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Using GeoGebra in generalization processes of geometrical challenging problems[1]

Usando o GeoGebra em processos de generalização de problemas geométricos desafiadores

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ABSTRACT

We generalize in this work three geometrical challenging problems addressed in mathematics literature. In generalizations, we adopt the theoretical assumptions established for this process and use GeoGebra to build figures and animation. The proposed and solved generalizations establish natural links between some mathematics areas, highlighting the importance of generalization processes for constructing mathematical knowledge in undergraduate programs in mathematics teacher education. We conclude that the use of GeoGebra was essential to a comprehensive understanding of the structures for generalization.

Keywords: *geometrical theorems; Mathematics Teaching; geometry software.*

RESUMO

Neste trabalho, generalizamos três problemas geométricos desafiadores presentes na literatura matemática. Nas generalizações, adotamos os pressupostos teóricos estabelecidos para esse processo e empregamos o GeoGebra para construir figuras e animações. As generalizações propostas e solucionadas estabelecem conexões naturais entre algumas áreas da matemática, destacando a importância dos processos de generalização à construção do conhecimento matemático em cursos de graduação que preparam professores de matemática. Concluímos que o emprego do GeoGebra foi essencial à compreensão abrangente das estruturas para a generalização.

Palavras-chave: *teoremas geométricos; Ensino de Matemática; aplicativo de geometria.*

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Introduction

Vygotsky (1986) considers that every concept is the result of a generalization process. Thus, for him, all concepts learned by human beings, and here we highlight the concepts and mathematical properties, are internalized through a generalization process.

According to Dumitrascu (2017, p. 47):

> In the mathematics literature, generalization can be seen as a statement that is true for a whole category of objects; it can be understood as the process through which we obtain a general statement; or it can be the way to transfer knowledge from one setting to a different one.

In accordance with Hashemi et al. (2013), the generalization is one of the fundamental activities in learning mathematics, which needs to be further explored by individuals who teach and study mathematics. For Mason (1996), generalization is the heartbeat of mathematics. Davydov (1990) argues that the development of student generalization capacity is one of the main goals of mathematics, while Sriraman (2004) considers that the generalization begins with the construction of examples, within which plausible patterns are detected and lead to the formulation of theorems.

However, generalizing in mathematics, particularly in geometry (ALLEN, 1950; PARK; KIM, 2017), is generally not a trivial process. From the sum of the internal angles of a triangle (180°) to the sum of the internal angles of an n-sided convex polygon ($180°(n-2)$), the generalization occurs by a partition of the n-sided convex polygon into $n-2$ triangles; from the Pythagorean theorem ($x^2 + y^2 = z^2$) to the Fermat theorem ($x^n + y^n = z^n$), the generalization-required centuries of study and the creation of new areas in mathematics (SINGH, 2002).

In this way, we illustrate in this work the generalization process in geometry using the concepts of Sriraman (2004). In this process, we replaced the construction of examples by selecting three geometrical challenging problems from the book *Challenging problems in geometry* by Posamentier and Salkind (1996): the measure of the midsegment of a triangle, the section of the hypotenuse, and the section of an internal angle of a triangle. These three geometrical challenging problems are then transformed into theorems, which can be complemented with proofs without words (NELSEN, 1993; LAGO; NÓS, 2020; NÓS; FERNANDES, 2018, 2019) in the dynamic geometry software GeoGebra (2021) and can be approached in mathematics teacher training courses. The first of the three problems can be presented in high school math classes.

1. Midsegment of a triangle

Problem 1 (Challenging problem 3-7, page 12) If the measures of two sides and the included angle of a triangle are 7, $\sqrt{50}$, and 135°, respectively, find the measure of the segment joining the midpoints of the two given sides.

We can solve Problem 1 utilizing distinct strategies. Posamentier and Salkind (1996) propose using the Pythagorean theorem (theorem 55, page 243) and the triangle midsegment theorem (theorem 26, page 241).

Solution 1 Consider the triangle ABC, with sides $AB = c = 7$, $AC = b = \sqrt{50}$ and $BC = a$, angle $B\hat{A}C = 135°$, E, and F midpoints on sides $\overline{AC}$ and $\overline{AB}$, respectively, and point D, orthogonal projection of the vertex C of the triangle ABC on the extension of side $\overline{AB}$, as shown in Figure 1.

