A characterization of inducible mappings between hyperspaces

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Abstract. For fixed hyperspaces $\mathcal{H}(X)$ and $\mathcal{H}(Y)$ of metric continua X and Y, respectively, a mapping $g : \mathcal{H}(X) \to \mathcal{H}(Y)$ is called *inducible* provided that there exists a mapping $f : X \to Y$ such that $g(A) = \{f(a) : a \in A\}$, for every $A \in \mathcal{H}(X)$. In this paper, we present a characterization of inducible mappings between hyperspaces, compare it with the necessary and sufficient conditions under which a mapping between hyperspaces g is inducible given by J.J. Charatonik and W.J. Charatonik in 1998, and exhibit examples to show the independence among the conditions in both characterizations in all hyperespaces, some of them have not been considered in the known characterization, doing complete the study of this class of mappings.

Keywords: Continuum, hyperspace, induced mapping, inducible mapping.
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Una caracterización de funciones inducibles entre hiperespacios

Resumen. Dados dos hiperespacios fijos $\mathcal{H}(X)$ y $\mathcal{H}(Y)$ de continuos métricos X y Y, respectivamente, una función continua $g : \mathcal{H}(X) \to \mathcal{H}(Y)$ es *inducible* si existe una función continua $f : X \to Y$ tal que $g(A) = \{f(a) : a \in A\}$, para cada $A \in \mathcal{H}(X)$. En este trabajo presentamos una caracterización de funciones inducibles entre hiperespacios, la comparamos con las condiciones necesarias y suficientes bajo las cuales una función continua entre hiperespacios es inducible, dada por J.J. Charatonik y W.J. Charatonik en 1998, y damos ejemplos que muestran la independencia entre las condiciones en ambas caracterizaciones en todos los hiperespacios, algunos de ellos no habían sido considerados en la caracterización ya conocida, haciendo completo el estudio de esta clase de funciones continuas.

Palabras clave: Continuo, función inducible, función inducida, hiperespacio.

1. Introduction

A *continuum* is a nonempty, compact, connected metric space. Given a continuum X and a positive integer n, consider the following hyperspaces of X:

$$2^X = \{A \subseteq X : A \text{ is a closed subset of } X \text{ and } A \neq \emptyset\},$$
$$\mathcal{C}_n(X) = \{A \in 2^X : A \text{ has at most } n \text{ components}\},$$
$$\mathcal{C}_\infty(X) = \{A \in 2^X : A \text{ has finitely many components}\},$$
$$\mathcal{F}_n(X) = \{A \in 2^X : A \text{ has at most } n \text{ points}\} \text{ and}$$
$$\mathcal{F}_\infty(X) = \{A \in 2^X : A \text{ has finitely many points}\}.$$

All these hyperspaces are considered with the Hausdorff metric (see [4, p. 11]). In this paper, the phrase $\mathcal{H}(X)$ *is a hyperspace of a continuum* X means that $\mathcal{H}(X)$ is one of the hyperspaces defined above for X. Readers specially interested in the structure of these hyperspaces are referred to [4].

The word *mapping* stands for a continuous function. Given a mapping between continua $f : X \to Y$ and a hyperspaces $\mathcal{H}(X)$ and $\mathcal{H}(Y)$ of X and of Y, respectively, the $\mathcal{H}$-*induced function by* f, $\mathcal{H}(f) : \mathcal{H}(X) \to \mathcal{H}(Y)$, is defined by

$$\mathcal{H}(f)(A) = \{f(a) : a \in A\}, \text{ for every } A \in \mathcal{H}(X).$$

By [5, 5.10.1 of Theorem 5.10, p. 170], the continuity of f implies that of the induced function to the hyperspace 2^X, 2^f, and since each $\mathcal{H}(f) = 2^f|_{\mathcal{H}(X)}$, we deduce that $\mathcal{H}(f)$ is also a mapping.

A mapping g between hyperspaces $\mathcal{H}(X)$ and $\mathcal{H}(Y)$ of continua X and Y, respectively, is called *inducible* provided that there exists a mapping $f : X \to Y$ such that $g = \mathcal{H}(f)$.

