Stability of Equilibrium Solutions of a Nonlinear Reaction-Diffusion Equation

Estabilidad de Soluciones de Equilibrio de una Ecuación de Reacción-Difusión no Lineal

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Abstract. In the present work, it is analyzed existence and stability of equilibrium solutions of the following nonlinear reaction-diffusion equation:

$$u_t = \alpha u_{xx} + wu + k\ln(u^2)u.$$

Explicit formulas for a family of equilibrium solutions to the former equation which decay to zero at infinity are provided. The instability of those solutions is obtained by detailed spectral analysis of the linear operator which approximates the solutions of the equation around the equilibrium solutions. A result about the instability of any non-trivial equilibrium solution of the equation is also established.

Keywords: Reaction-Diffusion equation, equilibrium solutions, stability.

Resumen. En el presente trabajo, se analiza la existencia y estabilidad de soluciones de equilibrio de la siguiente ecuación de reacción-difusión no lineal:

$$u_t = \alpha u_{xx} + wu + k\ln(u^2)u.$$

Se proporcionan fórmulas explícitas para una família de soluciones de equilibrio de la ecuación anterior que decaen a cero en infinito. La inestabilidad de esas soluciones se obtienen mediante el análisis espectral detallado del operador lineal que aproxima las soluciones de la ecuación alrededor de las soluciones de equilibrio. También se establece un resultado sobre la inestabilidad de cualquier solución de equilibrio no trivial de la ecuación.

Palabras claves: Ecuación de reacción-difusión, soluciones de equilibrio, estabilidad.

Mathematics Subject Classification: 35K57, 35B35, 35P99, 35B09.

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1. Introduction

Due to its diverse applications, the general reaction diffusion equation,

$$u_t = \alpha u_{xx} + f(u), \qquad (x,t) \in \mathbb{R} \times \mathbb{R}^+ \tag{1}$$

has received considerable attention from the scientific community in the last decades. Here $u(x,t) \in \mathbb{R}$, α is a positive real parameter and f denotes a real-valued function that, in most cases, satisfies some regularity conditions. If the reaction term f vanishes, then the equation represents a pure diffusion process. The corresponding equation is called Fick second law. The choice $f(u) = u(1-u)$ yields Fisher equation that was originally used to describe the spreading of biological populations [2], the Newell-Whitehead-Segel equation with $f(u) = u(1-u^2)$ to describe Rayleigh-Bénard convection [[12],[13]], the more general Zeldovich equation with $f(u) = u(1-u)(u-b)$ and $0 < b < 1$ that arises in combustion theory [14], and its particular degenerate case with $f(u) = u^2 - u^3$ that is sometimes referred to as the Zeldovich equation as well. From the mathematical point of view, problems like the existence of solutions for the Cauchy problem, the existence of global solutions, the existence of global attractor, the existence of particular solutions and its stability have been widely studied, [[6],[3],[4],[5],[10],[9]]. In particular, the problem of existence and stability of equilibrium solutions of the equation (1) for specific functions f have been also addressed, see for instance the recent work in [10], where the case $f(u) = wu + u^3 + u^5$ was dealt. The latter work was partially generalized in [11].

It is to be recalled that an equilibrium solution of the equation (1) is a solution of the equation that is independent of the variable t. Thus, $u(x,t) = \phi(x)$ is an equilibrium solution of the equation (1), if ϕ satisfies the following second order ordinary differential equation

$$\alpha\phi'' + f(\phi) = 0. \tag{2}$$

Let X be a Hilbert or Banach space where the Cauchy problem

$$\begin{cases} u_t = \alpha u_{xx} + f(u), \\ u(0) = u_0 \in X \end{cases} \tag{3}$$

is well posed. An equilibrium solution $\phi \in X$ is said to be stable in $X = (X, ||\cdot||_X)$, if for all $\epsilon > 0$, there exists $\delta > 0$ such that

$$\text{If} \quad ||u_0 - \phi|| < \delta, \quad \text{then} \quad ||u(t) - \phi|| < \epsilon, \quad \text{for all } t > 0.$$

Here, u denotes the solution of the Cauchy problem in (3) with $u(0) = u_0 \in X$. Otherwise, the equilibrium solution ϕ is said to be unstable.

