Topological Global and Partial Actions: Some Properties and Examples

Acciones Globales y Parciales Topológicas: Algunas Propiedades y Ejemplos

Jesús Ávila[1,a], **Fabián Molina**[2,b]

Abstract. In this work we study the global and partial group actions of topological groups on topological spaces. We present the basic concepts and their properties, together with enough examples to understand the theory. We introduce the concept of globalization of a topological partial action and we show that any topological partial action arises from the restriction of a minimal globalization, which is called an enveloping action. Finally, we explicitly show several examples of topological partial actions and we construct their enveloping actions in full detail.

Keywords: Global action, partial action, topological group.

Resumen. En este trabajo estudiamos las acciones globales y parciales de grupos topológicos sobre espacios topológicos. Presentamos los conceptos básicos y sus propiedades, junto con suficientes ejemplos para entender la teoría. Introducimos el concepto de globalización de una acción parcial topológica y mostramos que cualquier acción parcial topológica proviene de la restricción de una globalización minimal, la cual es llamada acción envolvente. Finalmente, mostramos explícitamente varios ejemplos de acciones parciales topológicas y construimos detalladamente sus acciones envolventes.

Palabras claves: Acción global, acción parcial, grupo topológico.

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1. Introduction

Historically, the works of Lagrange and Galois on the solvability of an algebraic equation by radicals originated the concept of group action on a set. Group actions have because been used successfully throughout the twentieth century in almost all areas of mathematics (see [9, 11]).

[1]Departamento de Matemáticas y Estadística, Universidad del Tolima, Ibagué, Colombia
[2]Estudiante Doctorado en Matemáticas y Estadística, Universidad de Oviedo, Oviedo, España
[a]javila@ut.edu.co
[b]molinafabian@uniovi.es

The theory of partial actions was developed in the last eighteen years and it has been shown to be an excellent generalisation of global actions. The theory of partial actions was defined and studied by Exel in [6], and later in [1] and [10], where the study of the globalizations of a partial action was initiated. These actions have been fundamental to obtain new results in rings (see [5, 7]), in metric and topological spaces (see [2, 3]), in homotopy theory ([13]), in operator theory and in dynamical systems, among many others (see [10, 4]).

Kellendonk and Lawson in [10] introduced the notion of globalization of a partial action and proved that any partial group action on a set possesses a unique minimal globalization, which is called an enveloping action. This concept became relevant for any study on partial actions (see [4]).

The purpose of this paper is to study topological partial actions. Although this concept was introduced by Abadie in [2] with some properties and general examples, we believe that it is necessary to explain certain important details to facilitate the understanding of this topic. In particular, we will explicitly develop several concrete examples and we will also make several proofs, which were not included in [2]. In addition, we introduce the concept of globalization of a topological partial action, which was not considered in [2]. This concept of globalization naturally leads to the enveloping of a topological partial action, which was considered in [2]. This paper is organised in the following manner. In Section 2, we present the topological global actions, some of their properties and we also give several interesting examples. In Section 3, we study the topological partial actions and show in detail several results of [2]. Moreover, we include several examples of topological partial actions, which are fully developed. Finally, in Section 4 we introduce the concept of globalization of a topological partial action and we prove that any topological partial action arises from a minimal globalization, which is called an enveloping action. In addition, we explicitly construct the enveloping actions of the examples given in the previous section.

2. Topological Global Actions

In this section we present the topological global actions, including a description of some of their properties and several examples.

Definition 2.1 ([8])**.** Let G be a topological group with identity element e and X a topological space. A topological global action of G on X is a function $\varphi : G \times X \to X$ which satisfies:

1. $\varphi(e, x) = x$ for each $x \in X$.
2. $\varphi(gh, x) = \varphi(g, \varphi(h, x))$, for every pair $g, h \in G$ and each $x \in X$.
3. φ is continuous with the product topology on $G \times X$.

In this case, we say that G acts globally and topologically on X. Also, if $x \in X$ and $A \subseteq X$, the $G-$orbit (or simply, orbit) of x with respect to φ is the set $G(x) = \{\varphi(g, x) : g \in G\}$ and the $A-$ orbit with respect to φ is the set $G(A) = \bigcup_{x \in A} G(x)$.

Example 2.2.

1. Given a topological group G and a topological space X, the projection $\pi_X : G \times X \to X$ is a global topological action of G on X.

2. The topological group $(\mathbb{Z}, +)$ with the discrete topology, acts globally and topologically on S^1 with the function $\beta : \mathbb{Z} \times S^1 \to S^1$ given by $\beta(k, z) = e^{2\pi i k\theta} z$, for some fixed angle θ, each pair $k \in \mathbb{Z}$, and $z \in S^1$. In fact, 1. and 2. are easy to check. For 3., given $(k, z) \in \mathbb{Z} \times S^1$ and a basic open set $U \subseteq S^1$ such that $\beta(k, z) \in U$, there exists $r > 0$ such that $U = B(\beta(k, z); r) \cap S^1$ where $B(\beta(k, z); r) \subseteq \mathbb{C}$ is the ball with center in $\beta(k, z)$ and radius r. Thus, we define $V = B(z; r) \cap S^1$ and so $z \in V$. Now, if $w \in \beta(\{k\} \times V)$, then $w = \beta(k, t)$ for some $t \in V$. That is, $\|t\| = 1$ and $t \in B(z; r)$. Therefore, $\|t - z\| < r$ and thus

$$\|w - \beta(k, z)\| = \|e^{2\pi i k\theta} t - e^{2\pi i k\theta} z\| = \|e^{2\pi i k\theta}\|\|t - z\| < r.$$

Moreover, $\|w\| = \|e^{2\pi i k\theta}\|\|t\| = 1$ and thus $w \in B(\beta(k, z); r) \cap S^1 = U$. That is, $\beta(\{k\} \times V) \subseteq U$ and consequently $\{k\} \times V$ is an open set of $\mathbb{Z} \times S^1$ such that $(k, z) \in \{k\} \times V$ and $\beta(\{k\} \times V) \subseteq U$. Hence, β is continuous.

3. The group $GL_n(\mathbb{R})$ with the topology induced by

$$\|A\| = \sup\left\{\frac{\|Ax\|}{\|x\|} : x \in \mathbb{R}^n, \|x\| \neq 0\right\},$$

acts globally and topologically on $\mathbb{R}^n$ with the function $\psi(A, x) = Ax$, for each pair $A \in GL_n(\mathbb{R})$ and $x \in \mathbb{R}^n$. In fact, 1. and 2. are verified immediately. For 3., given $\epsilon > 0$, $B \in GL_n(\mathbb{R})$ and $y \in \mathbb{R}^n$, let us define $\delta = \min\{1, \epsilon/(1 + \|y\| + \|B\|)\}$. Then for each pair $A \in GL_n(\mathbb{R})$ and $x \in \mathbb{R}^n$ such that $\|(A, x) - (B, y)\| < \delta$ we have

$$\begin{aligned}
\|\psi(A, x) - \psi(B, y)\| &= \|Ax - By\| \\
&= \|(A - B)(x - y) + (A - B)y + B(x - y)\| \\
&\leq \|(A - B)(x - y)\| + \|(A - B)y\| + \|B(x - y)\| \\
&\leq \|A - B\|\|x - y\| + \|A - B\|\|y\| + \|B\|\|x - y\| \\
&< \delta^2 + \delta\|y\| + \delta\|B\| \\
&\leq \delta + \delta\|y\| + \delta\|B\| \\
&= \delta(1 + \|y\| + \|B\|) \leq \epsilon.
\end{aligned}$$

Thus $\psi(A, x) = Ax$ is continuous and consequently it is a topological global action of $GL_n(\mathbb{R})$ on $\mathbb{R}^n$.