In [2, Theorem 2.2, p. 7], the authors characterize the inducible mappings for the case that $\mathcal{H}(X) \in \{2^X, \mathcal{C}_1(X)\}$. After, this characterization is extended for the case that $\mathcal{H}(X) \in \{\mathcal{C}_n(X) : n \in \mathbb{N}\}$ in [3, Theorem 49, p. 802]. Finally, in [1, Theorem 5.2, p. 256], the characterization of this class of mapping is generalized for every hyperspace $\mathcal{H}(X)$ listed above. However, the authors do not show that those three necessary and sufficient conditions are independent in the sense that each two of them do not imply the other one when $\mathcal{H}(X) \in \{\mathcal{C}_n(X), \mathcal{F}_n(X), \mathcal{F}_\infty(X) : n \geq 2\}$. On the other hand, an important tool in all these characterizations is an order in the set of all mappings between hyperspaces defined in [2, p. 6], so they are based in an extrinsic property of mappings between hyperspaces. In the current paper, a new characterization of the inducible mappings, using only intrinsic properties, will be given, and we present examples to prove the independence of our conditions and the previous ones to do complete the study of this class of mappings.

The study of mappings between hyperspace has been focused to induced mappings. The characterizations of inducible mappings provide information on mappings between hyperspace that could be used to produce a suitable mapping between hyperspace satisfying a specific property that has not been possible to find between continua, due to the relation between the induced mapping and the mapping between the ground spaces that induces it.

2. Preliminaries and auxiliary results

The symbol $\mathbb{N}$ denotes the set of all positive integers. A *relation* from a set X to a set Y is a subset of $X \times Y$. The domain of a relation φ, represented by $\mathrm{dom}(\varphi)$, is the set $\{x \in X : (\{x\} \times Y) \cap \varphi \neq \emptyset\}$. For a subset A of X, let $\varphi[A] = \{y \in Y : (A \times \{y\}) \cap \varphi \neq \emptyset\}$. Finally, recall that a function f from X to Y can be related with the relation $\{(x, f(x)) : x \in X\}$, and a relation φ is a function provided that the condition $(a, b), (a, c) \in \varphi$ implies that $b = c$.

Let X be a continuum. Define

$$\mathfrak{H}(X) = \{2^X, \mathcal{C}_\infty(X), \mathcal{F}_\infty(X)\} \cup \{\mathcal{F}_n(X) : n \in \mathbb{N} - \{1\}\} \cup \{\mathcal{C}_n(X) : n \in \mathbb{N}\}.$$

Let $x \in X$ and $\mathcal{H}(X) \in \mathfrak{H}(X)$. The set $\{A \in \mathcal{H}(X) : x \in A\}$ is denoted by $\mathcal{H}(x, X)$.

Let X and Y be continua, and let $\mathcal{H}(X) \in \mathfrak{H}(X)$ and $\mathcal{H}(Y) \in \mathfrak{H}(Y)$. Given a mapping $g : \mathcal{H}(X) \to \mathcal{H}(Y)$, define the relation φ_g from X to Y as follows:

$$(x, y) \in \varphi_g \text{ if, and only if, } \{g(A) : A \in \mathcal{H}(x, X)\} \subseteq \mathcal{H}(y, Y).$$

Lemma 2.1. *Let X and Y be continua and let $\mathcal{H}(X) \in \mathfrak{H}(X)$ and $\mathcal{H}(Y) \in \mathfrak{H}(Y)$. If $g : \mathcal{H}(X) \to \mathcal{H}(Y)$ is a mapping, then $\varphi_g[A] \subseteq g(A)$, for every $A \in \mathcal{H}(X)$.*

Proof. Let $A \in \mathcal{H}(X)$ and let $y \in \varphi_g[A]$. Then, there exists $x \in A$ such that $(x, y) \in \varphi_g$. So, by the definition of φ_g, we obtain that $y \in g(A)$. ☑

3. Main result

The following result allows us to characterize all mapping between hyperspace that are induced by a mapping between the ground spaces.

