A view of symplectic Lie algebras from quadratic Poisson algebras

Una mirada a las álgebras de Lie simplecticas desde las álgebras de Poisson cuadráticas

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Abstract. Using the concept of double extension, Benayadi [2] showed how to construct a new quadratic algebra $(\mathfrak{g}(\mathcal{A}), T)$ given a quadratic algebra $(\mathcal{A}, B)$. With both algebras and an invertible skew-symmetric algebra D over $\mathcal{A}$, he endowed $(\mathcal{A}, B)$ with a simplectic structure through a bilinear form ω, obtaining a simplectic algebra $(\mathfrak{g}(\mathcal{A}), T, \Omega)$. Our purpose in this short communication is to show the construction given by Benayadi and present the complete development of each one of his assertions. We remark that this communication does not have original results and it was made as a result of the undergraduated work titled "Construcción de álgebras de Lie simplécticas desde álgebras de Poisson cuadráticas"[5], which was awarded as the best mathematics undergraduated thesis in the XXVI Contest at Universidad Nacional de Colombia, Sede Bogotá. The work was written by the first author under the direction of the second author.

Keywords: Lie algebra, Poisson algebra, quadratic algebra, symplectic algebra.

Resumen. A partir del concepto de doble extensión, Benayadi [2] mostró cómo construir una nueva álgebra cuadrática $(\mathfrak{g}(\mathcal{A}), T)$, a partir de un álgebra cuadrática dada $(\mathcal{A}, B)$. Con estas dos álgebras y una derivación invertible anti-simétrica D sobre $\mathcal{A}$, él dotó a $(\mathcal{A}, B)$ de una estructura simpléctica a traves de una forma bilineal ω, obteniendo así una álgebra simpléctica $(\mathfrak{g}(\mathcal{A}), T, \Omega)$. Nuestro propósito en esta comunicación corta es mostrar la construcción dada por Benayadi y presentar el desarrollo completo de cada una de sus afirmaciones. Resaltamos que esta comunicación no tiene resultados originales, y además fue obtenida del trabajo de grado de pregrado titulado "Construcción de álgebras de Lie simplécticas desde álgebras de Poisson cuadráticas"[5], el cual resultó ganador del Concurso de Mejores Trabajos de Grado de Pregrado en su versión XXVI, en la Universidad Nacional de Colombia, Sede Bogotá. El trabajo fue realizado por el primer autor bajo la dirección del segundo autor.

Palabras claves: álgebra de Lie, álgebra de Poisson, álgebra cuadrática, álgebra simpléctica.

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1. Introduction

Historically, the importance of Lie algebras and Poisson algebras belongs to the physical field, since they allow us to describe certain behaviors in the universe, which might be related to classical mechanics and even quantum theory. One of the classical examples may be observed in the energy conservation phenomenon. This illustrates the possibility of formulating physical problems in a purely algebraic way.

The beginning of Lie theory lies in Felix Klein's thoughts, for whom the space geometry is determined by its symmetry groups. Therefore, Euclid, Riemann and Grothendieck's notions of space and geometry were taken to the supersymmetry world in physics and, by the hand of this geometry, the Lie algebras have been taken to a more general notion such as the Lie superalgebras[1]. Besides, the bilinear form concept arises when the characterization of semisimple algebras was made, since Killing's bilinear form (also known as Cartan-Killing's), is non-degenerated if and only if the algebra is semisimple. Before, it would be shown that starting from a semisimple Lie algebra, the algebra turns out to be quadratic (behind this, over some structures induce a metric or pseudo-metric according to the properties that are fulfilled). With all this in mind, the main goal is to find bilinear forms that satisfy certain conditions over the algebra, with the purpose of endowing with quadratic or also symplectic structure. Concerning the Poisson algebras, this kind of structures appear in the deformation idea of classical mechanics, quantum mechanics, based in the quantum group notion. Just as Lie algebras, the Poisson algebras are a tool in the solution of physical problems, which are a part of the Hamiltonian approach of classical mechanics, and motivate the use of commutators in quantum mechanics (see [4] for more details). In Example 2.16 we will remember the Poisson brackets in the formulation of the Hamiltonian mechanics.

In this paper we present complete details of the construction of the symplectic algebras from admissible Poisson algebras, following the treatment formulated by Benayadi (also, following the ideas given in [1]). The crucial point is to get a compatibility between the bilinear form already defined and some operators named derivations. Given that the admissible Poisson structure generates two types of algebras $\mathcal{A}^+$ a symmetric commutative algebra and $\mathcal{A}^-$ a Lie algebra, using the double extension definition it is possible to endow a higher dimension algebra with Lie simplectic quadratic structure, an idea that is hidden behind this definition is the T^*-extension notion. It should be pointed out that during the development, in every case and example that is going to be postulated, we will consider finite dimensional algebras over a field $\mathbb{K}$. Besides, given that the absence of symbols and letter makes impossible some definitions, we will use similar notation for some products, as in the case of $\circ$. In these cases we will give their relevant explanation.

[1]Wilhelm Killing and Elie Cartan did pioneering work on this matter. For example in 1894, Cartan classified the simple Lie algebras which have an important role in the representation theory of semisimple Lie algebras.

2. Preliminary results

In this section we remind some facts about Lie algebras and Poisson algebras.

2.1. Lie algebras

Definition 2.1. Let $\mathfrak{g}$ be a vector space. A Lie bracket over $\mathfrak{g}$ is a bilinear map $[-,-] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$, with the following properties:

(i) $[x, y] = -[y, x]$, for $x, y \in \mathfrak{g}$

(ii) $[x, [y, z]] + [y, [z, x]] + [z, [x, y]] = 0$, for $x, y, z \in \mathfrak{g}$ (also called Jacobi identity).

The pair $(\mathfrak{g}, [-,-])$ is called a *Lie algebra*.

One of the first examples of Lie algebras is determined by the cross for vectors in the space. Thus, if $[-,-] : \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}^3$, where $[x, y] = x \times y$ (*cross product*), then the pair $(\mathbb{R}^3, [-,-])$ is a Lie algebra. In fact, let $\{e_1, e_2, e_3\}$ be the canonical basis of $\mathbb{R}^3$, due to the determinant is a $3-$ *lineal* map and besides when we interchange rows, we know that the determinant reverses the sign, we have that $[e_i, e_i] = e_i \times e_i = 0$ y $[e_i, e_j] = -[e_j, e_i]$, so $[-,-]$ satisfied the definition 2.1. As $[e_1, e_2] = e_3, [e_1, e_3] = -e_2, [e_2, e_3] = e_1$, then $[e_1, [e_2, e_3]]+[e_2, [e_3, e_1]]+[e_3, [e_1, e_2]] = [e_1, e_1]+[e_2, e_2]+[e_3, e_3] = 0$. Thus, the item (ii) of the definition 2.1 is satisfied. We conclude that the pair $(\mathbb{R}^3, [-,-])$ is a Lie algebra. Note that the previous development does not depend of the choice of the basis of $\mathbb{R}^3$.

Definition 2.2. A vector space $\mathcal{A}$ with a bilinear map $\cdot : \mathcal{A} \times \mathcal{A} \to \mathcal{A}$ is called an *associative algebra*, if $a.(b.c) = (a.b).c$, for any $a, b, c \in \mathcal{A}$. We define the *commutator* as $[a, b] = a.b - b.a$, with $a, b \in \mathcal{A}$.

Proposition 2.3. *If $\mathcal{A}$ is an associative algebra and the binary map $\cdot : \mathcal{A} \times \mathcal{A} \to \mathcal{A}$ defined before, then $\mathcal{A}_L := (\mathcal{A}, [-,-])$ is a Lie algebra.*

Proof. Let's see that the previous commutator defines a Lie algebra over $\mathcal{A}$ (we consider $x.y := xy$ for each $x, y \in \mathcal{A}$). Let $x, y, z \in \mathcal{A}$, then the first property is achieved since $[x, y] = xy - yx = -(yx - xy) = -[y, x]$. The second property holds since

$$\begin{aligned}
[[x,y],z]+[[y,z],x] + [[z,x],y] &= [xy-yx,z] + [yz-zy,x] + [zx-xz,y] \\
&= (xy-yx)z - z(xy-yx) + (yz-zy)x - x(yz-zy) \\
&\quad + (zx-xz)y - y(zx-xz) \\
&= xyz - yxz - zxy + zyx + yzx - zyx - xyz + xzy + zxy - xzy \\
&\quad - yzx + yxz \\
&= 0.
\end{aligned} \tag{1}$$

Therefore, the couple $\mathcal{A}_L := (\mathcal{A}, [\cdot, \cdot])$ is a Lie algebra. $\square$

Definition 2.4. (i) Let $\mathfrak{g}$ and $\mathfrak{h}$ be two Lie algebras. A linear map $\alpha : \mathfrak{g} \to \mathfrak{h}$ is called a *homomorphism*, if $\alpha([x, y]) = [\alpha(x), \alpha(y)]$, for all $x, y \in \mathfrak{g}$.

(ii) An *isomorphism* of Lie algebras is a homomorphism α for which there is a homomorphism $\beta : \mathfrak{h} \to \mathfrak{g}$ such that $\alpha \circ \beta = \mathrm{id}_{\mathfrak{h}}$.

(iii) A *representation* of a Lie algebra $\mathfrak{g}$ over the vector space V is a homomorphism $\alpha : \mathfrak{g} \to \mathfrak{gl}_n(V)$. We write (α, V) the representation α of $\mathfrak{g}$ over V.

(iv) Let $\mathfrak{g}$ be a Lie algebra and U, V subsets of $\mathfrak{g}$. We write $[U, V] := \mathrm{span}\{[u, v] \mid u \in U, v \in V\}$ for the smallest subspace which contains all brackets $[u, v]$ with $u \in U$ and $v \in V$.

(v) A subspace $\mathfrak{h}$ of $\mathfrak{g}$ is called a *Lie subalgebra*, if $[\mathfrak{h}, \mathfrak{h}] \subseteq \mathfrak{h}$. We write $\mathfrak{h} < \mathfrak{g}$. If $[\mathfrak{g}, \mathfrak{h}] \subseteq \mathfrak{h}$, we call $\mathfrak{h}$ an *ideal* of $\mathfrak{g}$ and we write $\mathfrak{h} \trianglelefteq \mathfrak{g}$.

(vi) A Lie algebra $\mathfrak{g}$ is *abelian* if $[\mathfrak{g}, \mathfrak{g}] = \{0\}$, which means that all $[x, y] = 0$, for all $x, y \in \mathfrak{g}$.

We can see that if $\mathfrak{g}$ is a Lie algebra, its *center* $Z(\mathfrak{g}) := \{x \in \mathfrak{g} \mid [x, y] = 0,\ \forall y \in \mathfrak{g}\}$, is an ideal of $\mathfrak{g}$. Now, for every Lie algebra $\mathfrak{g}$, the subspace $[\mathfrak{g}, \mathfrak{g}]$ is an ideal called *commutator algebra of* $\mathfrak{g}$.

Definition 2.5. Let $(\mathcal{A}, .)$ be an algebra. A *derivation* of $\mathcal{A}$ is a $\mathbb{K}$-linear map $\delta : \mathcal{A} \to \mathcal{A}$ such that $\delta(x.y) = \delta(x).y + x.\delta(y)$. Based on the above facts, if $\mathfrak{g}$ is a Lie algebra, a linear map $\delta : \mathfrak{g} \to \mathfrak{g}$ is called a *derivation* if $\delta([x, y]) = [\delta(x), y] + [x, \delta(y)]$, for all $x, y \in \mathfrak{g}$. We define $\mathrm{der}(\mathfrak{g})$ as the *derivations set* of $\mathfrak{g}$.

