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SOME ORNSTEIN–UHLENBECK POTENTIALS FOR THE ONE–DIMENSIONAL SCHRÖDINGER OPERATOR PART II: POSITION-DEPENDENT DRIFT

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Abstract

We consider the Schrödinger operator on the unit circle, whose potential is an Ornstein – Uhlenbeck type process, with drift depending on its position. We describe the distribution of the periodic groundstate, based on the circular brownian motion measure. The results exposed here, have been mentioned, but not proved, in [7].

Keywords: Schrödinger Operator, Ornstein – Uhlenbeck Process, Periodic Groundstate, Circular Brownian Motion.

Resumen

Se considera el operador de Schrödinger en el círculo unitario, con un potencial de tipo Ornstein – Uhlenbeck, cuyo factor tendencial depende de la posición. Se describe la distribución del primer valor propio periódico, usando el movimiento browniano circular. Los resultados expuestos aquí, han sido mencionados, pero no demostrados, en [7].

Palabras clave: Operador de Schrödinger, proceso de Ornstein – Uhlenbeck, estado periódico, movimiento browniano circular.

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1. Introduction

The standard Ornstein–Uhlenbeck process is the diffusion q satisfying the stochastic differential equation

$$dq = db - mqdt,$$

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where m is a constant, and b is a standard Brownian motion. It is explicity given by[1]

$$q(t) = q(0)e^{-mt} + e^{-mt}\int_0^t e^{m\tau} db_\tau,$$

where the process $\int_0^t e^{m\tau} db_\tau$ can be written as $W\left(\frac{e^{2mt}-1}{2m}\right)$, for some Brownian motion W starting at 0. This allows the computation of the transition density for q, obtaining in particular that $\int p(1; a, a)da = (1 - e^{-m})^{-1} < \infty$. The process q can be made periodic by conditioning it to have $q(1) = q(0)$. The resulting finite-dimensional distributions are:

$$P[q_{t_0} \in da_0, ..., q_{t_{n-1}} \in da_{n-1}] = \left(1 - e^{-m}\right) \prod_{i=1}^{n} p(t_i - t_{i-1}; a_{i-1}, a_i) da_0 ... da_{n-1},$$

where $0 = t_0 < t_1 < ... < t_n = 1$, and $a_n = a_0$.
The coordinate process in $\mathcal{C}(S^1)$ is a periodic Ornstein - Uhlenbeck process under the probability measure dQ_* on $\mathcal{C}(S^1)$, defined by

$$\begin{aligned} E_*[\phi] &= (1 - e^{-m}) \int \left\{ \frac{\partial}{\partial x} E_a^{ou}[\phi(q), q(1) \le x] \right\}\Big|_{x=a} da \\ &= (1 - e^{-m}) \int E_a^{ou}[\phi(q) \mid q(1) = a]\, p(1; a, a) da. \end{aligned}$$

Moreover, by the Cameron–Martin formula one can get the following relation between dQ_* and the circular Brownian motion measure $d\mu_*$ on $\mathcal{C}(S^1)$:

$$\int \phi(q) dQ_* = E_*[\phi] = \sqrt{\frac{2}{\pi}} \sinh\left(\frac{m}{2}\right) \int \phi(q) e^{-\frac{m^2}{2}\int q^2} d\mu_*.$$

The details can be found in [5].
Remark: Taking $\phi \equiv 1$, this gives the corollary:

$$\int e^{-\frac{m^2}{2}\int_0^1 q^2} d\mu_* = \sqrt{\frac{\pi}{2}} \left[\sinh\left(\frac{m}{2}\right)\right]^{-1}.$$

With respect to the probability P_0, induced by μ_* on $H := \left\{q : \int_0^1 q = 0\right\}$, we get the following identity:

$$E^0\left[e^{-\frac{m^2}{2}\int_0^1 q^2}\right] = \frac{m}{2\sinh(\frac{m}{2})}. \tag{1}$$

The whole argument can be applied to the more general case $dq = -m(q)dt + db$, for any reasonable drift m. This is done in the next section. The case of $m = m(t)q$ is treated in [5].