Theorem 3.1. *Let X and Y be continua, and let $\mathcal{H}(X) \in \mathfrak{H}(X)$ and $\mathcal{H}(Y) \in \mathfrak{H}(Y)$. A mapping $g : \mathcal{H}(X) \to \mathcal{H}(Y)$ is inducible if, and only if, g satisfies each one of the following conditions:*

$(1)_g$ *φ_g is a function such that $\mathrm{dom}(\varphi_g) = X$; and*

$(2)_g$ *$g^{-1}[\mathcal{H}(y, Y)] \subseteq \{A \in \mathcal{H}(X) : y \in \varphi_g[A]\}$, for every $y \in Y$.*

Proof. Assume that there exists a mapping $f : X \to Y$ such that $g = \mathcal{H}(f)$. In order to see that $(1)_{\mathcal{H}(f)}$ is satisfied, we are going to verify that $f = \varphi_{\mathcal{H}(f)}$. The definition of $\mathcal{H}(f)$ implies that $f(x) \in \mathcal{H}(f)(A)$ for every $x \in X$ and for every $A \in \mathcal{H}(x, X)$. So, $\{(x, f(x)) : x \in X\} \subseteq \varphi_{\mathcal{H}(f)}$. Next, let $(w, z) \in \varphi_{\mathcal{H}(f)}$. From the definition of $\varphi_{\mathcal{H}(f)}$, we get that $z \in \mathcal{H}(f)(\{w\}) = \{f(w)\}$. Thus, $z = f(w)$. This proves that $f = \varphi_{\mathcal{H}(f)}$. Now, let $y \in Y$ and $B \in \mathcal{H}(f)^{-1}[\mathcal{H}(y, Y)]$. Hence, $y \in \mathcal{H}(f)(B) = \{f(b) : b \in B\} = \varphi_{\mathcal{H}(f)}[B]$. Thus, $(2)_{\mathcal{H}(f)}$ holds.

In order to show the second part, let $A \in \mathcal{H}(X)$ and let $y \in g(A)$ be arbitrary. Then, $A \in g^{-1}[\mathcal{H}(y, Y)]$. By $(2)_g$, we get that $y \in \varphi_g[A]$. So, $g(A) \subseteq \varphi_g[A]$. This and Lemma 2.1 together imply that $g(A) = \varphi_g[A]$ for every $A \in \mathcal{H}(X)$ and, since $(1)_g$ is true, we have that $g(\{x\}) = \varphi_g[\{x\}] = \{\varphi_g(x)\}$ for every $x \in X$. Thus, $\mathrm{dom}(\varphi_g) = X$ and, from the fact that X and $\mathcal{F}_1(X)$ are isometric, it follows that the continuity of g implies that of φ_g. Therefore, $\varphi_g : X \to Y$ is a mapping such that $g = \mathcal{H}(\varphi_g)$. ☑

4.   Comparation with the first characterization

In this section all the possible interrelations between the conditions of Theorem 3.1 and the conditions of [2, Theorem 2.2, p. 7] will be verified, obtaining one more characterization of inducible mappings more.

Define the order $\prec$ on the set of all mapping between hyperspaces $\mathcal{H}(X)$ and $\mathcal{H}(Y)$ as follows: $h \prec g$ if, and only if, $h(A) \subseteq g(A)$, for every $A \in \mathcal{H}(X)$ (see [2, p. 6]).

The result below is the general version of [2, Theorem 2.2, p. 7] presented in [1, Theorem 5.2, p. 256].

Theorem 4.1. *Let X and Y be continua, and let $\mathcal{H}(X) \in \mathfrak{H}(X)$ and $\mathcal{H}(Y) \in \mathfrak{H}(Y)$. A mapping $g : \mathcal{H}(X) \to \mathcal{H}(Y)$ is inducible if, and only if, each one of the following conditions is satisfied:*

$(I)_g$ $g[\mathcal{F}_1(X)] \subseteq \mathcal{F}_1(Y)$;

$(II)_g$ $A \subseteq B$ implies $g(A) \subseteq g(B)$, for every $A, B \in \mathcal{H}(X)$;

$(III)_g$ g is minimal with respect to the order $\prec$, i. e., if a mapping $h : \mathcal{H}(X) \to \mathcal{H}(Y)$ satisfies $(II)_h$ and $h \prec g$, then $h = g$.

