2-absorbing powerful ideals and related results

Ideales poderosos 2-absorbentes y resultados relacionados

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Abstract. Let R be an integral domain. In this paper, we will introduce the concepts of 2-absorbing powerful (resp. 2-absorbing powerful primary) ideals of R and obtain some related results. Also, we investigate a submodule N of an R-module M such that $Ann_R(N)$ and $(N :_R M)$ are 2-absorbing powerful (resp. 2-absorbing powerful primary) ideals of R.

Keywords: Powerful ideal, 2-absorbing powerful ideal, 2-absorbing powerful submodule, 2-absorbing powerful primary ideal, 2-absorbing powerful primary submodule.

Resumen. Sea R un dominio de integridad. En este artículo introducimos los conceptos de ideales poderosos 2-absorbentes (resp. ideales primarios poderosos 2-absorbentes) de R y obtenemos algunos resultados relacionados. Además, investigamos un submódulo N de un R-módulo M tal que $Ann_R(N)$ y $(N :_R M)$ son ideales poderosos 2-absorbentes (resp. ideales primarios poderosos 2-absorbentes) de R.

Palabras claves: ideal poderoso, ideal poderoso 2-absorbente, submódulo poderoso 2-absorbente, ideal primario poderoso 2-absorbente, submódulo primario poderoso 2-absorbente.

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1. Introduction

Throughout this paper, R will denote an integral domain with quotient field K. Further, $\mathbb{Z}$, $\mathbb{Q}$, and $\mathbb{N}$ will denote respectively the ring of integers, the field of rational numbers, and the set of natural numbers.

The concept of powerful ideals was introduced in [4]. A non-zero ideal I of R is said to be *powerful* if, whenever $xy \in I$ for elements $x, y \in K$, then $x \in R$ or $y \in R$.

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A proper ideal I of R is said to be *strongly prime* if, whenever $xy \in I$ for elements $x, y \in K$, then $x \in I$ or $y \in I$ [8].

The concept of 2-absorbing ideals was introduced in [3]. A proper ideal I of R is a *2-absorbing ideal* of R if whenever $a, b, c \in R$ and $abc \in I$, then $ab \in I$ or $ac \in I$ or $bc \in I$.

A 2-absorbing ideal I of R is said to be a *strongly 2-absorbing ideal* if, whenever $xyz \in I$ for elements $x, y, z \in K$, then we have either $xy \in I$ or $yz \in I$ or $xz \in I$ [2].

The purpose of this paper, is to introduce the concepts of 2-absorbing powerful (resp. 2-absorbing powerful primary) ideals of R and study some their basic properties. Moreover, we introduce and investigate the concepts of 2-absorbing powerful (resp. 2-absorbing copowerful) and 2-absorbing powerful primary (resp. 2-absorbing copowerful primary) submodules of an R-modules M.

2. 2-absorbing powerful ideals and submodules

Definition 2.1. We say that a non-zero ideal I of R is a *2-absorbing powerful ideal* if, whenever $xyz \in I$ for elements $x, y, z \in K$, we have either $xy \in R$ or $yz \in R$ or $xz \in R$.

Proposition 2.2. *If P is a strongly 2-absorbing ideal of R, then P is a 2-absorbing powerful ideal of R.*

Proof. This is clear. □

Question 2.3. *If I is a 2-absorbing powerful ideal of R, is then I a strongly 2-absorbing ideal of R?*

Proposition 2.4. *Let I be a powerful ideal of R. Then I is a 2-absorbing powerful ideal of R.*

Proof. Let $xyz \in I$ for some $x, y, z \in K$. Then by assumption, either $xy \in R$ or $z \in R$. If $xy \in R$, then we are done. If $z \in R$, then $zxyz \in I$. Thus again by assumption, either $zx \in R$ or $yz \in R$ as desired. □

Question 2.5. *If I is a 2-absorbing powerful ideal of R, is then I a powerful ideal of R?*

Theorem 2.6. *Let I be an ideal of R. Then the following statements are equivalent.*

(a) I is a 2-absorbing powerful ideal of R.

(b) For each $x, y \in K$ with $xy \notin R$ we have either $x^{-1}I \subseteq R$ or $y^{-1}I \subseteq R$.

Proof. $(a) \Rightarrow (b)$ Assume on the contrary that $x, y \in K$ with $xy \notin R$ and neither $x^{-1}I \nsubseteq R$ nor $y^{-1}I \nsubseteq R$. Then there exist $a, b \in I$ such that $x^{-1}a \notin R$ and $y^{-1}b \notin R$. Now as I is a 2-absorbing powerful ideal of R, we have $(x)(y)(x^{-1}y^{-1}a) = a \in I$ implies that $y^{-1}a \in R$. In a similar way we have $x^{-1}b \in R$. On the other hand,

$$a + b = (x)(y)(x^{-1}y^{-1}(a + b)) \in I$$

implies that either $xy \in R$ or $x^{-1}(a + b) \in R$ or $y^{-1}(a + b) \in R$. Therefore, either $xy \in R$ or $x^{-1}a \in R$ or $y^{-1}b \in R$, this is a contradiction.