Definition 2.6. (i) Let $\mathfrak{g}$ be a Lie algebra and $x \in \mathfrak{g}$. The map $\mathrm{ad} : \mathfrak{g} \to \mathfrak{gl}$, where $\mathrm{ad}_x : \mathfrak{g} \to \mathfrak{g}$, $\mathrm{ad}_x(y) := [x, y]$ is a derivation, and it is called *adjoint representation* of $\mathfrak{g}$. This kind of derivations are called *inner derivations*. The set of adjoint representations of $\mathfrak{g}$ is denoted as $\mathrm{ad}(\mathfrak{g}) := \{\mathrm{ad}_x \in \mathrm{der}(\mathfrak{g}) \mid x \in \mathfrak{g}\}$.

(ii) The map $\pi : \mathfrak{g} \to \mathrm{End}(\mathfrak{g}^*)$, such that every $x \in \mathfrak{g}$, $\pi(x) : \mathfrak{g}^* \to \mathfrak{g}^*$ defined as follow $\pi(x)(f)(y) = f(-\mathrm{ad}_x(y))$, for all $f \in \mathfrak{g}^*,\ y \in \mathfrak{g}$, is called *coadjoint representation* of $\mathfrak{g}$.

In fact, we can show that for $x \in \mathfrak{g}$, ad_x is a derivation. Also we can show that for every Lie algebra $\mathfrak{g}$, $\mathrm{der}(\mathfrak{g}) < \mathfrak{gl}(\mathfrak{g})$, and $\mathrm{ad}(\mathfrak{g}) \trianglelefteq \mathrm{der}(\mathfrak{g})$ is an ideal. In particular, $[D, \mathrm{ad}_x] = \mathrm{ad}_{Dx}, D \in \mathrm{der}(\mathfrak{g}),\ x \in \mathfrak{g}$ and $\ker(\mathrm{ad}) = Z(\mathfrak{g})$.

Proposition 2.7. *If $\mathfrak{g}$ is a Lie algebra and $x, y \in \mathfrak{g}$, then* $\mathrm{ad}_{[x,y]} = [\mathrm{ad}_x, \mathrm{ad}_y]$

Proof. Let $z \in \mathfrak{g}$, the map $\mathrm{ad}_{[x,y]}$ evaluated in z is given by the expressions $\mathrm{ad}_{[x,y]}(z) = [[x, y], z] = -[z, [x, y]] = [y, [z, x]] + [x, [y, z]] = [x, [y, z]] - [y, [x, z]] = \mathrm{ad}_x([y, z]) - \mathrm{ad}_y([x, z]) = \mathrm{ad}_x\mathrm{ad}_y(z) - \mathrm{ad}_y\mathrm{ad}_x(z) = (\mathrm{ad}_x\mathrm{ad}_y - \mathrm{ad}_y\mathrm{ad}_x)(z) = [\mathrm{ad}_x, \mathrm{ad}_y](z)$. □

2.2. Poisson algebras

With the aim of defining Poissson algebras, it is necessary to consider a vector space $\mathcal{A}$ with two different structures of algebra. One of them has a commutative and associative product, and the another product called Lie bracket. We need a notion of compatibility between these two products.

Definition 2.8. Consider a $\mathbb{K}$-vector space $\mathcal{A}$ equipped with two products $\cdot : \mathcal{A} \times \mathcal{A} \to \mathcal{A}$ and $[-,-] : \mathcal{A} \times \mathcal{A} \to \mathcal{A}$, such that

(i) $(\mathcal{A}, \cdot)$ is a commutative associative algebra over $\mathbb{K}$, with unit 1;

(ii) $(\mathcal{A}, [-,-])$ is a Lie algebra over $\mathbb{K}$;

(iii) The two products are compatible, in the sense that

$$[x \cdot y, z] = x \cdot [y, z] + [x, z] \cdot y, \quad (\textit{Leibniz's rule}) \tag{2}$$

for all $x, y, z \in \mathcal{A}$. If (2) is satisfied, the Lie bracket $[-,-]$ is called a Poisson bracket.

In this way, $(\mathcal{A}, \cdot, [\cdot,\cdot])$ is called a *Poisson algebra*.

Definition 2.9. A bilinear map $\delta : \mathcal{A} \times \mathcal{A} \to \mathcal{A}$ is called a *biderivation* of $\mathcal{A}$, if it satisfies the following equalities $\delta(xy, z) = x\delta(y, z) + y\delta(x, z)$, and, $\delta(z, xy) = x\delta(z, y) + y\delta(z, x)$, for all $x, y, z \in \mathcal{A}$.

From the above definitions, we conclude that the Poisson bracket $[-,-]$ is an anticommutative biderivation.

Definition 2.10. Let $(\mathcal{A}_i, \cdot_i, [\cdot,\cdot]_i)$ be two Poisson algebras over $\mathbb{K}$, with $i = 1, 2$. A map $\phi : \mathcal{A}_1 \to \mathcal{A}_2$ which satisfies that for all $x, y \in \mathcal{A}_1$,

(i) $\phi(x \cdot_1 y) = \phi(x) \cdot_2 \phi(y)$,

(ii) $\phi([x, y]_1) = [\phi(x), \phi(y)]_2$,

is called a *homomorphism of Poisson algebras*.

When we talk about a subalgebra or an ideal, we will take into account that it is based on the associative multiplication. With the purpose to make a difference between Poisson subalgebras and ideals, we define the following:

Definition 2.11. (i) Let $\mathcal{A}$ be a Poisson algebra and U, V subsets of $\mathcal{A}$. We write $U \cdot V := \mathrm{span}\{u \cdot v : u \in U, v \in V\}$ for the smallest subspace which contains all products $u \cdot v$ with $u \in U$ and $v \in V$.

(ii) A subspace $\mathcal{B}$ of $\mathcal{A}$ is called *Poisson subalgebra*, if $(\mathcal{B}, \cdot)$ is subalgebra and $(\mathcal{B}, [\cdot,\cdot])$ is Lie subalgebra, that is $\mathcal{B} \cdot \mathcal{B} \subset \mathcal{B}$ and $[\mathcal{B}, \mathcal{B}] \subset \mathcal{B}$.

(iii) A subspace $\mathcal{J}$ of $\mathcal{A}$ is a *Poisson ideal*, if $\mathcal{J} \cdot \mathcal{A} \subset \mathcal{J}$ and $[\mathcal{J}, \mathcal{A}] \subset \mathcal{J}$.

Examples 2.12. (i) Every Lie algebra $(\mathcal{A}, [-,-])$ is a Poisson algebra with respect to the null product: $x \cdot y = 0$. In fact, for $(\mathcal{A}, .)$ an associative algebra, also both products are compatible: $[x.y, z] = [0, z] = 0 = x.[y, z] + y.[x, z]$. Therefore, $(\mathcal{A}, ., [-,-])$ is a Lie algebra.

(ii) Every associative algebra $(\mathcal{A}, .)$ is a Poisson algebra with respect to the null bracket: $[x, y] = 0$. This algebra is called a null Poisson algebra. Due to the Lie bracket is zero for all products, it is easy to check (i) and (ii) through the definition 2.1. Note that both products are compatible: $[x.y, z] = 0 = x.0 + y.0 = x.[y, z] + y.[x, z]$.

We remind that we can endow each associative algebra with structure of Lie algebra, defining a suitable bracket. By the same way, we can endow an associative algebra with structure of Poisson algebra.

Proposition 2.13. *Let $(\mathcal{A}, \cdot)$ be a commutative associative algebra. It is possible to endow $\mathcal{A}$ with structure of Poisson algebra defining the bracket $[x, y] = x.y - y.x$. Then, $(\mathcal{A}, \cdot, [-,-])$ is a Poisson algebra.*

Proof. From Proposition 2.3 we know that $(\mathcal{A}, [-,-])$ is a Lie algebra. Let we observe the compatibility between $\cdot$ and $[-,-]$, considering the product $x \cdot y = xy$. Given that $[ab, c] = (ab)c - c(ab) = a(bc) - a(cb) + (ac)b - (ca)b = a(bc - cb) + (ac - ca)b = a[b, c] + [a, c]b$, we have that $(\mathcal{A}, \cdot, [-,-])$ is a Poisson algebra. $\square$

Based on the previously definition, and defining a new multiplication $\circ$, it is possible to build another example, which is important to the development of the present work because by means of $\circ$, we are going to define the Poisson-admissible algebras.

Example 2.14. Let $(\mathcal{A}, .)$ be an associative algebra with unit $1_{\mathcal{A}}$, defining the products $x \circ y := \frac{1}{2}(xy + yx)$, $[x, y] = xy - yx$, and additionally $[z, x]y = y[z, x]$, for all $x, y, z \in \mathcal{A}$. The triple $(\mathcal{A}, \circ, [-,-])$ results being a Poisson algebra (see [2]). We note that $(\mathcal{A}, \circ)$ is commutative algebra, because the sum is commutative. Also is associative, since $[z, x]y = y[z, x]$, then $(zx - xz)y + y(xz - zx) = 0$, that is $zxy + yxz - yzx - xzy = 0$ (from now on, if the product is associative we omit the parentheses). Since

$$
\begin{aligned}
(x \circ y) \circ z - x \circ (y \circ z) &= \frac{1}{4}(xyz + zxy + yxz + zyx - xyz - yzx - xzy - zyx) \\
&= \frac{1}{4}(zxy + yxz - yzx - xzy) = 0,
\end{aligned}
$$

then $(x \circ y) \circ z = x \circ (y \circ z)$. In other words, the algebra is associative, whose unit is $1_{\mathcal{A}}$, because $x \circ 1_{\mathcal{A}} = \frac{1}{2}(x1_{\mathcal{A}} + 1_{\mathcal{A}}x) = \frac{1}{2}(2x) = x$, for all $x \in \mathcal{A}$.

On the other hand, the pair $(\mathcal{A}, [-,-])$ is a Lie algebra. The explanation is similar to that made in the proposition 2.13. In such a way, we just need to see that both multiplications are compatible:

$$
\begin{aligned}
[x \circ y, z] = &\left[\frac{1}{2}(xy - yx), z\right] = \frac{1}{2}\left([xy, z] + [yx, z]\right) \\
&= \frac{1}{2}\left(x[y, z] + [x, z]y + y[x, z] + [y, z]x\right) \\
&= \frac{1}{2}\left(x[y, z] + [y, z]x\right) + \frac{1}{2}\left(y[x, z] + [x, z]y\right) = x \circ [y, z] + [x, z] \circ y.
\end{aligned}
$$

As a consequence, $(\mathcal{A}, \circ, [-,-])$ is a Poisson algebra.

