Rational Bilinear Forms as Cocycles

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Abstract

In this article, some nontrivial $2-$cocycles are computed for crossed products of algebraic extensions of the field of rational numbers $\mathbb{Q}$ with cyclic groups of order 2 and 3, with finite groups of order 4, 6 and 8. The $2-$cocycles are applied for bilinear forms over finite-dimensional vector space whose scalars field is $\mathbb{Q}$ and few examples of these cocycles are presented in terms of matrices.

Keywords. Rational bilinear form, $2-$cocycle, crossed product, Galois extension, Galois Group of a polynomial.

1. Introduction. The authors of [1] studied the Hochschild (co)homology of a twisted group algebra $A *_f G$, where A/k is a simple algebraic extension of fields and G is a finite groupIn this case $E = A *_f G$ is a crossed product of an algebra with a group, where f is a $2-$cocyclic for E. The study of Hochschild homology of the crossed product $E = \dfrac{K[X]}{\langle X^2 \rangle} \rtimes_f C_2$ originated the concern of to compute the nontrivial cocycles of crossed products. We inquire if $k = \mathbb{Q}$ and $G = S_3$ (nonabelian group), is possible to find a nontrivial $2-$cocycle f such that $A \rtimes_f G$ is a crossed product for some $\mathbb{Q}-$algebra A.

In [2] is presented as example of polynomial not solvable by radicals $f(x) = 2x^5 - 5x^4 + 5$ over $\mathbb{Q}$ by means of Affirmation 2: If $f(x)$ is an irreducible polynomial in $\mathbb{Q}[x]$ of degree 5 that has exactly 2 nonreal roots in the field of the complex numbers, then Galois group of $f(x)$ is the symmetric group S_5.

Following the same arguments, it is noted that this affirmation holds for polynomials of degree prime $p \geq 3$, the Galois group of an irreducible polynomial $f(x)$ in $\mathbb{Q}[x]$ of degree 3 that has exactly 2 nonreal roots in the field of the complex numbers is the symmetric group S_3. Using this fact and the characterization of a crossed product [3, Corollary 4.6] an affirmative answer is given to the question (see Section 4).

The purpose of this article is to show examples of nontrivial $2-$cocycles for crossed products of the forms $\frac{\mathbb{Q}[X]}{\langle P \rangle} \rtimes_f C_2$, $\frac{\mathbb{Q}[X]}{\langle P \rangle} \rtimes_f C_3$ and $\frac{\mathbb{Q}[X]}{\langle P \rangle} \rtimes_f G_v$. The development of this work is done in the three sections:
In the second section, a crossed product of an algebra with a group is presented.
In the third section, we apply the term of the cochain complex "$2-$cocycle" for bilinear forms over finite-dimensional vector space whose scalars field is $\mathbb{Q}$ and are presented few examples of these cocycles in terms of matrices.
In the fourth section, we give examples of rational nontrivial $2-$cocycles for the crossed products $\frac{\mathbb{Q}[X]}{\langle P \rangle} \rtimes_f C_2$, $\frac{\mathbb{Q}[X]}{\langle P \rangle} \rtimes_f C_3$, $\frac{\mathbb{Q}[X]}{\langle P \rangle} \rtimes_f G_4$, $\frac{\mathbb{Q}[X]}{\langle P \rangle} \rtimes_f G_6$ and $\frac{\mathbb{Q}[X]}{\langle P \rangle} \rtimes_f G_8$. Furthermore, it is shown that a crossed product of the form $\frac{\mathbb{Q}[X]}{\langle P \rangle} \rtimes_f G_v$ can have 2-cycles which are nonrational.

2. Cross Product of an Algebra with a Group. Let G a group, K a field, A a $K-$algebra and A^* the group of units of A. Let $\sigma : G \to Aut_K(A)$, $g \mapsto \sigma(g)(a) = a^g \; \forall a \in A$, an action of G on A; and $f : G \times G \to A^*$, $(g, h) \mapsto f(g, h)$ a map .

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Definition 2.1. [1] *A crossed product, $A \rtimes_f G$, of a $K-$algebra A with a group G, is a $K-$vector space $A \otimes K[G]$ endowed with the multiplication*

$$(a\omega_g)(b\omega_h) = ab^g f(g,h)\omega_{gh},$$

satisfying the following 3 conditions :
    1) $f(1,g) = f(g,1) = 1$,
    2) $f(g_1,g_2)^{g_0} f(g_0, g_1 g_2) = f(g_0, g_1) f(g_0 g_1, g_2)$,
    3) $(a^{g_1})^{g_0} f(g_0, g_1) = f(g_0, g_1) a^{g_0 g_1}$.

Condition 1) is the condition of normality, 2) is the condition of $2-$cocycle and 3) is the twisted module condition.

Remark 2.1. $E = A \rtimes_f G$ is a Hopf crossed product of A by $H = K[G]$.

$H = K[G]$ is a Hopf algebra with canonical basis G (see [4, Example 2.2.11]), where the comultiplication Δ and counit ε, are given by $\Delta(g) = g \otimes g$ and $\varepsilon(g) = 1$, $\forall g \in G$.

By definition, $A \rtimes_f G$ is the $K-$vector space $A \otimes K[G]$ with multiplication given by $(a\omega_g)(b\omega_h) = ab^g f(g,h)\omega_{gh}$ satisfying the conditions of normality, of $2-$cocycle and twisted module. In terms of tensorial product $\otimes = \otimes_K$,

$$(a \otimes g)(b \otimes h) = ab^g f(g,h) \otimes gh, \qquad \text{since } \Delta(g) = g \otimes g,$$
$$= ab^{g^{(1)}} f(g^{(2)}, h^{(1)}) \otimes g^{(3)} h^{(2)}.$$

Since $\Delta(g) = g \otimes g$ and $\varepsilon(g) = 1$, the following 3 conditions are satisfied :
    1) $f(1,g) = f(g,1) = \varepsilon(g)1_A$,
    2) $f(g_1^{(1)}, g_2^{(1)})^{g_0^{(1)}} f(g_0^{(2)}, g_1^{(2)} g_2^{(2)}) = f(g_0^{(1)}, g_1^{(1)}) f(g_0^{(2)} g_1^{(2)}, g_2)$,
    3) $(a^{g_1^{(1)}})^{g_0^{(1)}} f(g_0^{(2)}, g_1^{(2)}) = f(g_0^{(1)}, g_1^{(1)}) a^{g_0^{(2)} g_1^{(2)}}$.

It is observed that the action σ of G on A, induces a weak action $\#$ of $K[G]$ on A through $\#(g,a) = \sigma(g)(a)$ for $g \in G$; while that the map $f : G \times G \to A^*$ induces a $K-$bilinear map $f : K[G] \times K[G] \to A$. Therefore, according to [5] $E = A\#_f K[G]$; that is, E is a Hopf crossed product of A by $K[G]$.

3. Rational Bilinear Forms. In this section, the matrix of a rational bilinear form in an ordered basis $\mathcal{B}$ of a vector space $\mathbb{V}$ of dimension n is introduced, and the concept of normal $2-$cocycle over $\mathbb{V}$ is given.

Definition 3.1. *Let $\mathbb{V}$ be a vector space over the field of the rational numbers $\mathbb{Q}$. A rational bilinear form on $\mathbb{V}$ is a function $f : \mathbb{V} \times \mathbb{V} \to \mathbb{Q}$, which assigns to each ordered pair of vectors $u, v \in \mathbb{V}$ a scalar $f(u,v) \in \mathbb{Q}$ such that:*
    i) $f(\lambda u_1 + u_2, v) = \lambda, f(u_1, v) + f(u_2, v)$,
    ii) $f(u, \lambda v_1 + v_2) = \lambda f(u, v_1) + f(u, v_2)$,
for all $u, u_1, u_2, v, v_1, v_2 \in \mathbb{V}$ and all scalar $\lambda \in \mathbb{Q}$.

Let $\mathbb{V}$ be a finite-dimensional vector space over the field $\mathbb{Q}$, $\dim \mathbb{V} = 2$ and $\mathcal{B} = \{e_1, e_2\}$ an ordered basis of $\mathbb{V}$. Let $u = x_1 e_1 + x_2 e_2$ and $v = y_1 e_1 + y_2 e_2$ be vectors in $\mathbb{V}$.

Example 3.1. *The function $f : \mathbb{V} \times \mathbb{V} \to \mathbb{Q}$ defined by*

(3.1)
$$f(u,v) = 2x_1 y_1 + 3x_2 y_2$$

is a rational bilinear form on $\mathbb{V}$.

If $u, u', v \in \mathbb{V}$ and $\lambda \in \mathbb{Q}$, then

$$f(\lambda u + u', v) = 2(\lambda x_1 + x_1')y_1 + 3(\lambda x_2 + x_2')y_2$$
$$= \lambda(2x_1 y_1 + 3x_2 y_2) + 2x_1' y_1 + 3x_2' y_2$$
$$= \lambda f(u,v) + f(u',v).$$

Similarly, $f(u, \lambda v + v') = \lambda f(u,v) + f(u, v')$ for all $u, v, v' \in \mathbb{V}$ and $\lambda \in \mathbb{Q}$.

Example 3.2. *The function $f : \mathbb{V} \times \mathbb{V} \to \mathbb{Q}$ defined by*

(3.2)
$$f(u,v) = x_1 y_1 + x_1 y_2 + x_2 y_1 + 3x_2 y_2$$

is a rational bilinear form on $\mathbb{V}$.

If $u, v, v' \in \mathbb{V}$ and $\lambda \in \mathbb{Q}$, then

$$f(u, \lambda v' + v) = x_1(\lambda y_1 + y_1') + x_1(\lambda y_2 + y_2') + x_2(\lambda y_1 + y_1') + 3x_2(\lambda y_2 + y_2')$$
$$= \lambda(x_1 y_1 + x_1 y_2 + x_2 y_1 + 3x_2 y_2) + (x_1 y_1' + x_1 y_2' + x_2 y_1' + 3x_2 y_2')$$
$$= \lambda f(u,v) + f(u, v').$$

Similarly, $f(\lambda u + u', v) = \lambda f(u, v) + f(u', v)$ for all $u, u', v \in \mathbb{V}$ and $\lambda \in \mathbb{Q}$.

Example 3.3. *If f is a rational bilinear form on $\mathbb{V}$, then f is completely determined with the four scalars $a_{ij} = f(e_i, e_j)$ by*

$$f(u, v) = \sum_{i,j=1}^{2} a_{ij} x_i y_j.$$

If X and Y are the coordinate matrices of u and v in the basis $\mathcal{B}$, and if A is the matrix 2×2 with entries $a_{ij} = f(e_i, e_j)$, then

$$(3.3) \qquad\qquad f(u, v) = X^t A Y .$$

The Example 3.3 can be generalized to describe all the rational bilinear forms on a finite-dimensional vector space $\mathbb{V}$ over the field $\mathbb{Q}$. Let $\mathcal{B} = \{e_1, \ldots, e_n\}$ be an ordered basis of $\mathbb{V}$ and f a rational bilinear form on $\mathbb{V}$. If $u = x_1 e_1 + \cdots + x_n e_n$, $v = y_1 e_1 + \cdots + y_n e_n \in \mathbb{V}$, then

$$f(u, v) = \sum_{i,j=1}^{n} a_{ij} x_i y_j = X^t A Y ,$$

where $a_{ij} = f(e_i, e_j)$; X and Y are the coordinate matrices of u and v in the basis $\mathcal{B}$.

Thus, every rational bilinear form on $\mathbb{V}$ is given by

$$(3.4) \qquad\qquad f(u, v) = [u]_{\mathcal{B}}^t A [v]_{\mathcal{B}},$$

for some $n \times n$ matrix A over $\mathbb{Q}$.

Conversely, if A is $n \times n$ matrix over $\mathbb{Q}$, then it is easy to check that (3.4) defines a rational bilinear form on $\mathbb{V}$ such that $a_{ij} = f(e_i, e_j)$ (see Example 3.1 and Example 3.2). Then the matrix A is called the matrix of the rational bilinear form f in the basis $\mathcal{B}$, and denoted by $[f]_{\mathcal{B}}$.

Let $L(\mathbb{V}, \mathbb{V}, \mathbb{Q})$ be the vector space of all rational bilinear forms on $\mathbb{V}$.

Theorem 3.1. [6, Theorem X.1] *Let $\mathbb{V}$ be a finite-dimensional vector space over the field $\mathbb{Q}$ and $dim\mathbb{V} = n$. For each ordered basis $\mathcal{B}$ of $\mathbb{V}$, the function that associates with each rational bilinear form on $\mathbb{V}$ its matrix in the ordered basis $\mathcal{B}$, $f \mapsto [f]_{\mathcal{B}}$, is an isomorphism of the space $L(\mathbb{V}, \mathbb{V}, \mathbb{Q})$ onto the space of $n \times n$ matrices over the field $\mathbb{Q}$.*

The formula for the coboundary map is (see [7, (1.5.1.1)])

$$\beta(f)(a_1, \ldots, a_{n+1}) = a_1 f(a_2, \ldots, a_{n+1})$$
$$+ \sum_{0 < i < n+1} (-1)^i f(a_1, \ldots, a_i a_{i+1}, \ldots a_{n+1})$$
$$(3.5) \qquad\qquad + (-1)^{n+1} f(a_1, \ldots, a_n) a_{n+1}.$$

Then f is a $2-$cocycle if $af(b, c) - f(ab, c) + f(a, bc) - f(a, b)c = 0$. If the right action is trivial, $f(a, b)c = f(a, b)$. Writing the left action with exponential notation, $af(b, c) = f(b, c)^a$. Thus, the condition of $2-$cocycle is $f(b, c)^a + f(a, bc) = f(a, b) + f(ab, c)$. Using the multiplicative notation for the above condition, it will be said that f satisfies the condition of $2-$cocycle if $f(b, c)^a f(a, bc) = f(a, b) f(ab, c)$.

Definition 3.2. *Let f be a rational bilinear form on $\mathbb{V}$, $\mathcal{B}$ a basis of $\mathbb{V}$ endowed with structure of group. If exists a left action of $\mathcal{B}$ on $\mathbb{Q}$, then f is a $2-$cocycle on $\mathbb{V}$ if the matrix of f in $\mathcal{B}$ satisfies the condition of $2-$cocycle and this condition admits multiplicative notation .*

Let $\mathcal{B} = G = \langle g \rangle = \{1, g\}$ be the group of automorphisms of the field $\mathbb{Q}(\sqrt{2})$, where $g(1) = 1$ and $g(\sqrt{2}) = -\sqrt{2}$. Then $\mathbb{V} = \mathbb{Q}[G]$ and $G = G(\mathbb{Q}(\sqrt{2})/\mathbb{Q})$ [8, Example 48.7]. In the case of rational bilinear forms on $\mathbb{V}$, it will use $2-$cocycle to mean $2-$cocycle on $\mathbb{V}$.