Theorem 4.2. *Let X and Y be continua, let $\mathcal{H}(X) \in \mathfrak{H}(X)$ and $\mathcal{H}(Y) \in \mathfrak{H}(Y)$, and let $g : \mathcal{H}(X) \to \mathcal{H}(Y)$ be a mapping. If $(I)_g$ and $(II)_g$ are satisfied, then $(1)_g$ holds.*

Proof. Let us show that $\varphi_g = \{(x, y) \in X \times Y : g(\{x\}) = \{y\}\}$.

First, let $(x, y) \in \varphi_g$ be arbitrary. Then $y \in g(\{x\})$. This and condition $(I)_g$ together imply that $g(\{x\}) = \{y\}$. So, the inclusion $\varphi_g \subseteq \{(x, y) \in X \times Y : g(\{x\}) = \{y\}\}$ holds.

Next, let $(x, y) \in X \times Y$ be such that $g(\{x\}) = \{y\}$ and let $A \in \mathcal{H}(x, X)$. A consequence of condition $(II)_g$ is that $g(\{x\}) \subseteq g(A)$. This implies that $y \in g(A)$. So, we conclude that $(x, y) \in \varphi_g$.

Therefore, φ_g is a function.   ☑

Example 5.1 proves that the converse of Theorem 4.2 is false.

Theorem 4.3. *Let X and Y be continua, let $\mathcal{H}(X) \in \mathfrak{H}(X)$ and $\mathcal{H}(Y) \in \mathfrak{H}(Y)$, and let $g : \mathcal{H}(X) \to \mathcal{H}(Y)$ be a mapping. If $(2)_g$ is satisfied, then $(II)_g$ holds.*

Proof. Let $A, B \in \mathcal{H}(X)$ be such that $A \subseteq B$ and let $y \in g(A)$. Condition $(2)_g$ guarantees that there exists $x \in A$ such that $(x, y) \in \varphi_g$. This and our assumption $A \subseteq B$ imply that $y \in \varphi_g[B]$. Finally, Lemma 2.1 ensures that $\varphi_g[B] \subseteq g(B)$. Then $y \in g(B)$.   ☑

The converse of Theorem 4.3 fails (see examples 5.5, 5.7 and 5.8).

Corollary 4.4. *Let X and Y be continua, and let $\mathcal{H}(X) \in \mathfrak{H}(X)$ and $\mathcal{H}(Y) \in \mathfrak{H}(Y)$. A mapping $g : \mathcal{H}(X) \to \mathcal{H}(Y)$ is inducible if, and only if, each one of the following conditions is satisfied:*

$(I)_g$ $g(\mathcal{F}_1(X)) \subseteq \mathcal{F}_1(Y)$;

$(2)_g$ $g^{-1}[\mathcal{H}(y,Y)] \subseteq \{A \in \mathcal{H}(X) : y \in \varphi_g[A]\}$, *for every* $y \in Y$.

Proof. First, by theorems 3.1 and 4.1, if g is inducible, then $(I)_g$ and $(2)_g$ holds.

Now, suppose that g satisfies the conditions $(I)_g$ and $(2)_g$. Theorem 4.3 implies that $(II)_g$ holds. Hence, by Theorem 4.2, we have that $(1)_g$ is satisfied. Finally, from Theorem 3.1, it follows that g is inducible. ☑

5. Examples

The aim of this section is to show that conditions $(1)_g$, $(2)_g$, $(I)_g$ and $(III)_g$ are independent in the sense that none of them is implied by the other. Also, it will be proved that condition $(II)_g$ does not imply any condition of Theorem 3.1 and $(II)_g$ is not implied by $(1)_g$.

Our first example proves that the conditions $(I)_g$, $(II)_g$, $(III)_g$ and $(2)_g$ are not implied by $(1)_g$.