A classical method to study the stability/instability of an equilibrium solution is based on the analysis of the spectral properties of the following linear operator

$$\mathcal{L}_\phi = \alpha\frac{d^2}{dx^2} + f'(\phi) \tag{4}$$

which is defined on certain Hilbert or Banach space. Roughly speaking, if the spectrum of the operator $\mathcal{L}_\phi$ intercepts the set $\{z \in \mathbb{C} : 0 < \mathrm{Re}(z)\}$, then the equilibrium solution is unstable. On the other hand, if the spectrum of the operator $\mathcal{L}_\phi$ is contained in the set $\{z \in \mathbb{C} : \mathrm{Re}(z) \leq b\}$ for some $b < 0$, then the equilibrium solution is stable. See theorems 5.1.1 and 5.1.3 in [6] for details.

In this work, it is studied the problem of existence and stability of equilibrium solutions of the equation (1) when $f(u) = wu + k\ln(u^2)u$, that is to say, the semilinear parabolic equation

$$u_t = \alpha u_{xx} + wu + k\ln(u^2)u, \tag{5}$$

where $\alpha > 0$, $w \in \mathbb{R}$ and $k > 0$.

Recently, equation (5) has been discussed in [1] when $\alpha = 1$ and $w = 0$, i.e.,

$$u_t = u_{xx} + k\ln(u^2)u. \tag{6}$$

In there, the authors mainly establish the existence of a class of initial conditions such that its associated solutions either grow super exponentially or decay to zero super exponentially, the problem of global well posedness of solutions is also addressed. It is worth to note that by doing some simple transformation, the solutions of equation (5) can be obtained from the solutions of equation (6) and vice versa. In fact, if $u = u(x,t)$ is a solution of the equation (5) then, $h(x,t) = e^{\frac{w}{2k}}u(\sqrt{\alpha}x, t)$ is a solution of the equation (6). Conversely, if $u = u(x,t)$ is a solution of the equation (6) then, $g(x,t) = e^{\frac{-w}{2k}}u(x/\sqrt{\alpha}, t)$ is a solution of the equation (5). Thus, the dynamic of equation (5) is as complicated as the dynamic of the equation (6).

As it has been noticed in [1], some technical difficulties appear due to the logarithmic term in the equation (5). For instance, since the function $g(u) = u\ln(u^2)$, $g(0) = 0$ is not a Lipschitz function around zero, then the problem of local well-posedness in neighborhoods of zero cannot be approached by using classical techniques. In addition, because of the undefined sign of the second term of the formal energy

$$\mathcal{E}[u](t) = \frac{1}{2}\int_{\mathbb{R}} (\alpha(u_x)^2 - wu^2)(x,t)dx + \int_{\mathbb{R}} \frac{k}{2}u^2(1 - \ln(u^2))(x,t)dx$$

associated to the equation (5), then obtaining global existence of solutions of (5) via energy method is a delicate issue.

On the other hand, the dynamic of the equation (5) is quite interesting in comparison with other similar equations. For instance, for $\alpha, k > 0$, $w \in \mathbb{R}$ and $\epsilon > 0$, the equation

$$u_t = \alpha u_{xx} + wu + ku^{1+\epsilon} \tag{7}$$

has positive solutions that blow up in finite time when the initial data satisfy a certain general condition. In contrast, proposition 1.2 in [1] guarantee the existence of global solutions for equation (5) when the initial data satisfy a specific, but general condition. Therefore, it does not matter if ϵ is small enough, the phenomenon of blow up always occurs for equation (7), but not for equation (5). Roughly speaking, the reason for the latter is that

$$\int_2^\infty \frac{1}{u^{1+\epsilon}}\, du < \infty \qquad \text{and} \qquad \int_2^\infty \frac{1}{u\ln(u)}\, du = \infty,$$

see [3] for details. As it was mentioned above, another interesting feature of the dynamic of the equation (5) is that there exists a class of solutions that decay to zero super exponentially and another one that grows super exponentially. Of course, that kind of behavior reveals certain instability of the flow associated to the equation (5). Understanding how exactly this instability happens around the equilibrium solution ϕ with

$$\lim_{|x|\to\infty} \phi(x) = 0$$

is the main goal of this manuscript. It will be also obtained the instability of any nontrivial equilibrium solution of the equation (5).

