# On Pure Hyperideals in Ordered Semihypergroups

### Sobre Hiperideales Puros en Semihipergrupos Ordenados

**Thawhat Changphas**[1,a], **Bijan Davvaz**[2,b]

**Abstract.** In this paper, the notions of pure hyperideal, weakly pure hyperideal and purely prime hyperideal in ordered semihypergroups are introduced and studied. We prove that the set of all purely prime hyperideals is topologized.

**Keywords:** Algebraic hyperstructure, ordered semihypergroup, weakly regular, pure hyperideal, weakly pure hyperideal, purely prime hyperideal, topology.

**Resumen.** En este artículo, las nociones de hiperideal puro, hiperideal debilmente puro e hiperideal puramente primo en semi-hipergrupos ordenados son introducidos y estudiados. Mostramos que el conjunto de todos los hiperideales primos puros está topologizado.

**Palabras claves:** Hiperestructura algebraica, semihipergrupo ordenado, debilmente regular, hiperideal puro, hiperideal debilmente puro, hiperideal puramente primo, topología.

Mathematics Subject Classification: 20N20, 06F05.

Recibido: junio de 2020 Aceptado: junio de 2021

## 1. Introduction and preliminaries

The notions of pure ideal, weakly pure ideal and purely prime ideal have been introduced and studied in several kinds of algebraic structures. For examples, Ahsan and Takahashi [1] considered them in semigroups (without order); Bashir and Shabir [4] discussed in ternary semigroups and Changphas and Sanborisoot [6] considered them in ordered semigroups, extending the results on semigroups (without order). The concept of algebraic hyperstructures was introduced in 1934 by Marty [16] and has been studied in the following decades and nowadays by many mathematicians. Semihypergroups are studied by many authors, for

[1]Department of Mathematics, Faculty of Science, Khon Kaen University, Khon Kaen, Thailand
[2]Department of Mathematics, Yazd University, Yazd, Iran
[a]thacha@kku.ac.th
[b]davvaz@yazd.ac.ir

example, Anvariyeh et al. [2, 3], Bonansinga and Corsini [5], Davvaz [8, 9], De Salvo et al. [17], Freni [10], Heidari et al. [12], Hila et al. [14, 13], Leoreanu [15], Yaqoob et al. [18]. The concept of ordering hypergroups investigated by Chvalina [7] as a special class of hypergroups and was studied by him and many others.

One of the main results is that the set of all purely prime ideals is topologized. The purpose of this paper is to define and study the concepts mentioned above on the structure which is called ordered semihypergroups. Note that the results on ordered semigroups become then special cases.

Let $S$ be a non-empty set. A mapping $\circ : S \times S \to \mathcal{P}^*(S)$, where $\mathcal{P}^*(S)$ denotes the family of all non-empty subsets of $S$, is called a *hyperoperation* on $S$. The couple $(S, \circ)$ is called a *hypergroupoid*. In the above definition, if $A$ and $B$ are two non-empty subsets of $S$ and $x \in S$, then we denote

$$A \circ B = \bigcup_{a \in A, b \in B} a \circ b, \; x \circ A = \{x\} \circ A \; \text{ and } \; A \circ x = A \circ \{x\}.$$

A hypergroupoid $(S, \circ)$ is called a *semihypergroup* if for every $x, y, z$ in $S$,

$$x \circ (y \circ z) = (x \circ y) \circ z.$$

That is,

$$\bigcup_{u \in y \circ z} x \circ u = \bigcup_{v \in x \circ y} v \circ z.$$

In [11], Heidari and Davvaz studied a semihypergroup $(S, \circ)$ endowed with a binary relation $\leq$, where $\leq$ is a partial order relation such that satisfies the monotone condition. Indeed, an *ordered semihypergroup* $(S, \circ, \leq)$ is a semihypergroup $(S, \circ)$ together with a partial order $\leq$ that is *compatible* with the hyperoperation, meaning that for any $x, y, z$ in $S$,

$$x \leq y \Rightarrow z \circ x \leq z \circ y \text{ and } x \circ z \leq y \circ z.$$

Here, $z \circ x \leq z \circ y$ means for any $a \in z \circ x$ there exists $b \in z \circ y$ such that $a \leq b$. The case $x \circ z \leq y \circ z$ is defined similarly.

A non-empty subset $A$ of an ordered semihypergroup $(S, \circ, \leq)$ is called a *subsemihypergroup* of $S$ if $A \circ A \subseteq A$.