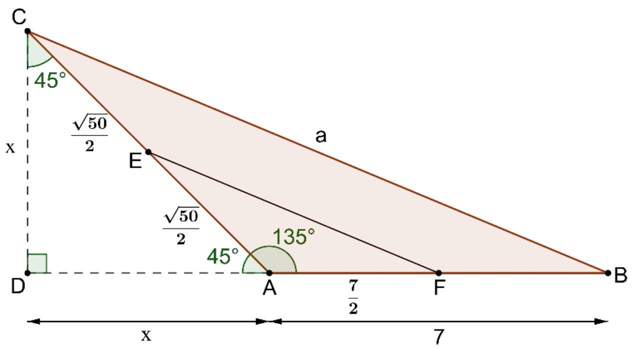


FIGURE 1: Challenging problem 1: the midpoint segment EF of the triangle ABC
SOURCE: Authors with GeoGebra

Applying the Pythagorean theorem (LOOMIS, 1968) in the isosceles right triangle ADC, with measurement sides $AD = CD = x$, we obtain that.

$$2x^2 = 50 \Longrightarrow x = 5. \tag{1}$$

Using (1) and the Pythagorean theorem in right triangle BDC - Figure 1, we have that

$$a^2 = x^2 + (7 + x)^2 \Longrightarrow a^2 = 25 + 144 \Longrightarrow a = 13,$$

where a is the measure of the hypotenuse of the right triangle BDC.

Since $\overline{EF}$ is a midsegment of the triangle ABC, applying the triangle midsegmente theorem, we conclude that

$$EF = \frac{a}{2} = \frac{13}{2}.$$

⁂

By fixing the measurements of the two sides of a triangle, as in Problem 1, and varying the measurement of the angle determined by those sides, we can visually check or make a proof without words of the triangle midsegment theorem using a dynamic geometry software. We built an animation in GeoGebra, which can be done in mathematics classroom, and we make it available at the following address:

https://www.GeoGebra.org/m/hamgfbgj.

This animation makes it easier to devise the generalization of Problem 1.

Theorem 1 (Generalization of Problem 1) If b and c are the measures of two sides of a triangle and θ is the angle determined by these two sides, then the measure of the segment whose ends are the midpoints of the sides with measures b and c is equal to

$$\frac{\sqrt{b^2 + c^2 - 2bc\,cos\theta}}{2}.$$

Proof Considers in the triangle ABC, with sides $AB = c$, $AC = b$, and $BC = a$: the angle $B\hat{A}C = \theta, 90° < \theta < 180°$; points E and F, respectively, midpoints of the sides $\overline{AC}$ and $\overline{AB}$, and point D, orthogonal projection of the vertex C of the triangle ABC on the extension of side $\overline{AB}$, as shown in Figure 2.

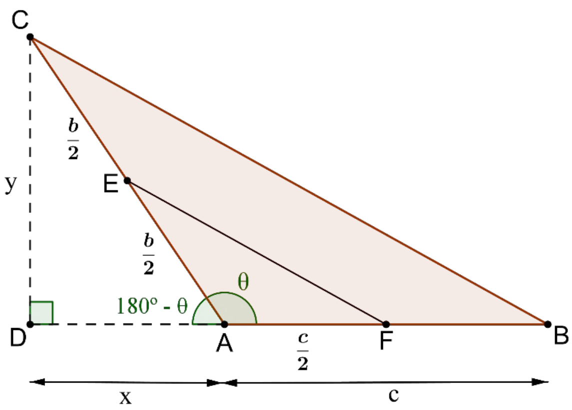


FIGURE 2: Generalization of challenging problem 1: the law of cosines
SOURCE: Authors with GeoGebra

Calculating trigonometric ratios in the right triangle ADC, where $AD = x$ and $CD = y$, and determining trigonometric transformations (HILL, 2019), we have

$$cos(180° - \theta) = \frac{x}{b} \Longrightarrow x = -b\,cos\theta, \quad (2)$$

$$sen(180° - \theta) = \frac{y}{b} \Longrightarrow y = b\,sen\theta. \quad (3)$$

Applying the Pythagorean theorem in the right triangle BDC - Figure 2, and using (2), and (3), and a trigonometric identity (HILL, 2019), we conclude that

$$a^2 = y^2 + (x + c)^2 \Rightarrow a^2 = (b\ sen\theta)^2 + (-b\ cos\theta + c)^2,$$
$$a^2 = b^2(sen^2\theta + cos^2\theta) + c^2 - 2bc\ cos\theta,$$
$$a^2 = b^2 + c^2 - 2bc\ cos\theta, \tag{4}$$
$$a = \sqrt{b^2 + c^2 - 2bc\ cos\theta},$$

where a is the measure of the hypotenuse of the right triangle BDC.

