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Properties of the $(n, m)-$fold hyperspace suspension of continua

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Abstract. Let $n, m \in \mathbb{N}$ with $m \leq n$ and X be a metric continuum. We consider the hyperspaces $C_n(X)$ (respectively, $F_n(X)$) of all nonempty closed subsets of X with at most n components (respectively, n points). The $(n, m)-$fold hyperspace suspension on X was introduced in 2018 by Anaya, Maya, and Vázquez-Juárez, to be the quotient space $C_n(X)/F_m(X)$ which is obtained from $C_n(X)$ by identifying $F_m(X)$ into a one-point set. In this paper we prove that $C_n(X)/F_m(X)$ contains an $n-$cell; $C_n(X)/F_m(X)$ has property (b); $C_n(X)/F_m(X)$ is unicoherent; $C_n(X)/F_m(X)$ is colocally connected; $C_n(X)/F_m(X)$ is aposyndetic; and $C_n(X)/F_m(X)$ is finitely aposyndetic.

Keywords: Aposyndesis, Cantor manifold, Continuum, Colocal connectedness, $(n, m)-$fold hyperspace suspension, Property (b), Unicoherent.

MSC2010: 54B20, 54F15.

Propiedades del $(n, m)-$ésimo hiperespacio suspensión de continuos

Resumen. Sean $n, m \in \mathbb{N}$ con $m \leq n$ y X un continuo métrico. Consideramos el hiperespacio de todos los subconjuntos cerrados, no vacíos de X con a lo más n componentes (respectivamente, n puntos) $C_n(X)$ (respectivamente, $F_n(X)$). El $(n, m)-$ésimo hiperespacio suspensión de X lo introdujeron, en 2018, Anaya, Maya y Vázquez-Juárez, como el espacio cociente $C_n(X)/F_m(X)$ que se obtiene de $C_n(X)$ al identificar $F_m(X)$ a un conjunto de un punto. En este artículo demostramos que $C_n(X)/F_m(X)$ contiene una $n-$celda; $C_n(X)/F_m(X)$ tiene la propiedad (b); $C_n(X)/F_m(X)$ es unicoherente; $C_n(X)/F_m(X)$ es colocalmente conexo; $C_n(X)/F_m(X)$ es aposindético y $C_n(X)/F_m(X)$ es finitamente aposindético.

Palabras clave: Aposindesis, Continuo, Colocalmente conexo, $(n, m)-$ésimo hiperespacio suspensión, Propiedad (b), Variedad de Cantor, Unicoherente.

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1. Introduction

Recently, the study of the $(n, m)-$fold hyperspace suspension of continua has been addressed in [1], [4], [5], [7]–[9], [13], [14], [16]–[18], [20], [21], [23].

A *continuum* is a nondegenerate compact connected metric space. A *subcontinuum* is a continuum contained in a continuum X. The set of positive integers is denoted by $\mathbb{N}$.

Given a continuum X and $n \in \mathbb{N}$, we consider the following hyperspaces of X:

$$2^X = \{A \subset X \colon A \text{ is a nonempty closed subset of } X\},$$

$$C_n(X) = \{A \in 2^X \colon A \text{ has at most } n \text{ components}\}, \text{ and}$$

$$F_n(X) = \{A \in 2^X \colon A \text{ has at most } n \text{ points}\}.$$

All these hyperspaces are metrized by the Hausdorff metric H [10, Theorem 2.2]. The hyperspaces $F_n(X)$ and $C_n(X)$ are called the *$n-$fold symmetric product of* X and the *$n-$fold hyperspace* of X, respectively, we will write $C(X)$ instead of $C_1(X)$. It is important to note that whenever X is a continuum, all these hyperspaces are continua (see [18, 1.8.8, 1.8.9, 1.8.12]).

