} &= \left(r\left(1 - \dfrac{S(T-\mathcal{T})}{K}\right)(S-M) - \beta I \right) S, \\[3mm] \dfrac{dI}{dT} &= \beta IS - \alpha I, \\[3mm] \dfrac{dR}{dT} &= \alpha I, \end{cases} \tag{3.1}$$

where $M < K$ and initial conditions: $S(\theta) = \Phi(\theta) \geq 0$, $I(\theta) = \Psi(\theta)$, $R(\theta) = \Upsilon(\theta)$, $\theta \in [-\mathcal{T}, 0]$, $\Phi, \Psi, \Upsilon \in C([-\mathcal{T}, 0], \mathbb{R}_+)$, $S(0) > 0$, $I(0) > 0$ and $R(0) > 0$. Here C denotes the Banach space of continuous functions mapping the interval $[-\mathcal{T}, 0]$ into $\mathbb{R}_+$.

From the above system (3.1) we can infer that S and I are free from the effect of R.

$$\begin{cases} \dfrac{dS}{dT} &= \left(r\left(1 - \dfrac{S(T-\mathcal{T})}{K}\right)(S-M) - \beta I \right) S, \\[3mm] \dfrac{dI}{dT} &= \beta IS - \alpha I, \end{cases} \tag{3.2}$$

with initial conditions: $S(\theta) = \Phi(\theta) \geq 0$, $I(\theta) = \Psi(\theta)$, $\theta \in [-\mathcal{T}, 0]$, $\Phi, \Psi \in C([-\mathcal{T}, 0], \mathbb{R}_+)$, $S(0) > 0$, $I(0) > 0$ and $R(0) > 0$. Here C denotes the Banach space of continuous functions mapping the interval $[-\mathcal{T}, 0]$ into $\mathbb{R}_+$. To reduce the number of parameters and determine which parameter combinations control the behavior of the system, we will dimension the system. For this, we choose the following chamge of variables and rescaling of time. Let be

$$(S, I, T, \mathcal{T}) = \left(Ks, \frac{rK}{\beta}i, \frac{1}{rK}t, \frac{1}{rK}\tau \right), \tag{3.3}$$

and the change of parameters (all positive)

$$(m, b, a) = \left(\frac{M}{K}, \frac{\beta}{r}, \frac{\alpha}{\beta K} \right),$$ (3.4)

where $m < 1$.
The new system is

$$\begin{cases} \dfrac{ds}{dt} = ((1 - s(t - \tau))(s - m) - i)\, s \\[4mm] \dfrac{di}{dt} = b(s - a)\, i \end{cases},$$ (3.5)

with the initial conditions $s(\theta) = \phi(\theta) \geq 0$, $i(\theta) = \psi(\theta)$, $\theta \in [-\tau, 0]$, $\phi, \psi \in C([-\tau, 0])$. Where C is the Banach space of continuous functions maping the interval $[-\tau, 0]$ into $\mathbb{R}_+^2$.
The change of coordinates (3.3) and reparametrization (3.4) define a biunique orientation-preserving diffeomorphism between trajectories of system (3.2) and trajectories of system (3.5), it means both systems are topological equivalent (solutions have same qualitative structure).

3.1. Stability Analysis. The stability of neutral delay-differential systems with a single delay via Routh–Hurwitz criteria is investigated [11], [12], and [13]. The reduced delayed model has the same steady states of the model without delay (2.5). it means, a minimum of three equilibrium points and a maximum of four.
Define the matrices

$$J = \begin{pmatrix} (1 - s_\tau)(2s - m) - i & -s \\ bi & b(s - a) \end{pmatrix}, \quad J_D = \begin{pmatrix} -(s - m)s & 0 \\ 0 & 0 \end{pmatrix},$$

with $s_\tau = s(t - \tau)$. The characteristic equation of the system (3.5) is

$$\lambda^2 + p\lambda + r + (s\lambda + q)e^{-\lambda\tau} = 0,$$

where

$$\begin{aligned} p &= -Tr(J) & &= -(1 - s_\tau)(2s - m) - i - b(s - a) \\ r &= Det(J) & &= [(1 - s_\tau)(2s - m) - i]b(s - a) + bis \\ s &= -Tr(J_D) & &= -s(s - m) \\ q &= Det(J + J_D) - Det(J) - Det(J_D) &&= 0. \end{aligned}$$