Since is $\overline{EF}$ a midsegment of the triangle ABC, we have by the triangle midsegment theorem that

$$EF = \frac{a}{2} = \frac{\sqrt{b^2 + c^2 - 2bc\ cos\theta}}{2}. \tag{5}$$

We can show that relations (4) and (5) remain true if $0° < \theta \leq 90°$.

□

The relation (4) is the law of cosines (HILL, 2019), and it can be applied directly to the triangle AEF - Figure 1, to solve Problem 1. However, we choose to deduce it through the generalization of Problem 1, thus showing that generalization processes can be used in mathematics classes as demonstration activities.

Using in relation (5) standard values of the first and second quadrants for the angle θ, we have

$$\theta = 30° \Rightarrow EF = \frac{\sqrt{b^2 + c^2 - bc\sqrt{3}}}{2},$$
$$\theta = 45° \Rightarrow EF = \frac{\sqrt{b^2 + c^2 - bc\sqrt{2}}}{2},$$
$$\theta = 60° \Rightarrow EF = \frac{\sqrt{b^2 + c^2 - bc\sqrt{1}}}{2},$$
$$\theta = 90° \Rightarrow EF = \frac{\sqrt{b^2 + c^2 - bc\sqrt{0}}}{2},$$
$$\theta = 120° \Rightarrow EF = \frac{\sqrt{b^2 + c^2 + bc\sqrt{1}}}{2},$$
$$\theta = 150° \Rightarrow EF = \frac{\sqrt{b^2 + c^2 + bc\sqrt{2}}}{2},$$
$$\theta = 150° \Rightarrow EF = \frac{\sqrt{b^2 + c^2 + bc\sqrt{3}}}{2}.$$

In relation with $\theta = 30°, 45°, 135°, 150°$, we have nested radicals, which allow us to discuss in the classroom the rules for denesting radicals (GKIOULEKAS, 2017; NÓS; SAITO; SANTOS, 2017).

2. Section of the hypotenuse

Problem 2 (Challenging problem 10-4, page 46) Prove that the sum of the squares of the distances from the vertex of the right angle, in a right triangle, to the trisection points along the hypotenuse, is equal to $5/9$ the square of the measure of the hypotenuse.

Posamentier and Salkind (1996) propose using Stewart's theorem (page 45) to solve challenging problem 2. They also propose in challenge 2 of problem 10-4 to predict the value of the sum of the squares for a quadrisection of the hypotenuse (NÓS; SAITO; OLIVEIRA, 2016).

Solution 2 Consider the right triangle ABC, with cathetus $AC = b$ and $AB = c$, and the cevians[5] d_1 and d_2, which trisect, respectively, the hypotenuse $BC = a$ at points T_1 and T_2, as shown in Figure 3.

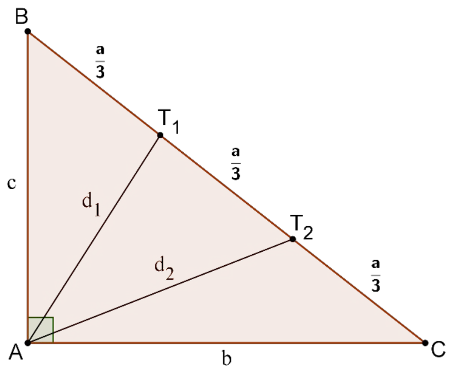


FIGURE 3: Challenging problem 2: the section of the hypotenuse in three congruent segments
SOURCE: Authors with GeoGebra

Applying Stewart's theorem to cevians d_1 and d_2, we obtain, respectively, that

$$\frac{b^2}{3} + \frac{2c^2}{3} - {d_1}^2 = \frac{2a^2}{9}, \tag{6}$$

$$\frac{2b^2}{3} + \frac{c^2}{3} - {d_2}^2 = \frac{2a^2}{9}. \tag{7}$$

Adding equations (6) and (7), and using the result from the application of the Pythagorean theorem in the triangle ABC, i.e. $b^2 + c^2 = a^2$, we conclude that

$$b^2 + c^2 - {d_1}^2 - {d_2}^2 = \frac{4}{9}a^2 \Longrightarrow {d_1}^2 + {d_2}^2 = a^2 - \frac{4}{9}a^2 = \frac{5}{9}a^2.$$

⁂

[5] Cevian is a line segment that joins a vertex of a triangle with a point on the opposite side (or its extension).