Note that if φ is a topological global action of G on X, then for each $g \in G$, the function $\varphi_g : X \to X$ defined by $\varphi_g(x) = \varphi(g, x)$ for each $x \in X$, is a homeomorphism. In fact, φ_g is bijective for every $g \in G$. Now, let $x \in X$, $g \in G$ and $U \subseteq X$ an open neighborhood of $\varphi_g(x)$. We then have that $\varphi(g, x) \in U$. Since φ is continuous, there are open sets $H \subseteq G$ and $V \subseteq X$ such that $(g, x) \in H \times V$ and $\varphi(H \times V) \subseteq U$. Thus, V is an open neighborhood of x such that $\varphi_g(V) = \varphi(\{g\} \times V) \subseteq \varphi(H \times V) \subseteq U$. Therefore, φ_g is continuous. Meanwhile, $\varphi_g^{-1} = \varphi_{g^{-1}}$ is a continuous function.

Furthermore, the following proposition shows that every topological global action of G on X determines a representation of G as homeomorphisms of X.

Proposition 2.3. *If φ is a topological global action of G on X and H_X is the set of all homeomorphisms of X, then the function $\Theta : G \to H_X$ given by $\Theta(g) = \varphi_g$ for each $g \in G$, is a continuous homomorphism of topological groups where H_X has the topology whose subbase is given by the sets*

$$S(x, U) = \{f \in H_X : f(x) \in U\},$$

where $x \in X$ and U is an open set of X.

Proof. It is clear that H_X is a topological group. Note that

$$\begin{aligned}\Theta(gh)(x) = \varphi_{gh}(x) = \varphi(gh, x) &= \varphi(g, \varphi(h, x)) \\ &= \varphi_g(\varphi_h(x)) = (\varphi_g \circ \varphi_h)(x) = (\Theta(g) \circ \Theta(h))(x).\end{aligned}$$

So $\Theta(gh) = \Theta(g) \circ \Theta(h)$. In addition, if $S(x, U)$ is an open neighborhood of $\Theta(g)$, then $\varphi_g \in S(x, U)$ and hence $\varphi_g(x) \in U$. Since φ is continuous, there are open neighborhoods $W \subseteq G$ and $V \subseteq X$ of g and x, respectively, such that $\varphi(W \times V) \subseteq U$. Note that if $\lambda \in \Theta(W)$, then $\lambda = \varphi_h$ for some $h \in W$ where $\lambda(x) \in \varphi_h(V)$. Furthermore, $\varphi_h(V) \subseteq \bigcup_{t \in W} \varphi_t(V) = \varphi(W \times V) \subseteq U$, then $\lambda(x) \in U$ and thus $\lambda \in S(x, U)$. Then, $g \in W$ and $\Theta(W) \subseteq S(x, U)$; that is, Θ is continuous. $\square$

Reciprocally, each representation of a topological group G by homeomorphisms of a topological space X determines a topological global action of G on X.

Proposition 2.4. *Let G be a topological group and X a topological space. If $\Omega : G \to H_X$ is a continuous homomorphism, then Ω induces a topological global action of G on X.*

Proof. Since $\Omega(g) : X \to X$ is a homeomorphism, then the function $\varphi : G \times X \to X$ given by $\varphi(g, x) = \Omega(g)(x)$, for every pair $g \in G$ and $x \in X$, is a topological global action. In fact, 1. $\varphi(e, x) = \Omega(e)(x) = i_X(x) = x$ for each $x \in X$. 2. $\varphi(gh, x) = \Omega(gh)(x) = (\Omega(g) \circ \Omega(h))(x) = \Omega(g)(\Omega(h)(x)) = \varphi(g, \varphi(h, x))$ for every pair $g, h \in G$ and for each $x \in X$. 3. Suppose that $U \subseteq X$ is an open neighborhood of $\varphi(g, x)$, then $\Omega(g)(x) \in U$. Thus, $\Omega(g) \in S(x, U)$

and since Ω is continuous, then there exists an open neighborhood $W \subseteq G$ of g such that $\Omega(W) \subseteq S(x, U)$. On the other hand, if $k \in W$ then $\Omega(k) \in S(x, U)$ and therefore $\Omega(k)(x) \in U$. Thus, there exists an open neighborhood V of x such that $\Omega(k)(V) \subseteq U$. Now, if $t \in \varphi(W \times V)$, then $t = \varphi(h, y)$ for some pair $h \in W$ and $y \in V$. Hence $t = \Omega(h)(y) \in \Omega(h)(V) \subseteq U$ and consequently $(g, x) \in W \times V$ and $\varphi(W \times V) \subseteq U$. That is, φ is continuous. □

3. Topological Partial Actions

The main aim of this section is to introduce the concept of topological partial action, show some examples and properties, and compare it with the concept of topological global action.

Definition 3.1 ([2])**.** Let G be a topological group and X a topological space. We say that α is a topological partial action of G on X, if there exists a class of pairs $\{(X_g, \alpha_g)\}_{g \in G}$ where X_g is an open set of X and $\alpha_g : X_{g^{-1}} \to X_g$ is a homeomorphism such that:

1. $X_e = X$ and $\alpha_e = id_X$.
2. $\alpha_h^{-1}(X_h \cap X_{g^{-1}}) \subseteq X_{(gh)^{-1}}$, for every pair $g, h \in G$.
3. $(\alpha_g \circ \alpha_h)(x) = \alpha_{gh}(x)$, for each $x \in \alpha_h^{-1}(X_h \cap X_{g^{-1}})$.
4. The set $\Gamma_\alpha = \{(g, x) \in G \times X : x \in X_{g^{-1}}\}$ is open in $G \times X$.
5. The function $\alpha : \Gamma_\alpha \to X$ given by $\alpha(g, x) = \alpha_g(x)$, is continuous.

In this case, we also say that G acts partially and topologically on X.

From the previous definition, we claim that all topological global action is a topological partial action. In fact, if φ is a topological global action of G on X, then $\{(X, \varphi_g)\}_{g \in G}$ is a collection of pairs that satisfies the conditions 1-5 of the Definition 3.1.

If α is a topological partial action of G on X, then there exists a class of pairs $\{(X_g, \alpha_g)\}_{g \in G}$ such that:

1. For each $x \in X$ we have that $(e, x) \in \Gamma_\alpha$ because $x \in X = X_e$. Hence $\alpha(e, x) = \alpha_e(x) = x$ for each $x \in X$.
2. If $x \in \alpha_h^{-1}(X_h \cap X_{g^{-1}})$, then $\alpha_h(x) \in X_h \cap X_{g^{-1}}$. Thus:
   (a) $x \in X_{h^{-1}}$ and so $(h, x) \in \Gamma_\alpha$.
   (b) $\alpha_h(x) \in X_{g^{-1}}$ and so $(g, \alpha_h(x)) \in \Gamma_\alpha$.

   Moreover, $x \in X_{(gh)^{-1}}$ where $(gh, x) \in \Gamma_\alpha$ and therefore

$$\alpha(gh, x) = \alpha_{gh}(x) = \alpha_g(\alpha_h(x)) = \alpha(g, \alpha(h, x)).$$

3. The function $\alpha : \Gamma_\alpha \to X$ given by $\alpha(g, x) = \alpha_g(x)$, is continuous.

Thus, α is a topological global action of G on X, if and only if, $\Gamma_\alpha = G \times X$.

The following example, created by the authors, shows a simple topological partial action, which is related to the reflection of the plane $\mathbb{R}^2$ over the $x-$axis.