Example 5.1. *Let* $\mathcal{H}([0,1]) \in \mathfrak{H}([0,1]) - \{\mathcal{C}_1([0,1])\}$ *and let* $g : \mathcal{H}([0,1]) \to \mathcal{H}([0,1])$ *be defined by*

$$g(A) = \{0, \min A\}$$

for each $A \in \mathcal{H}([0,1])$. *Notice that* g *is a mapping. First, to see that* $(1)_g$ *is satisfied, we shall prove that* $\varphi_g = [0,1] \times \{0\}$. *From the definition of* φ_g, *it follows that the inclusion* $[0,1] \times \{0\} \subseteq \varphi_g$ *holds. On the other hand, if* $(a,b) \in \varphi_g$, *then* $b \in \varphi_g(\{0,a\}) = \{0, \min\{0,a\}\} = \{0\}$. *Hence, we conclude that* $\varphi_g = [0,1] \times \{0\}$.

Next, we are going to argue that $(I)_g$, $(II)_g$, $(III)_g$ *and* $(2)_g$ *fail. Set* $B = \{1\}$ *and* $C = \{0,1\}$. *Observe that* $g(B) = \{0,1\}$ *and* $g(C) = \{0\}$. *Since* $B \in \mathcal{F}_1([0,1])$ *and* $g(B) \notin \mathcal{F}_1([0,1])$, *we have that* $(I)_g$ *fails. Notice that* B *is a subset of* C *but* $g(B)$ *is not contained in* $g(C)$. *Then* $(II)_g$ *is not satisfied. From this and Theorem 4.3, we infer that* $(2)_g$ *does not hold. Observe that the constant mapping* $h : \mathcal{H}([0,1]) \to \mathcal{H}([0,1])$ *defined by* $h(A) = \{0\}$ *for each* $A \in \mathcal{H}([0,1])$ *fulfils* $(II)_h$, $h \prec g$ *and* $h \neq g$. *Then* $(III)_g$ *fails.*

Finally, in the case that $\mathcal{H}([0,1]) = \mathcal{C}_1([0,1])$, *we define* $g : \mathcal{C}_1([0,1]) \to \mathcal{C}_1([0,1])$ *by*

$$g(A) = [0, \min A]$$

for each $A \in \mathcal{C}_1([0,1])$. *Similar arguments above prove that* $(1)_g$ *holds and* $(2)_g$, $(I)_g$, $(II)_g$ *and* $(III)_g$ *fail.*

The second example shows that each one of the following statements holds for each $\mathcal{H}(X) \in \mathfrak{H}(X)$:

(a) $(2)_g$ does not imply $(1)_g$.

(b) $(2)_g$ does not imply neither $(I)_g$ nor $(III)_g$.

(c) $(II)_g$ does not imply $(1)_g$.

(d) $(II)_g$ does not imply neither $(I)_g$ nor $(III)_g$.

Example 5.2. *Let X and Y be continua and let $\mathcal{H}(X) \in \mathfrak{H}(X)$. Fix $B \in \mathcal{H}(Y) - \mathcal{F}_1(Y)$. Let $g : \mathcal{H}(X) \to \mathcal{H}(Y)$ be the mapping defined by*

$$g(A) = B$$

for each $A \in \mathcal{H}(X)$. Notice that g is continuous. We are going to prove that $(2)_g$ and $(II)_g$ are satisfied and the conditions $(1)_g$, $(I)_g$ and $(III)_g$ are not.

Notice that $\varphi_g = X \times B$. Hence, φ_g is not a function and so $(1)_g$ is not satified. A consequence of the fact that $g[\mathcal{F}_1(X)] \cap \mathcal{F}_1(Y) = \emptyset$ is that $(I)_g$ does not hold. Now, let $y \in Y$. Notice that $\{A \in \mathcal{H}(X) : y \in \varphi_g[A]\} = \mathcal{H}(X)$ if $y \in B$, and $g^{-1}[\mathcal{H}(y,Y)] = \emptyset$ otherwise. Using the two last equalities, we get that g satisfies $(2)_g$. Theorem 4.3 guarantees that $(II)_g$ holds. Finally, fix $b \in B$. The mapping $h : \mathcal{H}(X) \to \mathcal{H}(Y)$ defined by $h(A) = \{b\}$ fulfils that $h \prec g$, $(II)_h$ holds and $h \neq g$. In conclusion, $(III)_g$ is not satisfied.