**Definition 1.1.** A non-empty subset $A$ of an ordered semihypergroup $(S, \circ, \leq)$ is called a *left* (respectively, *right*) *hyperideal* of $S$ if it satisfies the following conditions:

(i) $S \circ A \subseteq A$ (respectively, $A \circ S \subseteq A$);

(ii) if $x \in A$ and $y \in S$ is such that $y \leq x$, then $y \in A$.

If $A$ is both a left and a right hyperideal of $S$, then it is called a *two-sided hyperideal* of $S$, or simply a *hyperideal* of $S$.

Let $A$ be a non-empty subset of an ordered semihypergroup $(S, \circ, \leq)$. Define

$$(A] = \{x \in S \mid x \leq a \text{ for some } a \in A\}.$$

Note that the condition (ii) in Definition 1.1 is equivalent to $A = (A]$. If $A$ and $B$ are non-empty subsets of $S$, then

(1) $A \subseteq (A]$;

(2) $(A \cup B] = (A] \cup (B]$;

(3) $((A] \circ (B]] = (A \circ B]$;

(4) $(A] \circ (B] \subseteq (A \circ B]$.

*Remark* 1.2. Let $(S, \circ, \leq)$ be an ordered semihypergroup.

(1) If $A$ and $B$ are hyperideals of $S$, then $(A \circ B]$ is a hyperideal of $S$.

(2) Intersection of hyperideals of $S$ is a hyperideal of $S$ if it is nonempty.

(3) Union of hyperideals of $S$ is a hyperideal of $S$.

(4) Finite intersection of hyperideals of $S$ is a hyperideal of $S$.

For a non-empty subset $A$ of an ordered semihypergroup $(S, \circ, \leq)$, we denote by $(A)_l$ (respectively, $(A)_r, (A)$) the left (respectively, right, two-sided) hyperideal of $S$ generated by $A$.

**Lemma 1.3.** *If $A$ is a non-empty subset of an ordered semihypergroup $(S, \circ, \leq)$, then the following hold*:

(1) $(A)_l = (A \cup S \circ A]$;

(2) $(A)_r = (A \cup A \circ S]$;

(3) $(A) = (A \cup S \circ A \cup A \circ S \cup S \circ A \circ S]$.

***Proof***. By $A \subseteq (A)_l$ and $S \circ A \subseteq (A)_l$, it follows that $(A \cup S \circ A] \subseteq (A)_l$. Clearly, $(A \cup S \circ A] \neq \emptyset$. We have

$$S \circ (A \cup S \circ A] \subseteq (S \circ (A \cup S \circ A)] = (S \circ A \cup S \circ (S \circ A)] \subseteq (S \circ A] \subseteq (A \cup S \circ A].$$

Then, $(A \cup S \circ A]$ is a left hyperideal of $S$ containing $A$; hence $(A)_l \subseteq (A \cup S \circ A]$. This proves that (1) holds. The conditions (2) and (3) are proved similarly. $\square$

Let $(S, \circ, \leq)$ be an ordered semihypergroup. An element $a$ of $S$ is said to be *regular* if there exists $x$ in $S$ such that $a \in (a \circ x \circ a]$, and $S$ is called *regular* if every element of $S$ is regular. Note that $S$ is regular if and only if $a \in (a \circ S \circ a]$ for all $a$ in $S$.

## 2. Pure hyperideals

We introduce the notion of pure ideal in ordered semihypergroups as follows:

**Definition 2.1.** Let $(S, \circ, \leq)$ be an ordered semihypergroup. A hyperideal $A$ of $S$ is called a *left* (respectively, *right*) *pure ideal* if for any $x$ in $A$ there exists $y$ in $A$ such that $x \leq y \circ x$ (respectively, $x \leq x \circ y$). Similarly, we can define for $A$ to be a left and a right pure hyperideal of $S$.

**Equivalent Definition.** $x \in (A \circ x]$ (respectively, $x \in (x \circ A]$).

**Example 2.2.** Suppose that $S = \{a, b, c, d, e, f, g, h\}$. We consider the order semihypergroup $(S, \circ, \leq)$, where the hyperoperation $\circ$ is defined by the following table:

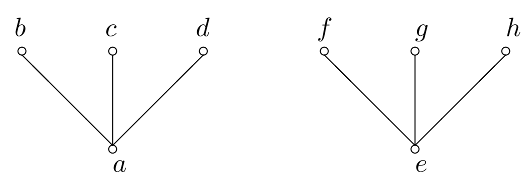
| $\circ$ | $a$ | $b$ | $c$ | $d$ | $e$ | $f$ | $g$ | $h$ |
|---|---|---|---|---|---|---|---|---|
| $a$ | $a$ | $a$ | $a$ | $a$ | $\{a,e\}$ | $\{a,e\}$ | $\{a,e\}$ | $\{a,e\}$ |
| $b$ | $a$ | $b$ | $c$ | $a$ | $\{a,e\}$ | $\{b,f\}$ | $\{c,g\}$ | $\{a,e\}$ |
| $c$ | $a$ | $a$ | $a$ | $a$ | $\{a,e\}$ | $\{a,e\}$ | $\{a,e\}$ | $\{a,e\}$ |
| $d$ | $a$ | $d$ | $a$ | $a$ | $\{a,e\}$ | $\{d,h\}$ | $\{a,e\}$ | $\{a,e\}$ |
| $e$ | $a$ | $a$ | $a$ | $a$ | $\{a,e\}$ | $\{a,e\}$ | $\{a,e\}$ | $\{a,e\}$ |
| $f$ | $a$ | $b$ | $c$ | $a$ | $\{a,e\}$ | $\{b,f\}$ | $\{c,g\}$ | $\{a,e\}$ |
| $g$ | $a$ | $a$ | $a$ | $a$ | $\{a,e\}$ | $\{a,e\}$ | $\{a,e\}$ | $\{a,e\}$ |
| $h$ | $a$ | $d$ | $a$ | $a$ | $\{a,e\}$ | $\{d,h\}$ | $\{a,e\}$ | $\{a,e\}$ |

and the order $\leq$ is defined by

$$\leq\, = \{(a,a),(b,b),(c,c),(d,d),(e,e),(f,f),(g,g),(h,h),$$
$$(a,b),(a,c),(a,d),(e,f),(e,g),(e,h)\}$$

The covering relation and the figure of $S$ are given by:

$$<\, = \{(a,b),(a,c),(a,d),(e,f),(e,g),(e,h)\}$$




It is easy to see that

$$\begin{aligned} I_1 &= \{a,e\}, \\ I_2 &= \{a,c,e\}, \\ I_3 &= \{a,c,e,g\}, \\ I_4 &= \{a,d,e,h\}, \\ I_5 &= \{a,c,d,e,g,h\} \end{aligned}$$

are hyperideals of $S$. We have

$$\begin{aligned} a &\leq a \circ a = a, & e &\leq e \circ e = \{e, a\}, \\ c &\leq b \circ c = c, & g &\leq b \circ g = \{c, g\}, \\ d &\leq d \circ b = d, & h &\leq h \circ f = \{d, h\}, \end{aligned}$$

and there is no $x$ in $S$ such that

$$c \leq c \circ x,\ g \leq g \circ x,\ d \leq x \circ d \text{ and } h \leq x \circ h.$$

Therefore,

(1) $I_1$ is a left and right pure hyperideal of $S$.

(2) $I_2$ and $I_3$ are left pure hyperideals of $S$, but they are not right pure hyperideals of $S$.

(3) $I_4$ is a right pure hyperideal of $S$, but it is not a left pure hyperideal of $S$.

(4) $I_5$ is not a left pure hyperideal of $S$ as well as a right pure hyperideal of $S$.

**Theorem 2.3.** *Let $A$ be a hyperideal of an ordered semihypergroup $(S, \circ, \leq)$. Then, $A$ is right pure if and only if $B \cap A = (B \circ A]$ for all right hyperideals $B$ of $S$.*

***Proof.*** Assume that a hyperideal $A$ of $S$ is right pure. Let $B$ be a right hyperideal of $S$. Since $B \circ A \subseteq B \circ S \subseteq B$, we have $(B \circ A] \subseteq B$. By $B \circ A \subseteq S \circ A \subseteq A$, it follows that $(B \circ A] \subseteq A$. Hence, $(B \circ A] \subseteq B \cap A$. If $x \in B \cap A$, then by assumption there exists $y$ in $A$ such that $x \leq x \circ y$; hence by $x \circ y \subseteq B \circ A$ we obtain $x \in (B \circ A]$. Thus, $B \cap A \subseteq (B \circ A]$.

Conversely, we assume that $B \cap A = (B \circ A]$ for all right hyperideals $B$ of $S$. Let $x \in A$. Since $(x \cup x \circ S]$ is a right hyperideal of $S$ and $S \circ A \subseteq A$, we have

$$(x \cup x \circ S] \cap A = ((x \cup x \circ S] \circ A] \subseteq (x \circ A \cup x \circ S \circ A] \subseteq (x \circ A].$$

Since $x \in (x \cup x \circ S] \cap A$, so $x \in (x \circ A]$. This proves that $A$ is a right pure hyperideal of $S$. □

**Definition 2.4.** An ordered semihypergroup $(S, \circ, \leq)$ is said to be *right weakly regular* if for any $x$ in $S$ there exist $y, z$ in $S$ such that $x \leq x \circ y \circ x \circ z$.