Now, beginning with two Poisson algebras and defining two new multiplications, it is possible to build a new Poisson algebra:

Example 2.15. If $(\mathcal{A}_1, \cdot_1, [-,-]_1)$ y $(\mathcal{A}_2, \cdot_2, [-,-]_2)$ are two Poisson algebras, then $(\mathcal{A}_1 \otimes \mathcal{A}_2, \cdot_3, [-,-]_3)$ also is a Poisson algebra, if we consider the products:

$$
\begin{aligned}
(a_1 \otimes a_2) \cdot_3 (b_1 \otimes b_2) &:= a_1 \cdot_1 b_1 \otimes a_2 \cdot_2 b_2, \\
[a_1 \otimes a_2, b_1 \otimes b_2]_3 &:= [a_1, b_1]_1 \otimes a_2 \cdot_2 b_2 + a_1 \cdot_1 b_1 \otimes [a_2, b_2]_2.
\end{aligned}
$$

From the previous fact, it is easy to see that $(\mathcal{A}_1 \otimes \mathcal{A}_2, \cdot_3)$ is a commutative associative algebra, and has as unit $1_{\mathcal{A}_1} \otimes 1_{\mathcal{A}_2}$, inasmuch as $(\mathcal{A}_1, \cdot_1)$ and $(\mathcal{A}_2 \cdot_2)$ are commutative and associative algebras with unit. Let $a_1 \otimes a_2, b_1 \otimes b_2, c_1 \otimes c_2$ be in $\mathcal{A}_1 \otimes \mathcal{A}_2$. For this example we consider $a_1 \cdot_1 b_1 := a_1 b_1$ y $a_2 \cdot_2 b_2 := a_2 b_2$. Then

$$
\begin{aligned}
-[b_1 \otimes b_2, a_1 \otimes a_2]_3 = &-\left([b_1, a_1]_1 \otimes b_2 a_2 + b_1 a_1 \otimes [b_2, a_2]_2\right) \\
&= [a_1, b_1]_1 \otimes a_2 b_2 + a_1 b_1 \otimes [a_2, b_2]_2 = [a_1 \otimes a_2, b_1 \otimes b_2]_3.
\end{aligned}
$$

Thus, $[-,-]_3$ is skew-symmetric.

$$
\begin{aligned}
[a_1 \otimes a_2, [b_1 \otimes b_2, c_1 \otimes c_2]_3]_3 &= [a_1 \otimes a_2, [b_1, c_1]_1 \otimes b_2 c_2 + b_1 c_1 \otimes [b_2, c_2]_2]_3 \\
&= [a_1, [b_1, c_1]]_1 \otimes a_2 b_2 c_2 + a_1 [b_1, c_1]_1 \otimes [a_2, b_2 c_2]_2 \\
&\quad + [a_1, b_1 c_1]_1 \otimes a_2 [b_2, c_2]_2 + a_1 b_1 c_1 \otimes [a_2, [b_2, c_2]]_2 \\
&= [a_1, [b_1, c_1]]_1 \otimes a_2 b_2 c_2 + a_1 [b_1, c_1]_1 \otimes b_2 [a_2, c_2]_2 \\
&\quad + a_1 [b_1, c_1]_1 \otimes [a_2, b_2]_2 c_2 + b_1 [a_1, c_1]_1 \otimes a_2 [b_2, c_2]_2 \\
&\quad + [a_1, b_1]_1 c_1 \otimes a_2 [b_2, c_2]_2 + a_1 b_1 c_1 \otimes [a_2, [b_2, c_2]]_2, \\
[c_1 \otimes c_2, [a_1 \otimes a_2, b_1 \otimes b_2]_3]_3 &= [c_1, [a_1, b_1]]_1 \otimes c_2 a_2 b_2 + c_1 [a_1, b_1]_1 \otimes a_2 [c_2, b_2]_2 \\
&\quad + c_1 [a_1, b_1]_1 \otimes [c_2, a_2]_2 b_2 + a_1 [c_1, b_1]_1 \otimes c_2 [a_2, b_2]_2 \\
&\quad + [c_1, a_1]_1 b_1 \otimes c_2 [a_2, b_2]_2 + c_1 a_1 b_1 \otimes [c_2, [a_2, b_2]]_2, \\
[b_1 \otimes b_2, [c_1 \otimes c_2, a_1 \otimes a_2]_3]_3 &= [b_1, [c_1, a_1]]_1 \otimes b_2 c_2 a_2 + b_1 [c_1, a_1]_1 \otimes c_2 [b_2, a_2]_2 \\
&\quad + b_1 [c_1, a_1]_1 \otimes [b_2, c_2]_2 a_2 + c_1 [b_1, a_1]_1 \otimes b_2 [c_2, a_2]_2 \\
&\quad + [b_1, c_1]_1 a_1 \otimes b_2 [c_2, a_2]_2 + b_1 c_1 a_1 \otimes [b_2, [c_2, a_2]]_2.
\end{aligned}
$$

Since $[-,-]_i$ is skew-symmetric and $\cdot_1$, $\cdot_2$ is a commutative product with $i \in 1, 2$, we have

$$
[a_1 \otimes a_2, [b_1 \otimes b_2, c_1 \otimes c_2]_3] + [c_1 \otimes c_2, [a_1 \otimes a_2, b_1 \otimes b_2]_3]_3 + [b_1 \otimes b_2, [c_1 \otimes c_2, a_1 \otimes a_2]_3]_3 = 0.
$$

In that way, $(\mathcal{A}, [-,-]_3)$ is a Lie algebra. Now, we have the next equalities

$$\begin{aligned}
[(a_1 \otimes a_2) \cdot_3 (b_1 \otimes b_2), c_1 \otimes c_2]_3 &= [a_1 b_1 \otimes a_2 b_2, c_1 \otimes c_2]_3 \\
&= [a_1 b_1, c_1]_1 \otimes a_2 b_2 c_2 + a_1 b_1 c_1 \otimes [a_2 b_2, c_2]_2 \\
(a_1 \otimes a_2) \cdot_3 [b_1 \otimes b_2, c_1 \otimes c_2]_3 &= (a_1 \otimes a_2) \cdot_3 ([b_1, c_1] \otimes b_2 c_2 + b_1 c_1 \otimes [b_2, c_2]) \\
&= a_1 [b_1, c_1]_1 \otimes a_2 b_2 c_2 + a_1 b_1 c_1 \otimes a_2 [b_2, c_2]_2 \\
[a_1 \otimes a_2, b_1 \otimes b_2] \cdot_3 (c_1 \otimes c_2) &= ([a_1, b_1]_1 \otimes a_2 b_2 + a_1 b_1 \otimes [a_2, b_2]) \cdot_3 (c_1 \otimes c_2) \\
&= [[a_1, b_1]_1, c_1] \otimes a_2 b_2 c_2 + a_1 b_1 c_1 \otimes [a_2, b_2]_2 c_2,
\end{aligned}$$

and the tensor product satisfies that $a_1 \otimes (a_2 + a_2') = a_1 \otimes a_2 + a_1 \otimes a_2'$. In addition, the brackets $[-,-]_1, [-,-]_2$ satisfy the Leibniz's rule (Definition 2),

$$\begin{aligned}
[(a_1 \otimes a_2) \cdot_3 (b_1 \otimes b_2), c_1 \otimes c_2]_3 = &\ (a_1 \otimes a_2) \cdot_3 [b_1 \otimes b_2, c_1 \otimes c_2]_3 \\
&+ [a_1 \otimes a_2, c_1 \otimes c_2] \cdot_3 (b_1 \otimes b_2)
\end{aligned}$$

Therefore, $(\mathcal{A}, \cdot_3, [-,-]_3)$ is a Poisson algebra.

In the case of Poisson algebras, the center with respect to the Poisson bracket is called the *Casimir set*. More precisely, if $\mathcal{A}$ is a Poisson algebra, the set $\operatorname{Cas}(\mathcal{A}) := \{x \in \mathcal{A} | [x, a] = 0,\ \forall a \in \mathcal{A}\}$, is called the *Casimir space*. We can show that if $\mathcal{A}$ is a Poisson algebra, then $\operatorname{Cas}(\mathcal{A})$ is a Poisson subalgebra of $\mathcal{A}$.

The Hamiltonian approach allows us to describe one of the more important examples of this kind of algebras, which are very useful in physics and were created to describe the Newton equation, and in general to understand the phenomenon of energy conservation (see [4] for more details).

Example 2.16. Let f and g two smooth functions over $\mathbb{R}^{2n}$ where $(\mathbf{x}, \mathbf{p}) \in \mathbb{R}^{2n}$ with $\mathbf{x}, \mathbf{p} \in \mathbb{R}^n$, $\mathbf{x} = (x_1, \ldots, x_n), \mathbf{p} = (p_1, \ldots, p_n)$. The space of all smooth functions of $\mathbb{R}^{2n}$ in $\mathbb{R}^{2n}$, with the product $[-,-] : \mathbb{R}^{2n} \times \mathbb{R}^{2n} \to \mathbb{R}^{2n}$, defined by

$$[f, g](\mathbf{x}, \mathbf{p}) = \sum_{j=1}^{n} \left(\frac{\partial f}{\partial x_j} \frac{\partial g}{\partial p_j} - \frac{\partial f}{\partial p_j} \frac{\partial g}{\partial x_j} \right) (\mathbf{x}, \mathbf{p}),$$

is a Poisson algebra. Let us show the details. Considering f, g and h smooth functions over $\mathbb{R}^{2n}$. First, we take the product between functions as: $fg(x) := f(x)g(x)$. Thereby, the algebra of smooth functions of $\mathbb{R}^{2n}$ over $\mathbb{R}^{2n}$ and the previous product, make a commutative associative algebra, whose unit is the identity function $\mathrm{id}_{\mathbb{R}^{2n}} : \mathbb{R}^{2n} \to \mathbb{R}^{2n}$, $\mathrm{id}_{\mathbb{R}^{2n}}(\mathbf{y}) = \mathbf{y}$, $\mathbf{y} \in \mathbb{R}^{2n}$. Since

$$[f, g] = \sum_{j=1}^{n} \left(\frac{\partial f}{\partial x_j} \frac{\partial g}{\partial p_j} - \frac{\partial f}{\partial p_j} \frac{\partial g}{\partial x_j} \right) = -\sum_{j=1}^{n} \left(\frac{\partial f}{\partial p_j} \frac{\partial g}{\partial x_j} - \frac{\partial f}{\partial x_j} \frac{\partial g}{\partial p_j} \right) = -[g, f],$$