Theorem 3.1. *The disease free steady state (eradication) $P_2(1, 0)$ of system (3.5) is*
  a) unstable if $\mathcal{R}_0 > 1$,
  b) locally asymptotically stable (LAS) if $\mathcal{R}_0 < 1$, and
  c) linearly neutrally stable if $\mathcal{R}_0 = 1$.
for time lag $\tau > 0$.

  Proof: The characteristic equation of the system (3.5) evaluated at $P_2(1, 0)$ is as follows:

$$\lambda^2 - b(1 - a)\lambda + [(1 - m)\lambda - b(1 - a)(1 - m)]e^{-\lambda\tau} = 0,$$

equivalent to

$$[\lambda - ba(\mathcal{R}_0 - 1)][\lambda + (1 - m)e^{-\lambda\tau}] = 0$$

  a) If $\mathcal{R}_0 > 1$
    The characteristic equation has one positive real root $\lambda_1 = ba(\mathcal{R}_0 - 1) > 0$, then the DFE (eradication) $P_2(1, 0)$ is unstable.
  b) If $\mathcal{R}_0 < 1$
    The characteristic equation has one negative real root $\lambda_1 = ba(\mathcal{R}_0 - 1) < 0$, and the other roots can be obtained from

$$\lambda + (1 - m)e^{-\lambda\tau} = 0$$

Assume that $Re(\lambda) \geq 0$ replacing $\lambda = A + iB,\ A, B \geq 0$ in the last equation, we get

$$A + iB = (m - 1)[e^{-A\tau} \cos(B\tau) + i \sin(B\tau)]$$

$$A = (m - 1)e^{-A\tau} \cos(B\tau) < m - 1 < 0,$$

a contradiction. Hence, if $\mathscr{R}_0 < 1$, the eigenvalues λ_2, λ_3 has negative real part. In consequence, the DFE $P_2(1, 0)$ is locally asymptoticalle stable.

c) If $\mathscr{R}_0 = 1$, $P_2(1, 0)$ has a simple zero eigenvalue, then P_2 is linearly neutrally stable. $\square$

$\square$

Theorem 3.2. *The disease free steady state $P_3(m, 0)$ of system (3.5) is unstable.*

- *If $\mathscr{R}_0 < \dfrac{1}{m}$, P_3 is a saddle point.*
- *If $\mathscr{R}_0 > \dfrac{1}{m}$, P_3 is a source node.*

Proof: The characteristic equation of the system (3.5) evaluated at $P_3\ (m, 0)$ is as follows:

$$\lambda^2 - [m(1 - m) + b(m - a)]\lambda + bm(1 - m)(m - a) = 0$$

$$(\lambda - (1 - m)m)(\lambda - b(m - a)) = 0$$

the eigenvalues are

$$\lambda_1 = (1 - m)m > 0, \quad \lambda_2 = b(m - a)$$

In consequence, P_3 is a saddle point if $\mathscr{R}_0 < \dfrac{1}{m}\ (\Rightarrow \lambda_2 < 0)$ and P_3 is a source node if $\mathscr{R}_0 > \dfrac{1}{m}\ (\Rightarrow \lambda_2 > 0)$.