The example below proves that, for each $\mathcal{H}([0,1]) \in \mathfrak{H}([0,1])$, there exists a mapping $g : \mathcal{H}([0,1]) \to \mathcal{H}([0,1])$ such that g satisfies $(I)_g$ and $(III)_g$, but not $(1)_g$, $(2)_g$ and $(II)_g$.

Example 5.3. *Let $\mathcal{H}([0,1]) \in \mathfrak{H}([0,1])$. Define $g : \mathcal{H}([0,1]) \to \mathcal{H}([0,1])$ as follows: for each $A \in \mathcal{H}([0,1])$, let*

$$g(A) = \{\min\{1, t + \max A - \min A\} : t \in A\}.$$

In order to prove that the function g is a mapping, define $l : 2^{[0,1]} \to 2^{[0,2]}$ by $l(A) = \{t + \max A - \min A : t \in A\}$ to get a mapping. Then $g(A) = \{\min\{1, s\} : s \in l(A)\}$, for every $A \in \mathcal{H}([0,1])$. Hence, g is a mapping. Let us argue that $(I)_g$ and $(III)_g$ are satisfied but $(1)_g$, $(2)_g$ and $(II)_g$ are not.

For each $x \in [0,1]$, we have that $g(\{x\}) = \{x\}$. Thus, the inclusion $g[\mathcal{F}_1([0,1])] \subseteq \mathcal{F}_1([0,1])$ holds. Hence $(I)_g$ is satisfied. Now, in order to see that $(III)_g$ holds and $(1)_g$, $(2)_g$ and $(II)_g$ do not hold, set $B = \{0\}$ and let $C \in \mathcal{H}([0,1])$ be such that $\min C = 0$ and $\max C = 1$. Observe that $g(B) = B$ and $g(C) = \{1\}$. Then $g(B)$ is not contained in $g(C)$. This shows that $(II)_g$ fails. Hence, from Theorem 4.3, it follows that $(2)_g$ is not true. Since there is no mapping $h : \mathcal{H}([0,1]) \to \mathcal{H}([0,1])$ satisfying $h \prec g$ and $(II)_h$, we have that $(III)_g$ is vacuously true. Finally, to prove that $(1)_g$ fails, let $b \in [0,1]$ be such that $(0,b) \in \varphi_g$. By the definition of φ_g, we deduce that $b \in g(B)$ and $b \in g(C)$. So, $b = 0$ and $b = 1$. Hence, φ_g is not a function and so $(1)_g$ does not hold.

The condition $(III)_g$ does not imply any of the conditions $(1)_g$, $(2)_g$ and $(II)_g$ for each $\mathcal{H}(X) \in \mathfrak{H}(X)$ (compare [2, Example 3.2]).

Example 5.4. *Let $\mathcal{H}([0,1]) \in \mathfrak{H}([0,1])$. Define $g : \mathcal{H}([0,1]) \to \mathcal{H}([0,1])$ by*

$$g(A) = \{\min A\}.$$

In [2, Example 3.2], the authors prove that $(I)_g$ and $(III)_g$ are satisfied, while the condition $(II)_g$ does not hold. Hence, from Theorem 4.3, it follows that g does not fulfil $(2)_g$.

In order to verify that $(1)_g$ does not hold, suppose to the contrary that φ_g is function. Let $b \in [0,1]$ be such that $(\frac{1}{2}, b) \in \varphi_g$. The definition of φ_g ensures that $b \in g(\{\frac{1}{2}\}) = \{\frac{1}{2}\}$, and so $b = \frac{1}{2}$. Now, take $A \in \mathcal{H}(\frac{1}{2}, [0,1]) \cap \mathcal{H}(0, [0,1])$. Then $b \in g(A) = \{0\}$. This is a contradiction. In conclusion, $(1)_g$ does not hold.

The last examples are not valid for all hyperspaces in $\mathfrak{H}(X)$. In each one of them, we indicate for which hyperspaces in $\mathfrak{H}(X)$ the example works.