2. The equation. What is known

We consider Hill's equation:

$$-y'' + qy = \lambda y, \tag{2}$$

the potential q being the periodic Ornstein–Ulenbeck process. We think of q as being any element in the set $\Omega = \mathcal{C}(S^1)$, and impose in Ω the probability measure Q_*. Let $\lambda_0(q)$ denote the periodic groundstate of (2), with potential q. In [5] we prove the following:

[1] See [9], or [6].

Theorem 1 *Under the transformation $Q = \lambda + p' + p^2$, the probability measure Q_*, restricted to $[\lambda_0 \geq \lambda]$, is transformed into the measure $dP_0 d\alpha$, according to the following: If ϕ is a bounded measurable function on Ω, then*

$$\int_{[\lambda_0 \geq \lambda]} \phi \, dQ_* = C \int_H \int_{I(p')}^{\infty} \phi(\lambda + p' + p^2) e^{-\frac{m^2}{2} \int (\lambda + p' + p^2)^2} G(\alpha, p') d\alpha dP_0(p'),$$

with $C = \frac{4}{\sqrt{2\pi}} \sinh(\frac{m}{2})$, $p = \alpha + \int_0^t p'$, $I(p') = -\int_0^1 \int_0^t p'$, and

$$G(\alpha, p') = \exp \left[\int_0^1 (p'^3 - 2p^2 p'^2) dt \right] \sinh \left(\int_0^1 p \right). \tag{3}$$

Corollary 1 *The distribution of λ_0 under Q_* is given by*

$$Q_*[\lambda_0 \geq \lambda] = C \int_H \int_{I(p')}^{\infty} e^{-\frac{m^2}{2} \int (\lambda + p' + p^2)^2} G(\alpha, p') d\alpha dP_0(p').$$

3. The new potentials

Let us consider the diffusion q solving

$$dq = -m(q)dt + db,$$

where b is a standard Brownian motion under a probability measure P, and m is an odd function, with $m(q) > 0$ for $q > 0$ (to avoid explossion). Thus q has associated the infinitesimal generator

$$\frac{1}{2} \frac{d^2}{dq^2} - m(q) \frac{d}{dq}$$

under P.

We denote by $\hat{Q}$ the probability measure under which q is Brownian motion. Notice that, for $q(t) > 0$,

$$q(t) = b(t) - b(t_0) - \int_{t_0}^{t} m(q_s) ds \leq b(t) - b(t_0),$$

with $t_0 = \text{máx}\{s \leq t : Q(s) = 0\}$. Similarly, for $q(t) < 0$,

$$q(t) = b(t) - b(t_0) - \int_{t_0}^{t} m(q_s) ds \geq b(t) - b(t_0),$$

and therefore, in any case

$$|q(t)| \leq 2 \max_{0 \leq s \leq t} |b(t)|, \quad \text{for all } t \geq 0.$$

In particular $q(t)$ is defined for all $t \geq 0$.
Let $t_N = \inf\{t \geq 0 : |q(t)| = N\}$. On $[t_N \geq t]$ we have

$$\max_{0 \leq s \leq t} |q(s)| \leq N.$$

Since $m(q)$ is bounded on $[-N, N]$, we can apply Cameron–Martin to the process $q(t \wedge t_N)$, i.e. for any bounded ϕ,

$$E_a^P\{\phi(q)\chi_{[t_N \geq t]}\} = \hat{E}_a\{\phi(q)Z(t)\chi_{[t_N \geq t]}\},$$

with

$$Z(t) := \exp\left[-\int_0^t m(q_s)dq_s - \frac{1}{2}\int_0^t m^2(q_s)ds\right].$$

Since $t_N \uparrow \infty$, taking $\phi \equiv 1$ and using dominated convergence gives

$$\hat{E}_a\{Z(t)\} = \lim_{N\to\infty} \hat{E}_a\{Z(t)\chi_{[t_N \geq t]}\} = \lim_{N\to\infty} E_a^P\{\chi_{[t_N \geq t]}\} = 1.$$

Thus, we can apply Cameron–Martin to the process $q(t)$ itself.