. $\square$

Define the basic replacement ratio

$$\mathscr{R}_1 = \frac{a(m - 1)(2a - m - 1)}{2b(a - 1)(m - a)}$$

Theorem 3.3. *For $\tau > 0$, the endemic equilibrium point $P_4\ (a, (1 - a)(a - m))$ of system (3.5) is locally asymptotically stable if $\mathscr{R}_1 > 1$ and $2a - m - 1 > 0$.*

Proof: The characteristic equation of the system (3.5) evaluated at $P_4\ (a, (1 - a)(a - m))$ is as follows:

$$\lambda^2 - a(1 - a)\lambda + ab(1 - a)(a - m) + [a(a - m)]\lambda e^{-\lambda\tau} = 0 \tag{3.6}$$

If instability occurs for some value of $\mathscr{R}_1$, then the characteristic root of the equation (3.6) cross the imaginary axis.

Assume the $\lambda = \xi i,\ \xi > 0$ is a root of the characteristic equation, replacing, we get the equation

$$-\xi^2 + a(a - 1)\xi i + ab(a - 1)(m - a) + a(a - m)(\xi i)e^{-(\xi i)\tau} = 0$$

$$-\xi^2 + a(a - 1)\xi i + ab(a - 1)(m - a) + a(a - m)(\xi i)[\cos(\xi\tau) - i \sin(\xi\tau)] = 0$$

separating real and imaginary parts, we get

$$\begin{cases} a(a - m)\xi \sin(\xi\tau) &= \xi^2 - ab(a - 1)(m - a), \\ a(a - m)\xi \cos(\xi\tau) &= -a(a - 1)\xi. \end{cases}$$

squaring and adding both equations, we get

$$\xi^4 + [-a^2(m - a)^2 - 2ab(a - 1)(m - a) + a^2(a - 1)^2]\xi^2 + a^2b^2(a - 1)^2(m - a)^2 = 0.$$

If $x = \xi^2$, the last equation becomes

$$x^2 + Ax + B = 0,$$

where $B = a^2b^2(a - 1)^2(m - a)^2 > 0$ and with the conditions $\mathscr{R}_1 > 1$ and $2a - m - 1 > 0$, we have $A = a^2(m - 1)(2a - m - 1) - 2ab(a - 1)(m - a) > 0$.

Using the Routh Hurwitz criterion, the quadratic equation has two roots with negative real parts, if $A > 0$ and $B > 0$, but this is a contradiction with $\lambda = i\xi$.

Hence the endemic equilibrium P_4 is locally aymptotically stable for $\tau > 0$ if $\mathscr{R}_1 \leq 1$. $\square$

4. Sensitivity Analysis.

(a) LHS/PRCC Sensitivity analysis of $\mathscr{R}_0$.

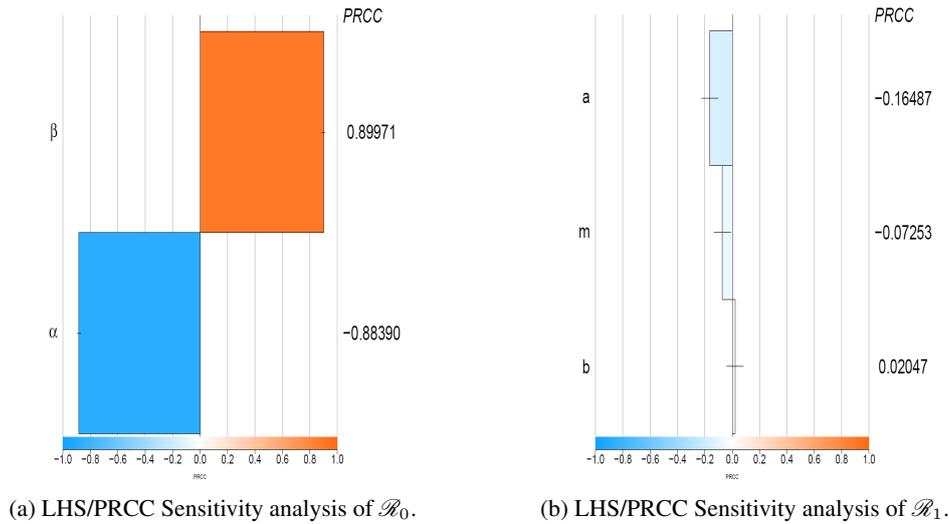(b) LHS/PRCC Sensitivity analysis of $\mathscr{R}_1$.