Example 3.4. *The rational bilinear form on $\mathbb{V}$*

$$f(u, v) = x_1 y_1 + x_1 y_2 + x_2 y_1 + 3 x_2 y_2$$

is a $2-$cocycle.

Since $[f]_{\mathcal{B}} = \begin{bmatrix} f(1, 1) & f(1, g) \\ f(g, 1) & f(g, g) \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 3 \end{bmatrix}$, the condition of $2-$cocycle is satisfied;

i.e., $f(b, c)^a f(a, bc) = f(a, b) f(ab, c)$ for all $a, b, c \in \mathcal{B}$. This is expressed with the following eight equalities:

(1) $1 = f(1,1)^1 f(1,(1)(1)) = f(1,1)f((1)(1),1)$,
(2) $1 = f(1,g)^1 f(1,(1)(g)) = f(1,1)f((1)(1),g)$,
(3) $1 = f(g,1)^1 f(1,(g)(1)) = f(1,g)f((1)(g),1)$,
(4) $3 = f(g,g)^1 f(1,(g)(g)) = f(1,g)f((1)(g),g)$,
(5) $1 = f(1,1)^g f(g,(1)(1)) = f(g,1)f((g)(1),1)$,
(6) $3 = f(1,g)^g f(g,(1)(g)) = f(g,1)f((g)(1),g)$,
(7) $3 = f(g,1)^g f(g,(g)(1)) = f(g,g)f((g)(g),1)$,
(8) $3 = f(g,g)^g f(g,g^2) = f(g,g)f(g^2,g)$.

The condition of $2-$cocycle admits multiplicative notation because $f(1,1)$, $f(1,g)$, $f(g,1)$ and $f(g,g) \in \mathbb{Q}^*$.

Example 3.5. *The rational bilinear form on* $\mathbb{V}$

$$f(u,v) = 2x_1 y_1 + 3x_2 y_2$$

is not a $2-$*cocycle.*

Since $[f]_\mathcal{B} = \begin{bmatrix} f(1,1) & f(1,g) \\ f(g,1) & f(g,g) \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$, the condition of $2-$cocycle is not satisfied because

:

(4) $6 = f(g,g)^1 f(1,(g)(g)) = f(1,g)f((1)(g),g) = 0$.

Example 3.6. *The rational bilinear form on* $\mathbb{V}$

$$f(u,v) = x_1 y_1 + x_1 y_2 + x_2 y_1$$

is not a $2-$*cocycle.*

Since $[f]_\mathcal{B} = \begin{bmatrix} f(1,1) & f(1,g) \\ f(g,1) & f(g,g) \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$, the condition of $2-$cocycle does not admit multiplicative notation because :

(8) $0 = f(g,g)^g f(g,g^2) = f(g,g)f(g^2,g)$ where $f(g,g) = 0$ does not have multiplicative inverse .

Example 3.7. *Find all the* 2×2 *matrices over* $\mathbb{Q}$ *which define by* (3.4) $2-$*cocycles on* $\mathbb{V}$.

Let f be a 2$-$cocycle on $\mathbb{V}$ such that $[f]_\mathcal{B} = \begin{bmatrix} f(1,1) & f(1,g) \\ f(g,1) & f(g,g) \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$. Then the

condition of $2-$cocycle is satisfied; i.e., $f(b,c)^a f(a,bc) = f(a,b)f(ab,c)$ for all $a,b,c \in \mathcal{B}$. This is expressed with the following eight equalities:

(1) $a_{11}{}^1 a_{11} = a_{11} a_{11}$,   (5) $a_{11}{}^g a_{21} = a_{21} a_{21}$,
(2) $a_{12}{}^1 a_{12} = a_{11} a_{12}$,   (6) $a_{12}{}^g a_{22} = a_{21} a_{22}$,
(3) $a_{21}{}^1 a_{12} = a_{12} a_{21}$,   (7) $a_{21}{}^g a_{22} = a_{22} a_{11}$,
(4) $a_{22}{}^1 a_{11} = a_{12} a_{22}$,   (8) $a_{22}{}^g a_{21} = a_{22} a_{12}$.

As the condition of $2-$cocycle admits multiplicative notation for f; a_{11}, a_{12}, a_{21} and $a_{22} \in \mathbb{Q}^*$. Since the action is trivial, from (6) by the cancellation law is obtained that $a_{12} = a_{21}$. Similarly from (7), $a_{21} = a_{11}$. It is noted that $a_{22} \neq 0$ is arbitrary, then setting $a_{11} = a_{12} = a_{21} = a$ and $a_{22} = b$; it follows that $[f]_\mathcal{B} = \begin{bmatrix} a & a \\ a & b \end{bmatrix}$ where a and $b \in \mathbb{Q}^*$.

Thus, in Example 3.4 $a = 1$ and $b = 3$.

Definition 3.3. *A* $2-$*cocycle* f *on* $\mathbb{V}$ *is said to be normal if* $[f]_\mathcal{B} = \begin{bmatrix} 1 & 1 \\ 1 & b \end{bmatrix}$ *donde* $b \in \mathbb{Q}^*$.

If in addition $b = 1$ *it is said to be the* $2-$*cocycle* f *is trivial. In the case in which* $b \neq 1$, f *is nontrivial* $2-$*cocycle.*

According to Example 3.4, $f(u,v) = x_1 y_1 + x_1 y_2 + x_2 y_1 + 3x_2 y_2$ is nontrivial normal $2-$cocycle.

Since $\mathbb{Q}^*$ is abelian group, $\mathbb{Q}^*$ may be regarded as a trivial left $G-$module for any group G (as in [9], section VI.1]). If G is a finite group, then $\mathbb{V} = \mathbb{Q}[G]$ is finite-dimensional vector space over the field $\mathbb{Q}$, whose basis is $\mathcal{B} = G$. Thus, there exists trivial left action of $\mathcal{B}$ on $\mathbb{Q}$; i.e., $r^g = r$ for all $r \in \mathbb{Q}$ and $g \in \mathcal{B}$. One may consider $\mathbb{V} = \mathbb{Q}[\mathbb{Z}_3]$ where $\mathbb{Z}_3 = \langle g \rangle = \{1, g, g^2\}$.

Example 3.8. *Find all the* 3×3 *matrices over* $\mathbb{Q}$ *which allow to define by* (3.4) *normal* $2-$*cocycles on* $\mathbb{V}$.

Let f be a normal $2-$cocycle on $\mathbb{V}$. Proceeding as in the Example 3.7, it results that $[f]_{\mathcal{B}} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & a & b \\ 1 & b & \frac{b}{a} \end{bmatrix}$

where a and $b \in \mathbb{Q}^*$.

Let $u = x_1 e_1 + x_2 e_2 + x_3 e_3$, $v = y_1 e_1 + y_2 e_2 + y_3 e_3$ be vectors in $\mathbb{V} = \mathbb{Q}[\mathbb{Z}_3]$. Taking $a = 1$, the above matrix yields

Example 3.9. *The rational bilinear form on* $\mathbb{V}$ *defined by*

$$f(u,v) = \begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & b \\ 1 & b & b \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$$

is a normal $2-$*cocycle.*

Example 3.10. *If one consider the case* $\mathbb{V} = \mathbb{Q}[\mathbb{Z}_4]$. *Then the matrix*

$$\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & a & b & c \\ 1 & b & \frac{bc}{a} & \frac{c}{a} \\ 1 & c & \frac{c}{a} & \frac{c}{b} \end{pmatrix}$$

where a, b *and* $c \in \mathbb{Q}^*$; *allows to define by* (3.4) *a normal* $2-$*cocycle on* $\mathbb{V}$.

4. Rational Nontrivial Cocycles for Crossed Products. Let G be a group, K a field and K^* the group multiplicative of units of K. It is interesting to note the sequence

$$H(G, K^*) \xrightarrow{d_1} H(G \times G, K^*) \xrightarrow{d_2} H(G \times G \times G, K^*) \xrightarrow{d_3} \cdots,$$

because $f \in H(G \times G, K^*)$ is a $2-$cocycle if $d_2(f) = 1$, one can then see that $d_2(f) : G \times G \times G \to K^*$ is such that

$$\begin{aligned} d_2(f)(a,b,c) &= [[f(b,c)]^a f(a,bc)][[f(a,b)]_c f(ab,c)]^{-1}, \text{ but right action is trivial} \\ &= [[f(b,c)]^a f(a,bc)][f(a,b)f(ab,c)]^{-1} \\ &= 1 . \end{aligned}$$

Thus, f is a $2-$cocycle if it satisfies $[f(b,c)]^a f(a,bc) = f(a,b)f(ab,c)$.

Denoting $H(G \times G, K^*)$ by $H^2(G, K^*)$, the existence of a $2-$cocycle in $H^2(G(M/K), M^*)$ compatible with the action is guaranteed by finding $f \in H^2(G(M/K), M^*)$ such that $[f(b,c)]^a f(a,bc) = f(a,b)f(ab,c)$ for all $a,b,c \in G$. But, for the map f to be $2-$cocycle with respect to a cross product, it is also required that f be normal and that it satisfies the twisted module condition as we will see next.

If M/K is finite and Galois extension of a field K & $G(M/K)$ is the Galois group of M over K, then is possible to find an element f of $H^2(G(M/K), M^*)$ such that $M \rtimes_f G(M/K)$ is a crossed product. That is, there exists $f \in H^2(G(M/K), M^*)$ a normal $2-$cocycle compatible with the action $\sigma : G(M/K) \to Aut(M), g \mapsto \sigma(g) : M \to M$ defined by $\sigma(g)(a) = a^g$; such that the twisted module condition holds [3, Corollary 4.6].

4.1. Cocycles for a crossed product $\frac{\mathbb{Q}[X]}{\langle P \rangle} \rtimes_f C_2$. Let be $M = \mathbb{Q}(\sqrt{2})$, $K = \mathbb{Q}$ and $G(M/K)$ is the Galois group of M over K. Let $P(X) = X^2 - 2$. Since $M = K(\Re_{P(X)})$, according to [2, Prop 3.1.5] it is known that M is Galois extension of $\mathbb{Q}$. Since $M = \{a_1 + a_2\sqrt{2} \mid a_1, a_2 \in \mathbb{Q}\}$, $\{1, \sqrt{2}\}$ is a basis of M over $\mathbb{Q}$. Then $G(M/K) = \{1, g\}$ is described by the table

(4.1)

	1	g
1	1	1
$\sqrt{2}$	$\sqrt{2}$	$-\sqrt{2}$

Since $G(M/K)$ is the Galois group of M over K, $G(M/K)$ acts (weakly) on M.

Clearly $M = K[\sqrt{2}] = \dfrac{K[X]}{\langle \operatorname{irr}(\gamma, K) \rangle}$, where $\operatorname{irr}(\gamma, K) = X^2 - 2$, is a field; so the group of units of M is

$M^* = M\backslash\{0\}$.

Let $f : G(M/K) \times G(M/K) \to M^*$ be a function such that $E = M \rtimes_f G(M/K)$ is a crossed product. The following three conditions are satisfied according to ([5, page 140] and [3, Corollary 4.6]) :

1) $f(1, g^i) = f(g^i, 1) = 1$,

2) $f(g^j, g^k)^{g^i} f(g^i, g^{j+k}) = f(g^i, g^j) f(g^{i+j}, g^k)$,

3) $((b_1 + b_2\sqrt{2})^{g^j})^{g^i} f(g^i, g^j) = f(g^i, g^j)(b_1 + b_2\sqrt{2})^{g^{i+j}}$, where $b_1 + b_2\sqrt{2} \in M$.

All the elements of $G(M/K) \times G(M/K) \times G(M/K)$ are in the following arrangement:

$$
\begin{array}{ll}
(1,1,1) & (1,1,g) \\
(1,g,1) & (1,g,g) \\
(g,1,1) & (g,1,g) \\
(g,g,1) & (g,g,g)
\end{array}
$$

The assertion 3) for f, $((b_1 + b_2\sqrt{2})^{g^j})^{g^i} f(g^i, g^j) = f(g^i, g^j)(b_1 + b_2\sqrt{2})^{g^{i+j}}$, is satisfied due to that the $K-$algebra M is commutative and $G(M/K) \times M \to M$,
$(g^i, a_1 + a_2\sqrt{2}) \mapsto (a_1 + a_2\sqrt{2})^{g^i}$ is an action of $G(M/K)$ on M.

Considering the above arrangement, the assertion 2) for f, $f(g^j, g^k)^{g^i} f(g^i, g^{j+k}) = f(g^i, g^j) f(g^{i+j}, g^k)$, is expanded to 8 equalities (see Example 3.4).

Since $f(g^j, g^k)^1 f(1, g^{j+k}) = f(1, g^j) f(g^j, g^k)$; $f(1, g^k)^{g^i} f(g^i, g^k) = f(g^i, 1) f(g^i, g^k)$ and
$f(g^j, 1)^{g^i} f(g^i, g^j) = f(g^i, g^j) f(g^{i+j}, 1)$; the following equality holds :

(8) $f(g, g)^g f(g, g^2) = f(g, g) f(g^2, g)$.

If $f(g, g) = a + b\sqrt{2}$, $f(g, g)^g = a - b\sqrt{2}$. Since $g^2 = 1$, from the equality (8) it is deduced that $f(g, g) = a$, where $a \in \mathbb{Q}^*$.

Setting $a = 3$ is obtained the matrix $\begin{pmatrix} 3 \end{pmatrix}$. Consequently, the rational bilinear form f such that

$$
[f]_\mathcal{B} = \begin{bmatrix} f(1,1) & f(1,g) \\ f(g,1) & f(g,g) \end{bmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 3 \end{pmatrix}
$$

is a nontrivial $2-$cocycle such that $A \rtimes_f C_2$ is a crossed product, where $A = \dfrac{\mathbb{Q}[X]}{\langle X^2 - 2 \rangle}$ and $C_2 = \langle g \rangle$ is the Galois group of $X^2 - 2$ over $\mathbb{Q}$.