$(b) \Rightarrow (a)$ Let $xyz \in I$ for some $x, y, z \in K$ and $xy \notin R$. Then by part (b), either $x^{-1}I \subseteq R$ or $y^{-1}I \subseteq R$. If $x^{-1}I \subseteq R$, then $yz = yzxx^{-1} = (yzx)x^{-1} \in x^{-1}I \subseteq R$. Similarly, if $y^{-1}I \subseteq R$, then we have $xz \in R$, as needed. □

Example 2.7. Consider an integral domain $\mathbb{Z}$, then $K = \mathbb{Q}$. Let n be a non-zero positive integer number, p_1, p_2, q_1, q_2 are distinct prime numbers such that $p_1, p_2 \not| n$. Then $(p_1/q_1)(p_2/q_2) \notin \mathbb{Z}$, $(q_1/p_1)(n\mathbb{Z}) \nsubseteq \mathbb{Z}$, and $(q_2/p_2)(n\mathbb{Z}) \nsubseteq \mathbb{Z}$ implies that $n\mathbb{Z}$ is not a 2-absorbing powerful ideal of $\mathbb{Z}$ by Theorem 2.7.

Proposition 2.8. *Let I be a 2-absorbing powerful ideal of R and Q be a prime ideal of R which is properly contained in I. Then I/Q is a 2-absorbing powerful ideal of R/Q.*

Proof. Let $\phi : R \to R/Q$ denote the canonical homomorphism. Suppose that $x_1 = \phi(y_1)/\phi(z_1)$ and $x_2 = \phi(y_2)/\phi(z_2)$ are elements of the quotient field of R/Q with $x_1 \notin R/Q$ and $x_2 \notin R/Q$ such that $x_1x_2 \notin R/Q$. Then $(y_1/z_1)(y_2/z_2) \notin R$. Hence if $a \in I$, we have $(z_1/y_1)a \in R$ or $(z_2/y_2)a \in R$ by using Theorem 2.7. We can assume that $(z_1/y_1)a \in R$. It follows that $(\phi(z_1)/\phi(y_1))\phi(a) \in R/Q$. Thus $x_1^{-1}(I/Q) \subseteq R/Q$, as needed. □

Proposition 2.9. *If $0 \neq J \subseteq I$ are ideals of R with I 2-absorbing powerful, then J is also 2-absorbing powerful.*

Proof. This is clear. □

Theorem 2.10. *Let I be a 2-absorbing powerful ideal of R. Then we have the following.*

(a) If J and H are ideals of R, then $JH \subseteq I$ or $I^2 \subseteq J \cup H$.

(b) If J and I are prime ideals of R, then J and I are comparable.

Proof. (a) Suppose J and H are ideals of R such that $JH \nsubseteq I$. Then there exist $a \in J$ and $b \in H$ such that $ab \in JH \setminus I$. Let $x, y \in I$. Then $(xy/ab)(a/x)(b/1) \in I$ implies that either $(a/x)(b/1) \in R$ or $(xy/ab)(a/x) \in R$ or $(xy/ab)(b/1) \in R$. Thus either $x(ab/x) \in xR \subseteq I$ or $b(y/b) \in bR \subseteq H$ or $a(xy/a) \in aR \subseteq J$. Hence, either $ab \in I$ or $y \in H$ or $xy \in J$. Since $ab \notin I$, we have either $y \in H$ or $xy \in J$. Therefore, $xy \in J \cup H$. This implies that $I^2 \subseteq J \cup H$, as desired.

(b) The result follows from the fact that $J^2 \subseteq I$ or $I^2 \subseteq J$ by part (a). □

Corollary 2.11. *Let m be a maximal 2-absorbing powerful ideal of R. Then R is a local ring with maximal ideal m.*

Proof. It follows from Theorem 2.11 (b). □

Example 2.12. If K is a field, then the maximal ideal (X^2, X^3) in $K[[X^2, X^3]]$ the ring of formal power series in the indeterminates X^2 and X^3 over K is a 2-absorbing powerful ideal that is not strongly prime by [6, Chap. 26, Example 2.1].