thus, the property of skew-symmetry is satisfied. Now, note that

$$\begin{aligned}
[f,[g,h]] &= \left[f, \sum_{j=1}^{n}\left(\frac{\partial g}{\partial x_j}\frac{\partial h}{\partial p_j} - \frac{\partial g}{\partial p_j}\frac{\partial h}{\partial x_j}\right)\right] \\
&= \sum_{j=1}^{n}\left[\frac{\partial f}{\partial x_j}\frac{\partial}{\partial p_j}\left(\frac{\partial g}{\partial x_j}\frac{\partial h}{\partial p_j} - \frac{\partial g}{\partial p_j}\frac{\partial h}{\partial x_j}\right) - \frac{\partial f}{\partial p_j}\frac{\partial}{\partial x_j}\left(\frac{\partial g}{\partial x_j}\frac{\partial h}{\partial p_j} - \frac{\partial g}{\partial p_j}\frac{\partial h}{\partial x_j}\right)\right] \\
&= \sum_{j=1}^{n}\Bigg[\frac{\partial f}{\partial x_j}\frac{\partial^2 g}{\partial p_j \partial x_j}\frac{\partial h}{\partial p_j} + \frac{\partial f}{\partial x_j}\frac{\partial g}{\partial x_j}\frac{\partial^2 h}{\partial p_j^2} - \frac{\partial f}{\partial x_j}\frac{\partial^2 g}{\partial p_j^2}\frac{\partial h}{\partial x_j} - \frac{\partial f}{\partial x_j}\frac{\partial g}{\partial p_j}\frac{\partial^2 h}{\partial p_j \partial x_j} \\
&\quad - \frac{\partial f}{\partial p_j}\frac{\partial^2 g}{\partial x_j^2}\frac{\partial h}{\partial p_j} - \frac{\partial f}{\partial p_j}\frac{\partial g}{\partial x_j}\frac{\partial^2 h}{\partial x_j \partial p_j} + \frac{\partial f}{\partial p_j}\frac{\partial^2 g}{\partial x_j \partial p_j}\frac{\partial h}{\partial x_j} + \frac{\partial f}{\partial p_j}\frac{\partial g}{\partial p_j}\frac{\partial^2 h}{\partial x_j^2}\Bigg], \\
[h,[f,g]] &= \sum_{j=1}^{n}\Bigg[\frac{\partial h}{\partial x_j}\frac{\partial^2 f}{\partial p_j \partial x_j}\frac{\partial g}{\partial p_j} + \frac{\partial h}{\partial x_j}\frac{\partial f}{\partial x_j}\frac{\partial^2 g}{\partial p_j^2} - \frac{\partial h}{\partial x_j}\frac{\partial^2 f}{\partial p_j^2}\frac{\partial g}{\partial x_j} - \frac{\partial h}{\partial x_j}\frac{\partial f}{\partial p_j}\frac{\partial^2 g}{\partial p_j \partial x_j} \\
&\quad - \frac{\partial h}{\partial p_j}\frac{\partial^2 f}{\partial x_j^2}\frac{\partial g}{\partial p_j} - \frac{\partial h}{\partial p_j}\frac{\partial f}{\partial x_j}\frac{\partial^2 g}{\partial x_j \partial p_j} + \frac{\partial h}{\partial p_j}\frac{\partial^2 f}{\partial x_j \partial p_j}\frac{\partial g}{\partial x_j} + \frac{\partial h}{\partial p_j}\frac{\partial f}{\partial p_j}\frac{\partial^2 g}{\partial x_j^2}\Bigg], \\
[g,[h,f]] &= \sum_{j=1}^{n}\Bigg[\frac{\partial g}{\partial x_j}\frac{\partial^2 h}{\partial p_j \partial x_j}\frac{\partial f}{\partial p_j} + \frac{\partial g}{\partial x_j}\frac{\partial h}{\partial x_j}\frac{\partial^2 f}{\partial p_j^2} - \frac{\partial g}{\partial x_j}\frac{\partial^2 h}{\partial p_j^2}\frac{\partial f}{\partial x_j} - \frac{\partial g}{\partial x_j}\frac{\partial h}{\partial p_j}\frac{\partial^2 f}{\partial p_j \partial x_j} \\
&\quad - \frac{\partial g}{\partial p_j}\frac{\partial^2 h}{\partial x_j^2}\frac{\partial f}{\partial p_j} - \frac{\partial g}{\partial p_j}\frac{\partial h}{\partial x_j}\frac{\partial^2 f}{\partial x_j \partial p_j} + \frac{\partial g}{\partial p_j}\frac{\partial^2 h}{\partial x_j \partial p_j}\frac{\partial f}{\partial x_j} + \frac{\partial g}{\partial p_j}\frac{\partial h}{\partial p_j}\frac{\partial^2 f}{\partial x_j^2}\Bigg],
\end{aligned}$$

and hence, we obtain the equality $[f,[g,h]]+[h,[f,g]]+[g,[h,f]]=0$. From this fact we conclude that the bracket $[-,-]$ generates a Lie algebra over the set of smooth functions over $\mathbb{R}^n$. Finally, we see that both multiplications are compatible:

$$\begin{aligned}
[f,gh] &= \sum_{j=1}^{n}\left[\frac{\partial f}{\partial x_j}\frac{\partial gh}{\partial p_j} - \frac{\partial f}{\partial p_j}\frac{\partial gh}{\partial x_j}\right] \\
&= \sum_{j=1}^{n}\left[\frac{\partial f}{\partial x_j}\left(\frac{\partial g}{\partial p_j}h + g\frac{\partial h}{\partial p_j}\right) - \frac{\partial f}{\partial p_j}\left(\frac{\partial g}{\partial x_j}h + g\frac{\partial h}{\partial x_j}\right)\right] \\
&= \sum_{j=1}^{n}\left[\frac{\partial f}{\partial x_j}\frac{\partial g}{\partial p_j} - \frac{\partial f}{\partial p_j}\frac{\partial g}{\partial x_j}\right]h + g\sum_{j=1}^{n}\left[\frac{\partial f}{\partial x_j}\frac{\partial h}{\partial p_j} - g\frac{\partial f}{\partial p_j}\frac{\partial h}{\partial x_j}\right] \\
&= [f,g]h + g[f,h].
\end{aligned}$$

As a result, we can conclude that the set of smooth functions over $\mathbb{R}^{2n}$ with the bracket $[-,-]$, is a Poisson algebra.

3. Quadratic and symplectic algebras

In this section we are going to define some kinds of algebras, which allows us to expose and develop the construction of symplectic Lie algebras. We need to make a transition by means of quadratic Lie algebras. We are going to write a pair of examples, which are very important to the key theorem. It is important to say that although the definitions have an algebraic approach, they also have a geometric overview (see [3]).

Definition 3.1. Let $(\mathcal{A}, .)$ be an algebra. Over $\mathcal{A}$ we can define two multiplications:

$$[x, y] := x.y - y.x; \quad x \circ y := \frac{1}{2}(x.y + y.x), \forall x, y \in \mathcal{A}.$$

The pair $(\mathcal{A}, .)$ is called a *Poisson-admissible algebra*, if $(\mathcal{A}, \circ, [-,-])$ is a Poisson algebra. From now on, we write $\mathcal{A}^- := (\mathcal{A}, [-,-])$ and $\mathcal{A}^+ := (\mathcal{A}, \circ)$.

Definition 3.2. (i) Let $(\mathcal{A}, .)$ be an algebra and $B : \mathcal{A} \times \mathcal{A} \to \mathbb{K}$ a bilinear form. We say that B is *invariant* (or *associative*), if $B(x.y, z) = B(x, y.z)$, for all $x, y, z \in \mathcal{A}$.

(ii) Let $B : \mathcal{A} \times \mathcal{A} \to \mathbb{K}$ be a bilinear form. We say that B is *nondegenerate*, if given $x \in \mathcal{A}$ the linear map fixing one component $B(x, -) : \mathcal{A} \to \mathbb{K}$ is an isomorphism. If the algebra is finite dimensional, the previous definition is equivalent to

$B(x, y) = 0,\ \forall y \in \mathcal{A}$ then $x = 0$ and $B(x, y) = 0,\ \forall x \in \mathcal{A}$ then $y = 0$. ([3]).

(iii) Let $(\mathfrak{g}, [-,-])$ be a Lie algebra and $B : \mathfrak{g} \times \mathfrak{g} \to \mathbb{K}$ a bilinear form. $(\mathfrak{g}, B)$ is called a *quadratic Lie algebra*, if B is symmetric, nondegenerate and invariant. In this case, B is called an invariant scalar product on $\mathfrak{g}$.

(iv) Let $(\mathcal{A}, \circ)$ an associative algebra and $B : \mathfrak{g} \times \mathfrak{g} \to \mathbb{K}$ a bilinear form. $(\mathcal{A}, B)$ is called a *symmetric algebra*, if B is symmetric, nondegenerate and invariant. In this case, B is called an invariant scalar product on $\mathcal{A}$.

(v) Let $(\mathcal{A}, .)$ be a Poisson-admissible algebra and $B : \mathfrak{g} \times \mathfrak{g} \to \mathbb{K}$ a bilinear form. $(\mathcal{A}, B)$ is called a *quadratic Poisson-admissible algebra*, if B is symmetric, nondegenerate and invariant. In this case, B is called an invariant scalar product on $\mathcal{A}$.

(vi) Let $(\mathcal{A}, \circ, [-,-])$ a Poisson algebra and $B : \mathfrak{g} \times \mathfrak{g} \to \mathbb{K}$ a bilinear form. $(\mathcal{A}, B)$ is called a *quadractic Poisson algebra*, if B is symmetric, nondegenerate and satisfies:

$$B([x, y], z) = B(x, [y, z]) \quad \text{and} \quad B(x \circ y, z) = B(x, y \circ z), \quad \forall a, b, c \in \mathcal{A}.$$

Remark 3.3. From the previous definition we have that:

(i) Let $(\mathcal{A}, .)$ be a Poisson-admissible algebra and $B : \mathcal{A} \times \mathcal{A} \to \mathbb{K}$ a bilinear form. The pair $(\mathcal{A}, B)$ is a quadratic algebra if and only if, $(\mathcal{A}^-, B)$ is a quadratic Lie algebra and $(\mathcal{A}^+, B)$ is a symmetric algebra.

(ii) $(\mathcal{A}, ., B)$ is a quadratic Poisson-admissible algebra if and only if, $(\mathcal{A}, \circ, [-,-], B)$ is a quadratic Poisson algebra (see the definition 3.1).

We are going to appreciate some examples of these algebras.

Example 3.4 (**Taken from [2]**). Let $(\mathcal{A}, \circ, [-,-])$ be a Poisson algebra and $\mathcal{A}^*$ the dual vector space $\mathcal{A}$. $(\mathcal{A}\oplus\mathcal{A}^*, \star, [\cdot,\cdot])$ is a Poisson algebra with multiplications given by

$$(x+f)\star(y+h) := x\circ y + h\circ L_x + f\circ L_y, \qquad [x+f, y+h] := [x,y] - h\circ \mathrm{ad}_x + f\circ \mathrm{ad}_y,$$

for all $(x,f),(y,h)\in\mathcal{A}\times\mathcal{A}^*$ and $L_x:\mathcal{A}\to\mathcal{A}$ where $L_x(a) := x\circ a$, with $a\in\mathcal{A}$ (the product between functions written by $\circ$ is the composition of functions). In addition, $(\mathcal{A}\oplus\mathcal{A}^*, B)$ is a quadratic Poisson algebra, if $B: (\mathcal{A}\oplus\mathcal{A}^*)\times(\mathcal{A}\oplus\mathcal{A}^*)\to\mathbb{K}$ is defined as $B(x+f, y+h) := f(y)+h(x)$, for all $(x,f),(y,h)\in\mathcal{A}\times\mathcal{A}^*$.

- So let us see that $(\mathcal{A}\oplus\mathcal{A}^*, \star)$ is a commutative and associative algebra:

$$\begin{aligned}[(x+f)\star(y+h)]\star(z+g) &= [x\circ y + h\circ L_x + f\circ L_y]\star(z+g)\\ &= (x\circ y)\circ z + g\circ L_{x\circ y} + h\circ L_x\circ L_z + f\circ L_y\circ L_z,\end{aligned} \tag{3}$$

$$\begin{aligned}(x+f)\star[(y+h)\star(z+g)] &= (x+f)\star[y\circ z + g\circ L_y + h\circ L_z]\\ &= x\circ(y\circ z) + g\circ L_y\circ L_x + h\circ L_z\circ L_x + f\circ L_{y\circ z}.\end{aligned} \tag{4}$$

  Let a be in $\mathcal{A}$, due to $x\circ y = \frac{1}{2}(x.y+y.x) = \frac{1}{2}(y.x+x.y) = y\circ x$, then $L_{x\circ y}(a) = x\circ y\circ a = y\circ x\circ a = L_y(x\circ a) = L_y\circ L_x(a)$, thus, $L_{x\circ y} = L_y\circ L_x = L_x\circ L_y$, for all $x,y\in\mathcal{A}$. Therefore, (3) is equal to (4). Note that the commutativity follows from the fact that $(\mathcal{A},\circ)$ is a commutative algebra:

$$(x+f)\star(y+h) = x\circ y + h\circ L_x + f\circ L_y = f\circ L_y + h\circ L_x + y\circ x = (y+h)\star(x+f).$$

  The unit is $1_{\mathcal{A}} + 0_{\mathcal{A}^*}\in\mathcal{A}\oplus\mathcal{A}^*$, where for all $x\in\mathcal{A}$, $0_{\mathcal{A}^*}(x) = 0\in\mathbb{K}$:

$$(x+f)\star(1_{\mathcal{A}}+0_{\mathcal{A}^*}) = x\circ 1_{\mathcal{A}} + 0_{\mathcal{A}^*}\circ L_x + f\circ L_{1_{\mathcal{A}}} = x + 0_{\mathcal{A}^*} + f = x+f.$$

- $(\mathcal{A}, [-,-])$ is a Lie algebra:

  Since $-[y+h, x+f] = -[y,x] + f\circ\mathrm{ad}_y - h\circ\mathrm{ad}_x = [x,y] - h\circ\mathrm{ad}_x + f\circ\mathrm{ad}_y = [x+f, y+h]$, the bracket is skew-symmetric.