In the case that $\mathcal{H}(X) \in \mathfrak{H}(X) - \{\mathcal{F}_n(X) : n \in \mathbb{N} \cup \{\infty\}\}$, the example below shows that $(II)_g$ does not imply $(2)_g$ and $(I)_g$ does not imply $(III)_g$ (compare [2, Example 3.1, p. 8]).

Example 5.5. *Let $\mathcal{H}([0,1]) \in \{2^{[0,1]}, \mathcal{C}_\infty([0,1])\} \cup \{\mathcal{C}_n([0,1]) : n \in \mathbb{N}\}$. Define $g : \mathcal{H}([0,1]) \to \mathcal{H}([0,1])$ by*

$$g(A) = \{x \in [0,1] : \text{ there exists } y \in A \text{ such that } |x-y| \leq \max A - \min A\}.$$

Notice that g is a mapping. Now, arguments in [2, Example 3.1, p. 8] prove that $(I)_g$ and $(II)_g$ hold and $(III)_g$ is not satisfied. Finally, by Theorem 4.1, g is not inducible. Therefore, by Corollary 4.4, the condition $(2)_g$ must fails.

The mapping g in the example below proves that $(I)_g$ does not implies $(III)_g$ when $\mathcal{H}(X) \in \{\mathcal{F}_n(X) : n \in (\mathbb{N} - \{2\}) \cup \{\infty\}\}$.

Example 5.6. *Let $\mathcal{H}([0,1]) \in \{\mathcal{F}_n([0,1]) : n \in (\mathbb{N} - \{2\}) \cup \{\infty\}\}$ and let $g : \mathcal{H}([0,1]) \to \mathcal{H}([0,1])$ be defined by $g(A) = \{0, \max A - \min A\}$. Then g is a mapping.*

First, if $t \in [0,1]$, then $g(\{t\}) = \{0, t-t\} = \{0\}$. So, $g[\mathcal{F}_1([0,1])] = \{\{0\}\} \subseteq \mathcal{F}_1([0,1])$. We obtain that $(I)_g$ is satisfied. Now, in order to prove that $(III)_g$ does not hold, let $h : \mathcal{H}([0,1]) \to \mathcal{H}([0,1])$ be defined by $h(A) = \{0\}$. We have that h is a mapping, $(II)_h$ is satisfied and $h \prec g$. Therefore, g is not minimal with respect to $\prec$, in other words, $(III)_g$ does not hold.

Our next example shows that $(2)_g$ is not implied by $(II)_g$ when $\mathcal{H}(X) = \mathcal{F}_n(X)$ for some $n \geq 2$.

Example 5.7. *Let $n \geq 2$. Define $g : \mathcal{F}_n([0,1]) \to \mathcal{F}_n([0,1])$ by*

$$g(A) = \begin{cases} \{0\}, & \text{if } |A| \leq n-1, \\ \{0, \min\{|x-y| : x, y \in A, x \neq y\}\}, & \text{if } |A| = n. \end{cases}$$

Notice that g is a mapping.

Now, observe that $g(A) \subseteq g(B)$ for each $A, B \in \mathcal{F}_n([0,1])$ such that $A \subseteq B$ and $g(\{x\}) = \{0\}$, for each $x \in [0,1]$. This means $(I)_g$ and $(II)_g$ hold. In light of Theorem 4.2, $(1)_g$ is satisfied. We have that $\varphi_g = [0,1] \times \{0\}$.

Notice that $\{\frac{m}{n-1} : m \in \{0, 1, \ldots, n-1\}\} \in g^{-1}[\mathcal{F}_n(\frac{1}{n-1}, [0,1])]$ and $\{A \in \mathcal{F}_n([0,1]) : \frac{1}{n-1} \in \varphi_g[A]\} = \emptyset$. Hence, $(2)_g$ fails.

Our next example exhibit a mapping g from $\mathcal{F}_\infty([0,1])$ into itself satisfying $(II)_g$, while $(2)_g$ fails.