4.2. Cocycles for a crossed product $\frac{\mathbb{Q}[X]}{\langle P \rangle} \rtimes_f C_3$. The polynomial $P(X) = X^3 - 7X + 7$ is irreducible in $\mathbb{Q}[X]$ by Eisenstein's criterion with $p = 7$. Using the techniques of calculus for polinomial function $P(x) = x^3 - 7x + 7$ as in the proof of [2, Affirmation 2], it is seen that $P(x)$ has three real roots.
Let be γ one of the roots of $P = P(X)$, $M = \mathbb{Q}(\gamma)$, $K = \mathbb{Q}$ and $G(M/K)$ is the Galois group of M over K. Since $M = K(\Re_{P(X)})$, it is known that M is Galois extension of K. Since $M = \{a_1 + a_2\gamma + a_3\gamma^2 \mid a_1, a_2, a_3 \in \mathbb{Q}\}$, $\{1, \gamma, \gamma^2\}$ is a basis of M over $\mathbb{Q}$. Then $G(M/K) = \{1, g, g^2\}$ is described by the table

(4.2)

	1	g	g^2
1	1	1	1
γ	γ	r_2	r_3
γ^2	γ^2	r_2^2	r_3^2

where $r_1 = \gamma$, $r_2 = 3\gamma^2 + 4\gamma - 14$ and $r_3 = -3\gamma^2 - 5\gamma + 14$ are roots of
$\mathrm{irr}(\gamma, K) = X^3 - 7X + 7$ [?, page 33].
Since $G(M/K)$ is the Galois group of M over K, $G(M/K)$ acts (weakly) on M.

Clearly $M = K(\gamma) = \dfrac{K[X]}{\langle \mathrm{irr}(\gamma, K) \rangle}$, where $\mathrm{irr}(\gamma, K) = X^3 - 7X + 7$, is a field; so the group of units of M is $M^* = M\backslash\{0\}$.

Let $f : G(M/K) \times G(M/K) \to M^*$ be a function such that $E = M \rtimes_f G(M/K)$ is a crossed product. Then the following three conditions are satisfied :

1) $f(1, g^i) = f(g^i, 1) = 1$,

2) $f(g^j, g^k)^{g^i} f(g^i, g^{j+k}) = f(g^i, g^j) f(g^{i+j}, g^k)$,

3) $((a_1 + a_2\gamma + a_3\gamma^2)^{g^j})^{g^i} f(g^i, g^j) = f(g^i, g^j)(a_1 + a_2\gamma + a_3\gamma^2)^{g^{i+j}}$,

where $a_1 + a_2\gamma + a_3\gamma^2 \in M$.

All the elements of $G(M/K) \times G(M/K) \times G(M/K)$ are in the following arrangement:

$$(1,1,1) \quad (1,1,g) \quad (1,1,g^2) \quad (1,g,1) \quad (1,g,g) \quad (1,g,g^2) \quad (1,g^2,1) \quad (1,g^2,g)$$
$$(1,g^2,g^2) \quad (g,1,1) \quad (g,1,g) \quad (g,1,g^2) \quad (g,g,1) \quad (g,g,g) \quad (g,g,g^2) \quad (g,g^2,1)$$
$$(g,g^2,g) \quad (g,g^2,g^2) \quad (g^2,1,1) \quad (g^2,1,g) \quad (g^2,1,g^2) \quad (g^2,g,1) \quad (g^2,g,g) \quad (g^2,g,g^2)$$
$$(g^2,g^2,1) \quad (g^2,g^2,g) \quad (g^2,g^2,g^2)$$

Considering this arrangement, the assertion 2) for f, $f(g^j, g^k)^{g^i} f(g^i, g^{j+k}) = f(g^i, g^j) f(g^{i+j}, g^k)$, is expanded to 27 equalities.

Since $f(g^j, g^k)^1 f(1, g^{j+k}) = f(1, g^j) f(g^j, g^k)$; $f(1, g^k)^{g^i} f(g^i, g^k) = f(g^i, 1) f(g^i, g^k)$ and $f(g^j, 1)^{g^i} f(g^i, g^j) = f(g^i, g^j) f(g^{i+j}, 1)$; the following 8 equalities hold :

(14) $f(g,g)^g f(g,g^2) = f(g,g) f(g^2, g)$,

(15) $f(g,g^2)^g = f(g,g) f(g^2, g^2)$,

(17) $f(g^2, g)^g = f(g, g^2)$,

(18) $f(g^2, g^2)^g f(g, g) = f(g, g^2)$,

(23) $f(g,g)^{g^2} f(g^2, g^2) = f(g^2, g)$,

(24) $f(g, g^2)^{g^2} = f(g^2, g)$,

(26) $f(g^2, g)^{g^2} = f(g^2, g^2) f(g, g)$,

(27) $f(g^2, g^2)^{g^2} f(g^2, g) = f(g^2, g^2) f(g, g^2)$.

If $a, b \in \mathbb{Q}^*$, then the matrix

$$\begin{pmatrix} f(g,g) & f(g,g^2) \\ f(g^2,g) & f(g^2,g^2) \end{pmatrix} = \begin{pmatrix} a & b \\ b & \frac{b}{a} \end{pmatrix}$$

presents a solution of the above system of 8 equations.

Therefore, the rational bilinear form f of Example 3.8 given by

$$[f]_{\mathcal{B}} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & a & b \\ 1 & b & \frac{b}{a} \end{bmatrix}$$

is a nontrivial $2-$cocycle such that $A \rtimes_f C_3$ is a crossed product, where $A = \dfrac{\mathbb{Q}[X]}{\langle X^3 - 7X + 7 \rangle}$ and $C_3 = \langle g \rangle$ is the Galois group of $X^3 - 7X + 7$ over $\mathbb{Q}$.

4.3. Cocycles for a crossed product $\frac{\mathbb{Q}[X]}{\langle P \rangle} \rtimes_f G_4$. Let be $M = \mathbb{Q}(\sqrt{2}, \sqrt{3})$, $K = \mathbb{Q}$ and $G(M/K)$ is the Galois group of M over K. According to Fraleigh [8, Example 53.3] it is known that M is Galois extension of $\mathbb{Q}$.

Since $M = \{a_1 + a_2\sqrt{2} + a_3\sqrt{3} + a_4\sqrt{6} \mid a_1, a_2, a_3, a_4 \in \mathbb{Q}\}$, then $G(M/K) = \{g_0, g_1, g_2, g_3\}$ is described by

(4.3)

	g_0	g_1	g_2	g_3
1	1	1	1	1
$\sqrt{2}$	$\sqrt{2}$	$-\sqrt{2}$	$\sqrt{2}$	$-\sqrt{2}$
$\sqrt{3}$	$\sqrt{3}$	$\sqrt{3}$	$-\sqrt{3}$	$-\sqrt{3}$
$\sqrt{6}$	$\sqrt{6}$	$-\sqrt{6}$	$-\sqrt{6}$	$\sqrt{6}$

Hence, the group table for $G(M/K)$ is given by

$$(4.4)$$

	g_0	g_1	g_2	g_3
g_0	g_0	g_1	g_2	g_3
g_1	g_1	g_0	g_3	g_2
g_2	g_2	g_3	g_0	g_1
g_3	g_3	g_2	g_1	$g_0 \cdot$

It is considered $\sigma : G(M/K) \to Aut(M)$, $g_i \mapsto \sigma(g_i) : M \to M$ defined by
$\sigma(g_i)(a_1 + a_2\sqrt{2} + a_3\sqrt{3} + a_4\sqrt{6}) = (a_1 + a_2\sqrt{2} + a_3\sqrt{3} + a_4\sqrt{6})^{g_i}$.
Recalls that $G(M/K) \times G(M/K) = \{(g_0,g_0),(g_0,g_1),(g_0,g_2),(g_0,g_3),(g_1,g_0),(g_1,g_1),$
$(g_1,g_2),(g_1,g_3),(g_2,g_0),(g_2,g_1),(g_2,g_2),(g_2,g_3),(g_3,g_0),(g_3,g_1),(g_3,g_2),(g_3,g_3)\}$.
If $(g_i,g_j) \in G(M/K) \times G(M/K)$, then $\sigma(g_i g_j) = \sigma(g_i)\sigma(g_j)$.

In fact, if $a_1 + a_2\sqrt{2} + a_3\sqrt{3} + a_4\sqrt{6} \in M$, then by (4.3) & (4.4):

$$\sigma(g_1 g_3)(a_1 + a_2\sqrt{2} + a_3\sqrt{3} + a_4\sqrt{6}) = \sigma(g_2)(a_1 + a_2\sqrt{2} + a_3\sqrt{3} + a_4\sqrt{6})$$
$$= a_1 + a_2\sqrt{2} - a_3\sqrt{3} - a_4\sqrt{6},$$

$$\sigma(g_1)\sigma(g_3)(a_1 + a_2\sqrt{2} + a_3\sqrt{3} + a_4\sqrt{6}) = \sigma(g_1)(a_1 - a_2\sqrt{2} - a_3\sqrt{3} + a_4\sqrt{6})$$
$$= a_1 + a_2\sqrt{2} - a_3\sqrt{3} - a_4\sqrt{6}.$$

The others equalities are checked similarly.

Since $G(M/K)$ is the Galois group of M over K, the map $\sigma(g_i)$ is a $K-$algebra automorphism of M.
That is; if $\lambda \in K$, $a_1 + a_2\sqrt{2} + a_3\sqrt{3} + a_4\sqrt{6}$ and $b_1 + b_2\sqrt{2} + b_3\sqrt{3} + b_4\sqrt{6} \in M$, then :

$$1)\sigma(g_i)(\lambda(a_1 + a_2\sqrt{2} + a_3\sqrt{3} + a_4\sqrt{6})) = \lambda\sigma(g_i)(a_1 + a_2\sqrt{2} + a_3\sqrt{3} + a_4\sqrt{6}),$$
$$2)\sigma(g_i)((a_1 + a_2\sqrt{2} + a_3\sqrt{3} + a_4\sqrt{6}) + (b_1 + b_2\sqrt{2} + b_3\sqrt{3} + b_4\sqrt{6}))$$
$$= \sigma(g_i)(a_1 + a_2\sqrt{2} + a_3\sqrt{3} + a_4\sqrt{6}) + \sigma(g_i)(b_1 + b_2\sqrt{2} + b_3\sqrt{3} + b_4\sqrt{6}),$$
$$3)\sigma(g_i)((a_1 + a_2\sqrt{2} + a_3\sqrt{3} + a_4\sqrt{6})(b_1 + b_2\sqrt{2} + b_3\sqrt{3} + b_4\sqrt{6}))$$
$$= \sigma(g_i)(a_1 + a_2\sqrt{2} + a_3\sqrt{3} + a_4\sqrt{6})\sigma(g_i)(b_1 + b_2\sqrt{2} + b_3\sqrt{3} + b_4\sqrt{6}).$$

Therefore, $G(M/K)$ acts (weakly) on M.

By the primitive element theorem [2, p 74] exists $\gamma \in M$ such that $M = K[\gamma] = \dfrac{K[X]}{\langle \mathrm{irr}(\gamma, K)\rangle}$ is a field;
so the group of units of M is $M^* = M\backslash\{0\}$.
For example $(1 + \sqrt{2} + \sqrt{3} + \sqrt{6})^{-1} = \dfrac{1}{2}(1 - \sqrt{2} - \sqrt{3} + \sqrt{6})$.
If $\gamma = 1 + \sqrt{2} + \sqrt{3} + \sqrt{6}$, then $\mathrm{irr}(\gamma, K) = X^4 - 4X^3 - 10X^2 - 20X - 23$.

Let $f : G(M/K) \times G(M/K) \to M^*$ be a function such that $E = M \rtimes_f G(M/K)$ is a crossed
product. Then the following three conditions are satisfied :
   1) $f(g_0, g_i) = f(g_i, g_0) = 1$,
   2) $f(g_j, g_k)^{g_i} f(g_i, g_j g_k) = f(g_i, g_j) f(g_i g_j, g_k)$,
   3) $((b_1 + b_2\sqrt{2} + b_3\sqrt{3} + b_4\sqrt{6})^{g_j})^{g_i} f(g_i, g_j) = f(g_i, g_j)(b_1 + b_2\sqrt{2} + b_3\sqrt{3} + b_4\sqrt{6})^{g_i g_j}$,
where $b_1 + b_2\sqrt{2} + b_3\sqrt{3} + b_4\sqrt{6} \in M$.
All the elements of $G(M/K) \times G(M/K) \times G(M/K)$ are the following:

(g_0,g_0,g_0)	(g_0,g_0,g_1)	(g_0,g_0,g_2)	(g_0,g_0,g_3)	(g_0,g_1,g_0)	(g_0,g_1,g_1)	(g_0,g_1,g_2)	(g_0,g_1,g_3)
(g_0,g_2,g_0)	(g_0,g_2,g_1)	(g_0,g_2,g_2)	(g_0,g_2,g_3)	(g_0,g_3,g_0)	(g_0,g_3,g_1)	(g_0,g_3,g_2)	(g_0,g_3,g_3)
(g_1,g_0,g_0)	(g_1,g_0,g_1)	(g_1,g_0,g_2)	(g_1,g_0,g_3)	(g_1,g_1,g_0)	(g_1,g_1,g_1)	(g_1,g_1,g_2)	(g_1,g_1,g_3)
(g_1,g_2,g_0)	(g_1,g_2,g_1)	(g_1,g_2,g_2)	(g_1,g_2,g_3)	(g_1,g_3,g_0)	(g_1,g_3,g_1)	(g_1,g_3,g_2)	(g_1,g_3,g_3)
(g_2,g_0,g_0)	(g_2,g_0,g_1)	(g_2,g_0,g_2)	(g_2,g_0,g_3)	(g_2,g_1,g_0)	(g_2,g_1,g_1)	(g_2,g_1,g_2)	(g_2,g_1,g_3)
(g_2,g_2,g_0)	(g_2,g_2,g_1)	(g_2,g_2,g_2)	(g_2,g_2,g_3)	(g_2,g_3,g_0)	(g_2,g_3,g_1)	(g_2,g_3,g_2)	(g_2,g_3,g_3)
(g_3,g_0,g_0)	(g_3,g_0,g_1)	(g_3,g_0,g_2)	(g_3,g_0,g_3)	(g_3,g_1,g_0)	(g_3,g_1,g_1)	(g_3,g_1,g_2)	(g_3,g_1,g_3)
(g_3,g_2,g_0)	(g_3,g_2,g_1)	(g_3,g_2,g_2)	(g_3,g_2,g_3)	(g_3,g_3,g_0)	(g_3,g_3,g_1)	(g_3,g_3,g_2)	(g_3,g_3,g_3)

Using the group table (4.4), $f(g_j, g_k)^{g_i} f(g_i, g_j g_k) = f(g_i, g_j) f(g_i g_j, g_k)$ is expanded to 64 equalities. Since $f(g_j, g_k)^{g_0} f(g_0, g_j g_k) = f(g_0, g_j) f(g_j, g_k)$; $f(g_0, g_k)^{g_i} f(g_i, g_k) = f(g_i, g_0) f(g_i, g_k)$ and $f(g_j, g_0)^{g_i} f(g_i, g_j) = f(g_i, g_j) f(g_i g_j, g_0)$; the following 27 equalities hold :