In this way of thinking, calculating the sum of the squares of the measures of the cevians that section the hypotenuse in congruent s $(s = 2,3,4, \dots)$ segments, we will find, respectively, the following fractions of the square of the measure of the hypotenuse:

$$\left\{\frac{1}{4}, \frac{5}{9}, \frac{7}{8}, \dots\right\}. \tag{8}$$

So, the question to be answered is whether we can establish the nth term of the sequence (8).

Theorem 2 (Generalization of Problem 2) If d_i, $i = 1,2, \dots, n$, are the measurements of the cevians with an end at the vertex of the right angle of a right triangle and which divide the hypotenuse in congruent $n + 1$ segments, then

$$\sum_{i=1}^{n} {d_i}^2 = \frac{n(2n+1)}{6(n+1)} a^2 , \tag{9}$$

where a is the measure of the hypotenuse of the right triangle.

Proof Let us consider the right triangle ABC, with cathetus $AC = b$ and $AB = c$, and cevians of measures d_i, $i = 1,2, \dots, n$, which section, respectively, the hypotenuse $BC = a$ at points $T_1, T_2, \dots, T_n$, as shown in Figure 4.

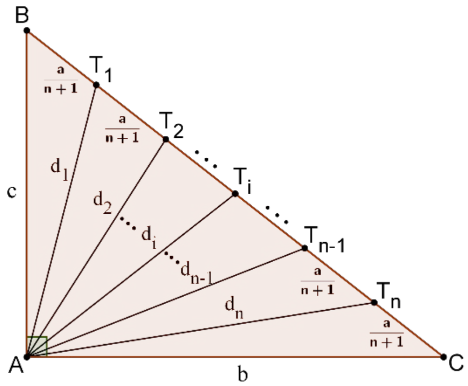


FIGURE 4: Generalization of challenging problem 2: the section of the hypotenuse in congruent $n + 1$ segments
SOURCE: Authors with GeoGebra

In right triangle ABC, applying Stewart's theorem for cevian d_i, $i = 1,2, \dots, n$, we have

$$b^2 \frac{a}{n+1} i + c^2 \frac{a}{n+1}(n+1-i) - {d_i}^2 a = a \frac{a}{n+1} i \frac{a}{n+1}(n+1-i),$$

$$\frac{i}{n+1} b^2 + \frac{n+1-i}{n+1} c^2 - {d_i}^2 = \frac{i(n+1-i)}{(n+1)^2} a^2. \tag{10}$$

Adding equation (10) in i and using discrete sum properties, we get

$$\frac{b^2}{n+1}\sum_{i=1}^{n} i + c^2\sum_{i=1}^{n}\left(1-\frac{i}{n+1}\right) - \sum_{i=1}^{n} {d_i}^2 = a^2\sum_{i=1}^{n}\left(\frac{i}{n+1}-\frac{i^2}{(n+1)^2}\right),$$

$$\frac{b^2}{n+1}\sum_{i=1}^{n} i + c^2\left(n-\frac{1}{n+1}\right)\sum_{i=1}^{n} i - \sum_{i=1}^{n} {d_i}^2 = a^2\left(\frac{1}{n+1}\sum_{i=1}^{n} i - \frac{1}{(n+1)^2}\sum_{i=1}^{n} i^2\right). \qquad (11)$$

From the sum of powers (WEISSTEIN, 2020), we know that

$$\sum_{i=1}^{n} i = \frac{n(n+1)}{2}, \qquad (12)$$

$$\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}. \qquad (13)$$

Thus, replacing (12) and (13) in (11), we obtain

$$\sum_{i=1}^{n} {d_i}^2 = \frac{n}{2}b^2 + \frac{n}{2}c^2 - a^2\left[\frac{n}{2} - \frac{n(2n+1)}{6(n+1)}\right],$$

$$\sum_{i=1}^{n} {d_i}^2 = \frac{n}{2}(b^2+c^2) - a^2\left[\frac{n}{2} - \frac{n(2n+1)}{6(n+1)}\right]. \qquad (14)$$

Using the Pythagorean theorem in (14) since the triangle ABC has a right angle at A, we conclude that

$$\sum_{i=1}^{n} {d_i}^2 = \frac{n}{2}a^2 - a^2\left[\frac{n}{2} - \frac{n(2n+1)}{6(n+1)}\right],$$

$$\sum_{i=1}^{n} {d_i}^2 = \frac{n(2n+1)}{6(n+1)}a^2.$$

$\square$

Equation (9) can be explored in GeoGebra. We build an animation for $n = 1,2,3$, and we make it available at

https://www.GeoGebra.org/m/tdd2hs8v.