Let X be a continuum and let $n, m \in \mathbb{N}$ be such that $m \leq n$. In 1979 Sam B. Nadler, Jr. introduced the *hyperspace suspension of a continuum* X as the quotient space $C(X)/F_1(X)$ obtained from $C(X)$ by shrinking $F_1(X)$ to a one-point set with the quotient topology, denoted by $HS(X)$, see [23]. Later, in 2004 Sergio Macías introduced the *$n-$fold hyperspace suspension of a continuum* X as the quotient space $C_n(X)/F_n(X)$, denoted by $HS_n(X)$, see [16]. Afterward in 2008, Juan Carlos Macías introduced the *$n-$fold pseudo-hyperspace suspension of a continuum* X as the quotient space $C_n(X)/F_1(X)$, denoted by $PHS_n(X)$, see [14]. Recently, in 2018 José G. Anaya, David Maya, and Francisco Vázquez-Juárez introduced the *$(n, m)-$fold hyperspace suspension of* X as the quotient space $C_n(X)/F_m(X)$ obtained from $C_n(X)$ by shrinking $F_m(X)$ to a one-point set with the quotient topology, denoted by $HS^n_m(X)$, see [1]. The fact that $HS^n_m(X)$ is a continuum follows from [24, Theorem 3.10]. The study of $(n, m)-$fold hyperspace suspension is, therefore, a generalization of the latter research.

The main topics of this paper are summed up in the following general problem.

Problem 1. Given a continuum X and $n, m \in \mathbb{N}$ satisfying that $m \leq n$, is there a topological property $\mathcal{P}$ that holds on $HS^n_m(X)$?

Related to Problem 1, the aim of this paper is to prove that:

(a) If X is a continuum and $n, m \in \mathbb{N}$ with $m \leq n$, then $HS^n_m(X)$ contains an $n-$cell (see Theorem 3.1).

(b) If X is a continuum and $n, m \in \mathbb{N}$ with $m \leq n$, then $HS^n_m(X)$ has property (b) (see Theorem 3.4).

(c) If X is a continuum and $n, m \in \mathbb{N}$ with $m \leq n$, then $HS^n_m(X)$ is unicoherent (see Theorem 3.5).

(d) If X is a continuum and $n, m \in \mathbb{N}$ with $m \leq n$, then $HS^n_m(X)$ is colocally connected (see Theorem 3.6).

(e) If X is a continuum and $n, m \in \mathbb{N}$ with $m \leq n$, then $HS^n_m(X)$ is aposyndetic (see Corollary 3.7).

(f) If X is a continuum and $n, m \in \mathbb{N}$ with $m \leq n$, then $HS^n_m(X)$ is finitely aposyndetic (see Theorem 3.8).

It is important to notice that those results that give a solution to Problem 1 are indeed generalizing Theorems 3.7, 4.1, and 4.2 as well as Corollary 4.3 and 4.4 proved by S. Macías in [16], respectively.

On the other hand, we present two results related to finite–dimensional Cantor manifolds, see Theorem 3.9 and Theorem 3.10.

2. Definitions and preliminary results

In this section, we present several results (with their references) that will be useful through this paper.

Given a subset A in a metric space X, $int_X(A)$ denotes the *interior* of A in X. If d is the metric of a continuum X, $\varepsilon > 0$, $A \subset X$, and $a \in X$, then the set $\{x \in X : d(a,x) < \varepsilon\}$ is denoted by $B_d(a,\varepsilon)$, or $B(a,\varepsilon)$ when there is no possibility of confusion. Let $N(\varepsilon, A) = \bigcup\{B(a,\varepsilon) : a \in A\}$. Given subsets $U_1, \ldots, U_r$ of X, with $r, n \in \mathbb{N}$, let

$$\langle U_1, \ldots, U_r \rangle_n = \{A \in C_n(X) : A \subset U_1 \cup \cdots \cup U_r \text{ and } A \cap U_i \neq \emptyset, \text{ for each } i \in \{1, \ldots, r\}\}.$$