Example 3.2. Let $G = \{1, -1\}$ (with the discrete topology) and $X = \{(x, y) \in \mathbb{R}^2 : y > 0\}$ (with the usual topology). If $\{(X_t, \alpha_t)\}_{t\in G}$ is the collection given by $X_1 = X$, $X_{-1} = \emptyset$, $\alpha_1 = i_X$ and $\alpha_{-1} = \emptyset$, then $\Gamma_\alpha = \{1\} \times X$ and the function $\alpha : \Gamma_\alpha \to X$ given by $\alpha(1, (x, y)) = (x, y)$ is a topological partial action of G on X.

In the next section it will be seen that the following example is closely related to the concept of suspension in dynamical systems [14].

Example 3.3 ([2])**.** Let X be a topological space and let $h : X \to X$ be a homeomorphism. If $\{(X_t, \alpha_t)\}_{t\in\mathbb{R}}$ is given by $X_t = X$ and $\alpha_t = h^t$ when $t \in \mathbb{Z}$, and $X_t = \emptyset$ and $\alpha_t = \emptyset$ when $t \notin \mathbb{Z}$. Then $\Gamma_\alpha = \mathbb{Z} \times X$ and the function $\alpha : \Gamma_\alpha \to X$ given by $\alpha(t, x) = h^t(x)$, is a topological partial action of $\mathbb{R}_d$ (with the discrete topology) on X.

The following example shows that topological partial actions appear naturally in differential geometry. Although this example is taken from [2], we develop all the details to prove that there is a topological partial action. In addition, we include a particular case to facilitate the understanding of these concepts.

Example 3.4 ([2])**.** The flow of a differentiable vector field is a topological partial action. More precisely, let X be a differentiable manifold of class C^2 and let $v : X \to TX$ be a vector field of class C^1 of X in the tangent bundle of X. For each $x \in X$, γ_x denotes the integral curve such that $\gamma_x(0) = x$ and $\gamma'_x(t) = v(\gamma_x(t))$ whose domain is a maximum open interval $I_x \subseteq \mathbb{R}$ containing 0. For each $t \in \mathbb{R}$, let us define $X_{-t} = \{x \in X : t \in I_x\}$ and $\alpha_t : X_{-t} \to X_t$ the function given by $\alpha_t(x) = \gamma_x(t)$. Then, $\Gamma_\alpha = \{(t, x) \in \mathbb{R} \times X : -t \in I_x\}$ and therefore the function $\alpha : \Gamma_\alpha \to X$ given by $\alpha(t, x) = \alpha_t(x)$ is a topological partial action of $\mathbb{R}$ on X. In fact,

1. For each $t \in \mathbb{R}$, X_t is an open set and α_t is a diffeomorphism. Then, α_t and α_t^{-1} are differentiable and thus they are continuous. So, α_t is a homeomorphism for each $t \in \mathbb{R}$.

2. Note that $X_0 = \{x \in X : 0 \in I_x\} = X$ and $\alpha_0(x) = \gamma_x(0) = x = id_X(x)$, for each $x \in X$.

3. If $s, t \in \mathbb{R}$, then $\alpha_t^{-1}(X_t \cap X_{-s}) \subseteq \alpha_t^{-1}(X_{-s}) = X_{-t} \cap X_{-(s+t)} \subseteq X_{-(s+t)}$.

4. If $x \in \alpha_t^{-1}(X_t \cap X_{-s})$, then $x \in X_{-(s+t)}$ and thus,

$$(\alpha_s \circ \alpha_t)(x) = \alpha_s(\alpha_t(x)) = \alpha_s(\gamma_x(t)) = \gamma_{\gamma_x(t)}(s) = \gamma_x(s + t) = \alpha_{s+t}(x).$$

5. If $(t,x) \in \Gamma_\alpha$, then $x \in X_{-t}$ and thus $t \in I_x$. So, $I_x \times X_{-t}$ is an open neighborhood of (t,x). Furthermore, $I_x \times X_{-t} \subseteq \Gamma_\alpha$. In fact, if $(s,y) \in I_x \times X_{-t}$, then $s \in I_x$, $y \in X_{-t}$ and thus,
$$y \in X_{-t} = \alpha_{-s}(X_s \cap X_{s-t}) \subseteq \alpha_{-s}(X_s) = \alpha_s^{-1}(X_s) = X_{-s}.$$
Then, $(s,y) \in \Gamma_\alpha$ and hence Γ_α is an open set.

6. The funtion $\alpha : \Gamma_\alpha \to X$ given by $\alpha(t,x) = \alpha_t(x)$ is continuous. Suppose that $(t,x) \in \Gamma_\alpha$ and let $U \subseteq X$ an open neighborhood of $\alpha(t,x)$. Since α_t is continuous, there is an open set $V_t \subseteq X_{t^{-1}}$ such that $x \in V_t$ and $\alpha_t(V_t) \subseteq U$. Hence, $t \in I_x$, $(t,x) \in I_x \times V_t \subseteq \Gamma_\alpha$, and $\alpha(I_x \times V_t) = \bigcup_{s \in I_x} \alpha_s(V_s) \subseteq U$.

In particular, let $X = \mathbb{R}$ and let $v : X \to TX$ be the vector field given by $v(w) = e^{-w}\partial/\partial w$. If α is the topological partial action described in the previous example, then $X_t = \{x \in \mathbb{R} : t < e^x\}$ and $\alpha_t(x) = \ln(t + e^x)$ (Figure 1). Moreover, note that $\Gamma_\alpha = \{(t,x) \in \mathbb{R}^2 : -t < e^x\} \subsetneq G \times X$ (Figure 2).

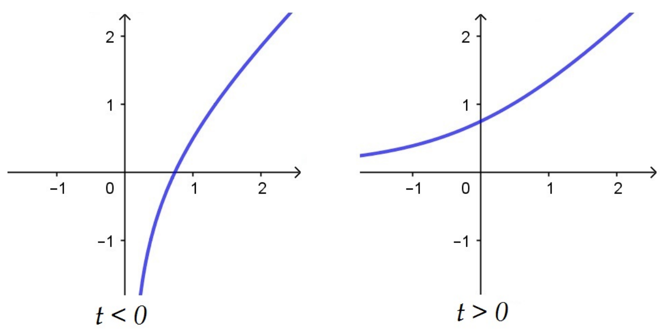


Figure 1. Graph of $\alpha_t(x)$. Particular Case of Example 3.4.

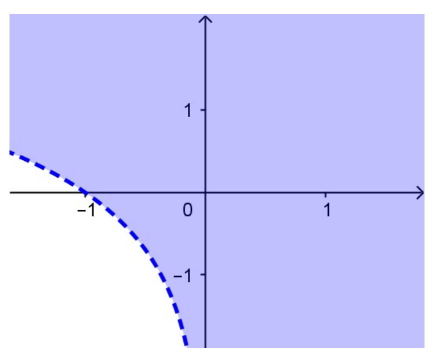


Figure 2. Graph of Γ_α. Particular Case of Example 3.4.

Note that all of the topological partial actions of the previous examples are not topological global actions. Abadie in [2] determines the sufficient conditions on the topological group and the topological spaces under which any topological partial action necessarily ends up being a topological global action. We present this result later on and we will include all of the details of the proof by showing clearly where the correspondence assumptions are used.