Proposition 2.13. *Let I be a 2-absorbing powerful ideal of R. If $x, y, z \in K$ and $xyz \in Rad(I)$, then there exists a positive integer m such that either $x^m y^m \in I$ or $x^m z^m \in I$ or $y^m z^m \in I$. In particular, if I is a 2-absorbing powerful ideal of R, then $Rad(I)$ is a 2-absorbing ideal of R.*

Proof. $xyz \in Rad(I)$ implies that $(xyz)^n \in I$ for some positive integer n. Thus

$$(x^{3n}/y^n z^n)(y^{3n}/x^n z^n)(z^{3n}/x^n y^n) = (xyz)^n \in I.$$

Now since I is a 2-absorbing powerful ideal of R, either $(x^{3n}/y^n z^n)(y^{3n}/x^n z^n) \in R$ or $(x^{3n}/y^n z^n)(z^{3n}/x^n y^n) \in R$ or $(y^{3n}/x^n z^n)(z^{3n}/x^n y^n) \in R$. Thus either $x^{2n}y^{2n}z^{2n}(x^{2n}y^{2n}/z^{2n}) \in x^{2n}y^{2n}z^{2n}R \subseteq I$ or $x^{2n}y^{2n}z^{2n}(x^{2n}z^{2n}/y^{2n}) \in x^{2n}y^{2n}z^{2n}R \subseteq I$ or $x^{2n}y^{2n}z^{2n}(z^{2n}y^{2n}/x^{2n}) \in x^{2n}y^{2n}z^{2n}R \subseteq I$. Therefore, either $x^{4n}y^{4n} \in I$ or $x^{4n}z^{4n} \in I$ or $z^{4n}y^{4n} \in I$ as needed. □

Proposition 2.14. *Let $\{I_\lambda\}_{\lambda \in \Lambda}$ be a chain of 2-absorbing powerful ideals of R. Then $\sum_{\lambda \in \Lambda} I_\lambda$ is a 2-absorbing powerful ideal of R.*

Proof. Suppose that $x, y \in K$ with $xy \notin R$ and we have $x^{-1}\sum_{\lambda \in \Lambda} I_\lambda \not\subseteq R$ and $y^{-1}\sum_{\lambda \in \Lambda} I_\lambda \not\subseteq R$. Then there exist $\alpha, \beta \in \Lambda$ such that $x^{-1}I_\alpha \not\subseteq R$ and $y^{-1}I_\beta \not\subseteq R$. Thus $y^{-1}I_\alpha \subseteq R$ and $x^{-1}I_\beta \subseteq R$. By assumption, $I_\alpha \subseteq I_\beta$ or $I_\beta \subseteq I_\alpha$. This implies that $x^{-1}I_\alpha \subseteq x^{-1}I_\beta \subseteq R$ or $y^{-1}I_\beta \subseteq y^{-1}I_\alpha \subseteq R$. This is a contradiction. □

Recall that a *chained ring* is any ring whose set of ideals is totally ordered by inclusion.

Corollary 2.15. *If R is a chained ring and contains a 2-absorbing powerful ideal, then R contains a unique largest 2-absorbing powerful ideal.*

Proof. This follows from Proposition 2.15. □

An R-module M is said to be a *multiplication module* if for every submodule N of M there exists an ideal I of R such that $N = IM$ [5].

Definition 2.16. We say that a non-zero submodule N of an R-module M is a *2-absorbing powerful* submodule of M if, $(N :_R M)$ is a 2-absorbing powerful ideal of R.

Proposition 2.17. *Let R be a chained ring and M be a faithful finitely generated multiplication R-module. If $\{N_i\}_{i\in I}$ is a family of 2-absorbing powerful submodules of M, then $\sum_{i\in I} N_i$ is a 2-absorbing powerful submodule of M.*

Proof. This follows from Corollary 2.16 and the fact that

$$(\sum_{i\in I}(N_i :_R M)M :_R M) = \sum_{i\in I}(N_i :_R M)$$

by [7, Theorem 3.1]. □

Recall that if K is the field of fractions of R, then an intermediate ring in the extension $R \subseteq K$ is called an *overring* of R.

Proposition 2.18. *Let I be a 2-absorbing powerful ideal of R, and let T be an overring of R. Then IT is a 2-absorbing powerful ideal of T.*

Proof. Let $x, y \in K \setminus T$ and $xy \notin T$. Then $x, y \notin R$ and $xy \notin R$. Thus by Theorem 2.7, either $x^{-1}I \subseteq R$ or $y^{-1}I \subseteq R$. Therefore, either $x^{-1}IT \subseteq T$ or $y^{-1}IT \subseteq T$. Hence IT is a 2-absorbing powerful ideal of T, again by Theorem 2.7. □

Theorem 2.19. *Let I be a 2-absorbing powerful ideal of R and let $T \neq K$ be an overring of R such that $IT \neq T$, then I^2T is a common ideal, and I^3T is 2-absorbing powerful in both rings.*

Proof. Let $x \in T$. If $(x^{-1})^2 \notin R$, then $xI^2 \subseteq xI \subseteq R$ by Theorem 2.7. Now let $(x^{-1})^2 \in R$. If $(x^{-1})^2 \in I$, then $1 = (x^{-1})^2x^2 \in IT$. This implies that $IT = T$, a contradiction. Thus $(x^{-1})^2 \notin I$. It follows that $I^2 \subseteq x^{-1}R \cup x^{-1}R = x^{-1}R$. Hence, again, $I^2x \subseteq R$. Therefore, I^2T is an ideal of R. Since $I^3T \subseteq I$, we have I^3T is a 2-absorbing powerful ideal of R by Proposition 2.10. Now I^3T is a 2-absorbing powerful ideal of T by Proposition 2.19. □