As mentioned above, solutions of equation (5) share some local similarities with the solutions of other reaction diffusion equations. However, it is not clear whether this equation models globally some specific problem in some area of natural sciences. This is also noticed in [1].

Next, it will be described the principal results of this manuscript. Regarding the existence of equilibrium solutions, it will be proven that the family of functions

$$\phi(x) = e^{\frac{k-w}{2k}} e^{\frac{-k(x+d)^2}{2\alpha}} \tag{8}$$

are solutions of the equation

$$\alpha\phi'' + w\phi + k\ln(\phi^2)\phi = 0. \tag{9}$$

In other words, the family of functions given in (8) are equilibrium solutions of the equation (5) that tend to zero at infinity. As it was noted above, the key point to conclude stability or instability of the equilibrium solutions given in (8) is the spectral analysis of the closure $\overline{\mathcal{L}}$ of the symmetric linear operator $\mathcal{L} : C_0^\infty(\mathbb{R}) \to L^2(\mathbb{R})$ given by

$$\mathcal{L}g = \alpha\frac{d^2}{dx^2}g + 3kg - \frac{k^2}{\alpha}x^2 g, \tag{10}$$

which is obtained from

$$\mathcal{L}_\phi = \alpha\frac{d^2}{dx^2} + w + 2k + k\ln(\phi^2), \tag{11}$$

by replacing the function ϕ given in (8) into the formula (11). Regarding this issue, it will be shown the following result on the spectrum of the self-adjoint operator $\overline{\mathcal{L}}$.

Theorem 1.1. *The spectrum of the self-adjoint operator $\overline{\mathcal{L}}$ is formed by the sequence of eigenvalues $\lambda_n = -2(n-1)k$, $n = 0, 1, 2, \ldots$, that is*

$$\lambda_0 = 2k,\ \lambda_1 = 0,\ \lambda_2 = -2k,\ \lambda_3 = -4k, \cdots, \lambda_n = -2(n-1)k, \cdots \tag{12}$$

each eigenvalue is simple and its corresponding eigenfunction is given by the formula (27).

Since k is a positive real number, then the former theorem implies that operator $\overline{\mathcal{L}}$ has a unique positive eigenvalue $\lambda_0 = 2k$. Then from the theorem 5.1.3 in [6], the following theorem holds.

Theorem 1.2. *The equilibrium solution ϕ given in (8) is unstable. Furthermore, the unstable manifold associated to this equilibrium solution has dimension one.*

Remark 1.3. It is to be noticed that the solutions of the equation in (5) around the equilibrium point ϕ can be approximated, at least formally, by the solutions of the linear equation $v_t = \mathcal{L}v$ when v is close to zero. In fact, by replacing $u(x,t) = \phi(x) + v(x,t)$ into the equation (5), we obtain that v satisfies the partial differential equation

$$v_t = F(\phi) + F'(\phi)v + 2kG(v,\phi) = \mathcal{L}v + 2kG(v,\phi),$$

where $F(u) = \alpha u_{xx} + wu + k\ln(u^2)u$, and the function G is given by

$$G(v,\phi) = \sum_{n=2}^{\infty} \frac{(-1)^n}{n(n-1)} \frac{v^n}{\phi^{n-1}} = (\phi+v)(\ln(1+v/\phi)-1), \quad \text{if} \quad |v(x,t)| \leq \phi(x).$$

Remark 1.4. As pointed out above, to prove the instability of an equilibrium solution ϕ by linearization method, it is enough to show that the linear operator $\mathcal{L}_\phi$ has an spectral value with positive real part. In some cases, this condition can be verified indirectly, neither knowing any explicit formula for the equilibrium solutions nor knowing the spectrum of the operator completely, see [8] for more details. Therefore, the equation discussed in this manuscript is quite particular, an explicit formula to the equilibriums solutions can be calculated (see (8)), and even, the spectrum of the operator $\mathcal{L}$ can be computed completely (see (12)).