Figure 4.1: LHS/PRCC Sensitivity analysis of threshold parameters.

Latin Hypercube Sampling/Partial Rank Correlation Coefficient (LHS/PRCC) sensitivity analysis is an efficient tool often employed in uncertainty analysis to explore the entire parameter space of a model with a minimum number of computer simulations [14], [15]. It involves the combination of two statistical techniques, Latin Hypercube Sampling (LHS), which was first introduced by [16] and further developed by [17], and Partial Rank Correlation Coefficient.

We checked that $\mathscr{R}_0$ is highly sensitive and positively correlated to the infection rate β (PRCC index: 0.89971), implying that an increase in the input flux will have a greater effect on $\mathscr{R}_1$ and the severity of an outbreak. The recovery rate α is also highly (PRCC index absolute value: 0.88390) and negatively correlated with $\mathscr{R}_1$, implying that decreasing the average infection time (i.e. increasing the recovery rate) will reduce both $\mathscr{R}_1$ and the spread of the outbreak (Fig, 2.2a)

The threshold number $\mathscr{R}_1$ is poorly sensitive and positively correlated to the parameter rate b (PRCC index: 0.02047), implying that an increase in the input flux will have a greater effect on $\mathscr{R}_1$ and the severity of an outbreak. The parameters a and m are also poorly (PRCC index absolute value: 0.0.16487 and 0.07253 respectively) and negatively correlated with $\mathscr{R}_1$, implying that decreasing the $a = 1/\mathscr{R}_0$ (increasing of $\mathscr{R}_0$) and m parameters will reduce $\mathscr{R}_1$ and the spread of the outbreak (Fig. 2.2b).

5. Simulations.

In order to corroborate the analytical results obtained in this work, we consider the most important equilibrium point, the endemic equilibrium point. Here, we provide some numerical simulations.

First, from the results in sections 2 and 3, about the basic reproduction number $\mathscr{R}_0$ and the basic number $\mathscr{R}_1$, determines that they are bifurcation parameters, keeping all other parameters and initial conditions constant (Fig. 5.1). The $\mathscr{R}_0 = 1$ plane determines the stability region of P_2 when cross to the instability. The $\mathscr{R}_1 = 1$ plane determines the stability region of P_4 when cross to the instability without consider the aditional condition of the sign of the expression $2a - m - 1$.

Second, suppose a non constant population, the rate a as a bifurcation parameter which determines changes in nature of local stability of the positive equilibrium point (endemic). The other parameter values are chosen in such a way that they show different scenarios. For the following simulations, the values are $b = 1.5, m = 0.2$.
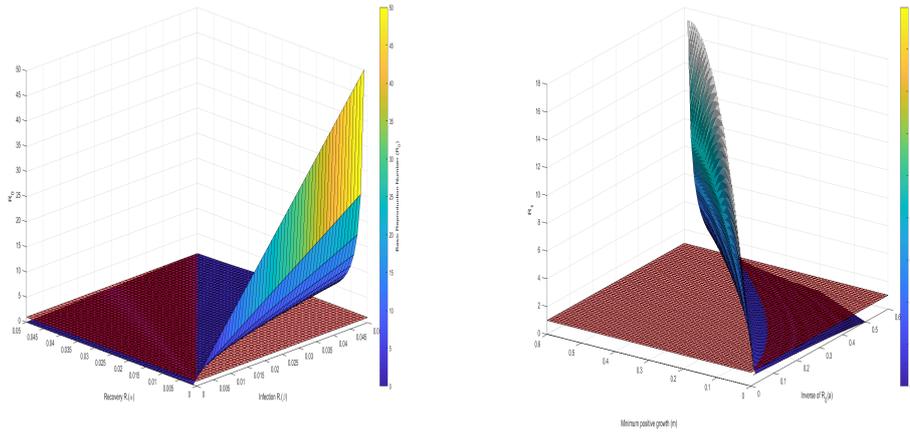
(a) Sensitivity analysis of $\mathscr{R}_0$ dependent on parameters α and β.