Theorem 3.5 ([2]). *If α is a topological partial action of G on a compact topological space X, then there exists an open subgroup $H \subseteq G$ such that $H \times X \subseteq \Gamma_\alpha$ and the restriction of α to $H \times X$ is a topological global action of H on X. Moreover, if G is connected, then α is a topological global action of G on X.*

Proof. Let $\{(X_g, \alpha_g)\}$ be the collection given by α. For each $y \in X$, let us define $A_y = \{g \in G : y \in X_{g^{-1}}\}$ and let $A = \bigcap\{A_y : y \in X\}$. Then:

1. If $x \in X$, then $x \in X_e$. Therefore, $e \in A_x$ and so $e \in A$.
2. If $x \in X$ and $g, h \in A$, then $g, h \in A_y$ for each $y \in X$. In particular, $g \in A_x$ and $h \in A_{\alpha_g(x)}$, thus $x \in X_{g^{-1}}$ and $\alpha_g(x) \in X_{h^{-1}}$. So $\alpha_g(x) \in X_g$ and consequently $\alpha_g(x) \in X_g \cap X_{h^{-1}}$. Hence, $x \in \alpha_g^{-1}(X_g \cap X_{h^{-1}}) \subseteq X_{(gh)^{-1}}$ and so $gh \in A_x$. Therefore, $gh \in A$.

Thus, A is a submonoid of G. In addition, for each $x \in X$ we have $(e, x) \in \Gamma_\alpha$ and so there are open neighborhoods $U_x \subseteq X$ of x and $W_x \subseteq G$ of e such that $W_x \times U_x \subseteq \Gamma_\alpha$. Consequently, there exists a symmetric open neighborhood V_x of e such that $V_x \subseteq W_x$.

Given that $\{U_x : x \in X\}$ is a covering of X, there are $x_1, ..., x_n \in X$ such that $X = \bigcup_{i=1}^n U_{x_i}$. If $V = \bigcap_{i=1}^n V_{x_i}$, then V is a symmetric open neighborhood of e. Furthermore, if $t \in V$ and $x \in X$, then $t \in V_{x_i}$ for every $i = 1, ..., n$ and $x \in U_{x_{i_0}}$ for some $1 \leq i_0 \leq n$. Therefore, $(t, x) \in V_{x_{i_0}} \times U_{x_{i_0}} \subseteq \Gamma_\alpha$ and so $x \in X_{t^{-1}}$. Hence $t \in A_x$ and consequently $V \subseteq A$.

Note that V generates an open subgroup contained in A. In fact, since A is a monoid then $V^m \subseteq A$ and it is an open set for every $m \in \mathbb{N}$. Thus, $H = \bigcup_{m=1}^\infty V^m$ is an open subgroup generated by V which is contained in A.

Now, if $(h, x) \in H \times X$, then $h \in A$ and therefore $h \in A_x$. Thus $x \in X_{h^{-1}}$, that is, $(h, x) \in \Gamma_\alpha$. Therefore, $H \times X \subseteq \Gamma_\alpha$. In addition, given that H is a group, then the restriction φ of α to $H \times X$ is a topological partial action of H on X. So the function $\varphi : H \times X \to X$ given by $\varphi(h, x) = \alpha_h(x)$ is well defined and it is a topological global action of H on X.

Finally, since H is an open subgroup, then for every $g \in G$ is fulfilled that gH is open (see [12] for details). From group theory, G is the disjoint union of H and the union of the left cosets gH where $g \notin H$. This implies that $H^c = \bigcup\{gH : g \notin H\}$ is open and therefore H is closed and not empty. Hence, if G is connected, then $H = G$ and so $\Gamma_\alpha = G \times X$. Thus, α is a topological global action of G on X. □

Corollary 3.6. *The flow of a differentiable vector field on a differentiable compact manifold is a topological global action.*

Example 3.7. Let $X = S^1$ and let $v : X \to TX$ be the vector field given by $v(z) = iz\partial/\partial z$. If α is the topological partial action described in Example 3.4, then $\alpha_t(x) = xe^{it}$ and $X_t = S^1$. Thus, $\Gamma_\alpha = \mathbb{R} \times S^1$ and the function $\alpha : \Gamma_\alpha \to S^1$ given by $\alpha(t, x) = xe^{it}$, is a topological global action of $\mathbb{R}$ on S^1.

The following proposition states that the restriction of a topological global action to an open set of a topological space is a topological partial action. This result is very important because from it arises the notion of globalization of a topological partial action, which will be studied in the next section. This result is in [2] without proof, but we include it here for the sake of completeness of this work.

Proposition 3.8 ([2])**.** *Let φ be a topological global action of G on X, S an open subset of X, and the collection given by $\{(S_g, \alpha_g)\}_{g\in G}$ with $S_g = S \cap \varphi_g(S)$ and $\alpha_g : S_{g^{-1}} \to S_g$ defined as $\alpha_g(x) = \varphi_g(x)$, for each $x \in S_{g^{-1}}$. If $\Gamma_\alpha = \{(g, x) \in G \times S : x \in S_{g^{-1}}\}$, then the function $\alpha : \Gamma_\alpha \to S$ given by $\alpha(g, x) = \alpha_g(x)$ is a topological partial action of G on S. This restriction of φ is called the induced topological partial action of G on S.*

Proof.

1. Since S is an open set, then $\varphi_g(S)$ is also an open set. Thus, $S_g = S\cap\varphi_g(S)$ is an open set of S, for each $g \in G$. Moreover, $\varphi_g : X \to X$ is a homeomorphism for each $g \in G$ and consequently $\alpha_g : S_{g^{-1}} \to S_g$ is also a homeomorphism.

2. Note that $S_e = S \cap \alpha_e(S) = S \cap S = S$. Furthermore, $\alpha_e(x) = \varphi_e(x) = x$, for each $x \in S$ and hence $\alpha_e = id_S$.

3. If $x \in S$ and $g, h \in G$ are such that $x \in \alpha_h^{-1}(S_h \cap S_{g^{-1}})$, then $x = \alpha_h^{-1}(y)$ for some $y \in S_h \cap S_{g^{-1}}$. Thus, $y \in S_{g^{-1}}$ and so $y \in \varphi_{g^{-1}}(S)$. Therefore, $y = \varphi_{g^{-1}}(z)$ for some $z \in S$. Consequetly,
$$x = \alpha_{h^{-1}}(y) = \varphi_{h^{-1}}(y) = \varphi_{h^{-1}}(\varphi_{g^{-1}}(z)) = \varphi_{h^{-1}g^{-1}}(z) = \varphi_{(gh)^{-1}}(z).$$
Hence $x \in \varphi_{(gh)^{-1}}(S)$ and so $x \in S \cap \varphi_{(gh)^{-1}}(S) = S_{(gh)^{-1}}$.

4. If $g, h \in G$ and $x \in \alpha_h^{-1}(S_h \cap S_{g^{-1}})$, then $x \in S_{(gh)^{-1}}$ and therefore
$$(\alpha_g \circ \alpha_h)(x) = (\varphi_g \circ \varphi_h)(x) = \varphi_{gh}(x) = \alpha_{gh}(x).$$

5. If $(g, x) \in \Gamma_\alpha$, then $x \in S_{g^{-1}}$ and thus $\varphi(g, x) = \alpha_g(x) \in S_g \subseteq S$. Since φ is continuous in $G \times X$, then it is also continuous in $G \times S$. Now, since S is an open neighborhood of $\varphi(g, x)$, then there are open sets $U \subseteq G$ and $V \subseteq S$ such that $(g, x) \in U \times V$ and $\varphi(U \times V) \subseteq S$, which implies that $U \times V \subseteq \Gamma_\alpha$. In fact, for each $t \in U$ we have that $\varphi_t(V) = \varphi(\{t\} \times V) \subseteq \varphi(U \times V) \subseteq S$ and thus $V \subseteq \varphi_t^{-1}(S) = \varphi_{t^{-1}}(S)$. Hence, if $(h, y) \in U \times V$, then $y \in V \subseteq \varphi_{h^{-1}}(S)$; that is, $y \in S_{h^{-1}}$ and therefore $(h, y) \in \Gamma_\alpha$. Consequently, Γ_α is open.