  So let us see that the bracket satisfies the Jacobi identity:

$$\begin{aligned}[x+f,[y+h,z+g]] &= [x+f, [y,z] - g\circ\mathrm{ad}_y + h\circ\mathrm{ad}_z]\\ &= [x,[y,z]] + g\circ\mathrm{ad}_y\circ\mathrm{ad}_x - h\circ\mathrm{ad}_z\circ\mathrm{ad}_x + f\circ\mathrm{ad}_{[y,z]},\\ [z+g,[x+f,y+h]] &= [z,[x,y]] + h\circ\mathrm{ad}_x\circ\mathrm{ad}_z - f\circ\mathrm{ad}_y\circ\mathrm{ad}_z + g\circ\mathrm{ad}_{[x,y]},\\ [y+h,[z+g,x+f]] &= [y,[z,x]] + f\circ\mathrm{ad}_z\circ\mathrm{ad}_y - g\circ\mathrm{ad}_x\circ\mathrm{ad}_y + h\circ\mathrm{ad}_{[z,x]}.\end{aligned}$$

  Since $\mathcal{A}$ is a Poisson algebra, the Jacobi identity is satisfied. Hence, Proposition 2.7 holds, that is to say $[x+f,[y+h,z+g]] + [z+g,[x+f,y+h]] + [y+h,[z+g,x+f]] = 0$.

- In order to see that $\star$ and $[-,-]$ are compatible, consider

$$\begin{aligned}[(x+f)\star(y+h),z+g] &= [x\circ y+h\circ L_x+f\circ L_y, z+g]\\ &= [x\circ y,z]-g\circ \mathrm{ad}_{x\circ y}+h\circ L_x\circ \mathrm{ad}_z+f\circ L_y\circ \mathrm{ad}_z,\end{aligned} \tag{5}$$

and

$$\begin{aligned}(x+f)\star[y+h,z+g]+[x+f,z+g]\star(y+h) &= (x+f)\star[y+h,z+g]\\ &\quad+[x+f,z+g]\star(y+h)\\ &= (x+f)\star([y,z]-g\circ \mathrm{ad}_y+h\circ \mathrm{ad}_z)+([x,z]-g\circ \mathrm{ad}_x+f\circ \mathrm{ad}_z)\star(y+h)\\ &= x\circ[y,z]-g\circ \mathrm{ad}_y\circ L_x+h\circ \mathrm{ad}_z\circ L_x+f\circ L_{[y,z]}+[x,z]\circ y\\ &\quad+h\circ L_{[x,z]}-g\circ \mathrm{ad}_x\circ L_y+f\circ \mathrm{ad}_z\circ L_y.\end{aligned} \tag{6}$$

We have that (5) and (6) are equal, because $\mathrm{ad}_{x\circ y}=\mathrm{ad}_y\circ L_x+\mathrm{ad}_x\circ L_y$, $L_x\circ \mathrm{ad}_z=\mathrm{ad}_z\circ L_x+L_{[x,z]}$, $L_y\circ \mathrm{ad}_z=L_{[y,z]}+\mathrm{ad}_z\circ L_y$, let a in $\mathcal{A}$:

$$\begin{aligned}\mathrm{ad}_{x\circ y}(a) &= [x\circ y,a]=\left[\frac{1}{2}(x.y+y.x),a\right]\\ &=\frac{1}{2}(x.y.a-a.x.y+y.x.a-a.y.x),\\ (\mathrm{ad}_y\circ L_x+\mathrm{ad}_x\circ L_y)(a) &= \mathrm{ad}_y(x\circ a)+\mathrm{ad}_x(y\circ a)\\ &= \left[y,\frac{1}{2}(x.a+a.x)\right]+\left[x,\frac{1}{2}(y.a+a.y)\right]\\ &= \frac{1}{2}(y.x.a-x.a.y+y.a.x-a.x.y)\\ &\quad+\frac{1}{2}(x.y.a-y.a.x+x.a.y-a.y.x)\\ &= \frac{1}{2}(y.x.a-a.x.y+x.y.a-a.y.x),\\ (L_x\circ \mathrm{ad}_z)(a) &= L_x([z,a])=x\circ[z,a],\\ (\mathrm{ad}_z\circ L_x+L_{[x,z]})(a) &= \mathrm{ad}_z(x\circ a)+[x,z]\circ a\\ &= [z,x\circ a]+[x,z]\circ a\\ &=x\circ[z,a]+[z,x]\circ a+[x,z]\circ a=x\circ[z,a].\end{aligned}$$

So, $(\mathcal{A}\oplus\mathcal{A}^*)$ is a Poisson algebra.

- Finally, we need to see that the bilinear form B is (i) symmetric, (ii) invariant with respect to the multiplications $\star$ as $[-,-]$ and (iii) nondegenerate:

  (i) $B(x+f,y+h)=f(y)+h(x)=h(x)+f(y)=B(h+y,f+x)$.

  (ii) Since the Poisson algebra has two multiplications, we need to see that B is invariant with respect to $\star$ and also with $[-,-]$:

$$\begin{aligned}
B((x+f)\star(y+h), z+g) &= B(x\circ y + h\circ L_x + f\circ L_y, z+g)\\
&= (h\circ L_x + f\circ L_y)(z) + g(x\circ y)\\
&= h(x\circ z) + f(y\circ z) + g(x\circ y),\\
B(x+f, (y+h)\star(z+g)) &= B(x+f, y\circ z + g\circ L_y + h\circ L_z)\\
&= f(y\circ z) + (g\circ L_y + h\circ L_z)(x)\\
&= f(y\circ z) + g(y\circ x) + h(z\circ x)\\
&= h(x\circ z) + f(y\circ z) + g(x\circ y).
\end{aligned}$$

(iii) Let (x,f) be in $\mathcal{A}\times\mathcal{A}^*$. Suppose that for all $(y,h)\in\mathcal{A}\times\mathcal{A}^*$, $B(x+f,y+h)=0$, thus, $f(y)+h(x)=0$. Let a be in $\mathcal{A}$, take an element $(a-x,f)\in\mathcal{A}\times\mathcal{A}^*$, then $0=B(x+f,(a-x)+f)=f(a-x)+f(x)=f(a)$, thus $f=0_{\mathcal{A}^*}$. In this way we have that $h(x)=0$ for all $h\in\mathcal{A}^*$, then $x=0_{\mathcal{A}}$. Therefore, the element $x+f=0+0_{\mathcal{A}^*}$. Consequently, B is nondegenerated. As a conclusion, $(\mathcal{A}\times\mathcal{A}^*,\star,[-,-],B)$ is a quadratic Poisson algebra.

Similarly, we can see that $(\mathcal{A}\oplus\mathcal{A}^*,\bowtie,[\cdot,\cdot],B)$ is a quadratic Poisson algebra, where for all $(x,f),(y,h)\in\mathcal{A}\times\mathcal{A}^*$, $(x+f\bowtie y+h)=x.y+h\circ R_x+f\circ L_y$, defining $R_x(a):=a\circ x$, with $a\in\mathcal{A}$.

Example 3.5. Let $(\mathcal{A},.,B)$ be a quadratic Poisson-admissible algebra, whose associate Poisson algebra is $(\mathcal{A},\cdot,[-,-],B)$ (see Definition 3.1) and $(\mathfrak{H},\star,\varphi)$ a symmetric commutative algebra. Let $a\otimes x, b\otimes y$ be in $\mathcal{A}\otimes\mathfrak{H}$. If we consider the product given by $(a\otimes x)\bullet(b\otimes y):=a.b\otimes x\star y$, then $(\mathcal{A}\otimes\mathfrak{H},\bullet)$ is a Poisson-admissible algebra. In the first place, we show that $(A\otimes\mathfrak{H},\ominus,[\cdot,\cdot])$ is a Poisson algebra:

$$(a\otimes x)\ominus(b\otimes y):=\frac{1}{2}[(a\otimes x)\bullet(b\otimes y)+(b\otimes y)\bullet(a\otimes x)]=\frac{1}{2}[a.b\otimes x\star y+b.a\otimes y\star x],$$
$$[a\otimes x,b\otimes y]:=(a\otimes x)\bullet(b\otimes y)-(b\otimes y)\bullet(a\otimes x)=a.b\otimes x\star y-b.a\otimes y\star x.$$

If $B\otimes\varphi:(\mathcal{A}\otimes\mathfrak{H})\otimes(\mathcal{A}\otimes\mathfrak{H})\to\mathbb{K}$, such that for all $(a,x),(b,y)\in\mathcal{A}\times\mathfrak{H}$:

$$B\otimes\varphi(a\otimes x,b\otimes y):=B(a,b)\varphi(x,y),$$

then $(\mathcal{A}\otimes\mathfrak{H},\bullet)$ is a quadratic Poisson-admissible algebra.

- Let us see that $(\mathcal{A}\otimes\mathfrak{H},\ominus)$ is a commutative and associative algebra with unit:

$$
\begin{aligned}
[(a\otimes x)\ominus(b\otimes y)]\ominus(c\otimes z) &= \left[\frac{1}{2}(a.b\otimes x\star y+b.a\otimes y\star x)\right]\ominus(x\otimes z)\\
&=\frac{1}{2}\left[\frac{1}{2}[(a.b).c\otimes(x\star y)\star z+c.(a.b)\otimes z\star(x\star y)]\right.\\
&\quad\left.+\frac{1}{2}[(b.a).c\otimes(y\star x)\star z+c.(b.a)\otimes z\star(y\star x)]\right]\\
&=\frac{1}{4}\left[a.b.c\otimes x\star y\star z+c.a.b\otimes z\star x\star y\right.\\
&\quad\left.+b.a.c\otimes y\star x\star z+c.b.a\otimes z\star y\star x\right]\\
&=\frac{1}{2}\left[\frac{1}{2}((a.b).c+c.(a.b))\right.\\
&\quad\left.+\frac{1}{2}((b.a).c+c.(b.a))\right]\otimes x\star y\star z\\
&=\frac{1}{2}[(a.b)\cdot c+(b.a)\cdot c]\otimes x\star y\star z\\
&=\frac{1}{2}[a.b+b.a]\cdot c\otimes x\star y\star z=a\cdot b\cdot c\otimes x\star y\star z,\\
(a\otimes x)\ominus[(b\otimes y)\ominus(c\otimes z)] &= (a\otimes x)\ominus\left[\frac{1}{2}(b.c\otimes y\star z+c.b\otimes z\star y)\right]\\
&=\frac{1}{2}\left[\frac{1}{2}[a.(b.c)\otimes x\star(y\star z)+(b.c).a\otimes(y\star z)\star x]\right.\\
&\quad\left.+\frac{1}{2}[a.(c.b)\otimes x\star(z\star y)+(c.b).a\otimes(z\star y)\star x]\right]\\
&=\frac{1}{4}[a.b.c\otimes x\star y\star z+b.c.a\otimes y\star z\star x\\
&\quad+a.c.b\otimes x\star z\star y+c.b.a\otimes z\star y\star x]\\
&=\frac{1}{2}\left[\frac{1}{2}(a.(b.c)+(b.c).a)\right.\\
&\quad\left.+\frac{1}{2}(a.(c.b)+(c.b).a)\right]\otimes x\star y\star z\\
&=\frac{1}{2}[a\cdot(b.c)+a\cdot(c.b)]\otimes x\star y\star z\\
&=a\cdot(\frac{1}{2}[b.c+c.b])\otimes x\star y\star z=a\cdot b\cdot c\otimes x\star y\star z.
\end{aligned}
$$