Now we define a probability measure Q_* (under which q will be periodic) by setting

$$\begin{aligned} E^*[\phi] &= C\int \left\{\frac{\partial}{\partial x}E_a^P[\phi(q), q(1) \leq x]\right\}\Big|_{x=a} da \\ &= C\int \left\{\frac{\partial}{\partial x}\hat{E}_a[\phi(q)Z(1), q(1) \leq x]\right\}\Big|_{x=a} da \\ &= C\int E^{00}\left[\phi(q+a)e^{-\frac{1}{2}\int_0^1 F(a+q_s)ds}\right]\frac{da}{\sqrt{2\pi}} \\ &= \frac{C}{\sqrt{2\pi}}\int \phi(q)e^{-\frac{1}{2}\int_0^1 F(q_s)ds}d\mu_*, \end{aligned}$$

where μ_* is the circular Brownian motion measure, $F(q) = -m'(q) + m^2(q)$ and the constant C is to be determined. We want to have

$$\int_{-\infty}^{\infty} E^{00}\left[e^{-\frac{1}{2}\int_0^1 F(a+q_s)ds}\right]da < \infty.$$

By Jensen's inequality and Fubini's theorem, it is enough that

$$\int_{-\infty}^{\infty} e^{-\frac{1}{2}F(y)}dy < \infty, \quad \text{ie.} \int_{-\infty}^{\infty} e^{\frac{1}{2}(m'-m^2)} < \infty. \tag{4}$$

If that is the case, then

$$dQ_* = C_0 e^{-\frac{1}{2}\int_0^1 F(q_s)ds}d\mu_*, \tag{5}$$

with

$$C_0 = \left[\int e^{-\frac{1}{2}\int_0^1 F(q_s)ds}d\mu_*\right]^{-1}.$$

4. Distribution of $\lambda_0(Q)$

We want to find the distribution of $\lambda_0(q)$ with respect to the probability measure Q_*. For this, we first express dQ_* in terms of $d\alpha dP_0(p')$. By the Cameron - Martin relation obtained above, it is enough to find an expression for $d\mu_*$ in terms of $d\alpha dP_0(p')$. More precisely, we consider the transformation

$$Q = \lambda + p' + p^2,$$

but instead of Q_*, we use μ_*. The calculations are done in [5], obtaining

$$d\mu_* = 2\exp\left[\int_0^1 ((p')^3 - 2p^2 (p')^2) dt\right] \sinh\left(\int_0^1 p\right) d\alpha dP_0.$$

In other words, for ϕ satisfying $\int |\phi| d\mu_* < \infty$,

$$\int_{[\lambda_0 \geq \lambda]} \phi(q) d\mu_* = 2\int_H \int_{I(p')}^{\infty} \phi(\lambda + p' + p^2) G(\alpha, p') d\alpha dP_0.$$

with $G(\alpha, p')$ given by

$$G(\alpha, p') = \exp\left[\int_0^1 ((p')^3 - 2p^2 (p')^2) dt\right] \sinh\left(\int_0^1 p\right).$$

It follows that

$$\int_{[\lambda_0 \geq \lambda]} \phi(q)\, dQ_* = 2C_0 \int_H \int_{I(p')}^{\infty} \hat{\phi}(\lambda) e^{-\frac{1}{2}\int_0^1 \hat{F} ds} G(\alpha, p') d\alpha d\mu_*,$$

where we have written $\hat{\phi}(\lambda)$ for $\phi(\lambda + p' + p^2)$, and similarly for $\widehat{F}$. In particular, the following theorem follows.

Theorem 2 *Let $m(q)$ be an odd function such that $m(q) > 0$ for $q > 0$, and satisfying (4). If Q_* is given by (5), then the distribution function $Q_*[\lambda_0 \geq \lambda]$ of $\lambda_0(Q)$ under Q_* is given by*

$$Q_*[\lambda_0 \geq \lambda] = 2C_0 \int_H \int_{I(p')}^{\infty} \exp\left[-\frac{1}{2}\int_0^1 F(\lambda + p'_s + p_s^2) ds\right] G(\alpha, p') d\alpha dP_0.$$

5. Joint Distribution

Consider the periodic diffusion coming from the infinitesimal operator

$$\frac{1}{2}\frac{d^2}{dq^2} - m(q)\frac{d}{dq},$$

under the probability measure Q_* defined by

$$dQ_* = C_0 \exp\left[-\frac{1}{2}\int_0^1 F(Q_s) ds\right] dP_*, \tag{6}$$

as explained in the last section, with $F = -m' + m^2$ and

$$C_0^{-1} = \int \exp\left[-\frac{1}{2}\int_0^1 F(Q_s) ds\right] dP_*.$$

We consider the transformation $(Q, a) \leftrightarrow (p', \alpha, \lambda)$ given by

$$Q = \lambda + p' + p^2, \quad a = \int_0^1 \left(\alpha + \int_0^t p'\right) dt.$$