It is known by [10, Theorem 1.2] that the family of all sets of the form $\langle U_1, \ldots, U_r \rangle_n$, where $r \in \mathbb{N}$ and each U_i is an open subset of X, is a basis for the topology in $C_n(X)$, known as *Vietoris topology.*

Recall that a useful tool in the theory of hyperspaces is the existence of order arcs. Given a continuum X, an *order arc* in 2^X is a continuous function $\alpha : [0,1] \to 2^X$ such that $\alpha(s) \subsetneq \alpha(t)$, for each $s, t \in [0,1]$ with $s < t$. If $A, B \in 2^X$ satisfy that $\alpha(0) = A$ and $\alpha(1) = B$, then we say that α is an *order arc from A to B*.

Lemma 2.1. *[22, (1.8)] Let $A, B \in 2^X$ be such that $A \neq B$. Then, the following two statements are equivalent:*

- *(a) there exists an order arc in 2^X from A to B,*
- *(b) $A \subset B$ and each component of B intersects A.*

An *arc* is any space homeomorphic to $[0,1]$. Given $n \in \mathbb{N}$, an *$n-$cell* is a space which is homeomorphic to $[0,1]^n$. A continuum is said to be *decomposable* provided it can be written as the union of two of its proper subcontinua.

Lemma 2.2. *[15, Theorem 3.4] Let X be a continuum and $n \in \mathbb{N}$. Then, $C_n(X)$ contains an $n-$cell.*

Lemma 2.3. *[15, Theorem 3.5] Let X be a continuum and $n \in \mathbb{N}$. If X contains n pairwise disjoint decomposable subcontinua, then $C_n(X)$ contains a $2n-$cell.*

Lemma 2.4. *[6, Proposition 1(a), p. 798] Let X be a continuum and $n \in \mathbb{N}$. If $V \subset X$ is an $n-$cell and U is an open set in X such that $U \cap V \neq \emptyset$, then there is an $n-$cell $\mathcal{T} \subset U \cap V$.*

Recall that, as in [**?**, p. 16], let A, B be two sets with equivalence relations R and S, respectively. A function $f : A \to B$ is said to be *relation-preserving* provided that aRa' implies $f(a)Sf(a')$.

Lemma 2.5. *[**?**, Theorem 4.3, p. 126] Let X, Y be spaces with equivalence relations R and S, respectively, and let $f : X \to Y$ be a relation-preserving, continuous function. Then, passing to the quotient, the function $f_* : X/R \to Y/S$ is also continuous.*

A continuum X has the *property (b)* provided that each continuous function from X into the unit circle $\mathcal{S}^1$ is homotopic to a constant function.

We say that a continuum X is *unicoherent* provided that for each pair A and B of subcontinua of X such that $X = A \cup B$, $A \cap B$ is connected.

Lemma 2.6. *[15, Theorem 4.7] Let X be a connected metric space. If X has the property (b), then X is unicoherent.*

Lemma 2.7. *[15, Theorem 4.8] Let X be a continuum and $n \in \mathbb{N}$. Then, $C_n(X)$ has the property (b). In particular, we have that $C_n(X)$ is unicoherent.*

Lemma 2.8. *[10, Theorem 19.7] If a continuum is contractible with respect to $\mathcal{S}^1$, then the continuum is unicoherent.*

A continuum is said to be *colocally connected* when each one of its points has a local base of open sets whose complement is connected.

The continuum X is *aposyndetic* if for each pair of points x and y of X, there exists a subcontinuum W of X such that $x \in int_X(W) \subset W \subset X - \{y\}$. A continuum X is *finitely aposyndetic* provided that for each finite subset F of X and each point $x \in X - F$, there exists a subcontinuum W of X such that $x \in int_X(W) \subset W \subset X - F$.

Lemma 2.9. *[2, Corollary 1] If X is an unicoherent and aposyndetic continuum, then X is finitely aposyndetic.*

We use the following notations: $\dim[X]$ stands for the dimension of X, $\dim_p[X]$ stands for the dimension of X at the point $p \in X$, as in [25, p. 5].