Proposition 2.20. *Suppose that T is an overring of R and that R and T share the non-zero ideal J. If J is 2-absorbing powerful ideal of T, then J^3 is a 2-absorbing powerful ideal of R.*

Proof. Let $x \in K$ and $x^2 \notin R$. If $x^2 \notin T$, then $x^{-1}J \subseteq T$. Thus $x^{-1}J^3 \subseteq J^2T \subseteq R$. Now assume that $x^2 \in T$. Since $x^2 \notin J$, we have $J^2 \subseteq xT$ by Theorem 2.11. Hence $x^{-1}J^3 \subseteq JT = J \subseteq R$, and the proof is complete. □

Proposition 2.21. *Let N be a 2-absorbing powerful submodule of an R-module M. Then we have the following.*

(a) Every submodule H of N such that $(H :_R M) \neq 0$ is a 2-absorbing powerful submodule of M.

(b) If $r \in K$ such that $r^{-1} \in R$ and $((N :_M r) :_R M) \neq 0$, then $(N :_M r)$ is a 2-absorbing powerful submodule of M.

(c) Let $f : M \to \acute{M}$ b a monomorphism of R-modules. Then N is a 2-absorbing powerful submodule of M if and only if $f(N)$ is a 2-absorbing powerful submodule of $f(M)$.

Proof. (a) This follows from Proposition 2.10 and the fact that $(H :_R M) \subseteq (N :_R M)$.

(b) Let $xyz \in ((N :_M r) :_R M)$ for some $x, y, z \in K$. Then $rxyz \in (N :_R M)$. Thus as N is a 2-absorbing powerful submodule, either $rxy \in R$ or $rxz \in R$ or $yz \in R$. Hence either $xy = r^{-1}rxy \in r^{-1}R \subseteq R$ or $xz = r^{-1}rxz \in r^{-1}R \subseteq R$ or $yz \in R$ as needed.

(c) This follows from the fact that $(N :_R M) = (f(N) :_R f(M))$. □

Definition 2.22. We say that a non-zero ideal I of R is a *semi powerful ideal* if, whenever $x^2 \in I$ for element $x \in K$, we have $x \in R$.

Remark 2.23. Clearly every powerful ideal of R is a semi powerful ideal of R. But as we see in the following example the converse is not true in general.

Example 2.24. Consider the integral domain $\mathbb{Z}$. Then $K = \mathbb{Q}$ and $(4/3)(3/2) = 2 \in 2\mathbb{Z}$ implies that $2\mathbb{Z}$ is not a powerful ideal of $\mathbb{Z}$. But $2\mathbb{Z}$ is a semi powerful ideal of $\mathbb{Z}$.

Example 2.25. Let $V = K + M$ be a rank one discrete valuation domain, where K is a field and $M = tV$ is the maximal ideal of V, and let $R = K + M^2$. Then M^2 is not a semi powerful ideal of R since $t^2 \in M^2$ but $t \notin R$.

Proposition 2.26. *(a) If P is a semi powerful and 2-absorbing powerful ideal of R, then P is a powerful ideal of R.*

(b) If P_1 and P_2 are semi powerful ideals of R, then $P_1 \cap P_2$ is a semi powerful ideal of R.

Proof. (a) Let P be a semi powerful and 2-absorbing powerful ideal of R and let $x \in K \setminus R$. Then $x^2 \notin P$. Since P is 2-absorbing powerful ideal $x^{-1}P \subseteq R$ by Theorem 2.7. Hence P is a powerful ideal of R by [4, 1.1].

(b) This is clear. □

Corollary 2.27. *Let P be a prime semi powerful 2-absorbing powerful ideal of R. Then P is a strongly 2-absorbing ideal of R.*

Proof. This follows from Proposition 2.27 and [4, 1.3]. □

Remark 2.28. In view of Proposition 2.27 and Corollary 2.28, if R is root closed, then the answers to questions 2.4 and 2.6 "Yes".