In section 2, it will be shown the existence of two different classes of periodic solutions of the equation (9). However, obtaining simple formulas to represent these solutions, or even obtaining explicit calculations of the spectrum of the linear operators associated with these solutions are extremely difficult tasks.

In the following section, it will be verified that periodic equilibrium solutions of the equation (5) do exist. Regarding these solutions, the following result will be proven.

Theorem 1.5. *Any nontrivial equilibrium solution of the equation (5) is unstable.*

This work is divided as follows. In section 2, the properties of the solutions of the equation in (9) will be discussed, in particular, it will be shown how to obtain the solutions given in (8). In section 3, based on two different approaches, a proof of the theorem 1.1 will be presented. In section 4, a proof of theorem 1.5 will be furnished.

2. Equilibrium solutions

In this section, it is discussed some qualitative properties of the equilibrium solution of the equation in (5). That is, solutions of the non-linear second order differential equation

$$\alpha\phi'' + w\phi + k\ln(\phi^2)\phi = 0. \tag{13}$$

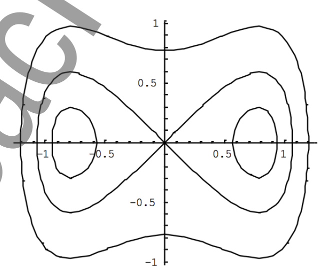
Multiplying the previous equation by ϕ and integrating, it is obtained that ϕ satisfies the following first order differential equation

$$[\phi']^2 + (r-s)\phi^2 + s\phi^2\ln(\phi^2) = c, \tag{14}$$

where $r = w\alpha$, $s = k\alpha$ and c is a constant of integration. Then, the behavior of the solutions of the equation in (13) can be analyzed from the level sets of the Hamiltonian function $\Psi : \mathbb{R}^2 \to \mathbb{R}$ defined by

$$\Psi(\phi, \phi') = [\phi']^2 + (r-s)\phi^2 + s\phi^2\ln(\phi^2).$$

The following figure contains three level sets of the function Ψ for $\alpha = 1$, $k = 1$ and $w = -1$,



Here, the horizontal axis describes values of the variable ϕ and the vertical axis describes values of the variable ϕ'. The previous figure represents three types of solutions of the equation (13), namely,

1. For $c < 0$ fixed, the two closed simple curves that look like ellipses represent positive and negative periodic solutions.

2. For $c = 0$, the curve that looks like the infinity symbol represents two solutions, a positive non-periodic solution that decays to zero at infinity (this solution can be computed explicitly, see the formula (16) below), and a negative non-periodic solution that decays to zero at infinity.

3. For $c > 0$ fixed, the external closed simple curve represents a periodic solution that takes both positive and negative values.

Now, it is deduced from the equation in (14) that, to obtain explicit formulas of the solutions represented in the figure above, it is necessary to solve the integral in the following equation:

$$\int \frac{d\phi}{\sqrt{c + (s-r)\phi^2 - s\phi^2 \ln(\phi^2)}} = x, \tag{15}$$

which is a difficult task for all values of the constant c. However, by assuming that ϕ and ϕ' decay to zero at infinity, then from (14), it is possible to conclude that the constant c must be zero. Then by replacing c by 0 into the equation (15) and integrating, it follows that

$$\phi(x) = e^{\frac{k-w}{2k}} e^{\frac{-k(x+d)^2}{2\alpha}} \tag{16}$$

are equilibrium solutions of the equation (5) that tend to zero as x tends to infinity. The constant d appearing in (16) is a constant of integration.

Remark 2.1. Other solutions of the equation (5) that can be calculated without technical difficulties are solutions that only depend on the variable t. In fact, by replacing $u(x,t) = g(t)$ in (5), it is obtained that g satisfies the following first order ordinary differential equation $g' = wg + k\ln(g^2)g$, whose general solution is given by $g(t) = e^{-\frac{\omega}{2k}} e^{se^{2kt}}$. Similarly, by assuming that $u(x,t) = h(t)\phi(x)$, is a solution to the equation given in (5) where ϕ is an equilibrium solution, then it is easy to check that h satisfies the following first order ordinary differential equation $h'(t) = 2k\ln(h(t))h(t)$, whose general solution is given by $h(t) = e^{se^{2kt}}$. Then, another explicit solution of the equation (5) is given by

$$u(x,t) = e^{se^{2kt}}\phi(x) = e^{se^{2kt}} e^{\frac{k-w}{2k}} e^{\frac{-k(x+d)^2}{2\alpha}}. \tag{17}$$

These solutions will provide us another way to prove the instability of the equilibrium solution ϕ given in (16).