Lemma 2.10. *[5, Theorem 3.1] If X is a finite–dimensional continuum and $n, m \in \mathbb{N}$ with $m \leq n$, then $\dim[C_n(X)]$ is finite if and only if $\dim[HS^n_m(X)]$ is finite. Moreover, if either $\dim[C_n(X)]$ is finite or $\dim[HS^n_m(X)]$ is finite, then $\dim[C_n(X)] = \dim[HS^n_m(X)]$.*

Lemma 2.11. *[12, Theorem 2.1] If X is a continuum such that* $\dim[X] = 2$*, then* $\dim[C(X)]$ *is infinite.*

Lemma 2.12. *[10, Theorem 72.5] If X is a continuum such that* $\dim[X] \geq 3$*, then* $\dim[C(X)]$ *is infinite.*

Lemma 2.13. *[3, Lemma 3.1] If X is a finite–dimensional continuum and $n \in \mathbb{N}$, then* $\dim[F_n(X)] \leq n \cdot \dim[X]$.

Lemma 2.14. *[25, 7.3] Let X, Y, Z be separable metric spaces such that $X = Y \cup Z$, where* $\dim[Y] \leq n$ *and* $\dim[Z] \leq n$*. If at least one of Y and Z is closed in X, then* $\dim[X] \leq n$.

A finite–dimensional continuum X is a *Cantor manifold* if for any subset A of X such that $\dim[A] \leq \dim[X] - 2$, then $X - A$ is connected.

Lemma 2.15. *[19, Theorem 4.6] The hyperspaces $C_n([0,1])$ and $C_n(\mathcal{S}^1)$ are $2n-$dimensional Cantor manifolds, for each $n \in \mathbb{N}$.*

A continuous function between continua X and Y is said to be *monotone* provided that point inverses are connected (equivalently if the inverse image of each subcontinuum of Y is connected).

For a continuum X and $n, m \in \mathbb{N}$ satisfying that $m \leq n$, the symbol $q_X^{(n,m)}$ denotes the natural projection $q_X^{(n,m)} \colon C_n(X) \to HS_m^n(X)$, and F_X^m denotes the element of $q_X^{(n,m)}(F_m(X))$. Notice that

$$q_X^{(n,m)}|_{C_n(X)-F_m(X)} \colon C_n(X) - F_m(X) \to HS_m^n(X) - \{F_X^m\} \tag{1}$$

is a homeomorphism.

We shall make use of other concepts not defined here, which will be taken as in [18].

3. Main Results

The following result extends [16, Theorem 3.7].

Theorem 3.1. *Let X be a continuum and $n, m \in \mathbb{N}$ with $m \leq n$. Then, $HS_m^n(X)$ contains an $n-$cell.*

Proof. By Lemma 2.2, $C_n(X)$ contains an $n-$cell $\mathcal{M}$. Moreover, since $C_n(X) - F_m(X)$ is a dense open subset of $C_n(X)$, we have that $((C_n(X) - F_m(X)) \cap \mathcal{M} \neq \emptyset$. By Lemma 2.4, there exists an $n-$cell $\mathcal{N}$ such that $\mathcal{N} \subset C_n(X) - F_m(X)$. Thus, by (1), $HS_m^n(X)$ contains an $n-$cell. ☑

The next result extends [16, Theorem 3.8].

Theorem 3.2. *If $n, m \in \mathbb{N}$ with $m \leq n$ and X is a continuum that contains n pairwise disjoint decomposable subcontinua, then $HS_m^n(X)$ contains a $2n-$cell.*

Proof. By Lemma 2.3, $C_n(X)$ contains a $2n-$cell $\mathcal{M}$. Moreover, since $C_n(X)-F_m(X)$ is a dense open subset of $C_n(X)$, we have that $((C_n(X)-F_m(X))\cap\mathcal{M}\neq\emptyset$. By Lemma 2.4, there exists a $2n-$cell $\mathcal{N}$ such that $\mathcal{N}\subset C_n(X)-F_m(X)$. Thus, by (1), $HS_m^n(X)$ contains a $2n-$cell. ☑

The following result extends [17, Theorem 4.1].