(22) $f(g_1, g_1)^{g_1} = f(g_1, g_1)$,

(23) $f(g_1, g_2)^{g_1} f(g_1, g_3) = f(g_1, g_1)$,

(24) $f(g_1, g_3)^{g_1} f(g_1, g_2) = f(g_1, g_1)$,

(26) $f(g_2, g_1)^{g_1} f(g_1, g_3) = f(g_1, g_2) f(g_3, g_1)$,

(27) $f(g_2, g_2)^{g_1} = f(g_1, g_2) f(g_3, g_2)$,

(28) $f(g_2, g_3)^{g_1} f(g_1, g_1) = f(g_1, g_2) f(g_3, g_3)$,

(30) $f(g_3, g_1)^{g_1} f(g_1, g_2) = f(g_1, g_3) f(g_2, g_1)$,

(31) $f(g_3, g_2)^{g_1} f(g_1, g_1) = f(g_1, g_3) f(g_2, g_2)$,

(32) $f(g_3, g_3)^{g_1} = f(g_1, g_3) f(g_2, g_3)$,

(38) $f(g_1, g_1)^{g_2} = f(g_2, g_1) f(g_3, g_1)$,

(39) $f(g_1, g_2)^{g_2} f(g_2, g_3) = f(g_2, g_1) f(g_3, g_2)$,

(40) $f(g_1, g_3)^{g_2} f(g_2, g_2) = f(g_2, g_1) f(g_3, g_3)$,

(42) $f(g_2, g_1)^{g_2} f(g_2, g_3) = f(g_2, g_2)$,

(43) $f(g_2, g_2)^{g_2} = f(g_2, g_2)$,

(44) $f(g_2, g_3)^{g_2} f(g_2, g_1) = f(g_2, g_2)$,

(46) $f(g_3, g_1)^{g_2} f(g_2, g_2) = f(g_2, g_3) f(g_1, g_1)$,

(47) $f(g_3, g_2)^{g_2} f(g_2, g_1) = f(g_2, g_3) f(g_1, g_2)$,

(48) $f(g_3, g_3)^{g_2} = f(g_2, g_3) f(g_1, g_3)$,

(54) $f(g_1, g_1)^{g_3} = f(g_3, g_1) f(g_2, g_1)$,

(55) $f(g_1, g_2)^{g_3} f(g_3, g_3) = f(g_3, g_1) f(g_2, g_2)$,

(56) $f(g_1, g_3)^{g_3} f(g_3, g_2) = f(g_3, g_1) f(g_2, g_3)$,

(58) $f(g_2, g_1)^{g_3} f(g_3, g_3) = f(g_3, g_2) f(g_1, g_1)$,

(59) $f(g_2, g_2)^{g_3} = f(g_3, g_2) f(g_1, g_2)$,

(60) $f(g_2, g_3)^{g_3} f(g_3, g_1) = f(g_3, g_2) f(g_1, g_3)$,

(62) $f(g_3, g_1)^{g_3} f(g_3, g_2) = f(g_3, g_3)$,

(63) $f(g_3, g_2)^{g_3} f(g_3, g_1) = f(g_3, g_3)$,

(64) $f(g_3, g_3)^{g_3} = f(g_3, g_3)$.

If $f(g_i, g_j) = a_{ij} + b_{ij}\sqrt{2} + c_{ij}\sqrt{3} + d_{ij}\sqrt{6}$, $f(g_1, g_1)^{g_1} = a_{11} - b_{11}\sqrt{2} + c_{11}\sqrt{3} - d_{11}\sqrt{6}$. From $f(g_1, g_1)^{g_1} = f(g_1, g_1)$, it follows that $f(g_1, g_1) = a_{11} + c_{11}\sqrt{3}$. Similarly from $f(g_2, g_2)^{g_2} = f(g_2, g_2)$ and $f(g_3, g_3)^{g_3} = f(g_3, g_3)$ it is obtained $f(g_2, g_2) = a_{22} + b_{22}\sqrt{2}$, $f(g_3, g_3) = a_{33} + d_{33}\sqrt{6}$ respectively. We note that the equations (27) with (59), (32) with (48) & (38) with (54) are the same. Then we must resolve a system of 24 equations.

By the condition of normality 1) of f, exist 9 variables $f(g_i, g_j)$ for $i, j : 1, 2, 3$. Furthermore, each of these variables depends on 4 variables. Thus, exist 36 variables. That is, one has a system of 24 equations in 36 variables.

Since $f(g_1, g_2)^{g_1} = a_{12} - b_{12}\sqrt{2} + c_{12}\sqrt{3} - d_{12}\sqrt{6}$; $f(g_1, g_3) = a_{13} + b_{13}\sqrt{2} + c_{13}\sqrt{3} + d_{13}\sqrt{6}$, and $(a+b\sqrt{2}+c\sqrt{3}+d\sqrt{6})(a'+b'\sqrt{2}+c'\sqrt{3}+d'\sqrt{6}) = (aa'+2bb'+3cc'+6dd')+(ab'+ba'+3cd'+3dc')\sqrt{2}+ (ac' + 2bd' + ca' + 2db')\sqrt{3} + (ad' + bc' + cb' + da')\sqrt{6}$, the equation $f(g_1, g_2)^{g_1} f(g_1, g_3) = f(g_1, g_1)$ is written as

$$\begin{cases} a_{12}a_{13} - 2b_{12}b_{13} + 3c_{12}c_{13} - 6d_{12}d_{13} = a_{11} \\ a_{12}b_{13} - b_{12}a_{13} + 3c_{12}d_{13} - 3d_{12}c_{13} = 0 \\ a_{12}c_{13} - 2b_{12}d_{13} + c_{12}a_{13} - 2d_{12}b_{13} = c_{11} \\ a_{12}d_{13} - b_{12}c_{13} + c_{12}b_{13} - d_{12}a_{13} = 0 \, . \end{cases}$$

Since $f(g_1, g_3)^{g_1} = a_{13} - b_{13}\sqrt{2} + c_{13}\sqrt{3} - d_{13}\sqrt{6}$ and $f(g_1, g_2) = a_{12} + b_{12}\sqrt{2} + c_{12}\sqrt{3} + d_{12}\sqrt{6}$, the equation $f(g_1, g_3)^{g_1} f(g_1, g_2) = f(g_1, g_1)$ is written as

$$\begin{cases} a_{13}a_{12} - 2b_{13}b_{12} + 3c_{13}c_{12} - 6d_{13}d_{12} = a_{11} \\ a_{13}b_{12} - b_{13}a_{12} + 3c_{13}d_{12} - 3d_{13}c_{12} = 0 \\ a_{13}c_{12} - 2b_{13}d_{12} + c_{13}a_{12} - 2d_{13}b_{12} = c_{11} \\ a_{13}d_{12} - b_{13}c_{12} + c_{13}b_{12} - d_{13}a_{12} = 0 \, . \end{cases}$$

Therefore the equations (23) with (24) are the same. Similarly, the equations (42) with (44) & (62) with (63) are the same. So, the system is reduced to one of 21 equations in 36 variables.

If a_{11}, a_{13} and $a_{22} \in \mathbb{Q}^*$, then the matrix

$$\begin{pmatrix} f(g_1, g_1) & f(g_1, g_2) & f(g_1, g_3) \\ f(g_2, g_1) & f(g_2, g_2) & f(g_2, g_3) \\ f(g_3, g_1) & f(g_3, g_2) & f(g_3, g_3) \end{pmatrix}$$

can be written as

$$
\begin{pmatrix}
a_{11} & \frac{a_{11}}{a_{13}} & a_{13} \\
\frac{a_{11}}{a_{13}} & a_{22} & \frac{a_{13}a_{22}}{a_{11}} \\
a_{13} & \frac{a_{13}a_{22}}{a_{11}} & \frac{a_{13}^2 a_{22}}{a_{11}}
\end{pmatrix}
\text{ or }
\begin{pmatrix}
a_{11} & \frac{a_{11}}{a_{13}} & a_{13} \\
-\frac{a_{11}}{a_{13}} & a_{22} & -\frac{a_{13}a_{22}}{a_{11}} \\
-a_{13} & \frac{a_{13}a_{22}}{a_{11}} & -\frac{a_{13}^2 a_{22}}{a_{11}}
\end{pmatrix}.
$$

This matrix presents a solution of the above system of 21 equations with 36 variables. If $a_{11} = a_{13} = 1$ & $a_{22} = 4$, it is obtained the matrix

$$
\begin{pmatrix}
1 & 1 & 1 \\
-1 & 4 & -4 \\
-1 & 4 & -4
\end{pmatrix}.
$$

Consequently, the rational bilinear form f such that

$$
[f]_{\mathcal{B}} =
\begin{pmatrix}
1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 \\
1 & -1 & 4 & -4 \\
1 & -1 & 4 & -4
\end{pmatrix}
$$

is a nontrivial $2-$cocycle such that $\frac{\mathbb{Q}[X]}{\langle P \rangle} \rtimes_f G_4$ is a crossed product, where $P = X^4 - 5X^2 + 6$ and G_4 is the Galois group of P over $\mathbb{Q}$.

4.3.1. Computing Nonrational $2-$cocycles. From (22) $f(g_1, g_1)^{g_1} = f(g_1, g_1)$, one has $f(g_1, g_1) = a_{11} + c_{11}\sqrt{3}$. If $f(g_1, g_3) = b_{13}\sqrt{2}$ where $b_{13} \in \mathbb{Q}^*$, then $a_{13} = c_{13} = d_{13} = 0$.

Since $f(g_1, g_2)^{g_1} f(g_1, g_3) = (a_{12} - b_{12}\sqrt{2} + c_{12}\sqrt{3} - d_{12}\sqrt{6})b_{13}\sqrt{2}$
$$= -2b_{12}b_{13} + a_{12}b_{13}\sqrt{2} - 2d_{12}b_{13}\sqrt{3} + c_{12}b_{13}\sqrt{6};$$

$f(g_1, g_1) = a_{11} + c_{11}\sqrt{3}$, from (23) $f(g_1, g_2)^{g_1} f(g_1, g_3) = f(g_1, g_1)$ one obtains $-2b_{12}b_{13} = a_{11}$, $a_{12} = 0$, $-2d_{12}b_{13} = c_{11}$ and $c_{12} = 0$. If $c_{11} = 0$, then $f(g_1, g_1) = a_{11}$ and $f(g_1, g_2) = -\frac{a_{11}}{2b_{13}}\sqrt{2}$.

Since $f(g_2, g_1)^{g_1} f(g_1, g_3) = (a_{21} - b_{21}\sqrt{2} + c_{21}\sqrt{3} - d_{21}\sqrt{6})b_{13}\sqrt{2}$
$$= -2b_{21}b_{13} + a_{21}b_{13}\sqrt{2} - 2d_{21}b_{13}\sqrt{3} + c_{21}b_{13}\sqrt{6};$$
$$f(g_1, g_2)f(g_3, g_1) = -\frac{a_{11}}{2b_{13}}\sqrt{2}(a_{31} + b_{31}\sqrt{2} + c_{31}\sqrt{3} + d_{31}\sqrt{6})$$
$$= -\frac{a_{11}b_{31}}{b_{13}} - \frac{a_{11}a_{31}}{2b_{13}}\sqrt{2} - \frac{a_{11}d_{31}}{b_{13}}\sqrt{3} - \frac{a_{11}c_{31}}{2b_{13}}\sqrt{6};$$

from (26) $f(g_2, g_1)^{g_1} f(g_1, g_3) = f(g_1, g_2)f(g_3, g_1)$ it follows that

(4.5) $2b_{21}b_{13} = \frac{a_{11}b_{31}}{b_{13}}$, $a_{21}b_{13} = -\frac{a_{11}a_{31}}{2b_{13}}$, $2d_{21}b_{13} = \frac{a_{11}d_{31}}{b_{13}}$ and $c_{21}b_{13} = -\frac{a_{11}c_{31}}{2b_{13}}$.

From (43) $f(g_2, g_2)^{g_2} = f(g_2, g_2)$, one obtains $f(g_2, g_2) = a_{22} + b_{22}\sqrt{2}$. If $b_{22} = 0$, $f(g_2, g_2) = a_{22}$.

Since $f(g_1, g_2)f(g_3, g_2) = -\frac{a_{11}}{2b_{13}}\sqrt{2}(a_{32} + b_{32}\sqrt{2} + c_{32}\sqrt{3} + d_{32}\sqrt{6})$
$$= -\frac{a_{11}b_{32}}{b_{13}} - \frac{a_{11}a_{32}}{2b_{13}}\sqrt{2} - \frac{a_{11}d_{32}}{b_{13}}\sqrt{3} - \frac{a_{11}c_{32}}{2b_{13}}\sqrt{6};$$

from (27) $f(g_2, g_2)^{g_1} = f(g_1, g_2)f(g_3, g_2)$ one obtains that $b_{32} = -\frac{a_{22}b_{13}}{a_{11}}$,

$a_{32} = c_{32} = d_{32} = 0$. Thus $f(g_3, g_2) = -\frac{a_{22}b_{13}}{a_{11}}\sqrt{2}$.