**Equivalent Definition.** $x \in (x \circ S \circ x \circ S]$.

*Remark* 2.5. Every regular ordered semihypergroup is right weakly regular.

**Theorem 2.6.** *The following are equivalent for an ordered semihypergroup $(S, \circ, \leq)$ :*

(1) $S$ *is right weakly regular;*

(2) $(A \circ A] = A$ *for all right hyperideals* $A$ *of* $S$*;*

(3) $B \cap A = (B \circ A]$ *for all right hyperideals* $B$ *and all hyperideals* $A$ *of* $S$*.*

***Proof.*** (1) $\Rightarrow$ (2). Assume that $S$ is right weakly regular. Let $A$ be a right hyperideal of $S$. Since $A \circ A \subseteq A \circ S \subseteq A$, we have $(A \circ A] \subseteq A$. Let $x \in A$. By assumption, there exist $y, z$ in $A$ such that $x \leq x \circ y \circ x \circ z$. Since $(x \circ y) \circ (x \circ z) \subseteq A \circ A$, we have $x \in (A \circ A]$, and so $A \subseteq (A \circ A]$. Hence, $(A \circ A] = A$.

(2) $\Rightarrow$ (1). Assume that $(A \circ A] = A$ for all right hyperideals $A$ of $S$. Let $x \in S$. Since $(x \cup x \circ S]$ is a right hyperideal of $S$, we have

$$\begin{aligned} (x \cup x \circ S] &= ((x \cup x \circ S] \circ (x \cup x \circ S]] \\ &\subseteq ((x \cup x \circ S) \circ (x \cup x \circ S)] \\ &= (x^2 \cup x^2 \circ S \cup x \circ S \circ x \cup x \circ S \circ x \circ S]. \end{aligned}$$

Then,

$$x \in (x^2 \cup x^2 \circ S \cup x \circ S \circ x \cup x \circ S \circ x \circ S],$$

hence $x \in (x \circ S \circ x \circ S]$. This proves that $S$ is right weakly regular.

(1) $\Rightarrow$ (3). Assume that $S$ is right weakly regular. Let $B$ and $A$ be a right hyperideal and a hyperideal of $S$, respectively. Since $B \circ A \subseteq B \circ S \subseteq B$, we have $(B \circ A] \subseteq B$. Similarly, $(B \circ A] \subseteq A$. Then, $(B \circ A] \subseteq B \cap A$. Let $x \in B \cap A$. We have $(x \circ S \circ x \circ S] \subseteq (B \circ A]$. By assumption, we get $x \in (x \circ S \circ x \circ S]$, hence $x \in (B \circ A]$. Thus, $B \cap A \subseteq (B \circ A]$, whence $B \cap A = (B \circ A]$.

(3) $\Rightarrow$ (1). Assume that $B \cap A = (B \circ A]$ for all right hyperideals $B$ and all hyperideals $A$ of $S$. To prove that $S$ is right weakly regular, let $x \in S$. We have $(x \cup x \circ S]$ and $(x \cup S \circ x \circ S]$ are right and (two-sided) hyperideals of $S$, respectively. Then,

$$\begin{aligned} & (x \cup x \circ S] \cap (x \cup S \circ x \circ S] \\ = \; & ((x \cup x \circ S] \circ (x \cup S \circ x \circ S]] \\ \subseteq \; & (x^2 \cup x \circ S \circ x \circ S \cup x \circ S \circ x \cup x \circ S \circ S \circ x \circ S], \end{aligned}$$

thus

$$x \in (x^2 \cup x \circ S \circ x \circ S \cup x \circ S \circ x \cup x \circ S \circ S \circ x \circ S].$$

This implies that $x \in (x \circ S \circ x \circ S]$, hence $S$ is right weakly regular. $\square$

**Theorem 2.7.** *An ordered semihypergroup* $(S, \circ, \leq)$ *is right weakly regular if and only if every hyperideal of* $S$ *is right pure.*

***Proof.*** This follows from Theorem 2.3 and Theorem 2.6. $\square$

**Theorem 2.8.** *Let* $(S, \circ, \leq)$ *be an ordered semihypergroup with zero* $0$*.*

(1) $\{0\}$ *is a right pure hyperideal of* $S$.

(2) *The union of any family of right pure hyperideals of* $S$ *is a right pure hyperideal of* $S$.

(3) *The finite intersection of right pure hyperideals of* $S$ *is a right pure hyperideal of* $S$.