To get the relation between dP_*da and $dP_0(p')d\alpha d\lambda$, it is enough to take care of the Jacobian. But this is done in the same way as for white noise (see [7]). It follows that, if $\int |\phi(Q,a)|dP_*da < \infty$, then[2]

$$\int \phi\, dP_*da = \int_H \int \left[\int \phi(\lambda + p' + p^2, \int p)d\lambda\right] E(\alpha, p')e^{-\int p} d\alpha dP_0, \tag{7}$$

with $p = \alpha + \int_0^t p'$ and

$$E(\alpha, p') = \exp\left[\int_0^1 (p'^3 - 2p^2 p'^2)\right] A(p).$$

Here we use the notation

$$A(p) = \int_0^1 e^{-2\int_0^x p} dx \int_0^1 e^{2\int_0^x p} dx.$$

Notice that $\alpha = I(p')$ if and only if $\int_0^1 p = 0$.
By (6) and (7), if $\int |\varphi(Q,a)|dQ_*da < \infty$, then

$$\int \varphi\ dQ_*da = C_0 \int_H \int \left[\int \hat{\varphi}(\lambda)\ e^{-\frac{1}{2}\int \hat{F}(\lambda)} d\lambda\right] e^{-\int p} E(\alpha, p')d\alpha dP_0,$$

where $\hat{\varphi}(\lambda)$ denotes $\varphi(\lambda + p' + p^2, \int p)$ and $\hat{F}(\lambda)$ denotes $F(\lambda + p'_s + p_s^2)$. In particular, for the classical Ornstein–Uhlenbeck case,

$$\int \varphi\ \ d\hat{Q}da = C \int_H \int \left[\int \hat{\varphi}\ e^{-\frac{m^2}{2}\int (\lambda + p' + p^2)^2} d\lambda\right] e^{-\int p} E(\alpha, p')d\alpha dP_0, \tag{8}$$

with $C = \sqrt{\frac{2}{\pi}} \sinh\left(\frac{m}{2}\right)$.

6. Induced Measure and Density of $\lambda_0(Q)$

The results of the last section will be used to find the density of $\lambda_0(Q)$ under P_* and Q_*. We first notice that the choice $\phi(Q) = \varphi(Q)\chi_{[0\le t\le h]}$ in (7) gives

$$\int \varphi\, dP_* = \int_H \frac{1}{h}\int^{+h} \left[\int_{-\infty}^{\infty} \varphi(\lambda + p' + p^2)d\lambda\right] e^{-\int_0^1 p} E(\alpha, p')d\alpha dP_0$$

and, as $h \to 0$,

$$\int \varphi(Q)dP_* = \int_H \left(\int_{-\infty}^{\infty} \varphi(\lambda + p' + p^2)d\lambda\right) E(p')dP_0.$$

Here, $E(p') := E(I(p'), p')$ and $p = I(p') + \int_0^t p'$.

Since $\int_0^1 Q$ is Lebesgue - distributed under P_*, taking $\varphi = \chi_{[0\le \int Q \le 1]}$ produces

$$\int_H E(p')dP_0 = 1, \tag{9}$$

[2] In the following dP_0 will denote $dP_0(p')$.

i.e.,

$$\int_H \exp\left[\int_0^1 (p'^3 - 2p^2p'^2)\right]\left(\int_0^1 e^{-2\int_0^x p}dx \int_0^1 e^{2\int_0^x p}dx\right) dP_0 = 1.$$

Since $\lambda_0(\lambda + p' + p^2) = \lambda$ when $\int p = 0$, we also have

$$\int \varphi(Q)\chi_{[\mu\le\lambda_0\le\rho]}dP_* = \int_H \left(\int_\mu^\rho \varphi(\lambda + p' + p^2)d\lambda\right) E(p')dP_0$$

for $-\infty \le \mu < \rho \le \infty$. This easily implies the following expression for the measure P_*^μ induced by P_* on $[\lambda_0 = \mu]$:

$$\int_{[\lambda_0=\mu]} \varphi(Q)dP_*^\mu = \int_H \varphi(\mu + p' + p^2)E(p')dP_0.$$