From (64) $f(g_3, g_3)^{g_3} = f(g_3, g_3)$, one has $f(g_3, g_3) = a_{33} + d_{33}\sqrt{6}$.

$$\text{Since } f(g_2, g_3)^{g_1} f(g_1, g_1) = (a_{23} - b_{23}\sqrt{2} + c_{23}\sqrt{3} - d_{23}\sqrt{6})a_{11}$$
$$= a_{23}a_{11} - b_{23}a_{11}\sqrt{2} + c_{23}a_{11}\sqrt{3} - d_{23}a_{11}\sqrt{6} \ ;$$
$$f(g_1, g_2)f(g_3, g_3) = (-\frac{a_{11}}{2b_{13}}\sqrt{2})(a_{33} + d_{33}\sqrt{6})$$
$$= -\frac{a_{11}a_{33}}{2b_{13}}\sqrt{2} - \frac{a_{11}d_{33}}{b_{13}}\sqrt{3};$$

from (28) $f(g_2, g_3)^{g_1} f(g_1, g_1) = f(g_1, g_2)f(g_3, g_3)$ one obtains that $a_{23} = d_{23} = 0$, $b_{23} = \frac{a_{33}}{2b_{13}}$, $c_{23} = -\frac{d_{33}}{b_{13}}$. Thus $f(g_2, g_3) = \frac{a_{33}}{2b_{13}}\sqrt{2} - \frac{d_{33}}{b_{13}}\sqrt{3}$.

$$\text{Since } f(g_3, g_2)^{g_1} f(g_1, g_1) = (-\frac{a_{22}b_{13}}{a_{11}}\sqrt{2})^{g_1} a_{11}$$
$$= a_{22}b_{13}\sqrt{2} \ ;$$

$f(g_1, g_3)f(g_2, g_2) = (b_{13}\sqrt{2})a_{22}$ from (31) $f(g_3, g_2)^{g_1} f(g_1, g_1) = f(g_1, g_3)f(g_2, g_2)$ one obtains the identity $a_{22}b_{13}\sqrt{2} = a_{22}b_{13}\sqrt{2}$.

$$\text{Since } f(g_3, g_3)^{g_1} = (a_{33} + d_{33}\sqrt{6})^{g_1} = a_{33} - d_{33}\sqrt{6} \ ;$$
$$f(g_1, g_3)f(g_2, g_3) = b_{13}\sqrt{2}(\frac{a_{33}}{2b_{13}}\sqrt{2} - \frac{d_{33}}{b_{13}}\sqrt{3}) = a_{33} - d_{33}\sqrt{6};$$

from (32) $f(g_3, g_3)^{g_1} = f(g_1, g_3)f(g_2, g_3)$ it is obtained the identity $a_{33} - d_{33}\sqrt{6} = a_{33} - d_{33}\sqrt{6}$.

$$\text{Since } f(g_1, g_2)^{g_2} f(g_2, g_3) = (-\frac{a_{11}}{2b_{31}}\sqrt{2})^{g_2}(\frac{a_{33}}{2b_{13}}\sqrt{2} - \frac{d_{33}}{b_{13}}\sqrt{3})$$
$$= -\frac{a_{11}a_{33}}{2b_{13}^2} + \frac{a_{11}d_{33}}{2b_{13}^2}\sqrt{6} \ ;$$
$$f(g_2, g_1)f(g_3, g_2) = (a_{21} + b_{21}\sqrt{2} + c_{21}\sqrt{3} + d_{21}\sqrt{6})(-\frac{a_{22}b_{13}}{a_{11}}\sqrt{2})$$
$$= -\frac{2b_{21}a_{22}b_{13}}{a_{11}} - \frac{a_{21}a_{22}b_{13}}{a_{11}}\sqrt{2} - \frac{2d_{21}a_{22}b_{13}}{a_{11}}\sqrt{3}$$
$$- \frac{c_{21}a_{22}b_{13}}{a_{11}}\sqrt{6}$$

from (39) $f(g_1, g_2)^{g_2} f(g_2, g_3) = f(g_2, g_1)f(g_3, g_2)$ one obtains that

(4.6) $a_{21} = d_{21} = 0$, $b_{21} = \frac{a_{11}^2 a_{33}}{4b_{13}^3 a_{22}}$ and $c_{21} = -\frac{a_{11}^2 d_{33}}{2b_{13}^3 a_{22}}$.

Thus $f(g_2, g_1) = b_{21}\sqrt{2} + c_{21}\sqrt{3}$. From (4.5) $a_{31} = d_{31} = 0$, so that $f(g_3, g_1) = b_{31}\sqrt{2} + c_{31}\sqrt{3}$.

$$\text{Since } f(g_1, g_1)^{g_1} = a_{11} \ ;$$
$$f(g_2, g_1)f(g_3, g_1) = (b_{21}\sqrt{2} + c_{21}\sqrt{3})(b_{31}\sqrt{2} + c_{31}\sqrt{3})$$
$$= (2b_{21}b_{31} + 3c_{21}c_{31}) + (b_{21}c_{31} + c_{21}b_{31})\sqrt{6};$$

from (38) $f(g_1, g_1)^{g_1} = f(g_2, g_1)f(g_3, g_1)$ one obtains the system

(4.7) $\begin{cases} 2b_{21}b_{31} + 3c_{21}c_{31} = a_{11} \\ b_{21}c_{31} + c_{21}b_{31} = 0 \ . \end{cases}$

$$\text{Since } f(g_1, g_3)^{g_2} f(g_2, g_2) = b_{13}a_{22}\sqrt{2} \ ;$$
$$f(g_2, g_1)f(g_3, g_3) = (b_{21}\sqrt{2} + c_{21}\sqrt{3})(a_{33} + d_{33}\sqrt{6})$$
$$= (b_{21}a_{33} + 3c_{21}d_{33})\sqrt{2} + (2b_{21}d_{33} + c_{21}a_{33})\sqrt{3};$$

from (40) $f(g_1,g_3)^{g_2}f(g_2,g_2) = f(g_2,g_1)f(g_3,g_3)$ one obtains the system

$$\begin{cases} b_{21}a_{33} + 3c_{21}d_{33} = b_{13}a_{22} \\ 2b_{21}d_{33} + c_{21}a_{33} = 0 \ . \end{cases}$$

Since $f(g_3,g_1)^{g_2}f(g_2,g_2) = (b_{31}\sqrt{2} - c_{31}\sqrt{3})a_{22} = b_{31}a_{22}\sqrt{2} - c_{31}a_{22}\sqrt{3}$;

$$f(g_2,g_3)f(g_1,g_1) = (\frac{a_{33}}{2b_{13}}\sqrt{2} - \frac{d_{33}}{b_{13}}\sqrt{3})a_{11} = \frac{a_{33}a_{11}}{2b_{13}}\sqrt{2} - \frac{d_{33}a_{11}}{b_{13}}\sqrt{3};$$

from (46) $f(g_3,g_1)^{g_2}f(g_2,g_2) = f(g_2,g_3)f(g_1,g_1)$ one obtains that

(4.8) $$b_{31} = \frac{a_{33}a_{11}}{2a_{22}b_{13}} \ , \ c_{31} = \frac{d_{33}a_{11}}{a_{22}b_{13}} \ .$$

Since $f(g_1,g_3)^{g_3}f(g_3,g_2) = (-b_{13}\sqrt{2})(-\frac{a_{22}b_{13}}{a_{11}}\sqrt{2}) = 2\frac{a_{22}b_{13}^2}{a_{11}}$;

$$f(g_3,g_1)f(g_2,g_3) = (b_{31}\sqrt{2} + c_{31}\sqrt{3})(\frac{a_{33}}{2b_{13}}\sqrt{2} - \frac{d_{33}}{b_{13}}\sqrt{3})$$

$$= (\frac{b_{31}a_{33}}{b_{13}} - 3\frac{c_{31}d_{33}}{b_{13}}) + (-\frac{b_{31}d_{33}}{b_{13}} + \frac{c_{31}a_{33}}{2b_{13}})\sqrt{6};$$

from (56) $f(g_1,g_3)^{g_3}f(g_3,g_2) = f(g_3,g_1)f(g_2,g_3)$ one obtains

$$\begin{cases} b_{31}a_{33} - 3c_{31}d_{33} = 2\frac{a_{22}b_{13}^3}{a_{11}} \\ -b_{31}d_{33} + \frac{c_{31}a_{33}}{2} = 0 \ . \end{cases}$$

Recalling (4.6) : $b_{21} = \frac{a_{11}^2 a_{33}}{4b_{13}^3 a_{22}}, c_{21} = -\frac{a_{11}^2 d_{33}}{2b_{13}^3 a_{22}}$.

From (4.8) $b_{31} = \frac{a_{33}a_{11}}{2a_{22}b_{13}}, c_{31} = \frac{d_{33}a_{11}}{a_{22}b_{13}}$. If one replaces these values in the system (4.7)

$$\begin{cases} 2b_{21}b_{31} + 3c_{21}c_{31} = a_{11} \\ b_{21}c_{31} + c_{21}b_{31} = 0, \end{cases}$$

one obtains the main equation

$$\frac{a_{11}^2 a_{33}^2}{4b_{13}^4 a_{22}^2} - \frac{3}{2}\frac{a_{11}^2 d_{33}^2}{b_{13}^4 a_{22}^2} = 1 \ .$$

If $d_{33} = 0$, $a_{33} = \pm\frac{2b_{13}^2 a_{22}}{a_{11}}, c_{21} = c_{31} = 0$.

By taking $a_{33} = \frac{2b_{13}^2 a_{22}}{a_{11}}; b_{21} = \frac{a_{11}}{2b_{13}}, b_{31} = b_{13}$.

Thus, the matrix

$$\begin{pmatrix} f(g_1,g_1) & f(g_1,g_2) & f(g_1,g_3) \\ f(g_2,g_1) & f(g_2,g_2) & f(g_2,g_3) \\ f(g_3,g_1) & f(g_3,g_2) & f(g_3,g_3) \end{pmatrix}$$

is

$$\begin{pmatrix} a_{11} & -\frac{a_{11}}{2b_{13}}\sqrt{2} & b_{13}\sqrt{2} \\ \frac{a_{11}}{2b_{13}}\sqrt{2} & a_{22} & \frac{b_{13}a_{22}}{a_{11}}\sqrt{2} \\ b_{13}\sqrt{2} & -\frac{a_{22}b_{13}}{a_{11}}\sqrt{2} & \frac{2b_{13}^2 a_{22}}{a_{11}} \end{pmatrix} \ .$$

If $a_{11} = b_{13} = a_{22} = 1$, one obtains the matrix

$$\begin{pmatrix} 1 & -\frac{1}{2}\sqrt{2} & \sqrt{2} \\ \frac{1}{2}\sqrt{2} & 1 & \sqrt{2} \\ \sqrt{2} & -\sqrt{2} & 2 \end{pmatrix}.$$

Therefore, the bilinear map f such that

$$[f]_{\mathcal{B}} = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -\frac{1}{2}\sqrt{2} & \sqrt{2} \\ 1 & \frac{1}{2}\sqrt{2} & 1 & \sqrt{2} \\ 1 & \sqrt{2} & -\sqrt{2} & 2 \end{pmatrix}$$

is a nontrivial $2-$cocycle such that $\frac{\mathbb{Q}[X]}{\langle P \rangle} \rtimes_f G_4$ is a crossed product, where $P = X^4 - 5X^2 + 6$ and G_4 is the Galois group of P over $\mathbb{Q}$.

4.3.2. Cocycles for a crossed product $\frac{\mathbb{Q}[X]}{\langle P \rangle} \rtimes_f C_2$. Let $M = \mathbb{Q}(\sqrt{2}, \sqrt{3})$ and $K = \mathbb{Q}$. According to Fraleigh [8, Example 53.3] it is known that M is Galois extension of $\mathbb{Q}$. Thus $M^* = M \backslash \{0\}$.
Since $M = \{a_1 + a_2\sqrt{2} + a_3\sqrt{3} + a_4\sqrt{6} \mid a_1, a_2, a_3, a_4 \in \mathbb{Q}\}$, $C_2 = \{1, g\} = \langle g \rangle$ is described by

(4.9)

	1	g
1	1	1
$\sqrt{2}$	$\sqrt{2}$	$\sqrt{2}$
$\sqrt{3}$	$\sqrt{3}$	$-\sqrt{3}$
$\sqrt{6}$	$\sqrt{6}$	$-\sqrt{6}$

It is considered $\sigma : C_2 \to Aut(M)$, $g^i \mapsto \sigma(g^i) : M \to M$ defined by
$\sigma(g^i)(a_1 + a_2\sqrt{2} + a_3\sqrt{3} + a_4\sqrt{6}) = (a_1 + a_2\sqrt{2} + a_3\sqrt{3} + a_4\sqrt{6})^{g^i}$.
Recalls that $C_2 \times C_2 = \{(1,1), (1,g), (g,1), (g,g)\}$.
If $(g^i, g^j) \in C_2 \times C_2$, then $\sigma(g^i g^j) = \sigma(g^i)\sigma(g^j)$.
In fact, if $a_1 + a_2\sqrt{2} + a_3\sqrt{3} + a_4\sqrt{6} \in M$, then by (4.9):

$$\sigma(gg)(a_1 + a_2\sqrt{2} + a_3\sqrt{3} + a_4\sqrt{6}) = \sigma(1)(a_1 + a_2\sqrt{2} + a_3\sqrt{3} + a_4\sqrt{6})$$
$$= a_1 + a_2\sqrt{2} + a_3\sqrt{3} + a_4\sqrt{6} \,,$$
$$\sigma(g)\sigma(g)(a_1 + a_2\sqrt{2} + a_3\sqrt{3} + a_4\sqrt{6}) = \sigma(g)(a_1 + a_2\sqrt{2} - a_3\sqrt{3} - a_4\sqrt{6})$$
$$= a_1 + a_2\sqrt{2} + a_3\sqrt{3} + a_4\sqrt{6}.$$

The others equalities are verified easily.
The map $\sigma(g)$ is a $K-$algebra automorphism of M.
If $\lambda \in K$, $a_1 + a_2\sqrt{2} + a_3\sqrt{3} + a_4\sqrt{6}$ and $b_1 + b_2\sqrt{2} + b_3\sqrt{3} + b_4\sqrt{6} \in M$, then :

1) $\sigma(g)(\lambda(a_1 + a_2\sqrt{2} + a_3\sqrt{3} + a_4\sqrt{6})) = \lambda\sigma(g)(a_1 + a_2\sqrt{2} + a_3\sqrt{3} + a_4\sqrt{6})$,

2) $\sigma(g)((a_1 + a_2\sqrt{2} + a_3\sqrt{3} + a_4\sqrt{6}) + (b_1 + b_2\sqrt{2} + b_3\sqrt{3} + b_4\sqrt{6}))$

= $\sigma(g)(a_1 + a_2\sqrt{2} + a_3\sqrt{3} + a_4\sqrt{6}) + \sigma(g)(b_1 + b_2\sqrt{2} + b_3\sqrt{3} + b_4\sqrt{6})$,

3) $\sigma(g)((a_1 + a_2\sqrt{2} + a_3\sqrt{3} + a_4\sqrt{6})(b_1 + b_2\sqrt{2} + b_3\sqrt{3} + b_4\sqrt{6}))$

= $\sigma(g)(a_1 + a_2\sqrt{2} + a_3\sqrt{3} + a_4\sqrt{6})\sigma(g)(b_1 + b_2\sqrt{2} + b_3\sqrt{3} + b_4\sqrt{6}).$

Therefore, C_2 acts (weakly) on M.

Example 4.1. *Find the value of $\alpha \in M^*$ such that*

(4.10)
$$f(g^i, g^j) = \begin{cases} 1 & \text{if } i + j < 2 \\ \alpha & \text{if } i + j \geq 2 \,, \end{cases}$$

where $0 \le i, j < 2$; $g^i, g^j \in C_2 = \{1, g\} = \langle g \rangle$ and $f : C_2 \times C_2 \to M^*$ is a $2-$cocycle of the crossed product $M \rtimes_f C_2$.