***Proof.*** (1) This is obvious.

(2) Let $\{A_i \mid i \in I\}$ be an indexed family of right pure hyperideals of $S$. We have $\bigcup_{i\in I} A_i$ is a hyperideal of $S$. Let $x \in \bigcup_{i\in I} A_i$. Then, $x \in A_j$ for some $j$ in $I$. Since $A_j$ is right pure, there exists $y$ in $A_j$ such that $x \leq x \circ y$. We have $y \in A_j \subseteq \bigcup_{i\in I} A_i$; hence $\bigcup_{i\in I} A_i$ is right pure.

(3) Let $\{A_1, A_2, \ldots, A_n\}$ be a finite indexed family of right pure hyperideals of $S$. Then, $\bigcap_{i=1}^n A_i$ is a hyperideal of $S$. Let $x \in \bigcap_{i=1}^n A_i$. For $k \in \{1, 2, \ldots, n\}$, there exists $y_k \in A_k$ such that $x \leq x \circ y_k$. We have

$$x \leq x \circ y_n \circ \cdots \circ y_2 \circ y_1.$$

Since $y_n \cdots y_2 \circ y_1 \in \bigcap_{i=1}^n A_i$, we conclude that $\bigcap_{i=1}^n A_i$ is right pure. □

**Theorem 2.9.** *Let* $(S, \circ, \leq)$ *be an ordered semihypergroup with zero* $0$ *and* $A$ *a hyperideal of* $S$. *Then,* $A$ *contains the largest right pure hyperideal of* $S$ *(called the pure part of* $A$*), denoted by* $\mathcal{S}(A)$.

***Proof.*** Clearly, $\{0\}$ is a right pure hyperideal of $S$ contained in $A$. Then, the union of all right pure hyperideals of $S$ contained in $A$ exists, and it is the largest right pure hyperideal of $S$ contained in $A$. □

**Theorem 2.10.** *Let* $(S, \circ, \leq)$ *be an ordered semihypergroup with zero* $0$. *Let* $A, B$ *and* $A_i, i \in I$ *be hyperideals of* $S$.

(1) $\mathcal{S}(A \cap B) = \mathcal{S}(A) \cap \mathcal{S}(B)$.

(2) $\bigcup\limits_{i\in I} \mathcal{S}(A_i) \subseteq \mathcal{S}(\bigcup\limits_{i\in I} A_i)$.

***Proof.*** (1) Since $\mathcal{S}(A) \subseteq A$ and $\mathcal{S}(B) \subseteq B$, we have $\mathcal{S}(A) \cap \mathcal{S}(B) \subseteq A \cap B$. Hence, $\mathcal{S}(A) \cap \mathcal{S}(B) \subseteq \mathcal{S}(A \cap B)$. Since $\mathcal{S}(A \cap B) \subseteq A \cap B \subseteq A$, we get $\mathcal{S}(A \cap B) \subseteq \mathcal{S}(A)$. Similarly, $\mathcal{S}(A \cap B) \subseteq \mathcal{S}(B)$. Then, $\mathcal{S}(A \cap B) \subseteq \mathcal{S}(A) \cap \mathcal{S}(B)$, whence $\mathcal{S}(A \cap B) = \mathcal{S}(A) \cap \mathcal{S}(B)$.

(2) Since $\mathcal{S}(A_i) \subseteq A_i$ for all $i \in I$, we have $\bigcup_{i\in I} \mathcal{S}(A_i) \subseteq \bigcup_{i\in I} A_i$. Then, $\bigcup_{i\in I} \mathcal{S}(A_i) \subseteq \mathcal{S}(\bigcup_{i\in I} A_i)$. □

**Definition 2.11.** A right pure hyperideal $A$ of an ordered semihypergroup $(S, \circ, \leq)$ is said to be *purely maximal* if for any proper right pure hyperideal $B$ of $S$, $A \subseteq B$ implies $A = B$.

**Example 2.12.** In Example 2.2, the right pure hyperideal $I_4$ is purely maximal.

**Definition 2.13.** Let $A$ be a proper right pure hyperideal of an ordered semihypergroup $(S, \circ, \leq)$. Then, $A$ is called *purely prime* if for any right pure hyperideals $B_1, B_2$ of $S$, $B_1 \cap B_2 \subseteq A$ implies $B_1 \subseteq A$ or $B_2 \subseteq A$.