Proposition 2.29. *Let S be a multiplicatively closed subset of R. If I is a 2-absorbing powerful ideal of R such that $S \cap I = \emptyset$, then $S^{-1}I$ is a 2-absorbing powerful ideal of $S^{-1}R$.*

Proof. Assume that $a, b, c \in K$ such that $abc \in S^{-1}I$. Then there exist $s, t \in S$ such that $(sa)(tb)c = stabc \in I$. Since I is a 2-absorbing powerful ideal of R, this implies that either $(sa)c \in R$ or $(tb)c \in R$ or $(sa)(tb) = stab \in R$ for some $n, m \geq 1$. Thus $ac = (sa)c/s \in s^{-1}R$ or $bc = (tb)c/t \in s^{-1}R$ or $ab = (sa)(tb)/st \in s^{-1}R$ as needed. □

Proposition 2.30. *Let N_1, N_2 be two submodules of an R-module M with $(N_1 :_R M)$ and $(N_2 :_R M)$ 2-absorbing powerful ideals of R. Then $N_1 \cap N_2$ is a 2-absorbing powerful submodule of M.*

Proof. Since $(N_1 \cap N_2 :_R M) = (N_1 :_R M) \cap (N_2 :_R M)$, the result follows from Proposition 2.10. □

Proposition 2.31. *Let $\{K_i\}_{i \in I}$ be a chain of 2-absorbing powerful submodules of an R-module M. Then $\cap_{i \in I} K_i$ is a 2-absorbing powerful submodule of M.*

Proof. Clearly, $(\cap_{i \in I} K_i :_R M) \neq 0$ since R is a domain. Let $a, b, c \in K$ and $abc \in (\cap_{i \in I} K_i :_R M) = \cap_{i \in I}(K_i :_R M)$. Assume contrary that $ab \notin R$, $bc \notin R$, and $ac \notin R$. Then $ab \notin \cap_{i \in I}(K_i :_R M)$, $bc \notin \cap_{i \in I}(K_i :_R M)$, and $ac \notin \cap_{i \in I}(K_i :_R M)$. Then there are $m, n, t \in I$ where $ab \notin (K_n :_R M)$, $bc \notin (K_m :_R M)$, and $ac \notin (K_t :_R M)$. Since $\{K_i\}_{i \in I}$ is a chain, we can assume that $K_m \subseteq K_n \subseteq K_t$. Then

$$(K_m :_R M) \subseteq (K_n :_R M) \subseteq (K_t :_R M).$$

As $abc \in (K_m :_R M)$, we have $ab \in R$ or $ac \in R$ or $bc \in R$. In any cases, we have a contradiction. □

Definition 2.32. We say that a 2-absorbing powerful submodule N of an R-module M is a *minimal 2-absorbing powerful submodule* of a submodule H of M, if $H \subseteq N$ and there does not exist a 2-absorbing powerful submodule T of M such that $H \subset T \subset N$.

It should be noted that a minimal 2-absorbing powerful submodule of M means that a minimal 2-absorbing powerful submodule of the submodule 0 of M.

Lemma 2.33. *Let M be an R-module. Then every 2-absorbing powerful submodule of M contains a minimal 2-absorbing powerful submodule of M.*

Proof. This is proved easily by using Zorn's Lemma and Proposition 2.32. □

Theorem 2.34. *Let M be a Noetherian R-module. Then M contains a finite number of minimal 2-absorbing powerful submodules.*

Proof. Suppose that the result is false. Let Σ denote the collection of all proper submodules N of M such that the module M/N has an infinite number of minimal 2-absorbing powerful submodules. Since $0 \in \Sigma$, we have $\Sigma \neq \emptyset$. Therefore Σ has a maximal member T, since M is a Noetherian R-module.

Clearly, T is not a 2-absorbing powerful submodule. Therefore, there exist $a, b, c \in K$ such that $abc(M/T) = 0$ but $ab \notin R$, $ac \notin R$, and $bc \notin R$. Hence, $ab(M/T) \neq 0$, $ac(M/T) \neq 0$, and $bc(M/T) \neq 0$. The maximality of T implies that $M/(T + abM)$, $M/(T + acM)$, and $M/(T + bcM)$ have only finitely many minimal 2-absorbing powerful submodules. Suppose P/T is a minimal 2-absorbing powerful submodule of M/T. So $abcM \subseteq T \subseteq P$, which implies that $abM \subseteq P$ or $acM \subseteq P$ or $bcM \subseteq P$. Thus $P/(T + abM)$ is a minimal 2-absorbing powerful submodule of $M/(T+abM)$ or $P/(T+bcM)$ is a minimal 2-absorbing powerful submodule of $M/(T+bcM)$ or $P/(T+acM)$ is a minimal 2-absorbing powerful submodule of $M/(T+acM)$. Thus, there are only a finite number of possibilities for the submodule P. This is a contradiction. □

Proposition 2.35. *Let N be a submodule of a finitely generated R-module M and S be a multiplicatively closed subset of R. If N is a 2-absorbing powerful submodule and $(N :_R M) \cap S = \emptyset$, then $S^{-1}N$ is a 2-absorbing powerful $S^{-1}R$-submodule of $S^{-1}M$.*

Proof. As M is finitely generated, $(S^{-1}N :_{S^{-1}R} S^{-1}M) = S^{-1}(N :_R M)$ by [9, 9.12]. Now the result follows from Proposition 2.30. □

Definition 2.36. We say that an R-module M is a *2-absorbing copowerful* if, $Ann_R(M)$ is a 2-absorbing powerful ideal of R.