The commutativity of $\ominus$ follows from the fact that the sum is commutative. Thus:

$$
\begin{aligned}
(a\otimes x)\ominus(b\otimes y) &= \frac{1}{2}(a.b\otimes x\star y+b.a\otimes y\star x)\\
&=\frac{1}{2}(b.a\otimes y\star x+a.b\otimes x\star y)=(b\otimes y)\ominus(a\otimes x).
\end{aligned}
$$

In addition, the unit is $1_{\mathcal{A}} \otimes 1_{\mathfrak{H}}$, because

$$(a \otimes x) \ominus (1_{\mathcal{A}} \otimes 1_{\mathfrak{H}}) = \frac{1}{2}(a.1_{\mathcal{A}} \otimes x \star 1_{\mathfrak{H}} + 1_{\mathcal{A}}.a \otimes 1_{\mathfrak{H}} \star x) = \frac{1}{2}2(a \otimes x) = a \otimes x.$$

- Secondly, we will see that $(\mathcal{A} \otimes \mathfrak{H}, \ominus, [\cdot, \cdot])$ is a Lie algebra. It is easy to verify the skew-symmetric of the bracket. Note that

$$\begin{aligned}
[a \otimes x, [b \otimes y, c \otimes z]] &= [a \otimes x, b.c \otimes y \star z - c.b \otimes z \star y] \\
&= a.(b.c) \otimes x \star (y \star z) - (b.c).a \otimes (y \star z) \star x \\
&\quad - a.(c.b) \otimes x \star (z \star y) + (c.b).a \otimes (z \star y) \star x \\
&= [(a.(b.c) - (b.c).a + (c.b).a - a.(c.b))] \otimes x \star y \star z \\
&= [a.b.c - b.c.a + c.b.a - a.c.b] \otimes x \star y \star z, \\
[c \otimes z, [a \otimes x, b \otimes y]] &= [c.a.b - a.b.c + b.a.c - c.b.a] \otimes x \star y \star z, \\
[b \otimes y, [c \otimes z, a \otimes x]] &= [b.c.a - c.a.b + a.c.b - b.a.c] \otimes x \star y \star z.
\end{aligned}$$

then, $[a \otimes x, [b \otimes y, c \otimes z]] + [c \otimes z, [a \otimes x, b \otimes y]] + [b \otimes y, [c \otimes z, a \otimes x]] = 0$.

- The products $\ominus, [-, -]$ are compatible. We have

$$\begin{aligned}
[(a \otimes x) \ominus (b \otimes y), c \otimes z] &= \left[\frac{1}{2}(a.b \otimes x \star y + b.a \otimes y \star x), c \otimes z\right] \\
&= \frac{1}{2}(a.b.c \otimes x \star y \star z - c.a.b \otimes z \star x \star y \\
&\quad + b.a.c \otimes y \star x \star z \\
&\quad - c.b.a \otimes z \star y \star x),
\end{aligned}$$

and,

$$\begin{aligned}
&[(a \otimes x), (c \otimes z)] \ominus (b \otimes y) + (a \otimes x) \ominus [(b \otimes y), (c \otimes z)] \\
&= (a.c \otimes x \star z - c.a \otimes z \star x) \ominus (b \otimes y) \\
&\quad + (a \otimes x) \ominus (b.c \otimes y \star z - c.b \otimes z \star y) \\
&= \frac{1}{2}(a.c.b \otimes x \star z \star y + b.a.c \otimes y \star x \star z \\
&\quad - c.a.b \otimes z \star x \star y - b.c.a \otimes y \star z \star x) \\
&+ \frac{1}{2}(a.b.c \otimes x \star y \star z + b.c.a \otimes y \star z \star x \\
&\quad - a.c.b \otimes x \star z \star y - c.b.a \otimes z \star y \star x) \\
&= \frac{1}{2}(b.a.c \otimes y \star x \star z - c.a.b \otimes z \star x \star y \\
&\quad + a.b.c \otimes x \star y \star z - c.b.a \otimes z \star y \star x).
\end{aligned}$$

The previous terms are equal due to $\star$ is commutative.

- $(\mathcal{A} \otimes \mathfrak{H}, \bullet)$ is a quadratic Poisson-admissible algebra. We only need to see that $B \otimes \varphi$ is (i) symmetric, (ii) invariant with respect to $\star$ as a $[-, -]$, and (iii) nondegenerate:

(i) Since the maps B and φ are symmetric:

$$B\otimes\varphi(b\otimes y,a\otimes x)=B(b,a)\varphi(y,x)=B(a,b)\varphi(x,y)=B\otimes\varphi(a\otimes x,b\otimes y).$$

(ii) $B\otimes\varphi$ is associative with respect to the multiplications $\ominus$, $[-,-]$:

$$\begin{aligned}
B\otimes\varphi((a\otimes x)\ominus(b\otimes y),c\otimes z)&=B\otimes\varphi(\frac{1}{2}(a.b\otimes x\star y+b.a\otimes y\star x),c\otimes z)\\
&=\frac{1}{2}\left(B(a.b,c)\varphi(x\star y,z)+B(b.a,c)\varphi(y\star x,z)\right),\\
B\otimes\varphi(a\otimes x,(b\otimes y)\ominus(c\otimes z))&=B\otimes\varphi(a\otimes x,\frac{1}{2}(b.c\otimes y\star z+c.b\otimes z\star y))\\
&=\frac{1}{2}(B(a,b.c)\varphi(x,y\star z)+B(a,c.b)\varphi(x,z\star y))\\
&=\frac{1}{2}(B(a.b,c)\varphi(x\star y,z)+B(a.c,b)\varphi(x\star y,z))\\
&=\frac{1}{2}(B(a.b,c)\varphi(x\star y,z)+B(b,a.c)\varphi(x\star y,z)),\\
B\otimes\varphi([a\otimes x,b\otimes y],c\otimes z)&=B\otimes\varphi(a.b\otimes x\star y-b.a\otimes y\star x,c\otimes z)\\
&=B(a.b,c)\varphi(x\star y,z)-B(b.a,c)\varphi(y\star x,z),\\
B\otimes\varphi(a\otimes x,[b\otimes y,c\otimes z])&=B\otimes\varphi(a\otimes x,b.c\otimes y\star z-c.b\otimes z\star y)\\
&=B(a,b.c)\varphi(x,y\star z)-B(a,c.b)\varphi(x,z\star y)\\
&=B(a.b,c)\varphi(x\star y,z)-B(a.c,b)\varphi(x\star y,z).
\end{aligned}$$

Then, $B(a.c,b)=B(b,a.c)=B(b.a,c)$ and $\varphi(x,y\star z)=\varphi(x\star y,z)=\varphi(y\star x,z)$, keeping in mind that $\star$ is commutative.

(iii) $B\otimes\varphi$ is a nondegenerate map: Let $a\otimes x\in\mathcal{A}\otimes\mathfrak{H}$. We need to see that $B\otimes\varphi(a\otimes x,-):\mathcal{A}\otimes\mathfrak{H}\to\mathbb{K}$ is a linear map and an isomorphism. Since $a\otimes x\neq 0$, then $a\neq 0$ or $x\neq 0$ (the respective zero of each algebra is taken, depending of the case).

- If $B\otimes\varphi(a\otimes x,b\otimes y)=B(a,b)\varphi(x,y)=0$. By hypothesis , $B(a,-)$ and $\varphi(x,-)$ are isomorphism, then $B(a,b)=0$ o $\varphi(x,y)=0$. If $B(a,b)=0$, then $b=0$, hence $b\otimes y$ is zero of $\mathcal{A}\otimes\mathfrak{H}$. With a similar analysis, if $\varphi(x,y)=0$, then $b\otimes y$ is zero of $\mathcal{A}\otimes\mathfrak{H}$. Hence $B\otimes\varphi(a\otimes x,-)$ is injective.
- Let k be in $\mathbb{K}$, since $B(a,-)$ and $\varphi(y,-)$ are surjective, and also we know that $1\in\mathbb{K}$, then exists $b\in\mathcal{A}$ such that $B(a,b)=k$ and $y\in\mathfrak{H}$ such that $\varphi(x,y)=1$. So, $k=k1=B(a,b)\varphi(x,y)=B\otimes\varphi(a\otimes x,b\otimes y)$. Therefore $B\otimes\varphi(a\otimes x,-)$ is surjective.

From the previous fact $(\mathcal{A}\otimes\mathfrak{H},\ominus,[-,-],B\otimes\varphi)$ is a quadratic Poisson algebra. Therefore, $(\mathcal{A}\otimes\mathfrak{H},\bullet,B\otimes\varphi)$ is a quadratic Poisson-admissible algebra.

Definition 3.6. Let $(\mathcal{A},.)$ be an algebra and $\omega:\mathcal{A}\times\mathcal{A}\to\mathbb{K}$ a bilinear form. We say that $(\mathcal{A},\omega)$ is a ***symplectic algebra*** (or that ω endow of symplectic structure to $(\mathcal{A},.)$), if the following conditions hold:

1. ω is skew-symmetric, if $\omega(x,y) = -\omega(y,z)$, for all $x,y \in \mathcal{A}$;
2. ω is nondegenerate;
3. $\omega(x.y,z) + \omega(y.z,x) + \omega(z.x,y) = 0$, for all $x,y,z \in \mathcal{A}$.

Definition 3.7. If $(\mathcal{A},.)$ is an algebra, B a scalar product over $\mathcal{A}$ and ω endow of symplectic structure to $\mathcal{A}$. $(\mathcal{A},B,\omega)$ is called a ***quadratic symplectic algebra***. If $(\mathcal{A},.)$ is an associative algebra also, we say that $(\mathcal{A},B,\omega)$ is a quadratic symplectic algebra.