6. The function $\alpha : \Gamma_\alpha \to S$, given by $\alpha(g,x) = \alpha_g(x)$ is continuous. In fact, if $(g,x) \in \Gamma_\alpha$ and $W \subseteq S$ is an open neighborhood of $\alpha(g,x)$, then $x \in S_{g^{-1}}$ and $\alpha(g,x) = \varphi(g,x)$. Hence, W is an open neighborhood of $\varphi(g,x)$ and thus there are open sets $U \subseteq G$ and $V \subseteq S$ such that $(g,x) \in U \times V$ and $\varphi(U \times V) \subseteq W$. Finally, by the previous paragraph, $U \times V \subseteq \Gamma_\alpha$ and therefore $\alpha(U \times V) = \varphi(U \times V) \subseteq W$.

$\square$

4. Enveloping Action of a Topological Partial Action

In this section, we introduce the concept of globalization of a topological partial action and prove that any topological partial action arises from a minimal globalization, which is called an enveloping action. Although this last concept was introduced in [2], we give several definitions and prove some results to clarify the difference between globalization and an enveloping action. Finally, we construct the enveloping actions of the examples given in Section 3.

Definition 4.1 ([2])**.** Let α and β be topological partial actions of G on X and Y, respectively. We say that α and β are equivalent, if there exists a homeomorphism $f : X \to Y$ such that $f(X_g) \subseteq Y_g$ and $\beta_g(f(x)) = f(\alpha_g(x))$, for every pair $g \in G$ and $x \in X_{g^{-1}}$.

Definition 4.2. Given a topological partial action α of G on X, we say that (Y, ψ, j) is a globalization of α, if ψ is a topological global action of G on Y and $j : X \to Y$ is an injective continuous function, such that, $j(X)$ is an open subset of Y and the induced topological partial action of G on $j(X)$ and α are equivalent.

Proposition 4.3 ([2])**.** *If α is a topological partial action of G on X, then the relation $\sim$ on $G \times X$ given by $(g,x) \sim (h,y)$ if, and only if, $x \in X_{g^{-1}h}$ and $\alpha_{h^{-1}g}(x) = y$, is an equivalence relation.*

Theorem 4.4 ([2])**.** *Let $\sim$ be the equivalence relation of the previous proposition. If $X^{\mathbf{e}} = (G \times X)/\sim$ and $q : G \times X \to X^{\mathbf{e}}$ is the quotient function, then:*

1. *The function $\iota : X \to X^{\mathbf{e}}$ given by $\iota(x) = q(e,x)$ is injective and open. Moreover, the set $\iota(X)$ is open in $X^{\mathbf{e}}$.*
2. *The function $\alpha^{\mathbf{e}} : G \times X^{\mathbf{e}} \to X^{\mathbf{e}}$ given by $\alpha^{\mathbf{e}}(g, q(h,x)) = q(gh, x)$, is a topological global action of G on $X^{\mathbf{e}}$.*
3. *$(X^{\mathbf{e}}, \alpha^{\mathbf{e}}, \iota)$ is a globalization of α.*

Proof. 1. If $x, y \in X$ and $\iota(x) = \iota(y)$, then $q(e,x) = q(e,y)$; that is, $(e,x) \sim (e,y)$. Thus, $x = \alpha_e(x) = y$. Now, suppose that $U \subseteq X$ is an open set. Since $\alpha : \Gamma_\alpha \to X$ is a topological partial action of G on X, it is continuous. Thus,

$$\begin{aligned} q^{-1}(\iota(U)) &= \{(t,x) : q(t,x) \in \iota(U)\} \\ &= \{(t,x) : q(t,x) = q(e,y) \text{ for some } y \in U\} \\ &= \{(t,x) : x \in X_{t^{-1}}, \alpha_t(x) \in U\} \\ &= \alpha^{-1}(U), \end{aligned}$$

is an open subset of Γ_α. Since Γ_α is open in $G \times X$, then so is $q^{-1}(\iota(U))$. Finally, for the quotient topology we have that $\iota(U)$ is open in $X^{\mathbf{e}}$. In particular, $\iota(X)$ is open in $X^{\mathbf{e}}$.

2. It is easy to verify that $\alpha^{\mathbf{e}}$ is well defined. Moreover,

   (a) $\alpha^{\mathbf{e}}(e, q(h,x)) = q(eh,x) = q(h,x)$, for every pair $h \in G$ and $x \in X$.

   (b) If $s, t \in G$, then for every pair $h \in G$ and $x \in X$ we have that

   $$\begin{aligned} \alpha^{\mathbf{e}}(st, q(h,x)) &= q((st)h,x) = q(s(th),x) \\ &= \alpha^{\mathbf{e}}(s, q(th,x)) = \alpha^{\mathbf{e}}(s, \alpha^{\mathbf{e}}(t, q(h,x))). \end{aligned}$$

   (c) Before to prove that $\alpha^{\mathbf{e}}$ is continuous, let us $\gamma : G \times (G \times X) \to G \times X$ given by $\gamma(g,(h,x)) = (gh,x)$ for every pair $g,h \in G$ and for each $x \in X$. Then γ is a topological global action of G on $G \times X$. In fact,

      i. $\gamma(e,(h,x)) = (eh,x) = (h,x)$, for every pair $h \in G$ and $x \in X$.

      ii. If $s,t \in G$, then for every pair $h \in G$ and $x \in X$ is fulfilled that

      $$\begin{aligned} \gamma(s, \gamma(t,(h,x))) &= \gamma(s,(th,x)) = (s(th),x) \\ &= ((st)h,x) = \gamma(st,(h,x)) \end{aligned}$$

      iii. Now, if $g \in G$, $(h,x) \in G \times X$ and $P \subseteq G \times X$ is an open neighborhood of $\gamma(g,(h,x))$. Then, $(gh,x) \in P$ and there are open neighborhoods $Q \subseteq G$ and $R \subseteq X$ of gh and x, respectively such that $Q \times R \subseteq P$. Furthermore, there are open neighborhoods Q_1 and Q_2 of g and h, respectively, such that $Q_1 Q_2 \subseteq Q$. Thus $Q_2 \times R$ is an open neighborhood of (h,x) and therefore, $Q_1 \times (Q_2 \times R)$ is an open neighborhood of $(g,(h,x))$ such that

      $$\gamma(Q_1 \times (Q_2 \times R)) = (Q_1 Q_2) \times R \subseteq Q \times R \subseteq P.$$

      Thus, γ is continuous.

   From the previous observation of Proposition 2.3, for each $g \in G$, the function $\gamma_g : (G \times X) \to (G \times X)$, given by $\gamma_g(h,x) = \gamma(g,(h,x)) = (gh,x)$ for every $(h,x) \in G \times X$, is a homeomorphism of $G \times X$. With this, we claim that the quotient function q is open in $G \times X$.