By a 2-absorbing copowerful submodule of a module we mean a submodule which is a 2-absorbing copowerful module.

Proposition 2.37. *Let N_1, N_2 be two 2-absorbing copowerful submodules of an R-module M. Then $N_1 + N_2$ is a 2-absorbing powerful submodule of M.*

Proof. Since $Ann_R(N_1 + N_2) = Ann_R(N_1) \cap Ann_R(N_2)$, the result follows from Proposition 2.10. □

Proposition 2.38. *Let M be an R-module. Then we have the following.*

(a) If N is a 2-absorbing copowerful submodule of M and $r \in K$ such that $r^{-1} \in R$, $rN \subseteq M$, and $Ann_R(rN) \neq 0$, then rN is a 2-absorbing copowerful submodule of M.

(b) Let $f : M \to \acute{M}$ be a monomorphism of R-modules. Then N is a 2-absorbing copowerful submodule of M if and only if $f(N)$ is a 2-absorbing copowerful submodule of $f(M)$.

(c) If N is a 2-absorbing copowerful submodule of M, then every submodule H of M such that $Ann_R(H) \neq 0$ and $N \subseteq H$ is a 2-absorbing copowerful submodule of M.

Proof. (a) Let $xyz \in Ann_R(rN)$ for some $x, y, z \in K$. Then $rxyz \in Ann_R(N)$. Thus as N is a 2-absorbing copowerful submodule, either $rxy \in Ann_R(N)$ or $rxz \in Ann_R(N)$ or $yz \in Ann_R(N)$. Hence either $xy = r^{-1}rxy \in r^{-1}Ann_R(N)$

$\subseteq Ann_R(N)$ or $xz = r^{-1}rxz \in r^{-1}Ann_R(N) \subseteq Ann_R(N)$ or $yz \in Ann_R(N)$ as needed.

(b) This follows from the fact that $Ann_R(N) = Ann_R(f(N))$.

(c) This follows from Proposition 2.10 and the fact that $Ann_R(H) \subseteq Ann_R(N)$. □

Proposition 2.39. *Let N be a finitely generated submodule of an R-module M and S be a multiplicatively closed subset of R. If N is a 2-absorbing copowerful submodule and $Ann_R(N) \cap S = \emptyset$, then $S^{-1}N$ is a 2-absorbing copowerful $S^{-1}R$-submodule of $S^{-1}M$.*

Proof. As N is finitely generated, $Ann_{S^{-1}R}(S^{-1}(N)) = S^{-1}(Ann_R(N))$ by [9, 9.12]. Now the result follows from Proposition 2.30. □

Proposition 2.40. *Let $\{K_i\}_{i\in I}$ be a chain of strongly 2-absorbing submodules of an R-module M. Then $\cup_{i\in I}K_i$ is a 2-absorbing copowerful submodule of M.*

Proof. Clearly, $Ann_R(\cup_{i\in I}K_i) \neq 0$. Let $a, b, c \in K$ and $abc \in Ann_R(\cup_{i\in I}K_i) = \cap_{i\in I}Ann_R(K_i)$. Assume contrary that $ab \notin \cap_{i\in I}Ann_R(K_i)$, $bc \notin \cap_{i\in I} Ann_R(K_i)$ and $ac \notin \cap_{i\in I}Ann_R(K_i)$. Then there are $m, n, t \in I$ where $ab \notin Ann_R(K_n)$, $bc \notin Ann_R(K_m)$, and $ac \notin Ann_R(K_t)$. Since $\{K_i\}_{i\in I}$ is a chain, we can assume that $K_m \subseteq K_n \subseteq K_t$. Then

$$Ann_R(K_t) \subseteq Ann_R(K_n) \subseteq Ann_R(K_m).$$

As $abc \in Ann_R(K_t)$, we have $ab \in Ann_R(K_t)$ or $ac \in Ann_R(K_t)$ or $bc \in Ann_R(K_t)$. In any case, we have a contradiction. □

Definition 2.41. We say that a 2-absorbing copowerful submodule N of an R-module M is a *Maximal 2-absorbing copowerful submodule* of a submodule H of M, if $N \subseteq H$ and there does not exist a 2-absorbing copowerful submodule T of M such that $N \subset T \subset H$.