Theorem 3.3. *Let X be a continuum and $n,m,s\in\mathbb{N}$ with $m\leq s<n$. Then, $HS_m^s(X)$ can be embedded in $HS_m^n(X)$.*

Proof. Let $i_{s,n}:C_s(X)\to C_n(X)$ be the inclusion function, $q_X^{(s,m)}:C_s(X)\to HS_m^s(X)$ and $q_X^{(n,m)}:C_n(X)\to HS_m^n(X)$ be quotient functions. We denote $q_X^{(s,m)}(F_m(X))=F_X^{(s,m)}$ and $q_X^{(n,m)}(F_m(X))=F_X^{(n,m)}$. Since

$$\{\{A\}:A\in C_n(X)-F_m(X)\}\cup\{F_m(X)\}\text{ and }\{\{B\}:B\in C_s(X)-F_m(X)\}\cup\{F_m(X)\}$$

are partitions of $C_n(X)$ and $C_s(X)$, respectively; then $i_{s,n}$ is a relation-preserving and continuous. Now, let $h_{s,n}:HS_m^s(X)\to HS_m^n(X)$ be given by

$$h_{s,n}(\mathcal{A})=\begin{cases} F_X^{(n,m)}, & \text{if } \mathcal{A}=F_X^{(s,m)};\\ q_X^{(n,m)}(i_{s,m}((q_X^{(s,m)})^{-1}(\mathcal{A}))), & \text{if } \mathcal{A}\neq F_X^{(s,m)}.\end{cases}$$

Notice that $h_{s,n}$ is a continuous function by Lemma 2.5. Moreover, as $h_{s,n}$ is defined, it is clear that $h_{s,n}$ is a one-to-one function. Since the spaces are compact, $h_{s,n}$ is an embedding. ☑

The next result extends [16, Theorem 4.1].

Theorem 3.4. *Let X be a continuum and $n,m\in\mathbb{N}$ with $m\leq n$. Then, $HS_m^n(X)$ has property (b).*

Proof. Let $\mathcal{A}\in HS_m^n(X)$. If $\mathcal{A}=F_X^m$, then $(q_X^{(n,m)})^{-1}(\mathcal{A})=F_m(X)$ which is a connected subset of $C_n(X)$. On the other hand, if $\mathcal{A}\neq F_X^m$, using relation (1), then $(q_X^{(n,m)})^{-1}(\mathcal{A})$ is a one-point set. Hence, $(q_X^{(n,m)})^{-1}(\mathcal{A})$ is a connected subset of $C_n(X)$. Therefore, $q_X^{(n,m)}$ is a monotone function. By Lemma 2.7, $C_n(X)$ has property (b). Since $q_X^{(n,m)}(C_n(X))=HS_m^n(X)$ and [11, Theorem 2, p.434], we conclude that $HS_m^n(X)$ has the property (b). ☑

Theorem 3.5. *Let X be a continuum and $n,m\in\mathbb{N}$ with $m\leq n$. Then, $HS_m^n(X)$ is unicoherent.*

Proof. Applying Theorem 3.4 and Lemma 2.6, the result follows. ☑

The following result extends [16, Theorem 4.2].

Theorem 3.6. *Let X be a continuum and $n,m\in\mathbb{N}$ with $m\leq n$. Then, $HS_m^n(X)$ is colocally connected.*

Proof. Case $n = m = 1$ is already proved in [4, Theorem 4.1].

Suppose $n \geq 2$ and let $\mathcal{A} \in HS_m^n(X)$. We are going to consider three cases:

Case 1. $\mathcal{A} = F_X^m$.