(b) Sensitivity analysis of $\mathscr{R}_1$ dependent on parameters m and a.

Figure 5.1: Sensitivity analysis of threshold parameters dependent on two parameters.

We have the following results:

- For $a = 0.7$ (lower basic reproduction number $\mathscr{R}_0 = 1.4285 > 1$) and $2a - m - 1 = 0.2 > 0$, the simulations in Fig. 5.2a, 5.2b correspond to the stable dynamics of the positive equilibrium point $P_4(0.7, 0.15)$.
- For $a = 0.3$ (higher basic reproduction number $\mathscr{R}_0 = 3.327$) and $2a - m - 1 < 0$, the simulations in Fig. 5.2c, 5.2d correspond to an unstable dynamics of the positive equilibrium point $P_4(0.3, 0.07)$.
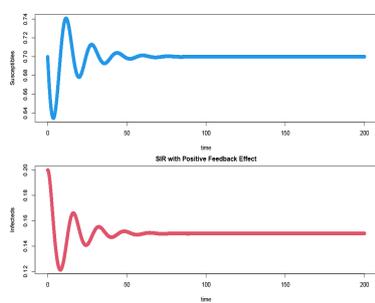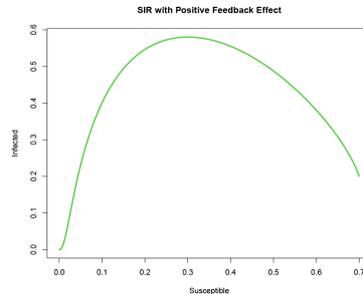

(a) Stable Endemic point with $a = 0.7$
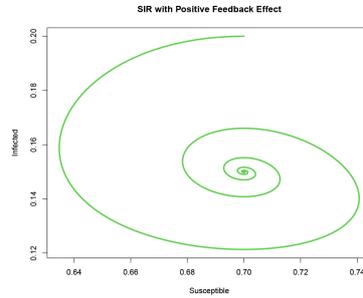
(b) Stable endemic point with $a = 0.7$
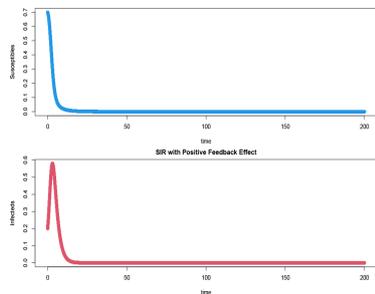
(c) Unstable endemic point with $a = 0.3$

(d) Unstable Endemic point with $a = 0.3$

Figure 5.2: Non delayed SIR model with Positive Feedback effect. For different values of parameter a and constant parameter values: $b = 1.5$ and $m = 0.2$, $\tau = 0$ and initial conditions $s(0) = 0.7$, $i(0) = 0.2$.

All the results are according to the non delayed SIR model with positive feedback effect (Allee effect type) results in Theorem 2.3. We can check that the susceptible and infected population oscillate over time and finally tends to the steady state (endemic) $P_4(0.3, 0.07)$ when the parameter $a = 0.7$ and tends to the extinction steady state $P_1(0, 0)$ when $a = 0.3$. (Fig. 5.2).

Third, in (Fig. 5.3), according to the delayed SIR model with positive feedback effect (Allee effect type): we check that the solution trajectory tends towards a positive state when $a = 0.7$ and towards the total extinction state when $a = 0.3$ with and additional condition.