Since $\alpha \in M^*$, then $\alpha = a + b\sqrt{2} + c\sqrt{3} + d\sqrt{6}$ where $a^2 + b^2 + c^2 + d^2 \neq 0$. From the definition of crossed product, the following three conditions are satisfied :

1) $f(1, g^i) = f(g^i, 1) = 1$,
2) $f(g^j, g^k)^{g^i} f(g^i, g^{j+k}) = f(g^i, g^j) f(g^{i+j}, g^k)$,
3) $((b_1 + b_2\sqrt{2} + b_3\sqrt{3} + b_4\sqrt{6})^{g^j})^{g^i} f(g^i, g^j) = f(g^i, g^j)(b_1 + b_2\sqrt{2} + b_3\sqrt{3} + b_4\sqrt{6})^{g^{i+j}}$,

where $b_1 + b_2\sqrt{2} + b_3\sqrt{3} + b_4\sqrt{6} \in M$.

$$C_2 \times C_2 \times C_2 = \{1, g\} \times \{1, g\} \times \{1, g\}$$
$$= \{(1,1,1), (1,1,g), (1,g,1), (1,g,g), (g,1,1), (g,1,g), (g,g,1), (g,g,g)\}.$$

Considering this set the assertion 2) for f, $f(g^j, g^k)^{g^i} f(g^i, g^{j+k}) = f(g^i, g^j) f(g^{i+j}, g^k)$, is expanded to the following 8 equalities :

(1) $1 = f(1,1)^1 f(1, (1)(1)) = f(1,1)f((1)(1), 1)$,
(2) $1 = f(1,g)^1 f(1, (1)(g)) = f(1,1)f((1)(1), g)$,
(3) $1 = f(g,1)^1 f(1, (g)(1)) = f(1,g)f((1)(g), 1)$,
(4) $\alpha = f(g,g)^1 f(1, (g)(g)) = f(1,g)f((1)(g), g)$,
(5) $1 = f(1,1)^g f(g, (1)(1)) = f(g,1)f((g)(1), 1)$,
(6) $\alpha = f(1,g)^g f(g, (1)(g)) = f(g,1)f((g)(1), g)$,
(7) $\alpha = f(g,1)^g f(g, (g)(1)) = f(g,g)f((g)(g), 1)$,
(8) $\alpha^g = f(g,g)^g f(g, g^2) = f(g,g)f(g^2, g) = \alpha$,

where $\alpha = a + b\sqrt{2} + c\sqrt{3} + d\sqrt{6}$, $\alpha^g = a + b\sqrt{2} - c\sqrt{3} - d\sqrt{6}$.
From the equality (8), it is deduced that $\alpha = a + b\sqrt{2} \; \square$

Setting $a = 3$ & $b = 5$ is obtained the matrix

$$\left(\; 3 + 5\sqrt{2} \; \right).$$

Therefore, the bilinear map f such that

$$[f]_{\mathcal{B}} = \begin{pmatrix} 1 & 1 \\ 1 & 3 + 5\sqrt{2} \end{pmatrix}$$

is a nontrivial $2-$cocycle such that $A \rtimes_f C_2$ is a crossed product, where $A = \dfrac{\mathbb{Q}[X]}{\langle X^4 - 5X^2 + 6 \rangle}$ and C_2 is the subgroup $\langle g \rangle$ of the Galois group of $X^4 - 5X^2 + 6$ over $\mathbb{Q}$.

4.4. Cocycles for a crossed product $\frac{\mathbb{Q}[X]}{\langle P \rangle} \rtimes_f G_6$. Let $\zeta = \zeta_3 = -\frac{1}{2} + \frac{\sqrt{3}}{2}i$ be the primitive $3-$th root of the unity in the field of complex numbers $\mathbb{C}$. Let be $M = \mathbb{Q}(\sqrt[3]{2}, \zeta_3)$, $K = \mathbb{Q}$ and $G(M/K)$ is the Galois group of M over K. According to [8, Example 50.9] it is known that M is Galois extension of $\mathbb{Q}$. The proof of [8, Theorem 31.4] then shows that $\{1, \sqrt[3]{2}, \sqrt[3]{2^2}, \zeta, \zeta\sqrt[3]{2}, \zeta\sqrt[3]{2^2}\}$ is a basis for M over $\mathbb{Q}$. Thus $M = \{a_1 + a_2\sqrt[3]{2} + a_3\sqrt[3]{2^2} + a_4\zeta + a_5\zeta\sqrt[3]{2} + a_6\zeta\sqrt[3]{2^2} \mid a_i \in \mathbb{Q}\}$. Then $G(M/K) = \{g_0, g_1, g_2, g_3, g_4, g_5\}$ is described as follows

(4.11)

	g_0	g_1	g_2	g_3	g_4	g_5
1	1	1	1	1	1	1
$\sqrt[3]{2}$	$\sqrt[3]{2}$	$\zeta\sqrt[3]{2}$	$\zeta^2\sqrt[3]{2}$	$\sqrt[3]{2}$	$\zeta\sqrt[3]{2}$	$\zeta^2\sqrt[3]{2}$
$\sqrt[3]{2^2}$	$\sqrt[3]{2^2}$	$\zeta^2\sqrt[3]{2^2}$	$\zeta\sqrt[3]{2^2}$	$\sqrt[3]{2^2}$	$\zeta^2\sqrt[3]{2^2}$	$\zeta\sqrt[3]{2^2}$
ζ	ζ	ζ	ζ	ζ^2	ζ^2	ζ^2
$\zeta\sqrt[3]{2}$	$\zeta\sqrt[3]{2}$	$\zeta^2\sqrt[3]{2}$	$\sqrt[3]{2}$	$\zeta^2\sqrt[3]{2}$	$\sqrt[3]{2}$	$\zeta\sqrt[3]{2}$
$\zeta\sqrt[3]{2^2}$	$\zeta\sqrt[3]{2^2}$	$\sqrt[3]{2^2}$	$\zeta^2\sqrt[3]{2^2}$	$\zeta^2\sqrt[3]{2^2}$	$\zeta\sqrt[3]{2^2}$	$\sqrt[3]{2^2}$

Hence, the group table for $G(M/K)$ is given by

(4.12)

	g_0	g_1	g_2	g_3	g_4	g_5
g_0	g_0	g_1	g_2	g_3	g_4	g_5
g_1	g_1	g_2	g_0	g_4	g_5	g_3
g_2	g_2	g_0	g_1	g_5	g_3	g_4
g_3	g_3	g_5	g_4	g_0	g_2	g_1
g_4	g_4	g_3	g_5	g_1	g_0	g_2
g_5	g_5	g_4	g_3	g_2	g_1	g_0 .

It is considered $\sigma : G(M/K) \to Aut(M)$, $g_i \mapsto \sigma(g_i) : M \to M$ defined by
$\sigma(g_i)(a_1 + a_2\sqrt[3]{2} + a_3\sqrt[3]{2^2} + a_4\zeta + a_5\zeta\sqrt[3]{2} + a_6\zeta\sqrt[3]{2^2}) = (a_1 + a_2\sqrt[3]{2} + a_3\sqrt[3]{2^2} + a_4\zeta + a_5\zeta\sqrt[3]{2} + a_6\zeta\sqrt[3]{2^2})^{g_i}$.
Recalls that
$G(M/K) \times G(M/K) = \{(g_0, g_0), (g_0, g_1), (g_0, g_2), (g_0, g_3), (g_0, g_4), (g_0, g_5), \ldots,$
$(g_5, g_0), (g_5, g_1), (g_5, g_2), (g_5, g_3), (g_5, g_4), (g_5, g_5)\}$.
If $(g_i, g_j) \in G(M/K) \times G(M/K)$, then $\sigma(g_i g_j) = \sigma(g_i)\sigma(g_j)$.
In fact, if $a = a_1 + a_2\sqrt[3]{2} + a_3\sqrt[3]{2^2} + a_4\zeta + a_5\zeta\sqrt[3]{2} + a_6\zeta\sqrt[3]{2^2} \in M$, then by (4.11) & (4.12):

$$\sigma(g_1 g_2)(a) = \sigma(g_0)(a) = a \,,$$
$$\sigma(g_1)\sigma(g_2)(a) = \sigma(g_1)(a_1 + a_2\zeta^2\sqrt[3]{2} + a_3\zeta\sqrt[3]{2^2} + a_4\zeta + a_5\sqrt[3]{2} + a_6\zeta^2\sqrt[3]{2^2}) = a \,.$$

The others equalities are checked similarly.
Since $G(M/K)$ is the Galois group of M over K, the map $\sigma(g_i)$ is a $K-$algebra automorphism of M.
That is; if $\lambda \in K$, $b = b_1 + b_2\sqrt[3]{2} + b_3\sqrt[3]{2^2} + b_4\zeta + b_5\zeta\sqrt[3]{2} + b_6\zeta\sqrt[3]{2^2} \in M$, then :

$$1)\sigma(g_i)(\lambda a) = \lambda\sigma(g_i)(a),$$
$$2)\sigma(g_i)(a + b) = \sigma(g_i)(a) + \sigma(g_i)(b),$$
$$3)\sigma(g_i)(ab) = \sigma(g_i)(a)\sigma(g_i)(b) \,.$$

Therefore, $G(M/K)$ acts (weakly) on M.

By the primitive element theorem [2, p 74] exists $\gamma \in M$ such that $M = K[\gamma] = \dfrac{K[X]}{\langle irr(\gamma, K)\rangle}$ is a field;
so the group of units of M is $M^* = M\backslash\{0\}$.

Let $f : G(M/K) \times G(M/K) \to M^*$ be a function such that $E = M \rtimes_f G(M/K)$ is a crossed product. Then the following three conditions are satisfied according to Blattner, Cohen, and Montgomery [3, Corollary 4.6] :
    1) $f(g_0, g_i) = f(g_i, g_0) = 1$,
    2) $f(g_j, g_k)^{g_i} f(g_i, g_j g_k) = f(g_i, g_j) f(g_i g_j, g_k)$,
    3) $(b^{g_j})^{g_i} f(g_i, g_j) = f(g_i, g_j) b^{g_i g_j}$, where $b \in M$.
The set $G(M/K) \times G(M/K) \times G(M/K)$ has $6^3 = 216$ elements .
The assertion 3) for f, $(b^{g_j})^{g_i} f(g_i, g_j) = f(g_i, g_j) b^{g_i g_j}$, is satisfied due to that the $K-$algebra M is commutative and $G(M/K) \times M \to M$, $(g_i, a) \mapsto a^{g_i}$ is an action of $G(M/K)$ on M.
Using the group table (4.12), $f(g_j, g_k)^{g_i} f(g_i, g_j g_k) = f(g_i, g_j) f(g_i g_j, g_k)$ is expanded to 216 equalities. Since $f(g_j, g_k)^{g_0} f(g_0, g_j g_k) = f(g_0, g_j) f(g_j, g_k)$; $f(g_0, g_k)^{g_i} f(g_i, g_k) = f(g_i, g_0) f(g_i, g_k)$ and $f(g_j, g_0)^{g_i} f(g_i, g_j) = f(g_i, g_j) f(g_i g_j, g_0)$; the following 125 equalities hold :

(44) $f(g_1, g_1)^{g_1} f(g_1, g_2) = f(g_1, g_1) f(g_2, g_1)$,
(45) $f(g_1, g_2)^{g_1} = f(g_1, g_1) f(g_2, g_2)$,
(46) $f(g_1, g_3)^{g_1} f(g_1, g_4) = f(g_1, g_1) f(g_2, g_3)$,
(47) $f(g_1, g_4)^{g_1} f(g_1, g_5) = f(g_1, g_1) f(g_2, g_4)$,
(48) $f(g_1, g_5)^{g_1} f(g_1, g_3) = f(g_1, g_1) f(g_2, g_5)$,
(50) $f(g_2, g_1)^{g_1} = f(g_1, g_2)$,
(51) $f(g_2, g_2)^{g_1} f(g_1, g_1) = f(g_1, g_2)$,
(52) $f(g_2, g_3)^{g_1} f(g_1, g_5) = f(g_1, g_2)$,
(53) $f(g_2, g_4)^{g_1} f(g_1, g_3) = f(g_1, g_2)$,

(54) $f(g_2, g_5)^{g_1} f(g_1, g_4) = f(g_1, g_2)$,
(56) $f(g_3, g_1)^{g_1} f(g_1, g_5) = f(g_1, g_3) f(g_4, g_1)$,
(57) $f(g_3, g_2)^{g_1} f(g_1, g_4) = f(g_1, g_3) f(g_4, g_2)$,
(58) $f(g_3, g_3)^{g_1} = f(g_1, g_3) f(g_4, g_3)$,
(59) $f(g_3, g_4)^{g_1} f(g_1, g_2) = f(g_1, g_3) f(g_4, g_4)$,
(60) $f(g_3, g_5)^{g_1} f(g_1, g_1) = f(g_1, g_3) f(g_4, g_5)$,
(62) $f(g_4, g_1)^{g_1} f(g_1, g_3) = f(g_1, g_4) f(g_5, g_1)$,
(63) $f(g_4, g_2)^{g_1} f(g_1, g_5) = f(g_1, g_4) f(g_5, g_2)$,
(64) $f(g_4, g_3)^{g_1} f(g_1, g_1) = f(g_1, g_4) f(g_5, g_3)$,