3. Section of an internal angle of a triangle

Problem 3 (Challenging problem 10-6, page 46) Prove that in any triangle the square of the measure of the internal bisector of any angle is equal to the product of the measures of the sides forming the bisected angle decreased by the product of the measures of the segments of the side to which this bisector is drawn.

Posamentier and Salkind (1996) propose two strategies to solve challenging problem 3: use Stewart's theorem (page 45) and, subsequently, the internal bisector theorem (theorem 47, page 242); or use properties of the inscribed quadrilateral (theorems 36a and 37, page 242) to establish similar triangles and, soon after, the intersecting chords theorem (theorem 52, page 243). The second strategy minimizes algebraic work in the generalization of challenging problem 3.

Solution 3 Considers: the triangle ABC, with sides $AB = c$, $AC = b$, and $BC = a$; the segment $\overline{AD}$ with $D \in \overline{BC}$ of measure d_1, which divides the angle $B\hat{A}C$ in two congruent angles of measure α, and divides the side $\overline{BC}$ in the segments $BD = \kappa_1$ and $DC = \kappa_2$; the point A_1, belonging to the extension of the segment $\overline{AD}$ and the circumference that circumscribes the triangle ABC, as shown in Figure 5.

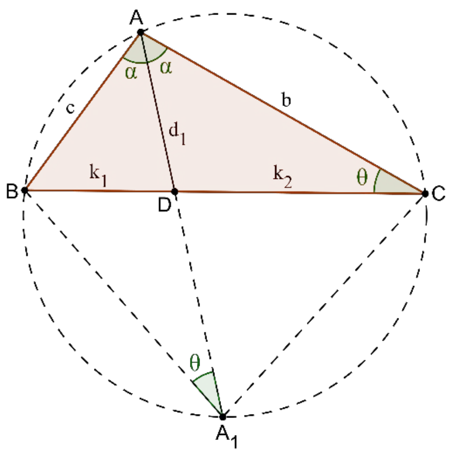


FIGURE 5: Challenging problem 3: the bisector of the internal angle $\hat{A}$ of the triangle ABC
SOURCE: Authors with GeoGebra

Due to the properties of the inscribed quadrilateral, we have $A\widehat{A_1}B \equiv A\hat{C}B = \theta$. So, by the case AA (angle-angle), the triangles ABA_1 and ADC are similar. Thus

$$\Delta ABA_1 \sim \Delta ADC \Longrightarrow \frac{AC}{AD} = \frac{AA_1}{AB},$$

$$AD(AD + DA_1) = AC.AB \Longrightarrow AD^2 = AC.AB - AD.DA_1. \qquad (15)$$

Applying the intersection chords theorem, we obtain

$$AD.DA_1 = BD.DC. \tag{16}$$

Replacing (16) in (15), we conclude that

$$AD^2 = AC.AB - BD.DC,$$

$$d_1{}^2 = bc - \kappa_1\kappa_2.$$

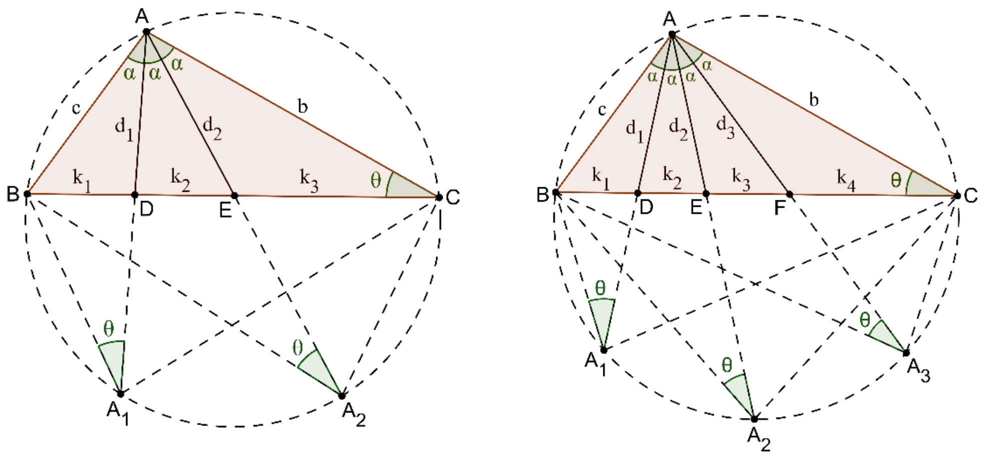
⁂

To generalize challenging problem 3 we should detect patterns (SRIRAMAN, 2004). Let us begin by analyzing two particular cases: the square of the measure of the segments that divide an internal angle of a triangle in three and four congruent angles.