Lemma 2.42. *Let M be an R-module. Then every 2-absorbing copowerful submodule of M is contained in a maximal 2-absorbing copowerful submodule of M.*

Proof. This is proved easily by using Zorn's Lemma and Proposition 2.41. □

Theorem 2.43. *Let M be an Artinian R-module. Then every non-zero submodule of M has only a finite number of maximal 2-absorbing copowerful submodules.*

Proof. Suppose that the result is false. Let Σ denote the collection of all non-zero submodules N of M such that the module N has an infinite number of maximal 2-absorbing copowerful submodules. Since $M \in \Sigma$, we have $\Sigma \neq \emptyset$. Therefore Σ has a minimal member T, since M is a Artinian R-module. Clearly, T is not a 2-absorbing copowerful submodule. Therefore, there exist $a, b, c \in K$ such that $abc(T) = 0$ but $ab(T) \neq 0$, $ac(T) \neq 0$, and $bc(T) \neq 0$. The minimality of T implies that $T \cap (0 :_M ab)$, $T \cap (0 :_M ac)$, and $T \cap (0 :_M bc)$ have only finitely many maximal 2-absorbing copowerful submodules. Suppose P is a maximal 2-absorbing copowerful submodule of T. So $P \subseteq T \subseteq (0 :_M abc)$, which implies that $P \subseteq (0 :_M ab)$ or $P \subseteq (0 :_M ac)$ or $P \subseteq (0 :_M bc)$. Thus P is a maximal 2-absorbing copowerful submodule of $T \cap (0 :_M ab)$ or P is a maximal 2-absorbing copowerful submodule of $T \cap (0 :_M bc)$ or P is a maximal 2-absorbing copowerful submodule of $T \cap (0 :_M ac)$. Thus, there are only a finite number of possibilities for the submodule T. This is a contradiction. □

3. 2-absorbing powerful primary ideals and submodules

Definition 3.1. We say that an ideal I of R is a 2-absorbing powerful primary whenever $xyz \in I$ for elements $x, y, z \in K$ we have either $xy \in R$ or $(yz)^n \in R$ or $(xz)^m \in R$ for some $n, m \in \mathbb{N}$.

R is said to be *root closed* if, whenever $x \in K$ and $x^n \in R$ for some integer $n \geq 1$, then $x \in R$ [1].

Proposition 3.2. *Let I be a 2-absorbing powerful. Then I is a 2-absorbing powerful primary ideal of R. The converse hold if R is root closed.*

Proof. This is clear. □

Proposition 3.3. *Let I be a 2-absorbing powerful primary ideal of R. Then $\sqrt{I}$ is a 2-absorbing powerful primary ideal of R.*

Proof. Let $a, b, c \in K$ such that $abc \in \sqrt{I}$, $(bc)^n \notin R$, $(ac)^n \notin R$ for each $n \in \mathbb{N}$. Since $abc \in \sqrt{I}$, there exists a positive integer n such that $(abc)^n = a^n b^n b^n \in I$. Since I is 2-absorbing powerful primary and $(bc)^n \notin R$, $(ac)^n \notin R$ we conclude that $ab \in R$ thus $\sqrt{I}$ is a 2-absorbing powerful ideal. □

Theorem 3.4. *Let I be a 2-absorbing powerful primary ideal of R. Then we have the following.*

(a) If J and H are radical ideals of R, then $JH \subseteq I$ or $I^2 \subseteq J \cup H$.

(b) If J and I are prime ideals of R, then J and I are comparable.

Proof. (a) Suppose that J and H are radical ideals of R such that $JH \not\subseteq I$. Then there exist $a \in J$ and $b \in H$ such that $ab \in JH \setminus I$. Let $x, y \in I$. Then $(xy/ab)(a/x)(b/1) \in I$ implies that either $(a/x)(b/1) \in R$ or $((xy/ab)(a/x))^n \in R$ or $((xy/ab)(b/1))^m \in R$ for some $n, m \geq 1$. Thus either $x(ab/x) \in xR \subseteq I$ or $(b(y/b))^n \in b^n I \subseteq b^n R \subseteq H$ or $(a(xy/a))^m \in a^m I \subseteq a^m R \subseteq J$. Hence, either $ab \in I$ or $y^n \in H$ or $(xy)^m \in J$. Since $ab \notin I$, we have either $y \in \sqrt{H} = H$ or $xy \in \sqrt{J} = J$. Therefore, $xy \in J \cup H$. This implies that $I^2 \subseteq J \cup H$, as desired.

(b) The result follows from the fact that $J^2 \subseteq I$ or $I^2 \subseteq J$ by part (a). □

Corollary 3.5. *Let m be a maximal 2-absorbing powerful primary ideal of R. Then R is a local ring with maximal ideal m.*

Proof. It follows from Theorem 3.4. □

Proposition 3.6. *Let S be a multiplicatively closed subset of R. If I is a 2-absorbing powerful primary ideal of R such that $S \cap I = \emptyset$, then $S^{-1}I$ is a 2-absorbing powerful primary ideal of $S^{-1}R$.*

Proof. Assume that $a, b, c \in K$ such that $abc \in S^{-1}I$. Then there exist $s, t \in S$ such that $(sa)(tb)c = stabc \in I$. Since I is a 2-absorbing powerful primary ideal of R, this implies that either $(sa)c \in R$ or $((tb)c)^n \in R$ or $((sa)(tb))^m = (stab)^m \in R$ for some $n, m \geq 1$. Thus $ac = (sa)c/s \in s^{-1}R$ or $(bc)^n = ((tb)c/t)^n \in s^{-1}R$ or $(ab)^m = ((sa)(tb)/st)^m \in s^{-1}R$ as needed. □

Proposition 3.7. *If $0 \neq J \subseteq I$ are ideals of R with I 2-absorbing powerful primary, then J is also 2-absorbing powerful primary.*

Proof. This is clear. □

Definition 3.8. We say that a submodule N of an R-module M is a *2-absorbing powerful primary* if, $(N :_R M)$ is a 2-absorbing powerful primary ideal of R.