(65) $f(g_4,g_4)^{g_1} = f(g_1,g_4)f(g_5,g_4),$

(66) $f(g_4,g_5)^{g_1}f(g_1,g_2) = f(g_1,g_4)f(g_5,g_5),$

(68) $f(g_5,g_1)^{g_1}f(g_1,g_4) = f(g_1,g_5)f(g_3,g_1),$

(69) $f(g_5,g_2)^{g_1}f(g_1,g_3) = f(g_1,g_5)f(g_3,g_2),$

(70) $f(g_5,g_3)^{g_1}f(g_1,g_2) = f(g_1,g_5)f(g_3,g_3),$

(71) $f(g_5,g_4)^{g_1}f(g_1,g_1) = f(g_1,g_5)f(g_3,g_4),$

(72) $f(g_5,g_5)^{g_1} = f(g_1,g_5)f(g_3,g_5),$

(80) $f(g_1,g_1)^{g_2}f(g_2,g_2) = f(g_2,g_1)f(g_0,g_1),$

(81) $f(g_1,g_2)^{g_2} = f(g_2,g_1),$

(82) $f(g_1,g_3)^{g_2}f(g_2,g_4) = f(g_2,g_1),$

(83) $f(g_1,g_4)^{g_2}f(g_2,g_5) = f(g_2,g_1),$

(84) $f(g_1,g_5)^{g_2}f(g_2,g_3) = f(g_2,g_1),$

(86) $f(g_2,g_1)^{g_2} = f(g_2,g_2)f(g_1,g_1),$

(87) $f(g_2,g_2)^{g_2}f(g_2,g_1) = f(g_2,g_2)f(g_1,g_2),$

(88) $f(g_2,g_3)^{g_2}f(g_2,g_5) = f(g_2,g_2)f(g_1,g_3),$

(89) $f(g_2,g_4)^{g_2}f(g_2,g_3) = f(g_2,g_2)f(g_1,g_4),$

(90) $f(g_2,g_5)^{g_2}f(g_2,g_4) = f(g_2,g_2)f(g_1,g_5),$

(92) $f(g_3,g_1)^{g_2}f(g_2,g_5) = f(g_2,g_3)f(g_5,g_1),$

(93) $f(g_3,g_2)^{g_2}f(g_2,g_4) = f(g_2,g_3)f(g_5,g_2),$

(94) $f(g_3,g_3)^{g_2} = f(g_2,g_3)f(g_5,g_3),$

(95) $f(g_3,g_4)^{g_2}f(g_2,g_2) = f(g_2,g_3)f(g_5,g_4),$

(96) $f(g_3,g_5)^{g_2}f(g_2,g_1) = f(g_2,g_3)f(g_5,g_5),$

(98) $f(g_4,g_1)^{g_2}f(g_2,g_3) = f(g_2,g_4)f(g_3,g_1),$

(99) $f(g_4,g_2)^{g_2}f(g_2,g_5) = f(g_2,g_4)f(g_3,g_2),$

(100) $f(g_4,g_3)^{g_2}f(g_2,g_1) = f(g_2,g_4)f(g_3,g_3),$

(101) $f(g_4,g_4)^{g_2} = f(g_2,g_4)f(g_3,g_4),$

(102) $f(g_4,g_5)^{g_2}f(g_2,g_2) = f(g_2,g_4)f(g_3,g_5),$

(104) $f(g_5,g_1)^{g_2}f(g_2,g_4) = f(g_2,g_5)f(g_4,g_1),$

(105) $f(g_5,g_2)^{g_2}f(g_2,g_3) = f(g_2,g_5)f(g_4,g_2),$

(106) $f(g_5,g_3)^{g_2}f(g_2,g_2) = f(g_2,g_5)f(g_4,g_3),$

(107) $f(g_5,g_4)^{g_2}f(g_2,g_1) = f(g_2,g_5)f(g_4,g_4),$

(108) $f(g_5,g_5)^{g_2} = f(g_2,g_5)f(g_4,g_5),$

(116) $f(g_1,g_1)^{g_3}f(g_3,g_2) = f(g_3,g_1)f(g_5,g_1),$

(117) $f(g_1,g_2)^{g_3} = f(g_3,g_1)f(g_5,g_2),$

(118) $f(g_1,g_3)^{g_3}f(g_3,g_4) = f(g_3,g_1)f(g_5,g_3),$

(119) $f(g_1,g_4)^{g_3}f(g_3,g_5) = f(g_3,g_1)f(g_5,g_4),$

(120) $f(g_1,g_5)^{g_3}f(g_3,g_3) = f(g_3,g_1)f(g_5,g_5),$

(122) $f(g_2,g_1)^{g_3} = f(g_3,g_2)f(g_4,g_1),$

(123) $f(g_2,g_2)^{g_3}f(g_3,g_1) = f(g_3,g_2)f(g_4,g_2),$

(124) $f(g_2,g_3)^{g_3}f(g_3,g_5) = f(g_3,g_2)f(g_4,g_3),$

(125) $f(g_2,g_4)^{g_3}f(g_3,g_3) = f(g_3,g_2)f(g_4,g_4),$

(126) $f(g_2,g_5)^{g_3}f(g_3,g_4) = f(g_3,g_2)f(g_4,g_5),$

(128) $f(g_3,g_1)^{g_3}f(g_3,g_5) = f(g_3,g_3),$

(129) $f(g_3,g_2)^{g_3}f(g_3,g_4) = f(g_3,g_3),$

(130) $f(g_3,g_3)^{g_3} = f(g_3,g_3),$

(131) $f(g_3,g_4)^{g_3}f(g_3,g_2) = f(g_3,g_3),$

(132) $f(g_3,g_5)^{g_3}f(g_3,g_1) = f(g_3,g_3),$

(134) $f(g_4,g_1)^{g_3}f(g_3,g_3) = f(g_3,g_4)f(g_2,g_1),$

(135) $f(g_4,g_2)^{g_3}f(g_3,g_5) = f(g_3,g_4)f(g_2,g_2),$

(136) $f(g_4,g_3)^{g_3}f(g_3,g_1) = f(g_3,g_4)f(g_2,g_3),$

(137) $f(g_4,g_4)^{g_3} = f(g_3,g_4)f(g_2,g_4),$

(138) $f(g_4,g_5)^{g_3}f(g_3,g_2) = f(g_3,g_4)f(g_2,g_5),$

(140) $f(g_5,g_1)^{g_3}f(g_3,g_4) = f(g_3,g_5)f(g_1,g_1),$

(141) $f(g_5,g_2)^{g_3}f(g_3,g_3) = f(g_3,g_5)f(g_1,g_2),$

(142) $f(g_5,g_3)^{g_3}f(g_3,g_2) = f(g_3,g_5)f(g_1,g_3),$

(143) $f(g_5,g_4)^{g_3}f(g_2,g_1) = f(g_3,g_5)f(g_1,g_4),$

(144) $f(g_5,g_5)^{g_3} = f(g_3,g_5)f(g_1,g_5),$

(152) $f(g_1,g_1)^{g_4}f(g_4,g_2) = f(g_4,g_1)f(g_3,g_1),$

(153) $f(g_1,g_2)^{g_4} = f(g_4,g_1)f(g_3,g_2),$

(154) $f(g_1,g_3)^{g_4}f(g_4,g_4) = f(g_4,g_1)f(g_3,g_3),$

(155) $f(g_1,g_4)^{g_4}f(g_4,g_5) = f(g_4,g_1)f(g_3,g_4),$

(156) $f(g_1,g_5)^{g_4}f(g_4,g_3) = f(g_4,g_1)f(g_3,g_5),$

(158) $f(g_2,g_1)^{g_4} = f(g_4,g_2)f(g_5,g_1),$

(159) $f(g_2,g_2)^{g_4}f(g_4,g_1) = f(g_4,g_2)f(g_5,g_2),$

(160) $f(g_2,g_3)^{g_4}f(g_4,g_5) = f(g_4,g_2)f(g_5,g_3),$

(161) $f(g_2,g_4)^{g_4}f(g_4,g_3) = f(g_4,g_2)f(g_5,g_4),$

(162) $f(g_2,g_5)^{g_4}f(g_4,g_4) = f(g_4,g_2)f(g_5,g_5),$

(164) $f(g_3,g_1)^{g_4}f(g_4,g_5) = f(g_4,g_3)f(g_1,g_1),$

(165) $f(g_3,g_2)^{g_4}f(g_4,g_4) = f(g_4,g_3)f(g_1,g_2),$

(166) $f(g_3,g_3)^{g_4} = f(g_4,g_3)f(g_1,g_3),$

(167) $f(g_3,g_4)^{g_4}f(g_4,g_2) = f(g_4,g_3)f(g_1,g_4),$

(168) $f(g_3,g_5)^{g_4}f(g_4,g_1) = f(g_4,g_3)f(g_1,g_5),$

(170) $f(g_4,g_1)^{g_4}f(g_4,g_3) = f(g_4,g_4),$

(171) $f(g_4,g_2)^{g_4}f(g_4,g_5) = f(g_4,g_4),$

(172) $f(g_4,g_3)^{g_4}f(g_4,g_1) = f(g_4,g_4),$

(173) $f(g_4,g_4)^{g_4} = f(g_4,g_4),$

(174) $f(g_4,g_5)^{g_4}f(g_4,g_2) = f(g_4,g_4),$

(176) $f(g_5,g_1)^{g_4}f(g_4,g_4) = f(g_4,g_5)f(g_2,g_1),$

(177) $f(g_5,g_2)^{g_4}f(g_4,g_3) = f(g_4,g_5)f(g_2,g_2),$

(178) $f(g_5,g_3)^{g_4}f(g_4,g_2) = f(g_4,g_5)f(g_2,g_3),$

(179) $f(g_5,g_4)^{g_4}f(g_4,g_1) = f(g_4,g_5)f(g_2,g_4),$

(180) $f(g_5,g_5)^{g_4} = f(g_4,g_5)f(g_2,g_5),$

(188) $f(g_1,g_1)^{g_5}f(g_5,g_2) = f(g_5,g_1)f(g_4,g_1),$

(189) $f(g_1,g_2)^{g_5} = f(g_5,g_1)f(g_4,g_2),$

(190) $f(g_1,g_3)^{g_5}f(g_5,g_4) = f(g_5,g_1)f(g_4,g_3),$

(191) $f(g_1,g_4)^{g_5}f(g_5,g_5) = f(g_5,g_1)f(g_4,g_4),$

(192) $f(g_1,g_5)^{g_5}f(g_5,g_3) = f(g_5,g_1)f(g_4,g_5),$

(194) $f(g_2,g_1)^{g_5} = f(g_5,g_2)f(g_3,g_1),$

(195) $f(g_2,g_2)^{g_5}f(g_5,g_1) = f(g_5,g_2)f(g_3,g_2),$

(196) $f(g_2,g_3)^{g_5}f(g_5,g_5) = f(g_5,g_2)f(g_3,g_3),$

(197) $f(g_2,g_4)^{g_5}f(g_5,g_3) = f(g_5,g_2)f(g_3,g_4),$

(198) $f(g_2,g_5)^{g_5}f(g_5,g_4) = f(g_5,g_2)f(g_3,g_5),$

(200) $f(g_3,g_1)^{g_5}f(g_5,g_5) = f(g_5,g_3)f(g_2,g_1),$

(201) $f(g_3,g_2)^{g_5}f(g_5,g_4) = f(g_5,g_3)f(g_2,g_2),$

(202) $f(g_3,g_3)^{g_5} = f(g_5,g_3)f(g_2,g_3),$

(203) $f(g_3,g_4)^{g_5}f(g_5,g_2) = f(g_5,g_3)f(g_2,g_4),$

(204) $f(g_3,g_5)^{g_5}f(g_5,g_1) = f(g_5,g_3)f(g_2,g_5),$

(206) $f(g_4,g_1)^{g_5}f(g_5,g_3) = f(g_5,g_4)f(g_1,g_1),$

(207) $f(g_4,g_2)^{g_5}f(g_5,g_5) = f(g_5,g_4)f(g_1,g_2),$

(208) $f(g_4,g_3)^{g_5}f(g_5,g_1) = f(g_5,g_4)f(g_1,g_3),$

(209) $f(g_4,g_4)^{g_5} = f(g_5,g_4)f(g_1,g_4),$

(210) $f(g_4,g_5)^{g_5}f(g_5,g_2) = f(g_5,g_4)f(g_1,g_5),$

(212) $f(g_5,g_1)^{g_5}f(g_5,g_4) = f(g_5,g_5),$

(213) $f(g_5,g_2)^{g_5}f(g_5,g_3) = f(g_5,g_5),$

(214) $f(g_5,g_3)^{g_5}f(g_5,g_2) = f(g_5,g_5),$

(215) $f(g_5,g_4)^{g_5}f(g_5,g_1) = f(g_5,g_5),$

(216) $f(g_5,g_5)^{g_5} = f(g_5,g_5),$

If $f(g_i,g_j) = a_{ij} + b_{ij}\sqrt[3]{2} + c_{ij}\sqrt[3]{2^2} + d_{ij}\zeta + e_{ij}\zeta\sqrt[3]{2} + k_{ij}\zeta\sqrt[3]{2^2}$, $f(g_3,g_3)^{g_3} = (a_{33} - d_{33}) + (b_{33} - e_{33})\sqrt[3]{2} + (c_{33} - k_{33})\sqrt[3]{2^2} - d_{33}\zeta - e_{33}\zeta\sqrt[3]{2} - k_{33}\zeta\sqrt[3]{2^2}$. From $f(g_3,g_3)^{g_3} = f(g_3,g_3)$, it follows that $f(g_3,g_3) = a_{33} + b_{33}\sqrt[3]{2} + c_{33}\sqrt[3]{2^2}$.

Of course, by applying the condition of normality 1) of f, one may know that exist 25 variables $f(g_i,g_j)$ for $i,j : 1,2,3,4,5$. Furthermore, each of these variables depends on 6 variables. Thus, exist 150 variables.

That is, one has a system of 125 equations in 150 variables. It is noted that the equations (128) with (132), (129) with (131), (170) with (172), (171) with (174), (212) with (215) & (213) with (214) are the same. So, the system is reduced to one of 119 equations in 150 variables.

By following the procedure applied in computing nonrational $2-$cocycles, recalling that $(a + b\sqrt[3]{2} + c\sqrt[3]{2^2} + d\zeta + e\zeta\sqrt[3]{2} + f\zeta\sqrt[3]{2^2})(a' + b'\sqrt[3]{2} + c'\sqrt[3]{2^2} + d'\zeta + e'\zeta\sqrt[3]{2} + f'\zeta\sqrt[3]{2^2}) = (aa' + 2bc' + 2cb' - dd' - 2ef' - 2fe') + (ab' + ba' + 2cc' - de' - ed' - 2ff')\sqrt[3]{2} + (ac' + bb' + ca' - df' - ee' - fd')\sqrt[3]{2^2} + (ad' + 2bf' + 2ce' + d(a' - d') + 2e(c' - f') + 2f(b' - e'))\zeta + (ae' + bd' + 2cf' + d(b' - e') + e(a' - d') + 2f(c' - f'))\zeta\sqrt[3]{2} + (af' + be' + cd' + d(c' - f') + e(b' - c') + f(a' - d'))\zeta\sqrt[3]{2^2}$ and resolving the system of equations, one can obtain the cocycles f .