Proposition 3.8. *If $(\mathcal{A},B)$ is a quadratic algebra, ω is a symplectic structure on $\mathcal{A}$ if and only if exists a unique skew-symmetric (with respect to B) invertible derivation $(\mathcal{A},.,B)$ such that:*

$$\omega(x,y) := B(D(x),y), \quad \text{for all } x,y \in \mathcal{A}. \tag{7}$$

Proof. Given a quadratic algebra $(\mathcal{A},B)$, suppose that there exists a unique skew-symmetric (with respect to B) invertible derivation, such that $\omega(x,y) = B(D(x),y)$, for all $x,y \in \mathcal{A}$. Let us see that ω define a symplectic structure over $\mathcal{A}$:

- $-\omega(y,z) = -B(D(y),x) = -(-B(y,D(x))) = B(D(x),y) = \omega(x,y)$.
- Let $x \in \mathcal{A}$ be a nonzero element and $\omega(x,-) : \mathcal{A} \to \mathbb{K}$ defined as $\omega(x,-)(y) = \omega(x,y)$. We will see that $\omega(x,-)$ is an isomorphism:
  - Let y,z in $\mathcal{A}$, such that $\omega(x,y) = \omega(x,z)$. Due to $\omega(x,y) = B(D(x),y) = -B(x,D(y))$ and $\omega(x,z) = -B(x,D(z))$. Then $B(x,D(y)) = B(x,D(z))$, because $B(x,-)$ is an isomorphism, $D(y) = D(z)$. As D is invertible, $y = z$. We conclude that ω is injective.
  - Let k be in $\mathbb{K}$; due to B endow of quadratic structure to $\mathcal{A}$, B is a nondegenerate map and $D(x) \in \mathcal{A}$, so there exists $y \in \mathcal{A}$ such that $B(D(x),y) = k$. Thus, $\omega(x,y) = k$. We conclude that ω is surjective.
- Now, let us show that the condition 3 of Definition 3.6 is satisfied:

$$\begin{aligned}
\omega(x.y,z) + \omega(z.x,y) + \omega(y.z,x) = {} & B(x.D(y),z) + B(D(x).y,z) + B(z.D(x),y) \\
& + B(D(z).x,y) + B(y.D(z),x) + B(D(y).z,x) \\
= {} & B(z.x,D(y)) + B(D(x),y.z) + B(z.D(x),y) + B(D(z).x,y) \\
& + B(y.D(z),x) + B(D(y).z,x) \\
= {} & - B(z.D(x),y) - B(D(z).x,y) - B(x,y.D(z)) - B(x,D(y).z) \\
& + B(z.D(x),y) + B(D(z).x,y) + B(y.D(z),x) + B(D(y).z,x) = 0.
\end{aligned}$$

Let us see conversely. Given the symplectic structure $\omega(x,y) := B(D(x),y)$, where D is a skew-symmetric (with respect to B) invertible derivation. We need to see that D is unique. Assume there are D,D' invertible derivations, which satisfies expression (7). Then, $B(D(x),y)=\omega(x,y)=B(D'(x),y)$, because B is symmetric $B(y,D(x))=B(y,D'(x))$, as B is a nondegenerate form, we have that $B(y,-)$ is an isomorphism, then $D(x)=D'(x)$. Therefore, D is unique. $\square$

Proposition 3.9. *Let $(\mathcal{A},.)$ be a Poisson-admissible algebra. If D is a derivation on $(\mathcal{A},.)$, then D is a derivation on $(\mathcal{A},\circ,[-,-])$, that is to say, a derivation based on $\circ$ and $[-,-]$.*

Proof. Let x,y be elements of $\mathcal{A}$. We have the equalities:

$$\begin{aligned}
D(x\circ y) &= D(\frac{1}{2}(xy+yx)) = \frac{1}{2}(D(x)y+xD(y)+D(y)x+yD(x))\\
&= \frac{1}{2}(D(x)y+yD(x)+xD(y)+D(y)x) = D(x)\circ y + x\circ D(y)\\
D([x,y]) &= D(xy-yx) = D(x)y+xD(y)-D(y)x-yD(x)\\
&= D(x)y-yD(x)+xD(y)-D(y)x = [D(x),y]+[x,D(y)].
\end{aligned}$$

$\square$

The aim of Proposition 3.8 was to construct a symplectic algebra having a quadratic algebra. Next, we establish some examples which illustrate the proposition given previously.

Example 3.10. Let $(\mathcal{P},.)$ be a Poisson-admissible algebra, $\mathcal{O}:=X/\mathbb{K}[X]$ the ideal of $\mathbb{K}[X]$ generated by X and $\mathcal{R}:=\mathcal{O}/X^n\mathcal{O}$ with $n\in\mathbb{N}^*$ (where $\mathbb{N}$ are the natural numbers without the zero). Since $\mathcal{R}$ is an associative and commutative algebra is generated by the basis $\{\overline{X},\overline{X^2},\dots,\overline{X^n}\}$ as $\mathbb{K}$-module, we can conclude that $\widetilde{\mathcal{P}}:=\mathcal{P}\otimes\mathcal{R}$ is endowed with a multiplication defined by $(x\otimes\overline{P})\bullet(y\otimes\overline{Q}):=x.y\otimes\overline{PQ}$, $\forall x,y\in\mathcal{P}$, $\forall\overline{P},\overline{Q}\in\mathcal{R}$. Thus, it is a nilpotent Poisson-admissible algebra (see the example 3.5). And $(\widetilde{\mathcal{P}}\oplus\widetilde{\mathcal{P}}^*,\bowtie,B)$, whose multiplication $\bowtie$ and the map B are defined as

$$\begin{aligned}
((x\otimes\overline{X^i})+f)\bowtie((y\otimes\overline{X^j})+h) :=& (x\otimes\overline{X^i})\bullet(y\otimes\overline{X^j})+h\circ R_{x\otimes\overline{X^i}}+f\circ L_{y\otimes\overline{X^j}},\\
B((x\otimes\overline{X^i})+f,(y\otimes\overline{X^j})+h) :=& f(y\otimes\overline{X^j})+h(x\otimes\overline{X^i}),
\end{aligned}$$

$\forall(x\otimes\overline{X^i},f),(y\otimes\overline{X^j},h)\in\widetilde{\mathcal{P}}\times\widetilde{\mathcal{P}}^*$ make a quadratic Poisson-admissible algebra (see Example 3.5). From now on, we consider $\mathcal{A}:=\widetilde{\mathcal{P}}\oplus\widetilde{\mathcal{P}}^*$.

Having the endomorphism D of $\widetilde{\mathcal{P}}$ defined by $D(x\otimes\overline{X^i}):=ix\otimes\overline{X^i}$, $\forall x\in\mathcal{P}$, $\forall i\in\{1,\dots,n\}$, we obtain an invertible derivation of $\widetilde{\mathcal{P}}$. In fact, if $x\otimes\overline{X^i},y\otimes\overline{X^j}\in\widetilde{\mathcal{P}}$, we have $D((x\otimes\overline{X^i})\bullet(y\otimes\overline{X^j}))=D(xy\otimes\overline{X^{i+j}})=(i+j)xy\otimes\overline{X^{i+j}}$, and

$$\begin{aligned}
(x\otimes\overline{X^i})\bullet D(y\otimes\overline{X^j}) &+ D(x\otimes\overline{X^i})\bullet(y\otimes\overline{X^j})\\
&= (x\otimes\overline{X^i})\bullet(jy\otimes\overline{X^j})+(ix\otimes\overline{X^i})\bullet(y\otimes\overline{X^j})\\
&= jxy\otimes\overline{X^{i+j}}+ixy\otimes\overline{X^{i+j}} = (i+j)xy\otimes\overline{X^{i+j}}
\end{aligned}$$

Now, defining the endomorphism $\widetilde{D}$ of $\mathcal{A}$ as an invertible derivation of $\mathcal{A}$,

$$\widetilde{D}((x\otimes\overline{X^i})+f):=D(x\otimes\overline{X^i})-f\circ D,\ \forall(x\otimes\overline{X^i},f)\in\widetilde{P}\times\widetilde{P}^*,$$

which is skew-symmetric with respect to B, since

$$\begin{aligned}\widetilde{D}((x\otimes\overline{X^i}+f)\bowtie(y\otimes\overline{X^j}+h)) =&\widetilde{D}((x\otimes\overline{X^i})\bullet(y\otimes\overline{X^j})+h\circ R_{x\otimes\overline{X^i}}+f\circ L_{y\otimes\overline{X^j}})\\ =&\ D((x\otimes\overline{X^i})\bullet(y\otimes\overline{X^j}))\\ &-(h\circ R_{x\otimes\overline{X^i}}+f\circ L_{y\otimes\overline{X^j}})\circ D\\ =&\ D(x\otimes\overline{X^i})\bullet(y\otimes\overline{X^j})+(x\otimes\overline{X^i})\bullet D(y\otimes\overline{X^j})\\ &-h\circ R_{x\otimes\overline{X^i}}\circ D-f\circ L_{y\otimes\overline{X^j}}\circ D\\ =&\ ixy\otimes\overline{X^iX^j}+jxy\otimes\overline{XiX^j}-h\circ R_{x\otimes\overline{X^i}}\circ D\\ &-f\circ L_{y\otimes\overline{X^j}}\circ D,\end{aligned}$$

and,

$$\begin{aligned}&(x\otimes\overline{X^i}+f)\bowtie\widetilde{D}(y\otimes\overline{X^j}+h)+\widetilde{D}(x\otimes\overline{X^i}+f)\bowtie(y\otimes\overline{X^j}+h)\\ &\quad=(x\otimes\overline{X^i}+f)\bowtie(jy\otimes\overline{X^j}-h\circ D)+(ix\otimes\overline{X^i}-f\circ D)\bowtie(y\otimes\overline{X^j}+h)\\ &\quad=jxy\otimes\overline{X^iX^j}-h\circ D\circ R_{x\otimes\overline{X^i}}+f\circ L_{jy\otimes\overline{X^j}}+ixy\otimes\overline{X^iX^j}\\ &\qquad+h\circ R_{ix\otimes\overline{X^i}}-f\circ D\circ L_{y\otimes\overline{X^j}}\\ &\quad=ixy\otimes\overline{X^iX^j}+jxy\otimes\overline{X^iX^j}+h\circ(R_{ix\otimes\overline{X^i}}-D\circ R_{x\otimes\overline{X^i}})\\ &\qquad+f\circ(L_{jy\otimes\overline{X^j}}-D\circ L_{y\otimes\overline{X^j}})\end{aligned}$$

Finally, we see that $-R_{x\otimes\overline{X^i}}\circ D=R_{ix\otimes\overline{X^i}}-D\circ R_{x\otimes\overline{X^i}}$ and $-L_{y\otimes\overline{X^j}}\circ D=L_{jy\otimes\overline{X^j}}-D\circ L_{y\otimes\overline{X^j}}$. If $z\otimes\overline{X^k}\in\widetilde{\mathcal{P}}$, then

$$\begin{aligned}-R_{x\otimes\overline{X^i}}\circ D(z\otimes\overline{X^k})=&\ -D(z\otimes\overline{X^k})\bullet(x\otimes\overline{X^i})\\ (R_{ix\otimes\overline{X^i}}-D\circ R_{x\otimes\overline{X^i}})(z\otimes\overline{X^k})=&\ (z\otimes\overline{X^k})\bullet(ix\otimes\overline{X^i})\\ &-D((z\otimes\overline{X^k})\bullet(x\otimes\overline{X^i}))\\ =&\ (z\otimes\overline{X^k})\bullet D(x\otimes\overline{X^i})\\ &-D((z\otimes\overline{X^k})\bullet(x\otimes\overline{X^i}))\\ -L_{y\otimes\overline{X^j}}\circ D(z\otimes\overline{X^k})=&\ -(y\otimes\overline{X^i})\bullet D(z\otimes\overline{X^k})\\ (L_{jy\otimes\overline{X^j}}-D\circ L_{y\otimes\overline{X^j}})(z\otimes\overline{X^k})=&\ (jy\otimes\overline{X^j})\bullet(z\otimes\overline{X^k})\\ &-D((jy\otimes\overline{X^j})\bullet(z\otimes\overline{X^k}))\\ =&\ D(y\otimes\overline{X^j})\bullet(z\otimes\overline{X^k})\\ &-D((jy\otimes\overline{X^j})\bullet(z\otimes\overline{X^k}))\end{aligned}$$

The equality results follow from the fact that D is a derivation on $\widetilde{\mathcal{P}}$. Due to the previously written, we can extend it to an arbitrary polynomial $\overline{P(X)}\in\mathcal{R}$

since $\{\overline{X}, \overline{X^1}, \ldots, \overline{X^n}\}$ is a basis of $\mathbb{R}$, whence $\widetilde{D}$ is a derivation on $\widetilde{\mathcal{P}} \oplus \widetilde{\mathcal{P}}^*$. In conclusion, the bilinear form ω on $\mathcal{A}$ defined by:

$$\omega(x+f, y+h) := B(\widetilde{D}(x+f), y+h), \ \forall (x,f),(y,h) \in \widetilde{P} \times \widetilde{P}^*$$

generate a symplectic structure on $\mathcal{A}$ (see the Example 3.4). Consequently, $(\mathcal{A}, B, \omega)$ is a symplectic quadratic Poisson-admissible algebra.