- For $a = 0.7$ (lower basic reproduction number $\mathscr{R}_0 = 1.4285 > 1$), it is satisfied $\mathscr{R}_1 = -2.488 < 1$ and $2a - m - 1 = 0, 2 > 0$, in consequence the simulations in Fig. 5.2a, 5.2b correspond to the stable dynamics of the positive steady state $P_4(0.7, 0.15)$.
- For $a = 0.3$ (higher basic reproduction number $\mathscr{R}_0 = 3.333 > 1$), it is satisfied $\mathscr{R}_1 = 0.685 < 1$ but $2a - m - 1 = -0.6 < 0$, in consequence, the simulations in Fig. 5.2c, 5.2d correspond to an unstable dynamics of the positive steady state $P_4(0.3, 0.07)$ and tends to the total extinction state $P_1(0, 0)$.

We can check that the susceptible and infected population oscillate over time and tends to a positive equilibrium $P_4(0.7, 0.15)$, when the parameter $a = 0.7$, $2a - m - 1 > 0$ and $\mathscr{R}_1 < 1$, and tends to the total extinction steady state $P_0(0, 0)$ for $2a - m - 1 > 0$ and $\mathscr{R}_1 > 1$ as theoretically predicted in Theorem 3.3.

(a) Stable Endemic point with $a = 0.7$

(b) Stable endemic point with $a = 0.7$

(c) Unstable endemic point with $a = 0.3$
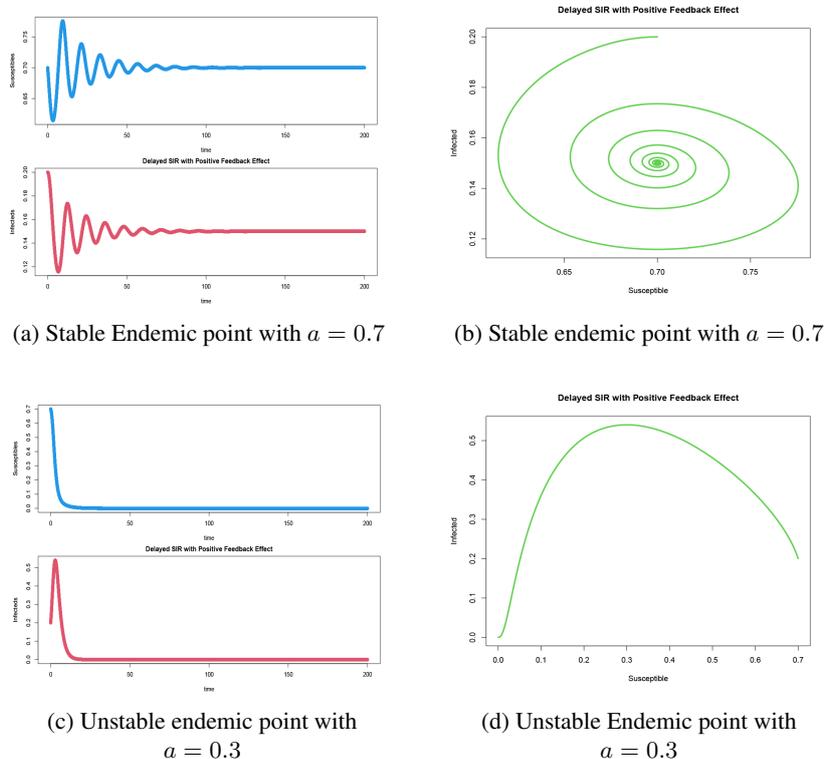
(d) Unstable Endemic point with $a = 0.3$

Figure 5.3: Delayed SIR model with Positive Feedback effect. For different values of parameter a and constant parameter values: $b = 1.5$ and $m = 0.2$, $\tau = 1.3$ and initial conditions $s(0) = 0.7$, $i(0) = 0.2$.