For the cevians d_1 and d_2 that divide the angle in three congruent angles, as illustrated in Figure 6(a), we conclude, using the same strategy, that

$$d_1{}^2 = bc\frac{d_1}{d_2} - \kappa_1(\kappa_2 + \kappa_3), \tag{17}$$

$$d_2{}^2 = bc\frac{d_2}{d_1} - \kappa_3(\kappa_1 + \kappa_2). \tag{18}$$



(a) (b)

FIGURE 6: Generalization of challenging problem 3: (a) cevians that divide angle $\hat{A}$ in three congruent angles; (b) cevians that divide angle $\hat{A}$ in four congruent angles
SOURCE: Authors with GeoGebra

Now, for the cevians d_1, d_2 and d_3 that divide the angle in four congruent angles, as shown in Figure 6(b), we find that

$$d_1{}^2 = bc\frac{d_1}{d_3} - \kappa_1(\kappa_2 + \kappa_3 + \kappa_4), \tag{19}$$

$$d_2{}^2 = bc\frac{d_2}{d_2} - (\kappa_1 + \kappa_2)(\kappa_3 + \kappa_4), \tag{20}$$

$$d_3{}^2 = bc\frac{d_3}{d_1} - \kappa_4(\kappa_1 + \kappa_2 + \kappa_3). \tag{21}$$

After/before proposing the generalization of Problem 3, we can observe the partition of an internal angle of a triangle in GeoGebra. We build an animation for two cases: bisection and trisection of the angle $B\hat{A}C$, and we make it available at

https://www.GeoGebra.org/m/fhhgsfja.

Equations (17)-(21) show that, in the generalization of challenging problem 3, the square of the measure of the cevians d_i cannot be expressed, except in particular cases as in equation (20), depending only on the measurements of the sides that determine the sectioned angle and of the segments determined by the cevians on the side opposite to the sectioned angle. When calculating the measures d_i, $i = 1,2,\dots,n$, it is necessary to solve a system of non-linear equations for $i \geq 2$.

Theorem 3 (Generalization of Problem 3) If d_i, $i = 1,2,\dots,n$, are the measures of the cevians that divide an internal angle of a triangle in congruent $n + 1$ angles, then

$$d_i{}^2 = bc\frac{d_i}{d_{n+1-i}} - (\kappa_1 + \kappa_2 + \cdots + \kappa_i)(\kappa_{i+1} + \dots + \kappa_{n+1}),$$

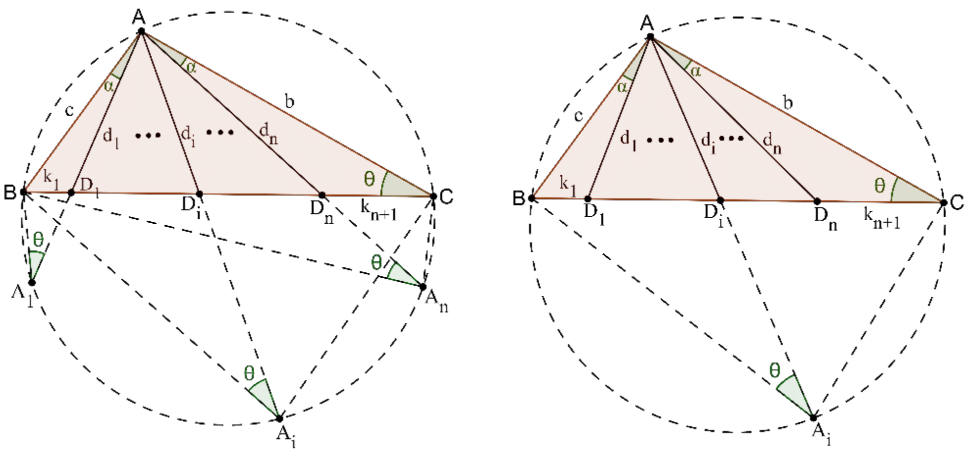
where b and c are the measurements of the sides that determine the sectioned angle, and $\kappa_1, \kappa_2, \dots, \kappa_{n+1}$ are the measurements of the segments determined by the cevians on the side opposite to the sectioned angle.