3. Spectral properties of $\mathcal{L}$

In this section, by using two different approaches, the spectral properties of the linear operator $\mathcal{L}$ are established. By defining $p = \frac{k}{\alpha} > 0$ and

$$H(p) = -\frac{d^2}{dx^2} + p^2x^2, \tag{18}$$

then, $\mathcal{L}$ can be rewritten in terms of the operator $H(p)$ as follows

$$\mathcal{L} = -\alpha H(p) + 3k, \tag{19}$$

so from (19), it follows that

$$\sigma\left(\mathcal{L}\right) = -\alpha\sigma\left(H(p)\right) + 3k, \tag{20}$$

where $\sigma\left(\mathcal{L}\right)$, $\sigma\left(H(p)\right)$ denote the spectrum of the operator $\mathcal{L}$ and $H(p)$ respectively. Now, the spectral properties of the operator $H := H(p)$ are dealt. First of all, it is to be noticed that the operator H is an essentially self-adjoint operator from $D(H) = C_0^\infty(\mathbb{R})$ to $L^2(\mathbb{R})$, that is, the closure of the symmetric operator H is self-adjoint. As a consequence of this

$$\sigma(H) \subseteq \mathbb{R}. \tag{21}$$

Moreover, the spectrum of H is purely discrete, all of the eigenvalues are positive, and infinity is the only possible accumulation point of eigenvalues, namely,

$$\lambda_0 < \lambda_1 \leq \lambda_2 \leq \lambda_3 \leq \lambda_4 \leq \cdots$$

see the theorem 10.7 and section 11.2 in [7] for details. Next, we proceed to calculate the eigenvalues of the operator $\mathcal{L}$ explicitly.

3.1. Algebraic approach

In this subsection we show that $\lambda_n = (2n-1)p \in \sigma(H)$, for every $n > 0$. In fact, considering I the identity operator, A and A^* the operators given by

$$A := \frac{d}{dx} + px, \qquad A^* := -\frac{d}{dx} + px, \tag{22}$$

the following table

[,]	I	A	A*	H
I	0	0	0	0
A	0	0	2pI	2pA
A*	0	-2pI	0	-2pA*
H	0	-2pA	2pA*	0

defines a Lie algebra of dimension four over the field $\mathbb{R}$. Here, the Lie bracket $[C, D]$ of any linear operators C and D is given by the usual commutator

$$[C, D] = C \circ D - D \circ C.$$

For instance the Lie bracket of the operators A and A^* is computed as follows

$$\begin{aligned}
[A, A^*] &= \left(\frac{d}{dx} + px\right) \circ \left(-\frac{d}{dx} + px\right) - \left(-\frac{d}{dx} + px\right) \circ \left(\frac{d}{dx} + px\right) \\
&= -\frac{d^2}{dx^2} + p + px\frac{d}{dx} + p^2x^2 - \left(-\frac{d^2}{dx^2} - p + px\frac{d}{dx} + p^2x^2\right) \\
&= 2pI.
\end{aligned}$$

Now, we claim that if λ is an eigenvalue of H, namely $Hg = \lambda g$, then $\lambda + 2p$ is an eigenvalue of H with eigenfunction A^*g. In fact,

$$\begin{aligned} HA^*g &= [H, A^*]g + A^*Hg \\ &= 2pA^*g + A^*\lambda g \\ &= (\lambda + 2p)A^*g \end{aligned} \tag{23}$$

Since $\lambda = p$ is an eigenvalue of H with eigenfunction $g = e^{-p\frac{x^2}{2}}$, then we obtain that the sequence

$$\lambda_1 = p, \lambda_2 = 3p, \lambda_3 = 5p, \cdots, \lambda_n = (2n-1)p, \cdots$$

are eigenvalures of the operator H, with

$$v_1 = g, v_2 = A^*g, v_3 = A^*A^*g, \cdots, v_n = A^* \cdots A^*g, \cdots$$

as its associated sequence of eigenfunctions. It is worth to note that the algebraic method described above does not let clear if there is any other eigenvalue in the spectrum of the operator H, it does not say anything about the multiplicity of each eigenvalue. Those important issues will be approached in the next subsection by applying a classical method for solving ordinary differential equations.