If $M \times N$ is a basic open subset of $G \times X$, we shall prove that $q^{-1}(q(M \times N))$ is open. In fact, if $(g, x) \in q^{-1}(q(M \times N))$, then $q(g, x) \in q(M \times N)$ and there exists $(h, y) \in M \times N$ such that $q(g, x) = q(h, y)$. So $x \in X_{g^{-1}h} = X_{(h^{-1}g)^{-1}e}$ and $y = \alpha_{h^{-1}g}(x) = \alpha_{e^{-1}(h^{-1}g)}(x) \in X_{h^{-1}g}$. This implies that $q(h^{-1}g, x) = q(e, y)$.
Since $y \in N$, then $q(h^{-1}g, x) = q(e, y) = \iota(y) \in \iota(N)$ and thus $(h^{-1}g, x) \in q^{-1}(\iota(N))$. Applying γ_h we obtain that

$$(g, x) \in \gamma_h(q^{-1}(\iota(N)))$$

where $\gamma_h(q^{-1}(\iota(N)))$ is open since ι is open, q is continuous and γ_h is a homeomorphism.
On the other hand, let us $(k, z) \in \gamma_h(q^{-1}(\iota(N)))$. Then

$$q(h^{-1}k, z) \in \iota(N)$$

and there exists $n \in N$ such that $q(h^{-1}k, z) = \iota(n) = q(e, n)$ and therefore $z \in X_{(h^{-1}k)^{-1}e} = X_{k^{-1}h}$ and $n = \alpha_{e^{-1}(h^{-1}k)}(z) = \alpha_{h^{-1}k}(z)$. Hence,

$$q(k, z) = q(h, n) \in q(M \times N)$$

and consequently, $(k, z) \in q^{-1}(q(M \times N))$. So, we have that

$$(g, x) \in \gamma_h(q^{-1}(\iota(N))) \subseteq q^{-1}(q(M \times N))$$

and thus $q^{-1}(q(M \times N))$ is open. By the quotient topology, $q(M \times N)$ is open in $X^{\mathbf{e}}$ and thus q is an open function.
Now, let us $g \in G$, $q(h, x) \in X^{\mathbf{e}}$ and $U \subseteq X^{\mathbf{e}}$ is an open neighborhood of $\alpha^{\mathbf{e}}(g, q(h, x))$. Then, $q(gh, x) \in U$ and since q is continuous, there are open neighborhoods $M \subseteq G$ and $N \subseteq X$ of gh and x, respectively such that $q(M \times N) \subseteq U$. In addition, there are open neighborhoods V_1 and V_2 of g and h, respectively, such that $V_1 V_2 \subseteq M$. Given that q is open, then $q(V_2 \times N)$ is an open neighborhood of $q(h, x)$. Therefore, $V_1 \times q(V_2 \times N)$ is an open neighborhood of $(g, q(h, x))$ such that

$$\alpha^{\mathbf{e}}(V_1 \times q(V_2 \times N)) = q(V_1 V_2 \times N) \subseteq q(M \times N) \subseteq U.$$

Thus, $\alpha^{\mathbf{e}}$ is continuous.

3. Given that $\alpha^{\mathbf{e}}$ is a topological global action of G on $X^{\mathbf{e}}$, the restriction of $\alpha^{\mathbf{e}}$ on $\iota(X)$ induces a topological partial action ϵ of G on $S = \iota(X)$ given by the collection $\{(S_g, \epsilon_g)\}_{g \in G}$ where $S_g = S \cap \alpha^{\mathbf{e}}_g(S)$ and $\epsilon_g : S_{g^{-1}} \to S_g$ is the homeomorphism defined as $\epsilon_g(\iota(x)) = \alpha^{\mathbf{e}}_g(\iota(x))$ for each $\iota(x) \in S_{g^{-1}}$.
Now, if $g \in G$ and $\iota(y) \in \iota(X_g)$, then $\iota(y) = \iota(\alpha_g(z))$ for some $z \in X_{g^{-1}} \subseteq X$ and so $\iota(y) = \alpha^{\mathbf{e}}_g(\iota(z)) \in \alpha^{\mathbf{e}}_g(S)$. Note that $X_g \subseteq X$, then $\iota(X_g) \subseteq S$ and so $\iota(y) \in S \cap \alpha^{\mathbf{e}}_g(S) = S_g$. Consequently, $\iota(X_g) \subseteq S_g$.

In addition, if $g \in G$ and $x \in X_{g^{-1}}$, then $\alpha_g(x) \in X_g$ and $\alpha_{g^{-1}e}(\alpha_g(x)) = x$. So $(e, \alpha_g(x)) \sim (g, x)$ and therefore $q(g, x) = q(e, \alpha_g(x)) = \iota(\alpha_g(x))$. Furthermore,

$$\epsilon_g(\iota(x)) = \alpha_g^{\mathbf{e}}(\iota(x)) = \alpha^{\mathbf{e}}(g, q(e, x)) = q(g, x) = \iota(\alpha_g(x)).$$

Finally, ι is the composition of the inclusion function $x \mapsto (e, x)$ and the quotient function q. Because these functions are continuous, then so is ι. Hence, $\iota : X \to \iota(X)$ is a homeomorphism such that $\iota(X_g) \subseteq S_g$ and $\iota(\alpha_g(x)) = \epsilon_g(\iota(x))$, for each $g \in G$ and $x \in X_{g^{-1}}$. Thus, α and ϵ are equivalent and therefore $(X^{\mathbf{e}}, \alpha^{\mathbf{e}}, \iota)$ is a globalization of α.

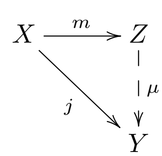
□

With the following definitions, we will prove very important properties of the globalization defined in the previous proposition.

Definition 4.5. Let φ and ψ be topological global actions of G on X and Y, respectively. We say that a continuous function $f : X \to Y$ is a $G-$morphism of X in Y (or simply $G-$morphism), if for every pair $x \in X$ and $g \in G$, it holds $f(\varphi(g, x)) = \psi(g, f(x))$.

Definition 4.6. If α is a topological partial action of G on X, we say that the globalization (Z, ϕ, m) of α is minimal, if for any globalization (Y, ψ, j) of α, there exists a unique $G-$morphism $\mu : Z \to Y$ such that $j = \mu \circ m$.

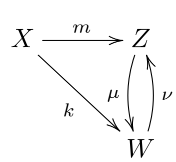
The existence of μ guarantees that the following diagram commutes:



Definition 4.7. Let φ and ψ be topological global actions of G on X and Y, respectively. We say that φ and ψ are equivalent, if there exists a homeomorphism $f : X \to Y$, which is a $G-$morphism.

Proposition 4.8. *If α is a topological partial action of G on X and (Z, ϕ, m) is a minimal globalization of α, then (Z, ϕ, m) is unique, except for equivalences.*

Proof. Let (W, ω, k) be a minimal globalization of α. Then, there are $G-$morphisms $\mu : Z \to W$ and $\nu : W \to Z$ such that $\mu \circ m = k$ and $\nu \circ k = m$:



Note that, $i_Z : Z \to Z$ is a $G-$morphism such that $i_Z \circ m = m$. Meanwhile, $\nu \circ \mu$ is a $G-$morphism of Z in Z such that $(\nu \circ \mu) \circ m = \nu \circ (\mu \circ m) = \nu \circ k = m$. Hence, $\nu \circ \mu = i_Z$. Analogously, $\mu \circ \nu = i_W$ and therefore $\nu = \mu^{-1}$. Now, since ν is continuous, then so is μ^{-1} and so $\mu : Z \to W$ is a homeomorphism. Finally, since $\mu(\phi(g, z)) = \omega(g, \mu(z))$, for every pair $g \in G$ and $z \in Z$, then ϕ and ω are equivalent and hence the proof is complete. □

Theorem 4.9. *If α is a topological partial action of G on X, then with the notations of the Theorem 4.4, $(X^{\mathbf{e}}, \alpha^{\mathbf{e}}, \iota)$ is a minimal globalization of α.*