**Theorem 2.14.** *Every purely maximal hyperideal of an ordered semihypergroup $(S, \circ, \leq)$ is purely prime.*

***Proof***. Let $A$ be a purely maximal hyperideal of $S$. Let $B$ and $C$ be right pure hyperideals of $S$ such that $B \cap C \subseteq A$ and $B \nsubseteq A$. Since $B \cup A$ is a right pure hyperideal such that $A \subset B \cup A$, so $S = B \cup A$. We have

$$C = C \cap S = C \cap (B \cup A) = (C \cap B) \cup (C \cap A) \subseteq A.$$

Then, $A$ is purely prime. □

**Theorem 2.15.** *Let $(S, \circ, \leq)$ be an ordered semihypergroup with zero. The pure part of any maximal hyperideal of $S$ is purely prime.*

***Proof***. Let $A$ be a maximal ideal of $S$. To show that $\mathcal{S}(A)$ is purely prime, let $B$ and $C$ be right pure hyperideals of $S$ such that $B \cap C \subseteq \mathcal{S}(A)$. If $B \subseteq A$, then $B \subseteq \mathcal{S}(A)$. Suppose that $B \nsubseteq A$. We have $B \cup A$ is an ideal of $S$. By maximality of $A$, $S = B \cup A$, and hence $C \subseteq A$. Thus, $C \subseteq \mathcal{S}(A)$. □

**Theorem 2.16.** *Let $(S, \circ, \leq)$ be an ordered semihypergroup and $A$ a right pure hyperideal of $S$. If $x \in S \setminus A$, then there exists a purely prime hyperideal $B$ of $S$ such that $A \subseteq B$ and $x \notin B$.*

***Proof***. Assume that $x \in S \setminus A$. Let

$$P = \{B \mid B \text{ is a right pure hyperideal of } S,\ A \subseteq B \text{ and } x \notin B\}.$$

We have $P \neq \emptyset$ since $A \in P$. Moreover, $P$ is a partially ordered set under the usual inclusion. Let $\{B_k \mid k \in K\}$ be any totally ordered subset of $P$. By Theorem 2.8, $\bigcup_{k \in K} B_k$ is a right pure hyperideal. Since $A \subseteq \bigcup_{k \in K} B_k$ and $x \notin \bigcup_{k \in K} B_k$, we obtain $\bigcup_{k \in K} B_k \in P$. By Zorn's lemma, $P$ has a maximal element, say $M$, such that $M$ is a right pure hyperideal, $A \subseteq M$ and $x \notin M$. We shall show that $M$ is purely prime. Suppose that $A_1$ and $A_2$ are right pure hyperideals of $S$ such that $A_1 \nsubseteq M$ and $A_2 \nsubseteq M$. Since $A_1$, $A_2$ and $M$ are right pure, so $A_1 \cup M$ and $A_2 \cup M$ are right pure hyperideals containing $M$. Since $x \in A_1 \cup M$ and $x \notin M$, we have $x \in A_1$. Similarly, $x \in A_2$. Hence, $x \in A_1 \cap A_2$. Thus, $A_1 \cap A_2 \nsubseteq M$. This shows that $M$ is purely prime. □

**Theorem 2.17.** *Any proper right pure hyperideal $A$ of an ordered semihypergroup $(S, \circ, \leq)$ is the intersection of all the purely prime hyperideals of $S$ containing $A$.*

***Proof***. Let $\{B_i \mid i \in I\}$ be the set of all purely prime hyperideals of $S$ containing $A$; by Theorem 2.16 this set is non-empty. We have $A \subseteq \bigcap_{i \in I} B_i$. The reverse inclusion follows by Theorem 2.16. □

# 3. Weakly pure hyperideals

In this section, we introduce the concept of weakly pure hyperideal in an ordered semihypergroup as follows:

**Definition 3.1.** Let $(S, \circ, \leq)$ be an ordered semihypergroup. A hyperideal $A$ of $S$ is called *left (resp. right) weakly pure* if $A \cap B = (A \circ B]$ (respectively, $A \cap B = (B \circ A]$) for all hyperideals $B$ of $S$.

*Remark* 3.2. Every left (right) pure two-sided hyperideal is left (right) weakly pure.

**Theorem 3.3.** *Let $(S, \circ, \leq)$ be an ordered semihypergroup with zero 0. If $A$ and $B$ are hyperideals of $S$, then*

$$\begin{aligned} B \circ A^{-1} &= \{s \in S \mid \forall x \in A, x \circ s \subseteq B\} \\ A_{-1} \circ B &= \{s \in S \mid \forall x \in A, s \circ x \subseteq B\} \end{aligned}$$

*are hyperideals of $S$.*

***Proof.*** Clearly, $0 \in B \circ A^{-1}$. Let $u, v \in S$ and $s \in B \circ A^{-1}$. Let $x \in A$. Since $x \circ u \subseteq A$, we have $(x \circ u) \circ s \subseteq B$, and hence

$$x \circ (u \circ s \circ v) = (x \circ u) \circ s \circ v \subseteq B.$$

Let $a \in B \circ A^{-1}$ and $b \in S$ be such that $b \leq a$. Let $y \in A$; then $y \circ b \leq y \circ a$. Since $y \circ a \subseteq B$, so $y \circ b \subseteq B$. This shows that $B \circ A^{-1}$ is a hyperideal of $S$.