Then the matrix on $\mathbb{Q}$

$$\begin{pmatrix} f(g_1,g_1) & f(g_1,g_2) & f(g_1,g_3) & f(g_1,g_4) & f(g_1,g_5) \\ f(g_2,g_1) & f(g_2,g_2) & f(g_2,g_3) & f(g_2,g_4) & f(g_2,g_5) \\ f(g_3,g_1) & f(g_3,g_2) & f(g_3,g_3) & f(g_3,g_4) & f(g_3,g_5) \\ f(g_4,g_1) & f(g_4,g_2) & f(g_4,g_3) & f(g_4,g_4) & f(g_4,g_5) \\ f(g_5,g_1) & f(g_5,g_2) & f(g_5,g_3) & f(g_5,g_4) & f(g_5,g_5) \end{pmatrix}$$

can be written as

$$\begin{pmatrix} a_{11} & a_{11}a_{22} & a_{13} & \frac{a_{11}^2 a_{22}}{a_{13}a_{15}} & a_{15} \\ a_{11}a_{22} & a_{22} & \frac{a_{11}a_{22}}{a_{15}} & \frac{a_{11}a_{22}}{a_{13}} & \frac{a_{13}a_{15}}{a_{11}} \\ \frac{a_{11}^4 a_{22}^2 a_{33}}{a_{13}^4 a_{15}a_{44}^2} & \frac{a_{11}a_{22}a_{33}}{a_{13}a_{44}} & a_{33} & \frac{a_{13}a_{44}}{a_{11}a_{22}} & \frac{a_{15}a_{33}^2}{a_{13}^2 a_{44}} \\ \frac{a_{13}a_{44}}{a_{33}} & \frac{a_{11}^3 a_{22}^2 a_{33}}{a_{13}^3 a_{15}a_{44}} & \frac{a_{33}}{a_{13}} & a_{44} & \frac{a_{13}^3 a_{15}a_{44}^2}{a_{11}^3 a_{22}^2 a_{33}} \\ \frac{a_{13}^3 a_{15}a_{44}}{a_{11}^2 a_{22}a_{33}} & \frac{a_{13}^4 a_{15}a_{44}^2}{a_{11}^3 a_{22}a_{33}^2} & \frac{a_{15}a_{33}}{a_{11}a_{22}} & \frac{a_{13}a_{15}a_{44}}{a_{11}^2 a_{22}} & \frac{a_{13}^4 a_{15}^2 a_{44}^2}{a_{11}^4 a_{22}^2 a_{33}} \end{pmatrix}$$

where $a_{13}^6 a_{44}^3 = a_{11}^4 a_{22}^2 a_{33}^3$ (from of (56) and (200)) .
This matrix presents a solution of the above system of 119 equations in 150 variables.
By setting $a_{33} = a_{44} = 3$, $a_{11} = a_{13} = a_{22} = 2$ & $a_{15} = 2$, one obtains the matrix

$$\begin{pmatrix} 2 & 4 & 2 & 2 & 2 \\ 4 & 2 & 2 & 2 & 2 \\ 2 & 2 & 3 & \frac{3}{2} & \frac{3}{2} \\ 2 & 2 & \frac{3}{2} & 3 & \frac{3}{2} \\ 2 & 2 & \frac{3}{2} & \frac{3}{2} & 3 \end{pmatrix}.$$

Therefore, the rational bilinear form f such that

$$(4.13) \qquad [f]_{\mathcal{B}} = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 2 & 4 & 2 & 2 & 2 \\ 1 & 4 & 2 & 2 & 2 & 2 \\ 1 & 2 & 2 & 3 & \frac{3}{2} & \frac{3}{2} \\ 1 & 2 & 2 & \frac{3}{2} & 3 & \frac{3}{2} \\ 1 & 2 & 2 & \frac{3}{2} & \frac{3}{2} & 3 \end{pmatrix}$$

is a required nontrivial $2-$cocycle such that $A \rtimes_f G$ is a crossed product, where $A = \dfrac{\mathbb{Q}[X]}{\langle X^3 - 2 \rangle}$ and $G = S_3$ is the Galois group of $X^3 - 2$ over $\mathbb{Q}$.

4.5. Cocycle for a crossed product $\dfrac{\mathbb{Q}[X]}{\langle P \rangle} \rtimes_f G_8$**.** Let $\zeta = \zeta_4 = i$ be the primitive $4-$th root of the unity in the field of complex numbers $\mathbb{C}$. Let be $M = \mathbb{Q}(\sqrt[4]{2}, \zeta_4)$, $K = \mathbb{Q}$ and $G(M/K)$ is the Galois

group of M over K. According to [8, Example 50.9] it is known that M is Galois extension of $\mathbb{Q}$. The proof of [8, Theorem 31.4] then shows that $\{1, \sqrt[4]{2}, \sqrt[4]{2^2}, \sqrt[4]{2^3}, \zeta, \zeta\sqrt[4]{2}, \zeta\sqrt[4]{2^2}, \zeta\sqrt[4]{2^3}\}$ is a basis for M over $\mathbb{Q}$. Thus $M = \{a_1 + a_2\sqrt[4]{2} + a_3\sqrt[4]{2^2} + a_4\sqrt[4]{2^3} + a_5\zeta + a_6\zeta\sqrt[4]{2} + a_7\zeta\sqrt[4]{2^2} + a_8\zeta\sqrt[4]{2^3} \mid a_i \in \mathbb{Q}\}$. Then $G(M/K) = \{g_0, g_1, g_2, g_3, g_4, g_5, g_6, g_7\}$ is described as follows

(4.14)

	g_0	g_1	g_2	g_3	g_4	g_5	g_6	g_7
1	1	1	1	1	1	1	1	1
$\sqrt[4]{2}$	$\sqrt[4]{2}$	$\zeta\sqrt[4]{2}$	$\zeta^2\sqrt[4]{2}$	$\zeta^3\sqrt[4]{2}$	$\sqrt[4]{2}$	$\zeta\sqrt[4]{2}$	$\zeta^2\sqrt[4]{2}$	$\zeta^3\sqrt[4]{2}$
$\sqrt[4]{2^2}$	$\sqrt[4]{2^2}$	$\zeta^2\sqrt[4]{2^2}$	$\sqrt[4]{2^2}$	$\zeta^2\sqrt[4]{2^2}$	$\sqrt[4]{2^2}$	$\zeta^2\sqrt[4]{2^2}$	$\sqrt[4]{2^2}$	$\zeta^2\sqrt[4]{2^2}$
$\sqrt[4]{2^3}$	$\sqrt[4]{2^3}$	$\zeta^3\sqrt[4]{2^3}$	$\zeta^2\sqrt[4]{2^3}$	$\zeta\sqrt[4]{2^3}$	$\sqrt[4]{2^3}$	$\zeta^3\sqrt[4]{2^3}$	$\zeta^2\sqrt[4]{2^3}$	$\zeta\sqrt[4]{2^3}$
ζ	ζ	ζ	ζ	ζ	$-\zeta$	$-\zeta$	$-\zeta$	$-\zeta$
$\zeta\sqrt[4]{2}$	$\zeta\sqrt[4]{2}$	$\zeta^2\sqrt[4]{2}$	$\zeta^3\sqrt[4]{2}$	$\sqrt[4]{2}$	$-\zeta\sqrt[4]{2}$	$-\zeta^2\sqrt[4]{2}$	$-\zeta^3\sqrt[4]{2}$	$-\sqrt[4]{2}$
$\zeta\sqrt[4]{2^2}$	$\zeta\sqrt[4]{2^2}$	$\zeta^3\sqrt[4]{2^2}$	$\zeta\sqrt[4]{2^2}$	$\zeta^3\sqrt[4]{2^2}$	$-\zeta\sqrt[4]{2^2}$	$-\zeta^3\sqrt[4]{2^2}$	$-\zeta\sqrt[4]{2^2}$	$-\zeta^3\sqrt[4]{2^2}$
$\zeta\sqrt[4]{2^3}$	$\zeta\sqrt[4]{2^3}$	$\sqrt[4]{2^3}$	$\zeta^3\sqrt[4]{2^3}$	$\zeta^2\sqrt[4]{2^3}$	$-\zeta\sqrt[4]{2^3}$	$-\sqrt[4]{2^3}$	$-\zeta^3\sqrt[4]{2^3}$	$-\zeta^2\sqrt[4]{2^3}$

Hence, the group table for $G(M/K)$ is given by

(4.15)

	g_0	g_1	g_2	g_3	g_4	g_5	g_6	g_7
g_0	g_0	g_1	g_2	g_3	g_4	g_5	g_6	g_7
g_1	g_1	g_2	g_3	g_0	g_5	g_6	g_7	g_4
g_2	g_2	g_3	g_0	g_1	g_6	g_7	g_4	g_5
g_3	g_3	g_0	g_1	g_2	g_7	g_4	g_5	g_6
g_4	g_4	g_7	g_6	g_5	g_0	g_3	g_2	g_1
g_5	g_5	g_4	g_7	g_6	g_1	g_0	g_3	g_2
g_6	g_6	g_5	g_4	g_7	g_2	g_1	g_0	g_3
g_7	g_7	g_6	g_5	g_4	g_3	g_2	g_1	g_0 ·

Since $G(M/K)$ is the Galois group of M over K, $G(M/K)$ acts (weakly) on M.

By the primitive element theorem exists $\gamma \in M$ such that $M = K[\gamma] = \dfrac{K[X]}{\langle \mathrm{irr}(\gamma, K)\rangle}$ is a field; so the group of units of M is $M^* = M\backslash\{0\}$.

Let $f : G(M/K) \times G(M/K) \to M^*$ be a function such that $E = M \rtimes_f G(M/K)$ is a crossed product. Then the following three conditions are satisfied :
   1) $f(g_0, g_i) = f(g_i, g_0) = 1$,
   2) $f(g_j, g_k)^{g_i} f(g_i, g_j g_k) = f(g_i, g_j)f(g_i g_j, g_k)$,
   3) $(b^{g_j})^{g_i} f(g_i, g_j) = f(g_i, g_j)b^{g_i g_j}$, where $b \in M$.
The set $G(M/K) \times G(M/K) \times G(M/K)$ has $8^3 = 512$ elements. Using the group table (4.15), $f(g_j, g_k)^{g_i} f(g_i, g_j g_k) = f(g_i, g_j)f(g_i g_j, g_k)$ is expanded to 512 equalities.
Since the assertion 3) is satisfied clearly, assuming that the assertions 1) and 2) are satisfied, and applying the procedure developed in Subsection 4.3 one can obtain the family of $2-$cocycles f for the crossed product $\frac{\mathbb{Q}[X]}{\langle X^4 - 2\rangle} \rtimes_f G_8$.

Thus, the matrix on $\mathbb{Q}$

$$
\begin{pmatrix}
f(g_1, g_1) & f(g_1, g_2) & f(g_1, g_3) & f(g_1, g_4) & f(g_1, g_5) & f(g_1, g_6) & f(g_1, g_7) \\
f(g_2, g_1) & f(g_2, g_2) & f(g_2, g_3) & f(g_2, g_4) & f(g_2, g_5) & f(g_2, g_6) & f(g_2, g_7) \\
f(g_3, g_1) & f(g_3, g_2) & f(g_3, g_3) & f(g_3, g_4) & f(g_3, g_5) & f(g_3, g_6) & f(g_3, g_7) \\
f(g_4, g_1) & f(g_4, g_2) & f(g_4, g_3) & f(g_4, g_4) & f(g_4, g_5) & f(g_4, g_6) & f(g_4, g_7) \\
f(g_5, g_1) & f(g_5, g_2) & f(g_5, g_3) & f(g_5, g_4) & f(g_5, g_5) & f(g_5, g_6) & f(g_5, g_7) \\
f(g_6, g_1) & f(g_6, g_2) & f(g_6, g_3) & f(g_6, g_4) & f(g_6, g_5) & f(g_6, g_6) & f(g_6, g_7) \\
f(g_7, g_1) & f(g_7, g_2) & f(g_7, g_3) & f(g_7, g_4) & f(g_7, g_5) & f(g_7, g_6) & f(g_7, g_7)
\end{pmatrix}
$$

can be written as

$$\begin{pmatrix}
4 & \frac{1}{2} & \frac{1}{2} & 1 & 1 & 1 & 1 \\
\frac{1}{2} & \frac{1}{16} & \frac{1}{8} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \\
\frac{1}{2} & \frac{1}{8} & 1 & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\
1 & \frac{1}{4} & \frac{1}{2} & 1 & 2 & 4 & 1 \\
1 & \frac{1}{4} & \frac{1}{2} & 1 & 1 & 2 & 4 \\
1 & \frac{1}{4} & \frac{1}{2} & 4 & 1 & 1 & 2 \\
1 & \frac{1}{4} & \frac{1}{2} & 2 & 4 & 1 & 1
\end{pmatrix}.$$

Consequently, the rational bilinear form f such that

$$[f]_{\mathcal{B}} = \begin{pmatrix}
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & 4 & \frac{1}{2} & \frac{1}{2} & 1 & 1 & 1 & 1 \\
1 & \frac{1}{2} & \frac{1}{16} & \frac{1}{8} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \\
1 & \frac{1}{2} & \frac{1}{8} & 1 & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\
1 & 1 & \frac{1}{4} & \frac{1}{2} & 1 & 2 & 4 & 1 \\
1 & 1 & \frac{1}{4} & \frac{1}{2} & 1 & 1 & 2 & 4 \\
1 & 1 & \frac{1}{4} & \frac{1}{2} & 4 & 1 & 1 & 2 \\
1 & 1 & \frac{1}{4} & \frac{1}{2} & 2 & 4 & 1 & 1
\end{pmatrix}.$$

is a nontrivial $2-$cocycle such that $A \rtimes_f G_8$ is a crossed product, where $A = \dfrac{\mathbb{Q}[X]}{\langle X^4 - 2 \rangle}$ and G_8 is the Galois group of $X^4 - 2$ over $\mathbb{Q}$.

5. Conclusions.

1. The cocycle f of the crossed product $\frac{\mathbb{Q}[X]}{\langle P \rangle} \rtimes_f G_4$ can take values outside of $\mathbb{Q}$ because $\sqrt{2} \in \mathbb{Q}(\sqrt{2}, \sqrt{3}) \backslash \mathbb{Q}$ (Subsubsection 4.3.1).

2. The cocycle f of the crossed product $\frac{\mathbb{Q}[X]}{\langle P \rangle} \rtimes_f C_2$ can take values outside of $\mathbb{Q}$ because $3 + 5\sqrt{2} \in \mathbb{Q}(\sqrt{2}) \backslash \mathbb{Q}$, where $C_2 = \langle g \rangle$ is a subgroup of G_4 (Example 4.1).

3. Considering $G = S_3$ as Galois group of the polynomial $X^3 - 2$ over $\mathbb{Q}$, A as the $\mathbb{Q}-$algebra $\dfrac{\mathbb{Q}[X]}{\langle X^3 - 2 \rangle}$, one can find a nontrivial $2-$cocycle f (see (4.13)) so that $A \rtimes_f G$ is a crossed product.

4. The problem of to find the family of rational $2-$cocycles in order that $E = M \rtimes_f G(M/\mathbb{Q})$ to be crossed product, it is reduced to the resolution of the system of algebraic equations with several variables over the field $\mathbb{Q}$, which is determined with the condition of normality and the condition of $2-$cocycle (Subsection 4.4).

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