3.2. Analytic approach

Now, the eigenvalues of the operator H are computed by analyzing the power series of the solutions of the differential equation (24) below. Recall that an eigenvalue of the operator H is a complex number λ such that there exists a non-trivial solution y of the ordinary differential equation

$$-y'' + p^2x^2y = \lambda y, \tag{24}$$

that satisfies

$$\lim_{x \to |\infty|} y(x) = 0.$$

Regarding the solutions of the equation (24), we have the following result,

Lemma 3.1. *The solutions of the ordinary differential equation (24) satisfy the following properties,*

1. ***Power series solutions.*** *The general solution of the equation (24) can be written as $y = y^0_\lambda + y^1_\lambda$, where the solutions y^0_λ, y^1_λ are real analytic functions given by*

$$y^0_\lambda(x) = e^{-\frac{px^2}{2}} \sum_{i=0}^{\infty} c_{2i}(\lambda)x^{2i}, \qquad y^1_\lambda(x) = e^{-\frac{px^2}{2}} \sum_{i=0}^{\infty} c_{2i+1}(\lambda)x^{2i+1},$$

and the coefficients $c_{2i}(\lambda) = c_{2i}$ and $c_{2i+1}(\lambda) = c_{2i+1}$ are given by

$$c_{2i} = \frac{(p-\lambda)(5p-\lambda)(9p-\lambda)\cdots((4i-3)p-\lambda)}{(2i)!}c_0, \tag{25}$$

$$c_{2i+1} = \frac{(3p-\lambda)(7p-\lambda)(11p-\lambda)\cdots((4i-1)p-\lambda)}{(2i+1)!}c_1, \tag{26}$$

for $i = 1, 2, \cdots$.

2. ***Bounded solutions.*** *If $\lambda = (2n+1)p$, $n = 0, 1, 2, \cdots$, then*

$$v_n(x) := \begin{cases} y^0_\lambda(x) = H_n(x)e^{-\frac{px^2}{2}}, & \text{if } n \text{ is even,} \\ y^1_\lambda(x) = H_n(x)e^{-\frac{px^2}{2}}, & \text{if } n \text{ is odd.} \end{cases} \tag{27}$$

where H_n denotes the well known Hermite nth-degree polynomial. In particular, we have that

$$\lim_{|x|\to\infty} y^j_\lambda(x) = 0, \qquad \text{for } j = 0, 1. \tag{28}$$

3. ***Unbounded solutions.*** *If $\lambda \neq 4n+1$, $n = 0, 1, 2, \cdots$, then*

$$\lim_{|x|\to\infty} y^0_\lambda(x) = \infty. \tag{29}$$

If $\lambda \neq 4n-1$, $n = 1, 2, \cdots$, then

$$\lim_{x\to\infty} y^1_\lambda(x) = \infty, \qquad \lim_{x\to-\infty} y^1_\lambda(x) = -\infty. \tag{30}$$

Proof. If $y(x) = e^{-p\frac{x^2}{2}}h(x)$ satisfies (24), then h must satisfy the ordinary differential equation

$$-h'' + 2pxh' = (\lambda - p)h, \tag{31}$$

so, if $h(x) = \sum_{i=0}^{\infty} c_i x^i$ is a solution of (31), then the coefficients c_i must satisfy the following recurrence relation

$$c_{i+2}(\lambda) = \frac{(2i+1)p - \lambda}{(i+2)(i+1)}c_i, \qquad \text{for} \quad i = 0, 1, 2, \cdots. \tag{32}$$

By solving the recurrence relation (32), we obtain the formulas given in (25) and (26). Now, (27) and (28) follow from (32) and the definition of the solutions y^0_λ, y^1_λ given above. To prove the item *3*, we first suppose $\lambda \neq 4m+1$, $m = 0, 1, 2, \cdots$. From the recurrence relation (32), we obtain that

$$c_{2i+2} = \frac{(4i+1)p - \lambda}{(2i+2)(2i+1)}c_{2i}, \qquad i = 0, 1, 2, \cdots. \tag{33}$$