Proof Consider triangle ABC, with sides $AB = c$, $AC = b$, and $BC = a$; the cevians $\overline{AD_\iota}, i = 1,2,\dots,n$, of measure d_i which divide the angle $B\hat{A}C$ in $n + 1$ congruent angles of measure α, and divide the segment $\overline{BC}$ in $n + 1$ segments of measure $BD_1 = \kappa_1, D_1D_2 = \kappa_2, \dots, D_{n-1}D_n = \kappa_n, D_nC = \kappa_{n+1}$; points $A_1, A_2, \dots, A_n$, respectively belonging to the extensions of the cevians $\overline{AD_1}, \overline{AD_2}, \dots, \overline{AD_n}$ and the circumference that circumscribes triangle ABC, as illustrated in Figure 7(a).

For any $i, i = 1,2,\dots,n$, we have by the property of the inscribed quadrilateral that $A\widehat{A_\iota}B \equiv A\hat{C}B = \theta$, as shown in Figure 7(b). Therefore, by the case AA (angle-angle), the triangles ABA_i and $AD_{n+1-i}C$, , $i = 1,2,\dots,n$, are similar regardless of the possible positions of A_i and D_{n+1-i}, as shown in Figure 8[6]. Thus

$$\Delta ABA_i \sim \Delta AD_{n+1-i}C \Longrightarrow \frac{AC}{AD_{n+1-i}} = \frac{AA_i}{AB}, \qquad i = 1,2,\dots,n,$$

[6] In Figure 8, Case II is obtained only if $i = \frac{n+1}{2}$ and n is odd.



(a) (b)

FIGURE 7: Generalization of challenging problem 3: (a) cevians of measure $d_i, i = 1,2, \dots, n$, that divide angle $B\hat{A}C$ in $n+1$ congruent angles; (b) congruent angles $A\hat{A_i}B$ and $A\hat{C}B$

SOURCE: Authors with GeoGebra

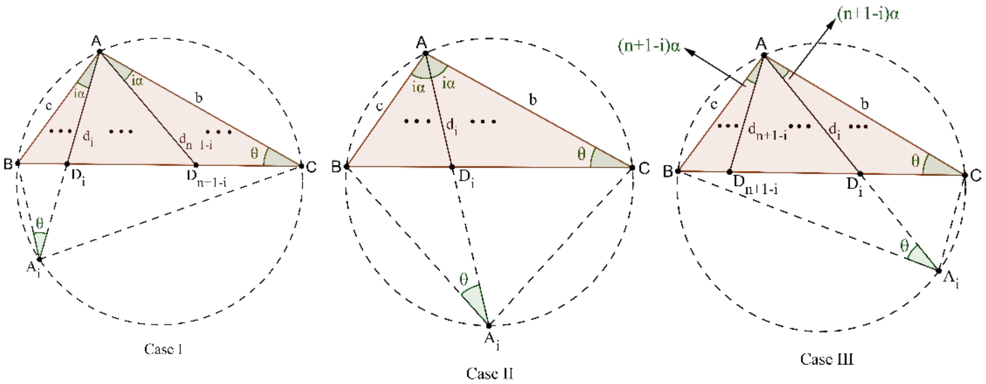


FIGURE 8: Possible triangles ABA_i and $AD_{n+1-i}C$ in the generalization of challenging problem 3

SOURCE: Authors with GeoGebra

$$AD_{n+1-i}(AD_i + D_iA_i) = AC.AB. \tag{22}$$

Using the intersecting chords theorem, we obtain

$$AD_i.D_iA_i = BD_i.D_iC, \qquad i = 1,2, \dots, n,$$

$$D_iA_i = \frac{BD_i.D_iC}{AD_i}. \tag{23}$$

Replacing (23) in (22), we conclude that

$$AD_i{}^2 = AC.AB.\frac{AD_i}{AD_{n+1-i}} - BD_i.D_iC, \qquad i = 1,2,\dots,n,$$

$$d_i{}^2 = bc\frac{d_i}{d_{n+1-i}} - (\kappa_1 + \kappa_2 + \cdots + \kappa_i)(\kappa_{i+1} + \dots + \kappa_{n+1}).$$

□

In the generalization of challenging problem 3, we found that, given the measures of the sides $AB = c$ and $AC = b$, which determine the sectioned angle of the triangle ABC, and the measures $\kappa_1, \kappa_2, \dots, \kappa_n, \kappa_{n+1}$ of the segments determined by the cevians on the opposite side $\overline{BC}$ to the sectioned angle, it is possible to calculate the measurements d_i of cevians by solving the following system of non-linear equations:

$$d_i{}^2 = bc\frac{d_i}{d_{n+1-i}} - (\kappa_1 + \kappa_2 + \cdots + \kappa_i)(\kappa_{i+1} + \dots + \kappa_{n+1}), \quad i = 1,2,\dots,n. \qquad (24)$$