For any $\varepsilon > 0$, let $\mathcal{U}_\varepsilon = B_H(F_m(X), \varepsilon)$. Notice that $\{q_X^{(n,m)}(\mathcal{U}_\varepsilon) \colon \varepsilon > 0\}$ forms a base of open sets about F_X^m. Fix $\varepsilon > 0$. Let $\mathcal{B} \in HS_m^n(X) - q_X^{(n,m)}(\mathcal{U}_\varepsilon)$. Thus, $(q_X^{(n,m)})^{-1}(\mathcal{B}) \in C_n(X) - \mathcal{U}_\varepsilon$. By Lemma 2.1, there exists an order arc $\alpha : [0,1] \to C_n(X)$ such that $\alpha(0) = (q_X^{(n,m)})^{-1}(\mathcal{B})$ and $\alpha(1) = X$ and $\alpha([0,1]) \subset C_n(X) - \mathcal{U}_\varepsilon$. Notice that $q_X^{(n,m)} \circ \alpha : [0,1] \to HS_m^n(X)$ is an arc from $\mathcal{B}$ to $q_X^{(n,m)}(X)$ satisfying $(q_X^{(n,m)} \circ \alpha)([0,1]) \subset HS_m^n(X) - q_X^{(n,m)}(\mathcal{U}_\varepsilon)$, which implies that this space is arcwise connected.

Case 2. $\mathcal{A} = q_X^{(n,m)}(X)$.

For any $\varepsilon > 0$, let $\mathcal{U}_\varepsilon = B_H(X, \varepsilon)$. Observe that $\{q_X^{(n,m)}(\mathcal{U}_\varepsilon) \colon \varepsilon > 0\}$ forms a base of open sets about $q_X^{(n,m)}(X)$. Fix $\varepsilon > 0$. Let $\mathcal{B} \in HS_m^n(X) - q_X^{(n,m)}(\mathcal{U}_\varepsilon)$. Thus, $(q_X^{(n,m)})^{-1}(\mathcal{B}) \in C_n(X) - \mathcal{U}_\varepsilon$. Let $D \in F_m((q_X^{(n,m)})^{-1}(\mathcal{B}))$. By Lemma 2.1, there exists an order arc $\alpha : [0,1] \to C_n(X)$ such that $\alpha(0) = D$ and $\alpha(1) = (q_X^{(n,m)})^{-1}(\mathcal{B})$. Moreover, $\alpha([0,1]) \subset C_n(X) - \mathcal{U}_\varepsilon$. Hence, $q_X^{(n,m)} \circ \alpha : [0,1] \to C_n(X)$ is an arc such that $(q_X^{(n,m)} \circ \alpha)(0) = F_X^m$, $(q_X^{(n,m)} \circ \alpha)(1) = D$ and $(q_X^{(n,m)} \circ \alpha)([0,1]) \subset HS_m^n(X) - q_X^{(n,m)}(\mathcal{U}_\varepsilon)$. Therefore, $HS_m^n(X) - q_X^{(n,m)}(\mathcal{U}_\varepsilon)$ is an arcwise connected space.

Case 3. $\mathcal{A} \in HS_m^n(X) - \{F_X^m, q_X^{(n,m)}(X)\}$.