That $A_{-1} \circ B$ is a hyperideal of $S$ is proved similarly. □

**Theorem 3.4.** *Let $(S, \circ, \leq)$ be an ordered semihypergroup and $A$ a hyperideal of $S$. Then, $A$ is left (right) weakly pure if and only if $A \cap (B \circ A^{-1}) = A \cap B$ $(A \cap (A_{-1} \circ B) = A \cap B)$ for all hyperideals $B$ of $S$.*

***Proof.*** Assume that $A$ is left weakly pure. Let $B$ be a hyperideal of $S$. By Theorem 3.3, $B \circ A^{-1}$ is a hyperideal of $S$, and thus $A(\cap B \circ A^{-1}) \subseteq (A \circ (B \circ A^{-1})]$. Since $A \circ (B \circ A^{-1}) \subseteq A \circ S \subseteq A$, we have $(A \circ (B \circ A^{-1})] \subseteq (A] = A$. Let $t \in (A \circ (B \circ A^{-1})]$ be such that $t \leq x \circ y$ for some $x$ in $A$ and $y$ in $B \circ A^{-1}$. By the definition of $B \circ A^{-1}$, $x \circ y \subseteq B$. Then, $t \in B$. This proves that $A \cap B \circ A^{-1} \subseteq A \cap B$. Let $x \in A \cap B$. Since $a \circ x \subseteq B$ for any $a$ in $A$, we have $x \in B \circ A^{-1}$. We get $x \in A \cap (B \circ A^{-1})$. Hence, $A \cap B \subseteq A \cap (B \circ A^{-1})$.

Conversely, assume that $(B \circ A^{-1}) \cap A = A \cap B$ for all hyperideals $B$ of $S$. To show that $A$ is left weakly pure, let $C$ be any hyperideal of $S$. We shall show that $A \cap C = (A \circ C]$. By assumption, $A \cap C = A \cap (C \circ A^{-1})$. Since $A \circ C \subseteq A \circ S \subseteq A$, we have $(A \circ C] \subseteq A$. Let $t \in (A \circ C]$ such that $t \leq x \circ y$ for some $x$ in $A$ and $y$ in $C$, and let $a \in A$. Since

$$a \circ (x \circ y) = (a \circ x) \circ y \subseteq C,$$

we obtain $x \circ y \subseteq C \circ A^{-1}$, and so $t \in C \circ A^{-1}$. Then, $(A \circ C] \subseteq C \circ A^{-1}$. This proves that $(A \circ C] \subseteq A \cap C$. For the reverse inclusion, we have $C \subseteq (A \circ C] \circ A^{-1}$ because $c \in C$ and $a \in A$ implies $a \circ c \subseteq A \circ C \subseteq (A \circ C]$. Then,

$$A \cap C \subseteq A \cap ((A \circ C] \circ A^{-1}) = A \cap (A \circ C] \subseteq (A \circ C].$$

The second half of this theorem can be proved similarly. □

**Theorem 3.5.** *The following are equivalent on an ordered semihypergroup* $(S, \circ, \leq)$ :

(1) *every hyperideal is left weakly pure;*

(2) *for every hyperideal* $A$ *of* $S$, $A \circ A = A$;

(3) *every hyperideal is right weakly pure.*

***Proof.*** This can be proved similarly as Proposition 4.4 in [4]. □

# 4. Pure spectrums

Let $(S, \circ, \leq)$ be an ordered semihypergroup such that $S \circ S = S$. The set of all right pure hyperideals of $S$ and the set of all proper pure prime hyperideals of $S$ will be denoted by $P(S)$ and $P'(S)$, respectively. For $A \in P(S)$, let

$$I_A = \{J \in P'(S) \mid A \not\subseteq J\} \text{ and } \tau(S) = \{I_A \mid A \in P(S)\}.$$

**Theorem 4.1.** $\tau(S)$ *forms a topology on* $P'(S)$.

***Proof.*** Since $\{0\}$ is a right pure hyperideal of $S$ and $I_{\{0\}} = \emptyset$, we have $\emptyset \in \tau(S)$. Since $S$ is a right pure hyperideal of itself and $I_S = P'(S)$, we obtain $P'(S) \in \tau(S)$. Now, if $\{I_{A_\alpha} \mid \alpha \in \Lambda\} \subseteq \tau(S)$, then $\bigcup_{\alpha \in \Lambda} I_{A_\alpha} = I_{\bigcup_{\alpha \in \Lambda} A_\alpha}$; hence $\bigcup_{\alpha \in \Lambda} I_{A_\alpha} \in \tau(S)$.