So, taking $c_0 \neq 0$, we deduce that $c_{2i} \neq 0$ for all $i = 1, 2, \cdots$, therefore, if $i \to \infty$ then $\frac{2ic_{2i+2}}{pc_{2i}} \to 2$. Thus, for $1 < \gamma < 2$, there exists $s \in \mathbb{N}$ such that

$$\frac{c_{2i+2}}{c_{2i}} > \frac{\gamma p}{2i} > 0, \qquad \text{for all } i \geq s. \tag{34}$$

In addition, choosing c_0 conveniently, we can consider $c_{2i} > 0$ for all $i \geq s$. Now, from (34), it follows that

$$\frac{c_{2s+2}}{c_{2s}} > \frac{\gamma p}{2s} \; ; \; \frac{c_{2s+4}}{c_{2s+2}} > \frac{\gamma p}{2s+2} \; ; \; \cdots \; ; \; \frac{c_{2s+2n}}{c_{2s+2(n-1)}} > \frac{\gamma p}{2s+2(n-1)}$$

by multiplying the previous $n \geq 1$ inequalities, we obtain the following two inequalities

$$\begin{aligned} \frac{c_{2s+2n}}{c_{2s}} &> \left(\frac{\gamma p}{2}\right)^n \frac{1}{s} \frac{1}{s+1} \cdots \frac{1}{s+n-1} \\ &> \left(\frac{\gamma p}{2}\right)^n \frac{1}{s+1} \frac{1}{s+2} \cdots \frac{1}{s+n}, \end{aligned} \tag{35}$$

for all $n \geq 1$. Now, multiplying (35) by x^{2n+2s} $(x \neq 0)$, we get that

$$c_{2s+2n} x^{2n+2s} > \left(\frac{\gamma p}{2}\right)^n \frac{1}{s+1} \frac{1}{s+2} \cdots \frac{1}{s+n} c_{2s} x^{2n+2s}$$

for all $n \geq 1$, then

$$\begin{aligned} \sum_{n=1}^{\infty} c_{2n+2s} x^{2n+2s} &> \sum_{n=1}^{\infty} \left(\frac{\gamma p}{2}\right)^n \frac{1}{s+1} \frac{1}{s+2} \cdots \frac{1}{s+n} c_{2s} x^{2n+2s} \\ &= c_{2s} s! \sum_{n=1}^{\infty} \left(\frac{\gamma p}{2}\right)^n \frac{1}{(s+n)!} x^{2(n+s)} \\ &= c_{2s} s! \left(\frac{2}{\gamma p}\right)^s \sum_{j=s+1}^{\infty} \left(\frac{\gamma p}{2}\right)^j \frac{1}{j!} x^{2j}. \end{aligned} \tag{36}$$

Hence, by using the power serie representation of the exponential function, it is deduced that

$$\sum_{n=1}^{\infty} c_{2n+2s} x^{2n+2s} > c_{2s} s! \left(\frac{2}{\gamma p}\right)^s \left[e^{\frac{\gamma p x^2}{2}} - \sum_{j=0}^{s} \frac{1}{j!} \left(\frac{\gamma p x^2}{2}\right)^j \right]. \tag{37}$$

The last inequality implies that

$$\sum_{i=0}^{\infty} c_{2i} x^{2i} > c_{2s} s! \left(\frac{2}{\gamma p}\right)^s \left[e^{\frac{\gamma p x^2}{2}} - \sum_{j=0}^{s} \frac{1}{j!} \left(\frac{\gamma p x^2}{2}\right)^j \right] + \sum_{i=0}^{s} c_{2i} x^{2i}.$$