Finally, for $a = 0.5995$ (medium basic reproduction number $\mathscr{R}_0 = 1.668 > 1$), the simulation in Fig. 5.4 show that solution trajectory tends to have a periodic dynamics around the positive steady state $P_4(0.5995, 0.1599)$.

Both models exhibit periodically behavior but in the delayed model SIR, the periodic dynamics is observed to accelerate.

(a) Without delay $\tau = 0$

(b) Without delay $\tau = 0$

(c) With delay $\tau = 1.3$
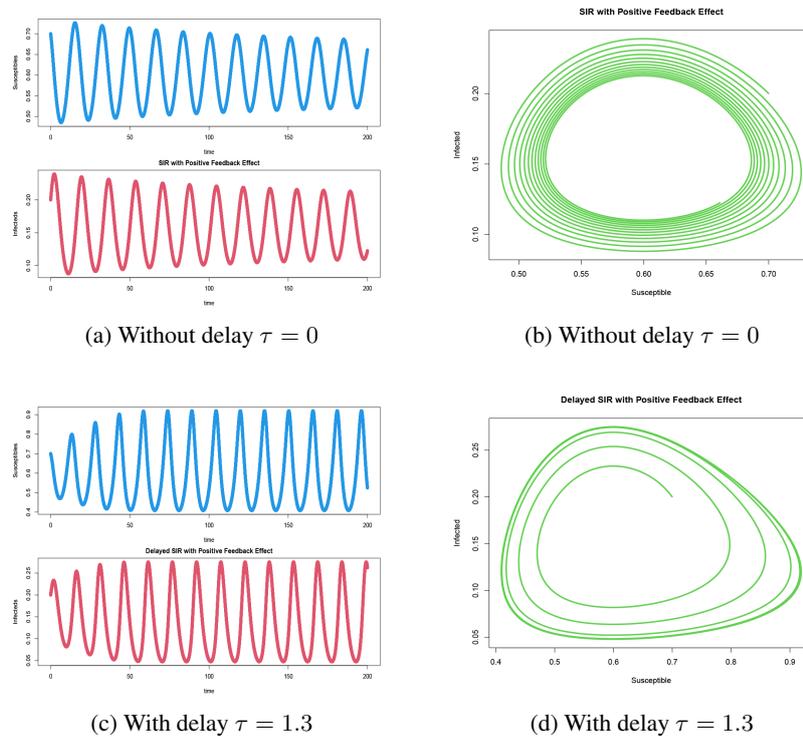

(d) With delay $\tau = 1.3$

Figure 5.4: The solution trajectory exhibits periodic behavior as theoretically predicted for $\tau = 0$ and $\tau > 0$. Constant parameter values: $a = 0.5995$, $b = 1.5$ and $m = 0.2$ and initial conditions $s(0) = 0.7$, $i(0) = 0.2$.

6. Conclusions. In this work, a system of nonlinear delay differential equations is utilized to investigate the dynamic behaviors of a SIR epidemic model with positive feedback effect with a delay. The system is analyzed by using bifurcation theory for local stability of the steady states. The conditions on the system parameters that ensure the existence of a periodic behavior are obtained. Our analysis find that the positive steady state of our system is asymptotically stable when the parameter a is part of a threshold parameter which is less than a critical value. By simulation, we check the existence of solutions with periodic behavior due to the threshold parameter and the positive feedback effect. It would be good to analyze the continuation of this existence for larger values of the delay parameter.

By analysis, we proved what happen with the stability of the SIR epidemic model with positive feedback effect without delay (previously studied) and then we appreciate numerically the switch going from stability to instability of the disease-free equilibrium of the system with delay.

The feedback (Density-dependent) effects which may either be positive or negative can play a key role in the population dynamics of species by modifying their population per capita growth rates.