Example 5.8. *Define* $g : \mathcal{F}_\infty([0,1]) \to \mathcal{F}_\infty([0,1])$ *by* $g(A) = \{|x - y| : x, y \in A\}$ *for each* $A \in \mathcal{F}_\infty([0,1])$. *Observe that* g *is a mapping.*

Next, from the definition of g, *it follows that* $(II)_g$ *is satisfied and* $[0,1] \times \{0\} \subseteq \varphi_g$. *Note that if* $(x, y) \in \varphi_g$, *then* $y \in g(\{x\}) = \{0\}$, *and so, we conclude that* $\varphi_g = [0,1] \times \{0\}$. *Then,* g *fulfils* $(1)_g$.

Finally, since $\{0, 1\} \in g^{-1}[\mathcal{F}_\infty(1, [0,1])]$ *and* $\{A \in \mathcal{F}_\infty([0,1]) : 1 \in \varphi_g[A]\} = \emptyset$, *we obtain that* $(2)_g$ *fails.*

The next example shows that there exists a continuum X for which the statement $(III)_g$ does not imply $(I)_g$ holds for every $\mathcal{H}(X) \in \mathfrak{H}(X) - \{\mathcal{C}_1(X)\}$.

Example 5.9. *Let* S^1 *be the set of all complex numbers having norm equal to 1. Consider the mapping* $\exp : \mathbb{R} \to S^1$ *defined by* $\exp(s) = \cos(2\pi s) + i\sin(2\pi s)$. *Let* $\mathcal{H}([0,1]) \in \mathfrak{H}([0,1]) - \{\mathcal{C}_1([0,1])\}$ *and* $\mathcal{H}(S^1) \in \mathfrak{H}(S^1) - \{\mathcal{C}_1(S^1)\}$. *Define* $g : \mathcal{H}([0,1]) \to \mathcal{H}(S^1)$ *by*

$$g(A) = \left\{ \exp\left(\frac{\max A + \min A}{2} \right), \exp\left(\frac{\max A + \min A + 1}{2} \right) \right\}.$$

Notice that g *is a mapping. Now, we will show that* g *satisfies* $(III)_g$, *while the condition* $(I)_g$ *does not hold.*

First, for each $x \in [0,1]$, *by the definition of* g, *we have that* $g(\{x\}) = \{\exp(x), \exp(x + \frac{1}{2})\}$. *So,* $(I)_g$ *does not hold.*

Now, let $h : \mathcal{H}([0,1]) \to \mathcal{H}(S^1)$ *be a mapping such that* $h \prec g$. *Let* $A \in \mathcal{H}(X)$ *be such that* $0 \in A$ *and* $\max A = \frac{1}{2}$. *Observe that the equality* $g(A) \cap g(\{0\}) = \{i, -i\} \cap \{1, -1\} = \emptyset$ *implies that* $h(A) \cap h(\{0\}) = \emptyset$. *This shows that* $(II)_h$ *fails. So we have that* $(III)_g$ *is vacuously true.*

Example described in [2, Example 3.4] can be used to argue that the condition $(III)_g$ does not imply $(I)_g$ when $\mathcal{H}(X) = \mathcal{C}_1(X)$.

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References

[1] Charatonik J.J., "Recent research in hyperspace theory", *Extracta Math.* 18 (2003), No. 2, 235-262.

[2] Charatonik J.J. and Charatonik W.J., "Inducible mappings between hyperspaces", *Bull. Pol. Acad. Sci. Math.* 46 (1998), No. 1, 5-9.

[3] Charatonik J.J., Illanes A. and Macías S., "Induced mapping on the hyperspaces $C_n(X)$ of a continuum X", *Houston J. Math.* 28 (2002), No. 4, 781-805.

[4] Illanes A. and Nadler, S.B.Jr., *Hyperspaces: Fundamentals and recent advances*, Monographs and Textbooks in Pure and Applied Math., Vol. 216, Marcel Dekker, Inc., New York, 1999.

[5] Michael E., "Topologies on spaces of subsets", *Trans. Amer. Math. Soc.* 71 (1951), 152-182. doi: 10.1090/s0002-9947-1951-0042109-4.