Proof. Let (Y, ψ, j) be a globalization of α. Note that if $g \in G$ and $x \in X$, then $q(g, x) \in X^{\mathbf{e}}$, $j(x) \in Y$ and $\psi(g, j(x)) \in Y$. Therefore, the function $\mu : X^{\mathbf{e}} \to Y$ given by $\mu(q(g, x)) = \psi(g, j(x))$ is well defined. In fact, if $q(g, x) = q(h, y)$, then $x \in X_{g^{-1}h}$ and $\alpha_{h^{-1}g}(x) = y$. So $j(\alpha_{h^{-1}g}(x)) = j(y)$. In addition, ψ induces a partial topological action β of G on $T = j(X)$ equivalent to α given by the collection $\{(T_g, \beta_g)\}_{g\in G}$ where $T_g = T \cap \psi_g(T)$ and $\beta_g : T_{g^{-1}} \to T_g$ is the homeomorphism defined as $\beta_g(t) = \psi_g(t)$ for each $t \in T_{g^{-1}}$. Hence,

$$\beta_{h^{-1}g}(j(x)) = j(\alpha_{h^{-1}g}(x)) = j(y).$$

Consequently,

$$(\beta_{h^{-1}} \circ \beta_g)(j(x)) = \psi_{h^{-1}}(\psi_g(j(x))) = \psi_{h^{-1}g}(j(x)) = \beta_{h^{-1}g}(j(x)) = j(y)$$

and so $\beta_g(j(x)) = \beta_h(j(y))$. Thus, $\psi(g, j(x)) = \psi(h, j(y))$, that is, $\mu(q(g, x)) = \mu(q(h, y))$.

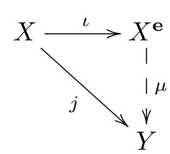
Now, suppose that $q(g, x) \in X^{\mathbf{e}}$ and $U \subseteq Y$ is an open neighborhood of $\mu(q(g, x)) = \psi(g, j(x))$. Since ψ is continuous, there are open sets $M \subseteq G$ and $N \subseteq Y$ such that $(g, j(x)) \in M \times N$ and $\psi(M \times N) \subseteq U$. Because the function j is continuous, there is an open $S \subseteq X$ such that $x \in S$ and $j(S) \subseteq N$. Moreover, given that the quotient function q is open, $V = q(M \times S) \subseteq X^{\mathbf{e}}$ is an open neighborhood of $q(g, x)$ such that $\mu(V) = \mu(q(M \times S)) = \psi(M \times j(S)) \subseteq \psi(M \times N) \subseteq U$. Therefore, μ is continuous. In addition, if $g, h \in G$ and $x \in X$, then

$$\begin{aligned}\mu(\alpha^{\mathbf{e}}(g, q(h, x))) = \mu(q(gh, x)) &= \psi(gh, j(x)) \\ &= \psi(g, \psi(h, j(x))) = \psi(g, \mu(q(h, x))).\end{aligned}$$

Thus, μ is a $G-$morphism. Furthermore

$$(\mu \circ \iota)(x) = \mu(\iota(x)) = \mu(q(e, x)) = \psi(e, j(x)) = j(x)$$

for each $x \in X$. Therefore, $j = \mu \circ \iota$ and thus the following diagram is commutative:



Finally, if $\nu : X^{\mathbf{e}} \to Y$ is a $G-$morphism such that $j = \nu \circ \iota$, then for each $q(g,x) \in X^{\mathbf{e}}$ we have

$$\begin{aligned}\mu(q(g,x)) = \psi(g, j(x)) &= \psi(g, \nu(\iota(x))) \\ &= \psi(g, \nu(q(e,x))) = \nu(\alpha^{\mathbf{e}}(g, q(e,x))) = \nu(q(g,x)).\end{aligned}$$

Thus, μ is unique and therefore $(X^{\mathbf{e}}, \alpha^{\mathbf{e}}, \iota)$ is the minimal globalization of α. □

Definition 4.10 ([2])**.** Let α be a topological partial action of G on X. The minimal globalization of α, given by $(X^{\mathbf{e}}, \alpha^{\mathbf{e}}, \iota)$ is called the enveloping action of α.

Now, we explicitly construct the enveloping actions of the examples given in the previous section. We include all of the details for the benefit of the reader.

Example 4.11. *Let $G = \{1, -1\}$, $X = \{(x,y) \in \mathbb{R}^2 : y > 0\}$, and α the topological partial action of G on X of Example 3.2. For $(a,b) \in X$, we have that $q(1,(a,b)) = \{(1,(a,b))\}$ and $q(-1,(a,b)) = \{(-1,(a,b))\}$. Thus,*

$$\begin{aligned}X^{\mathbf{e}} &= \{q(1,(x,y)) : (x,y) \in X\} \cup \{q(-1,(x,y)) : (x,y) \in X\} \\ &\cong \{(x,y) \in \mathbb{R}^2 : y > 0\} \cup \{(x,y) \in \mathbb{R}^2 : y < 0\} \text{ (Figure 3).}\end{aligned}$$

So $\alpha^{\mathbf{e}} : G \times X^{\mathbf{e}} \to X^{\mathbf{e}}$ is equivalent to $\alpha^{\mathbf{e}}(t,(x,y)) = (x, ty)$ and $\iota(X) \cong X$.

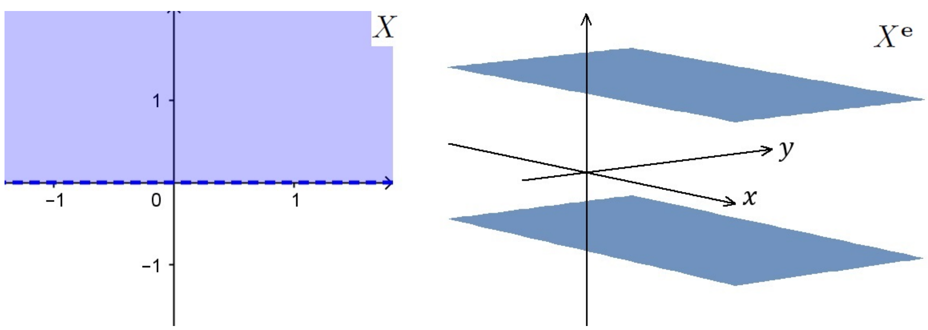


Figure 3. Graph of X and $X^{\mathbf{e}}$. Example 4.11.

Example 4.12. Let $G = \mathbb{R}_d$, $X = \mathbb{R}$, $h : X \to X$ the homeomorphism given by $h(x) = -x$ for each $x \in \mathbb{R}$, and α the topological partial action described in Example 3.3. Then, $\alpha^{\mathbf{e}} : \mathbb{R}_d \times X^{\mathbf{e}} \to X^{\mathbf{e}}$ is defined as $\alpha^{\mathbf{e}}(s, q(t,x)) = q(s+t, x)$. Furthermore,

1. If $t \in \mathbb{Z}$ and $x \in X$, then
   (a) If $t \in \mathbb{Z}$ is even, we have that $q(t,x) = q(0,x)$.
   (b) If $t \in \mathbb{Z}$ is odd, we have that $q(t,x) = q(0,-x)$.
2. If $t \notin \mathbb{Z}$ y $x \in X$, then there exist $n \in \mathbb{Z}$ and $s \in \mathbb{R}$ such that $t - s = n$, where $n = [|t|]$ and $0 < s < 1$. Thus,

(a) If n is even, we have that $q(t,x) = q(s,x)$.

(b) If n is odd, we have that $q(t,x) = q(s,-x)$.

Consequently, $X^{\mathbf{e}} \cong [0,1) \times \mathbb{R}$ (Figure 4) and $\iota(X) \cong \{(x,y) \in \mathbb{R}^2 : x = 0\}$.

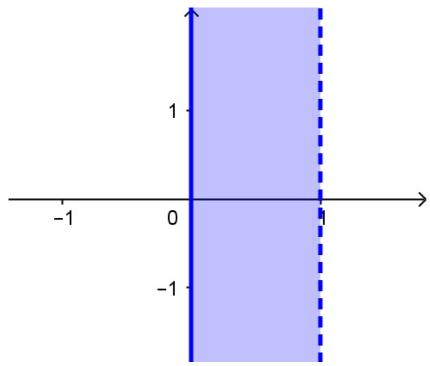


Figure 4. Enveloping space $X^{\mathbf{e}}$. Example 4.12.