The time delay on the reproduction of susceptible population affected by a positive feedback plays a significant role in controlling the infected population. We can see that when the threshold parameter $\mathscr{R}_1$ is less than the critical value, the reproduction of susceptibles is fast enough to control the infected population to some certain levels. This work let us to check the implications of the delayed positive feedback effect in a SIR model.

ORCID and License
Roxana López-Cruz https://orcid.org/0000-0002-7703-5784

References

[1] Goel K. Stability behavior of a nonlinear mathematical epidemic transmission model with time delay. Nonlinear Dynamics. 2019 Oct;98(2):1501-18.

[2] López-Cruz R. Global stability of an SAIRD epidemiological model with negative feedback. Advances in Continuous and Discrete Models. 2022 Dec;2022(1):1-3.

[3] Hilker FM, Langlais M, Malchow H. The Allee effect and infectious diseases: extinction, multistability, and the (dis-) appearance of oscillations. The American Naturalist. 2009 Jan;173(1):72-88.

[4] Hilker FM. Population collapse to extinction: the catastrophic combination of parasitism and Allee effect. Journal of Biological Dynamics. 2010 Jan 1;4(1):86-101.

[5] Lobov SA, Zharinov AI, Berdnikova ES, Kurganov DP, Kazantsev VB. Information feedback provokes multi-peak dynamics in the modern world epidemic spreads. Research Square; 2022. DOI: 10.21203/rs.3.rs-1496894/v1.

[6] Lv Y, Chen L, Chen F, Li Z. Stability and bifurcation in an SI epidemic model with additive Allee effect and time delay. International Journal of Bifurcation and Chaos. 2021 Mar 30;31(04):2150060.

[7] Thieme HR, Dhirasakdanon T, Han Z, Trevino R. Species decline and extinction: synergy of infectious disease and Allee effect?. Journal of Biological Dynamics. 2009 May 1;3(2-3):305-23.

[8] Usaini S, Lloyd AL, Anguelov R, Garba SM. Dynamical behavior of an epidemiological model with a demographic Allee effect. Mathematics and Computers in Simulation. 2017 Mar 1;133:311-25.

[9] Guo ZG, Li J, Li C, Liang J, Yan Y. Pattern Dynamics of Nonlocal Delay SI Epidemic Model with the Growth of the Susceptible following Logistic Mode. Complexity. 2020 Jul 26;2020.

[10] Zhang FF, Jin Z, Sun GQ. Bifurcation analysis of a delayed epidemic model. Applied Mathematics and Computation. 2010 Apr 1;216(3):753-67.

[11] Fukuda I, Kiri Y, Saito W, Ueda Y. Stability Criteria for the System of Delay Differential Equations and its Applications. Osaka Journal of Mathematics. 2022 Jan;59(1):235-51.

[12] Hu GD, Hu GD, Cahlon B. Algebraic criteria for stability of linear neutral systems with a single delay. Journal of Computational and Applied mathematics. 2001 Oct 1;135(1):125-33.

[13] Song Y, Han M, Wei J. Stability and Hopf bifurcation analysis on a simplified BAM neural network with delays. Physica D: Nonlinear Phenomena. 2005 Jan 15;200(3-4):185-204.

[14] Blower SM, Dowlatabadi H. Sensitivity and uncertainty analysis of complex models of disease transmission: an HIV model, as an example. International Statistical Review/Revue Internationale de Statistique. 1994 Aug 1:229-43.

[15] Marino S, Hogue IB, Ray CJ, Kirschner DE. A methodology for performing global uncertainty and sensitivity analysis in systems biology. Journal of theoretical biology. 2008 Sep 7;254(1):178-96.

[16] McKay MD, Beckman RJ, Conover WJ. A comparison of three methods for selecting values of input variables in the analysis of output from a computer code. Technometrics. 2000 Feb 1;42(1):55-61.

[17] Iman RL, Helton JC. An investigation of uncertainty and sensitivity analysis techniques for computer models. Risk analysis. 1988 Mar;8(1):71-90.