Because of (9), taking $\varphi \equiv 1$ implies:
Proposition. $\lambda_0(Q)$ is Lebesgue–distributed under P_*. The corresponding formulas for Q_* are

$$\int \varphi(Q) : dQ_* = C_0 \int_H \left[\int \hat{\varphi}(\lambda)e^{-\frac{1}{2}\int \hat{F}(\lambda)}d\lambda\right] E(p')dP_0 \tag{10}$$

and

$$\int_{[\lambda_0=\mu]} \varphi(Q)dQ_*^\mu = C_0 \int_H \hat{\varphi}(\mu)e^{-\frac{1}{2}\int \hat{F}(\mu)} E\left(p'\right) dP_0.$$

In particular, the density of λ_0 under Q_* is

$$f(\mu) = C_0 \int_H e^{-\frac{1}{2}\int F(\mu+p'+p^2)} E\left(p'\right) dP_0.$$

For the classical Ornstein–Uhlenbeck case,

$$\int_{[\lambda_0=\mu]} \varphi(Q)d\hat{Q}_\mu = \sqrt{\frac{2}{\pi}}\sinh\left(\frac{m}{2}\right) \int_H \hat{\varphi}(\mu)e^{-\frac{m^2}{2}\int_0^1(\mu+p'+p^2)^2} E\left(p'\right) dP_0$$

and

$$f(\mu) = \sqrt{\frac{2}{\pi}}\sinh(\frac{m}{2}) \int_H e^{-\frac{m^2}{2}\int_0^1(\mu+p'+p^2)^2} E\left(p'\right) dP_0.$$

7. Some identities

Now we deduce a series of identities involving the measures Q_*, P_* and P_0, based on the results of previous sections.

Let us start by applying (7) to $\phi(Q,a) = \psi(Q)\, e^{-a^2/2}$. The result is

$$\sqrt{2\pi}\int \psi(Q)dP_* = \int_H \int_{\mathbb{R}} \left[\int_{\mathbb{R}} \hat{\psi}(\lambda, p')d\lambda\right] e^{-\frac{1}{2}(\int p)^2 - \int p}E(\alpha, p')d\alpha dP_0,$$

which for $\psi(Q) = \chi_{[0\le\int Q\le 1]}$ gives

$$\int_H \int_{-\infty}^{\infty} e^{-\frac{1}{2}\left(\int p\right)^2 - \int p}E(\alpha, p')d\alpha dP_0 = \sqrt{2\pi}.$$

More generally,

$$\int_H \int_{-\infty}^{\infty} \varphi\left(\int p\right) e^{-\int p} E(\alpha, p') d\alpha dP_0 = \int_{-\infty}^{\infty} \varphi(x) dx.$$

Compare with (9). Next, take $\varphi = \varphi\left(\int Q\right)$ in (10) to obtain

$$\int \varphi\left(\int p\right) dQ_* = C_0 \int_H \left[\int \varphi\left(\lambda + \int p^2\right) e^{-\frac{1}{2}\int \hat{F}(\lambda)} d\lambda\right] E(p') dP_0. \tag{11}$$

In the Ornstein–Uhlenbeck, with $\varphi \equiv 1$, this gives

$$\int_H e^{-\frac{m^2}{2}\left[\int p^4 + \int |p'|^2 - \left(\int p^2\right)^2\right]} E(p') dP_0 = \frac{m}{2\sinh(\frac{m}{2})},$$

and (11) reduces to

$$\int \varphi\left(\int Q\right) d\hat{Q} = \frac{m}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \varphi(x) e^{-\frac{m^2}{2}x^2} dx,$$

i.e., $\int_0^1 Q \sim (0, m^{-2})$ under $\hat{Q}$. In a similar way, for $\varphi(Q,a) = \psi(\int Q)\varphi(a)$ in (8),

$$\int_H \int_{\mathbb{R}} e^{-\frac{m^2}{2}[\int p^4 + \int p'^2 - (\int p^2)^2]} \varphi\left(\int p\right) e^{-\int p} E(\alpha, p') d\alpha dP_0 = \frac{m}{2\sinh(\frac{m}{2})} \int_{\mathbb{R}} \varphi.$$

These identities seem to be very useful as a tool in solving some open problems on asymptotic behaviour of the density as $|\lambda| \to \infty$.

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