For any $\varepsilon > 0$, let $\mathcal{U}_\varepsilon = B_H((q_X^{(n,m)})^{-1}(\mathcal{A}), \varepsilon)$. Thus, $\{q_X^{(n,m)}(\mathcal{U}_\varepsilon) \colon \varepsilon > 0\}$ forms a base of open sets about $\mathcal{A}$. Fix $\varepsilon > 0$ such that $q_X^{(n,m)}(\mathcal{U}_\varepsilon) \cap \{F_X^m, q_X^{(n,m)}(X)\} = \emptyset$. Let $\mathcal{B} \in HS_m^n(X) - q_X^{(n,m)}(\mathcal{U}_\varepsilon)$. If $(q_X^{(n,m)})^{-1}(\mathcal{B}) \nsubseteq (q_X^{(n,m)})^{-1}(\mathcal{A})$, by Lemma 2.1 there exists an order arc $\alpha : [0,1] \to C_n(X)$ such that $\alpha(0) = (q_X^{(n,m)})^{-1}(\mathcal{B})$ and $\alpha(1) = X$. Thus, $\alpha([0,1]) \subset C_n(X) - B_H((q_X^{(n,m)})^{-1}(\mathcal{A}), \varepsilon)$. Hence, $q_X^{(n,m)} \circ \alpha$ is an arc from $\mathcal{B}$ to $q_X^{(n,m)}(X)$ such that $q_X^{(n,m)} \circ \alpha \subset HS_m^n(X) - \mathcal{U}_\varepsilon$, as desired.

On the other hand, suppose that $(q_X^{(n,m)})^{-1}(\mathcal{B}) \subset (q_X^{(n,m)})^{-1}(\mathcal{A})$. Let $D \in F_m((q_X^{(n,m)})^{-1}(\mathcal{B}))$. By Lemma 2.1, there exists an order arc $\beta : [0,1] \to C_n(X)$ such that $\beta(0) = D$ and $\beta(1) = (q_X^{(n,m)})^{-1}(\mathcal{B})$. Thus, $\beta([0,1])$ is contained in $C_n(X) - B_H((q_X^{(n,m)})^{-1}(\mathcal{A}), \varepsilon)$. Hence, $q_X^{(n,m)} \circ \beta : [0,1] \to HS_m^n(X)$ is an arc from F_X^m to $\mathcal{B}$ and $(q_X^{(n,m)} \circ \beta)([0,1]) \subset HS_m^n(X) - q_X^{(n,m)}(\mathcal{U}_\varepsilon)$. Therefore, the last space is arcwise connected. ☑

Since colocal connectedness implies aposyndesis, we have the next result which extends [16, Corollary 4.3].

Corollary 3.7. *Let X be a continuum and $n, m \in \mathbb{N}$ with $m \leq n$. Then, $HS_m^n(X)$ is aposyndetic.*

From this, we can prove the following result which extends [16, Corollary 4.4].

Theorem 3.8. *Let X be a continuum and $n, m \in \mathbb{N}$ with $m \leq n$. Then, $HS_m^n(X)$ is finitely aposyndetic.*

Proof. By Theorem 3.5, $HS_m^n(X)$ is unicoherent. By Corollary 3.7, $HS_m^n(X)$ is aposyndetic. Finally, by Lemma 2.9, any aposyndetic unicoherent continuum is finitely aposyndetic. ☑

The following result extends [16, Theorem 3.9].

Theorem 3.9. *Let X be a continuum and $n, m \in \mathbb{N}$ with $m \leq n$. If $C_n(X)$ is a finite–dimensional Cantor manifold such that $\dim[C_n(X)] \geq n+2$, then $HS_m^n(X)$ is a finite–dimensional Cantor manifold.*