The system of non-linear equations (24) is a decoupled system with two equations, because if is n odd, we get

$$d_{\frac{n+1}{2}}{}^2 = bc - \left(\kappa_1 + \kappa_2 + \cdots + \kappa_{\frac{n+1}{2}}\right)\left(\kappa_{\frac{n+1}{2}+1} + \dots + \kappa_{n+1}\right).$$

Additionally, then we have an even number of equations. Thus, to solve the system (24), it is sufficient to solve a system of non-linear equations with two equations and two unknowns, i.e. fixing $i, i = 1,2,\dots,n$, solve the following non-linear equation system:

$$\begin{cases} d_i{}^2 = bc\dfrac{d_i}{d_{n+1-i}} - (\kappa_1 + \kappa_2 + \cdots + \kappa_i)(\kappa_{i+1} + \dots + \kappa_{n+1}) \\ d_{n+1-i}{}^2 = bc\dfrac{d_{n+1-i}}{d_i} - (\kappa_1 + \kappa_2 + \cdots + \kappa_{n+1-i})(\kappa_{n+1-i+1} + \dots + \kappa_{n+1}) \end{cases}. \quad (25)$$

In Proposition 1, we prove that the system of non-linear equations (25) has a solution.

Proposition 1 The system of non-linear equations (25) has a solution.

Proof Let $x = d_i$ and $y = d_{n+1-i}$. Replacing x and y in the system (25) we obtain

$$\begin{cases} x^2 = bc\dfrac{x}{y} - (\kappa_1 + \kappa_2 + \cdots + \kappa_i)(\kappa_{i+1} + \dots + \kappa_{n+1}) \\ y^2 = bc\dfrac{y}{x} - (\kappa_1 + \kappa_2 + \cdots + \kappa_{n+1-i})(\kappa_{n+1-i+1} + \dots + \kappa_{n+1}) \end{cases}. \qquad (26)$$

Considering

$\alpha = \kappa_1 + \kappa_2 + \cdots + \kappa_i,\ \beta = \kappa_{i+1} + \ldots + \kappa_{n+1},$

$\gamma = \kappa_1 + \kappa_2 + \cdots + \kappa_{n+1-i}, \rho = \kappa_{n+1-i+1} + \ldots + \kappa_{n+1},$

we can rewrite system (26) as

$$\begin{cases} x^2 &= bc\dfrac{x}{y} - \alpha\beta \\ y^2 &= bc\dfrac{y}{x} - \gamma\rho \end{cases}. \tag{27}$$

Multiplying the first equation of the system (27) by y^2 and the second equation by $-x^2$, we obtain

$$\begin{cases} x^2y^2 &= bcxy - \alpha\beta y^2 \\ -x^2y^2 &= -bcxy - \gamma\rho x^2 \end{cases}. \tag{28}$$

The addition of the two equations of the system (28) results in $0 = -\alpha\beta y^2 + \gamma\rho x^2$.

Therefore, $y^2 = \frac{\gamma\rho}{\alpha\beta}x^2 \Longrightarrow y = \pm\sqrt{\frac{\gamma\rho}{\alpha\beta}}x.$

Since x and y are positive numbers, we conclude that

$$y = \sqrt{\frac{\gamma\rho}{\alpha\beta}}x. \tag{29}$$

Finally, replacing (29) in the second equation of the system (27), we obtain

$$x^2 = bc\sqrt{\frac{\alpha\beta}{\gamma\rho}} - \alpha\beta.$$

Thus showing that the system (25) has a solution since $bc > \sqrt{\alpha\beta\gamma\rho}$.

□

Concluding remarks

In this work we present the generalization of three geometrical challenging problems proposed in Challenging problems in geometry by Posamentier and Salkind (1996). In generalization procedures, we establish a link between some areas of mathematics, such as geometry and arithmetic, we involve some theorems of plane geometry, and we use the software GeoGebra to construct figures and animation.

It is important to emphasize that, as challenging problem 3 shows, the generalization process can lead to changes in the configuration of the expected result, that is, the theorem thesis proposed in cited case.

Following Hashemi et al. (2013), we hope that this work will motivate the main agents involved in the mathematics teaching-learning process, i.e. students and teachers, to establish generalization processes in the classroom in the analysis/investigation of mathematical properties, particularly in geometry, thus contributing to consolidation and expansion of mathematical knowledge.

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