Let $I_{A_1}, I_{A_2} \in \tau(S)$. We shall show that $I_{A_1} \cap I_{A_2} = I_{A_1 \cap A_2}$, therefore let $J \in I_{A_1} \cap I_{A_2}$. Then, $J \in P'(S)$, $A_1 \not\subseteq J$ and $A_2 \not\subseteq J$. Suppose that $A_1 \cap A_2 \subseteq J$. Since $J$ is pure prime, we have $A_1 \subseteq J$ or $A_2 \subseteq J$. This is a contradiction. Then, $J \in I_{A_1 \cap A_2}$, and thus $I_{A_1} \cap I_{A_2} \subseteq I_{A_1 \cap A_2}$. For the reverse inclusion, let $J \in I_{A_1 \cap A_2}$. Since $A_1 \cap A_2 \not\subseteq J$, it follows that $A_1 \not\subseteq J$ and $A_2 \not\subseteq J$. This implies that $J \in I_{A_1} \cap I_{A_2}$. Hence, $I_{A_1 \cap A_2} \subseteq I_{A_1} \cap I_{A_2}$. Therefore, $\tau(S)$ forms a topology on $P'(S)$. □

## References


[1] J. Ahsan and M. Takahashi, *Pure spectrum of a monoid with zero*, Kobe J. Mathematics **6** (1989), 163–182.

[2] S. M. Anvariyeh, S. Mirvakili, and B. Davvaz, *On* $\Gamma$*-hyperideals in* $\Gamma$*-semihypergroups*, Carpathian Journal of Mathematics **26** (2010), 11–23.

[3] ———, *Pawlak's approximations in* $\Gamma$*-semihypergroups*, Computers and Mathematics with Applications **60** (2010), 45–53.

[4] S. Bashir and M. Shabir, *Pure ideals in ternary semigroups*, Quasigroups and Related Systems **17** (2009), 149–160.

[5] P. Bonansinga and P. Corsini, *On semihypergroup and hypergroup homomorphisms*, Boll. Un. Mat. Ital. B **6** (1982), no. 1(2), 717–727.

[6] T. Changphas and J. Sanborisoot, *Pure ideals in ordered semigroups*, Kyungpook Mathematical Journal **54** (2014), 123–129.

[7] J. Chvalina, *Commutative hypergroups in the sense of marty and ordered sets*, General algebra and ordered sets (Horn-Lipova, 1994), 19–30.

[8] B. Davvaz, *Some results on congruences in semihypergroups*, Bull. Malays. Math. Soc. **23** (2000), no. 2, 53–58.

[9] ———, *Characterizations of sub-semihypergroups by various triangular norms*, Czechoslovak Mathematical Journal **55** (2005), no. 4, 923–932.

[10] D. Freni, *Minimal order semihypergroups of type* $U$ *on the right, II*, J. Algebra **340** (2011), 77–89.

[11] D. Heidari and B. Davvaz, *On ordered hyperstructures*, Politehn. Univ. Bucharest Sci. Bull. Ser. A Appl. Math. Phys. **73** (2011), no. 2, 85–96.

[12] D. Heidari, S. O. Dehkordi, and B. Davvaz, $\Gamma$*-Semihypergroups and their properties*, UPB Scientific Bulletin, Series A: Applied Mathematics and Physics **72** (2010), no. 1, 195–208.

[13] K. Hila, B. Davvaz, and J. Dine, *Study on the structure of* $\Gamma$*-semihypergroups*, Communications in Algebra **40** (2012), no. 8, 2932–2948.

[14] K. Hila, and B. Davvaz and K. Naka, *On Quasi-hyperideals in Semihypergroups*, Communications in Algebra **39** (2011), 4183–4194.

[15] V. Leoreanu, *About the simplifiable cyclic semihypergroups*, Ital. J. Pure Appl. Math. **7** (2000), 69–76.

[16] F. Marty, *Sur une generalization de la notion de groupe*, $8^{iem}$ congres Math. Scandinaves, Stockholm (1934), 45–49.

[17] M. De Salvo, D. Freni, and G. Lo Faro, *Fully simple semihypergroups*, J. Algebra **399** (2014), 358–377.

[18] N. Yaqoob, M. Aslam, B. Davvaz, and A. Borumand Saeid, *On rough* $(m, n)$ *bi-*$\Gamma$*-hyperideals in* $\Gamma$*-semihypergroups*, UPB Scientific Bulletin, Series A **75** (2013), no. 1, 119–128.