The enveloping action constructed in the previous example is known in dynamical systems as the suspension of h [14].

Example 4.13. *Let us suppose that $X = \mathbb{R}$ and let $v : X \to TX$ be the vector field given by $v(w) = e^{-w}\partial/\partial w$. If α is the topological partial action described in Example 3.4, then for every pair $t \in \mathbb{R}$ and $x \in X$ we have that*

$$q(t,x) = \{(s,y) \in \mathbb{R}^2 : s < e^x + t, y = \ln(e^x + t - s)\}.$$

Therefore, $X^{\mathbf{e}}$ is the set of all curves of the form $y = \ln(e^x + t - s)$. While $\alpha^{\mathbf{e}}(r, q(t,x))$ is the curve $y = \ln(e^x + t + r - s)$; that is, $\alpha^{\mathbf{e}}$ acts globally and topologically through horizontal translations of the curves of $X^{\mathbf{e}}$ (Figure 5).

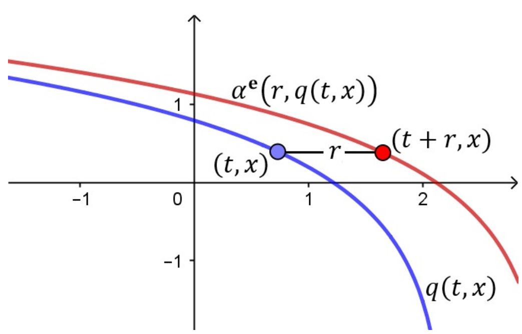


Figure 5. Graph of $q(t,x)$ and $\alpha^{\mathbf{e}}(r, q(t,x))$. Example 4.13.

Note that if α is a topological partial action of G on X and $x \in X$, then the orbit of $\iota(x)$ with respect to $\alpha^{\mathbf{e}}$ is the set $\{q(g,x) : g \in G\}$. Consequently, the orbit of $\iota(X)$ with respect to $\alpha^{\mathbf{e}}$ is $\bigcup_{x\in X}\{q(g,x) : g \in G\} = X^{\mathbf{e}}$. Therefore, X can be seen as an open subset of $X^{\mathbf{e}}$ and α as the induced topological partial action of $\alpha^{\mathbf{e}}$ on X. So X and $X^{\mathbf{e}}$ have the same local topological properties. However, its global topological properties are not preserved, as shown later on.

Example 4.14 ([2])**.** In the Example 4.11 we can see that X is connected, but the space $X^{\mathbf{e}}$ is not connected.

Example 4.15 ([2])**.** Consider the topological global action $\beta : \mathbb{Z} \times S^1 \to S^1$ given by $\beta(k,z) = e^{2\pi i k\theta}z$, X an open arc of S^1, and α the topological partial action given by the restriction of β on X. Since the enveloping action of α is minimal and unique except for equivalences, we have that $X^{\mathbf{e}} = S^1$. However, X and S^1 have a different fundamental group.

Example 4.16 ([2])**.** Suppose that $G = \{1,-1\}$, $X = [0,1]$, and α is the topological partial action of G on X given by $X_1 = X$, $X_{-1} = V$, $\alpha_1 = i_X$, $\alpha_{-1} = i_V$ where $V = (a,1]$ for some fixed $0 < a < 1$. If $G \times X$ has the product topology, then $X^{\mathbf{e}}$ is the topological space obtained by identifying the point $(1,t)$ with $(-1,t)$ for each $t \in (a,1]$ (Figure 6). Note that X is Hausdorff, but not there are disjoint open neighborhoods in $X^{\mathbf{e}}$ for $q(1,a)$ and $q(-1,a)$; that is, $X^{\mathbf{e}}$ is not Hausdorff.

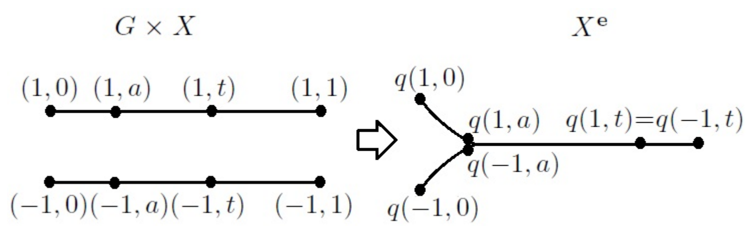


Figure 6. Enveloping space $X^{\mathbf{e}}$. Example 4.16.

Proposition 4.17. *Let α be a topological partial action of G on a connected topological space X. If X is dense in $X^{\mathbf{e}}$, then $X^{\mathbf{e}}$ is a connected space.*

Proof. If X is dense in $X^{\mathbf{e}}$, then $\overline{X} = X^{\mathbf{e}}$. Since X is connected and the closure of any connected is connected, we have that $X^{\mathbf{e}}$ is connected. □

The following result shows under what conditions the enveloping action turns out to be a Hausdorff space.

Proposition 4.18. *Let α be a topological partial action of G on a Hausdorff topological space X. If $K = \{(g,x,y) \in G \times X \times X : x,y \in X_{g^{-1}}, \alpha^{\mathbf{e}}_g(x) = \alpha^{\mathbf{e}}_g(y)\}$ is a closed subset of $G \times X \times X$, then the space $X^{\mathbf{e}}$ is Hausdorff, .*

Proof. Let $x^{\mathbf{e}}$ and $y^{\mathbf{e}}$ be distinct elements of $X^{\mathbf{e}}$ and $g \in G$. Then, $x^{\mathbf{e}} = \alpha^{\mathbf{e}}_g(x)$ and $y^{\mathbf{e}} = \alpha^{\mathbf{e}}_g(y)$ for some pair $x, y \in X \subseteq X^{\mathbf{e}}$ because $\alpha^{\mathbf{e}}_g : X^{\mathbf{e}} \to X^{\mathbf{e}}$ is bijective.

Therefore, $\alpha_g^{\mathbf{e}}(x) \neq \alpha_g^{\mathbf{e}}(y)$; that is, $(g, x, y) \notin K$. Since K^c is open, there are open sets $H \subseteq G$ and $M, N \subseteq X$ such that $(g, x, y) \in H \times M \times N \subseteq K^c$. So $x^{\mathbf{e}} = \alpha_g^{\mathbf{e}}(x) \in \alpha_g^{\mathbf{e}}(M)$ and $y^{\mathbf{e}} = \alpha_g^{\mathbf{e}}(y) \in \alpha_g^{\mathbf{e}}(N)$ where $\alpha_g^{\mathbf{e}}(M)$ and $\alpha_g^{\mathbf{e}}(N)$ are open sets because $\alpha_g^{\mathbf{e}}$ is a homeomorphism. Moreover, $H \times M \times N$ and K are disjoint and this implies that $\alpha_g^{\mathbf{e}}(M)$ and $\alpha_g^{\mathbf{e}}(N)$ are also disjoint. In fact, if $t \in \alpha_g^{\mathbf{e}}(M) \cap \alpha_g^{\mathbf{e}}(N)$, then $t = \alpha_g^{\mathbf{e}}(m)$ and $t = \alpha_g^{\mathbf{e}}(n)$ for some pair $m \in M$ and $n \in N$. So, $\alpha_g^{\mathbf{e}}(m) = \alpha_g^{\mathbf{e}}(n)$ and thus $(g, m, n) \in K$ which is a contradiction. Consequently, $X^{\mathbf{e}}$ is Hausdorff. □

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