Finally,

$$y_\lambda^0(x) > M e^{\frac{(\gamma-1)px^2}{2}} + \sum_{j=0}^{s} B(j,s) x^{2j} e^{-\frac{px^2}{2}},$$

where $B(j,s) = c_{2j} - M\frac{1}{j!}\left(\frac{\gamma p}{2}\right)^j$ and $M = c_{2s}s!\left(\frac{2}{\gamma p}\right)^s$. Hence, since $\gamma > 1$, then from the previous inequality, it is obtained that $\lim\limits_{|x|\to\infty} y_\lambda^0(x) = \infty$, which proves (29). Similar arguments prove (30). This finishes the proof of our lemma. $\square$

Now, since the operator H is self-adjoint, then the algebraic approach developed in section 3.1 and the lemma 3.1 allow us to conclude that

$$\sigma(H(p)) = \{p, 3p, 5p, \cdots, (2n+1)p, \cdots\},$$

for $n \geq 0$. Hence, from the formula (20) with $p = k/\alpha$, we obtain that

$$\sigma(\mathcal{L}) = \{2k, 0, -2k, -4k, \cdots, -2(n-1)k, \cdots\},$$

wich proves the theorem 1.1.

Remark 3.2. Let ψ be a real valued function of real variable and $\theta > 0$. By defining $U(\theta)\psi(x) = \theta^{1/2}\psi(\theta x)$, then U is a unitary representation of the multiplicative group $(\mathbb{R}^+, \cdot)$. In particular, $U(\theta)^{-1} = U(\theta^{-1})$. Furthermore, for $p > 0$ the operators $U(p^{-1/2})$ and $H(p)$ satisfy the following interesting relation

$$U(p^{-1/2})H(p)U(p^{-1/2})^{-1}\psi = pH(1)\psi.$$

In other words, the operator $H(p)$ is similar to the operator $H(1)$. Therefore,

$$\sigma(H(p)) = p\sigma(H(1)).$$

In addition, if r is an eigenvalue of the operator $H(1)$ with eigenfunction ψ, then rp is an eigenvalue of the operator $H(p)$ with associated eigenfunction $U(p^{1/2})\psi(x) = p^{1/4}\psi(p^{1/2}x)$.

4. Instability of any non trivial equilibrium

In this section, we give a simple proof on the instability of any nontrivial equilibrium solution to the equation (5). Although this proof does not make use of any spectral information of the linear operator $\mathcal{L}$ given in (4), it also does not give any information about the nature of the instability of the equilibrium: for instance, we cannot say anything about the dimension of the unstable manifold nor the dimension of the stable manifold.

If ϕ is an equilibrium solution of the equation given in (5), i.e., ϕ satisfies the following ordinary differential equation

$$\alpha\phi'' + w\phi + k\ln(\phi^2)\phi = 0,$$

then it is not difficult to check that

$$u(x,t) = e^{se^{2kt}}\phi(x), \qquad s \in \mathbb{R} \tag{38}$$

is a solution of the Cauchy problem

$$\begin{cases} u_t = \alpha u_{xx} + wu + k\ln(u^2)u, \\ u(x,0) = e^s\phi(x) \end{cases} \tag{39}$$

In addition,

$$||e^{se^{2kt}}\phi - \phi|| = |e^{se^{2kt}} - 1|||\phi|| = |se^{2kt} + s^2e^{4kt}/2 + \cdots|||\phi|| > se^{2kt}||\phi|| \tag{40}$$

for all $s > 0$. Hence, considering

$$u_n(0) = e^{\frac{1}{n}}\phi, \qquad t_n > \frac{1}{2k}\ln(n/||\phi||), \qquad \text{and} \qquad u_n(t) = e^{\frac{1}{n}e^{2kt}}\phi,$$

being the solution of the Cauchy problem in (39) with initial data $u_n(0) = e^{\frac{1}{n}}\phi \in X$, it follows that

$$\lim_{n\to\infty} ||u_n(0) - \phi|| = 0,$$

and

$$||u_n(t_n) - \phi|| > 1.$$

It implies that any non trivial equilibrium solution of the equation (5) is unstable on any Banach or Hilbert space where the Cauchy problem be well posed. In particular, all the equilibrium solutions described in section 2 are unstable. It proves the theorem 1.5. Finally, it is worth to notice that the solutions given in (38) satisfy the following properties: for $s > 0$ fixed, u grows super exponentially and for $s < 0$ fixed, u decays to zero super exponentially. Those properties are investigated in detail in [1].

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