Proof. Let $k = \dim[C_n(X)]$. According to Lemma 2.10, $\dim[HS_m^n(X)] = k$. Suppose $HS_m^n(X)$ is not a Cantor manifold. Hence, there exists a subset $\mathcal{A}$ of $HS_m^n(X)$ such that $\dim[\mathcal{A}] \leq k-2$ and $HS_m^n(X) - \mathcal{A}$ is not connected. Hence, there exist a separation $\mathcal{A}_1, \mathcal{A}_2$ of $HS_m^n(X) - \mathcal{A}$. Furthermore, by [**?**, (1.4), p. 43], there exist a closed subset $\mathcal{A}'$ of $\mathcal{A}$ and nonempty open subsets $\mathcal{D}, \mathcal{E}$ of $HS_m^n(X)$ such that $HS_m^n(X) - \mathcal{A}' = \mathcal{D} \cup \mathcal{E}$ where $\mathcal{D} \subset \mathcal{A}_1$ and $\mathcal{E} \subset \mathcal{A}_2$. Hence, $C_n(X) - (q_X^{(n,m)})^{-1}(\mathcal{A}') = (q_X^{(n,m)})^{-1}(\mathcal{D}) \cup (q_X^{(n,m)})^{-1}(\mathcal{E})$, where $(q_X^{(n,m)})^{-1}(\mathcal{D})$ and $(q_X^{(n,m)})^{-1}(\mathcal{E})$ are disjoint open subsets of $C_n(X)$. In order to reach a contradiction, we will see that $\dim[(q_X^{(n,m)})^{-1}(\mathcal{A}')] \leq k-2$ so that, $C_n(X)$ is not a Cantor manifold. Consider two cases.

Case 1. $F_X^m \notin \mathcal{A}'$.

Since $(q_X^{(n,m)})^{-1}(\mathcal{A}')$ is homeomorphic to $\mathcal{A}'$, it follows that $\dim[(q_X^{(n,m)})^{-1}(\mathcal{A}')] \leq k-2$.

Case 2. $F_X^m \in \mathcal{A}'$.

By Lemma 2.11 and Lemma 2.12, $\dim[X] = 1$. Observe that $(q_X^{(n,m)})^{-1}(\mathcal{A}') = (q_X^{(n,m)})^{-1}(\mathcal{A}' - \{F_X^m\}) \cup (q_X^{(n,m)})^{-1}(\{F_X^m\}) = (q_X^{(n,m)})^{-1}(\mathcal{A}' - \{F_X^m\}) \cup F_m(X)$. By Lemma 2.13, $\dim[F_m(X)] \leq m$. Since $m \leq n \leq k-2$ and $\dim[(q_X^{(n,m)})^{-1}(\mathcal{A}' - \{F_X^m\})] \leq \dim[\mathcal{A}'] \leq k-2$, by Lemma 2.14, we conclude that $\dim[(q_X^{(n,m)})^{-1}(\mathcal{A}')] \leq k-2$. ☑

The following result extends [16, Corollary 3.10].

Theorem 3.10. *Let $n, m \in \mathbb{N}$ be such that $m \leq n$. The hyperspaces $HS_m^n([0,1])$ and $HS_m^n(\mathcal{S}^1)$ are $2n-$dimensional Cantor manifolds.*

Proof. Case $n = m$ is already proved in [16, Corollary 3.10].

Suppose that $n > m$. By Lemma 2.15 we have that $C_n([0,1])$ and $C_n(\mathcal{S}^1)$ are $2n-$dimensional Cantor manifolds. Since $n \geq 2$ and $2n \geq n+2$, the result follows from Theorem 3.9. ☑

Question 3.11. For what continua X does the natural embedding in the proof of Theorem 3.3 embed $HS_m^s(X)$ as a retract of $HS_m^n(X)$? In particular, what about the case when X is $\mathcal{S}^1$?

Question 3.12. For what continua X, can $HS^s_m(X)$ be embedded in $HS^n_m(X)$ as a retract $(m \leq s < n)$?

According to [5, Theorem 4.4] which states that if X is a contractible continuum and $n, m \in \mathbb{N}$ with $m \leq n$, then $HS^n_m(X)$ is contractible, the following question arises:

Question 3.13. What continua X have the property that $HS^n_m(X)$ is contractible for each $n, m \in \mathbb{N}$ with $m \leq n$?

Question 3.14. [5, Question 7.5] If X is decomposable and $n, m \in \mathbb{N}$ with $m < n$, is $HS^n_m(X)$ locally arcwise connected at F^m_X?

Question 3.15. What continua X have the property that $HS^n_m(X)$ is pseudo$-$contractible for each $n, m \in \mathbb{N}$ with $m \leq n$?

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