Proposition 3.9. *Let N_1, N_2 be two 2-absorbing powerful primary submodules of an R-module M. Then $N_1 \cap N_2$ is a 2-absorbing powerful primary submodule of M.*

Proof. Since $(N_1 \cap N_2 :_R M) = (N_1 :_R M) \cap (N_2 :_R M)$, the result follows from Proposition 3.7. □

Proposition 3.10. *Let N be a submodule of a finitely generated R-module M and S a multiplicatively closed subset of R. If N is a 2-absorbing powerful primary submodule and $(N :_R M) \cap S = \emptyset$, then $S^{-1}N$ is a 2-absorbing powerful primary $S^{-1}R$-submodule of $S^{-1}M$.*

Proof. As M is finitely generated, $(S^{-1}N :_{S^{-1}R} S^{-1}M) = S^{-1}(N :_R M)$ by [9, 9.12]. Now the result follows from Proposition 3.6. □

Proposition 3.11. *Let N be a 2-absorbing powerful primary submodule of an R-module M. Then we have the following.*

(a) Every submodule H of N such that $(H :_R M) \neq 0$ is a 2-absorbing powerful primary submodule of M.

(b) If $r \in K$ such that $r^{-1} \in R$ and $((N :_M r) :_R M) \neq 0$, then $(N :_M r)$ is a 2-absorbing powerful primary submodule of M.

(c) If $f : M \to \acute{M}$ be a monomorphism of R-modules. Then N is a 2-absorbing powerful primary submodule of M if and only if $f(N)$ is a 2-absorbing powerful primary submodule of $f(M)$.

Proof. (a) This follows from Proposition 3.7 and the fact that $(H :_R M) \subseteq (N :_R M)$.

(b) Let $xyz \in ((N :_M r) :_R M)$ for some $x, y, z \in K$. Then $rxyz \in (N :_R M)$. Thus as N is a 2-absorbing powerful primary submodule, either $rxy \in R$ or $(rxz)^n \in R$ or $(yz)^m \in R$ for some $n, m \geq 1$. Hence either $xy = r^{-1}rxy \in r^{-1}R \subseteq R$ or $(xz)^n = (r^{-1}rxz)^n \in r^{-1}R \subseteq R$ or $(yz)^m \in R$, as needed.

(c) This follows from the fact that $(N :_R M) = (f(N) :_R f(M))$. □

Definition 3.12. We say that an R-module M is a *2-absorbing copowerful primary* if, $Ann_R(M)$ is a 2-absorbing powerful primary ideal of R.

By a 2-absorbing copowerful primary submodule of a module we mean a submodule which is a 2-absorbing copowerful primary module.

Proposition 3.13. *Let N_1, N_2 be two 2-absorbing copowerful primary submodules of an R-module M. Then $N_1 + N_2$ is a 2-absorbing powerful primary submodule of M.*

Proof. The proof is similar to that of Proposition 2.38. □

Proposition 3.14. *Let N be a finitely generated submodule of an R-module M and S be a multiplicatively closed subset of R. If N is a 2-absorbing copowerful primary submodule and $Ann_R(N) \cap S = \emptyset$, then $S^{-1}N$ is a 2-absorbing copowerful primary $S^{-1}R$-submodule of $S^{-1}M$.*

Proof. The proof is similar to that of Proposition 3.10. □

Proposition 3.15. *Let M be an R-module. Then we have the following.*

(a) If N is a 2-absorbing copowerful primary submodule of M and $r \in K$ such that $r^{-1} \in R$, $rN \subseteq M$, and $Ann_R(rN) \neq 0$, then rN is a 2-absorbing copowerful primary submodule of M.

(b) Let $f : M \to \acute{M}$ be a monomorphism of R-modules. Then N is a 2-absorbing copowerful primary submodule of M if and only if $f(N)$ is a 2-absorbing copowerful primary submodule of $f(M)$.

(c) If N is a 2-absorbing copowerful primary submodule of M, then every submodule H of M such that $Ann_R(H) \neq 0$ and $N \subseteq H$ is a 2-absorbing copowerful primary submodule of M.

Proof. The proof is similar to that of Proposition 2.39. □

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