4. Quadratic Lie algebras from Poisson–admissibles algebras

One of the most important structures is the quadratic Lie algebra. In this way, we know from Proposition 3.8 that for a quadratic algebra with an invertible derivation, we can endow the algebra with symplectic structure. Based on this proposition and starting with a quadratic Poisson algebra and an invertible derivation, we will construct a symplectic Lie algebra.

Definition 4.1. Let $\mathfrak{g}$ be a Lie algebra and $f : \mathfrak{g} \times \mathfrak{g} \to \mathbb{K}$ such that $f(x,x) = 0$, for all $x \in \mathfrak{g}$. We say that f is a *2-cocycle*, if f satisfies:

$$f(x,[y,z]) + f(z,[x,y]) + f(y,[z,x]) = 0 \quad \forall x,y,z \in \mathfrak{g}.$$

Proposition 4.2. *Let $\mathfrak{g}_1, \mathfrak{g}_2$ be two Lie algebras, B the invariant scalar product of $\mathfrak{g}_1$, $\varphi : \mathfrak{g}_2 \to \operatorname{der}_a(\mathfrak{g}_1, B)$, where*

$$\operatorname{der}_a(g_1, B) := \{f \in \operatorname{End}(\mathfrak{g}_1) : B(f(X_1), Y_1) = -B(X_1, f(Y_1)), \ X_1, Y_1 \in \mathfrak{g}_1\}.$$

Then $\psi(X_1, Y_1)(X_2) := B(\varphi(X_2)(X_1), Y_1)$ for $X_1, Y_1 \in \mathfrak{g}_1$ and $X_2 \in \mathfrak{g}_2$, is a 2-cocycle. Also, it endows with structure of $\mathfrak{g}_1$-module to $\mathfrak{g}_2$.

Proof. We just need to see the proof was made in the Proposition 3.8, due to both are solved in a similar way; where the derivation given is $D := \varphi(X_2)$, and $[-,-]$ the multiplication on which B is invariant. Therefore, we obtain that for all $X_1, Y_1, Z_1 \in \mathfrak{g}_1$:

$$\psi(X_1, [Y_1, Z_1]) + \psi(Z_1, [X_1, Y_1]) + \psi(Y_1, [Z_1, X_1]) = 0,$$

so, ψ is a 2-cocycle. $\square$

Definition 4.3. Let $\mathfrak{g}_1, \mathfrak{g}_2$ be two Lie algebras over a commutative ring $\mathbb{K}$. Suppose there is an action $f : \mathfrak{g}_1 \to \mathfrak{g}_2$. We define the *semi-direct product* $\mathfrak{g}_1 \ltimes_f \mathfrak{g}_2$ of $\mathfrak{g}_1$ and $\mathfrak{g}_2$ by means of $\mathbb{K}$-module $\mathfrak{g}_1 \oplus \mathfrak{g}_2$ with the bracket:

$$[(x_1+x_2),(y_1+y_2)] = [x_1,y_1] + ([x_2,y_2] + f(x_1)y_2 - f(y_1)x_2), \ x_1, y_1 \in \mathfrak{g}_1 \text{ y } x_2, y_2 \in \mathfrak{g}_2.$$

Following the order, to construct the algebras we define a double extension for the case of a quadratic Lie algebra.

Definition 4.4. Let $(\mathfrak{g}_1, [-,-]_1, B_1)$ be a quadratic Lie algebra, $(\mathfrak{g}_2, [-,-]_2)$ a Lie algebra, a homomorphism of Lie algebras $\varphi : \mathfrak{g}_2 \to \mathrm{der}_a(\mathfrak{g}_1, B_1)$ where $\mathrm{der}_a(\mathfrak{g}_1, B_1)$ is the set of skew-symmetric derivation of $\mathfrak{g}_1$ with respect to B_1. The map $\psi : \mathfrak{g}_1 \times \mathfrak{g}_1 \to g_2^*$ defined by $\psi(X_1, Y_1)(X_2) := B_1(\varphi(X_2)(X_1), Y_1)$, for all $X_1, Y_1 \in \mathfrak{g}_1$ y $X_2 \in g_2$; ψ is a 2-cocycle and $\mathfrak{g}_2$ has structure of $\mathfrak{g}_1$-module. In this way, the vector space $\mathfrak{g}_1 \oplus \mathfrak{g}_2^*$ endow with the multiplication:

$$[X_1 + f, Y_1 + h]_c := [X_1, Y_1]_1 + \psi(X_1, Y_1), \quad \forall X_1, Y_1 \in \mathfrak{g}_1, f, h \in g_2^*,$$

is a Lie algebra, which is called *central extension* of $\mathfrak{g}_1$ by means of ψ.

Let π be the co-adjoint representation of $\mathfrak{g}_2$. Given $X_2 \in \mathfrak{g}_2$, the endomorphism $\overline{\varphi}(X_2)$ defined by $\varphi(X_2)(X_1 + f) := \varphi(X_2)(X_1) + \pi(X_2)(f), \forall X_1 \in \mathfrak{g}_1, f \in \mathfrak{g}_2^*$, is a derivation of the Lie algebra $(\mathfrak{g}_1 \oplus g_2^*, [-,-]_c)$. In addition, $\overline{\varphi} : \mathfrak{g}_2 \to \mathrm{der}(\mathfrak{g}_1 \oplus \mathfrak{g}_2^*)$ is a homomorphism of Lie algebras. So $\mathfrak{g} := \mathfrak{g}_2 \ltimes_{\overline{\varphi}} (\mathfrak{g}_1 \oplus \mathfrak{g}_2^*)$ is the semi-direct product of $\mathfrak{g}_1 \oplus \mathfrak{g}_2^*$ by $\mathfrak{g}_2$ by means of $\overline{\varphi}$. We can consider $\mathfrak{g} = \mathfrak{g}_2 \oplus \mathfrak{g}_1 \oplus \mathfrak{g}_2^*$ and the bracket defined by (see Definition 4.3):

$$\begin{aligned}[X_2 + X_1 + f, Y_2 + Y_1 + h] =& [X_2, Y_2]_2 + ([X_1, Y_1]_1 + \varphi(X_2)(Y_1) - \varphi(Y_2)(X_1)) \\ &+ (\pi(X_2)(h) - \pi(Y_2)(f) + \psi(X_1, Y_1)),\end{aligned}$$

for all $(X_2, X_1, f), (Y_2, Y_1, h) \in \mathfrak{g}_2 \times \mathfrak{g}_1 \times \mathfrak{g}_2^*$. Thus, the pair $(\mathfrak{g}, [-,-])$ is a Lie algebra. Furthermore, if $\gamma : \mathfrak{g}_2 \times \mathfrak{g}_2 \to \mathbb{K}$ is an invariant, symmetric bilinear on $\mathfrak{g}_2$, it is easy to see that the bilinear form $B_\gamma : \mathfrak{g} \times \mathfrak{g} \to \mathbb{K}$ defined by:

$$B_\gamma(X_2 + X_1 + f, Y_2 + Y_1 + h) := \gamma(X_2, Y_2) + B_1(X_1, Y_1) + f(Y_2) + h(X_2),$$

for all $(X_2, X_1, f), (Y_2, Y_1, h) \in \mathfrak{g}_2 \times \mathfrak{g}_1 \times \mathfrak{g}_2^*$, is an invariant scalar product on $\mathfrak{g}$, so $(\mathfrak{g}, [-,-], B_\gamma)$ is a quadratic Lie algebra. The pair $(\mathfrak{g}, B_0)$ is called the *double extension* of $(\mathfrak{g}_1, [-,-]_1, B_1)$ by $\mathfrak{g}_2$ by means of φ.

Next, we see that the pair $(\mathfrak{g}_1 \oplus \mathfrak{g}_2^*, [-,-]_c)$ define a Lie algebra: let $(X_1, f), (Y_1, h), (Z_1, g)$ be in $\mathfrak{g}_1 \times \mathfrak{g}_2^*$,

$$-\psi(Y_1, X_1)(X_2) = -B_1(\varphi(X_2)(Y_1), X_1) = B_1(Y_1, \varphi(X_2)(X_1)) = \psi(X_1, Y_1)(X_2),$$

for the previous fact, we use that B_1 is skew-symmetric with respect to $\varphi(X_2)$, and moreover B_1 is symmetric.

Note that the multiplication $[-,-]_c$ is skew-symmetric:

$$\begin{aligned}-[Y_1 + h, X_1 + f]_c =& -[Y_1, X_1]_1 - \psi(Y_1, X_1) \\ =& [X_1, Y_1]_1 - \psi(Y_1, X_1) = [X_1, Y_1]_1 + \psi(X_1, Y_1) \\ =& [X_1 + f, Y_1 + h]_c.\end{aligned}$$

Since we have the equalities

$$\begin{aligned}[X_1 + f, [Y_1 + h, Z_1 + g]_c]_c &= [X_1 + f, [Y_1, Z_1]_1 + \psi(Y_1, Z_1)]_c \\ &= [X_1, [Y_1, Z_1]_1]_1 + \psi(X_1, [Y_1, Z_1]_1), \\ [Z_1 + f, [X_1 + h, Y_1 + g]_c]_c &= [Z_1, [X_1, Y_1]_1]_1 + \psi(Z_1, [X_1, Y_1]_1), \\ [Y_1 + f, [Z_1 + h, X_1 + g]_c]_c &= [Y_1, [Z_1, X_1]_1]_1 + \psi(Y_1, [Z_1, X_1]_1),\end{aligned}$$

and moreover

$$\begin{aligned}
\psi(X_1,[Y_1,Z_1]_1)+\psi(Z_1,[X_1,Y_1]_1)+\psi(Y_1,[Z_1,X_1]_1) &= B_1(\varphi(X_2)(X_1),[Y_1,Z_1]_1)\\
&\quad + B_1(\varphi(X_2)(Z_1),[X_1,Y_1]_1)+B_1(\varphi(X_2)(Y_1),[Z_1,X_1]_1)\\
&= -B_1(X_1,\varphi(X_2)([Y_1,Z_1]_1))-B_1(Z_1,\varphi(X_2)([X_1,Y_1]_1))\\
&\quad - B_1(Y_1,\varphi(X_2)([Z_1,X_1]_1))\\
&= -B_1(X_1,[\varphi(X_2)(Y_1),Z_1]_1)-B_1(X_1,[Y_1,\varphi(X_2)(Z_1)]_1)\\
&\quad - B_1(Z_1,[\varphi(X_2)(X_1),Y_1]_1)\\
&\quad - B_1(Z_1,[X_1,\varphi(X_2)(Y_1)]_1)-B_1(Y_1,[\varphi(X_2)(Z_1),X_1]_1)\\
&\quad - B_1(Y_1,[Z_1,\varphi(X_2)(X_1)]_1)\\
&= B_1(Y_1,[\varphi(X_2)(Z_1),X_1]_1)+B_1(Y_1,[Z_1,\varphi(X_2)(X_1)]_1)\\
&\quad - B_1(X_1,[Y_1,\varphi(X_2)(Z_1)]_1)\\
&\quad + B_1(X_1,[\varphi(X_2)(Y_1),Z_1]_1)+B_1(X_1,[Y_1,\varphi(X_2)(Z_1)]_1)\\
&\quad - B_1(Z_1,[X_1,\varphi(X_2)(X_1)]_1)\\
&\quad - B_1(Y_1,[\varphi(X_2)(Z_1),X_1]_1)-B_1(Y_1,[Z_1,\varphi(X_2)(X_1)]_1)=0,
\end{aligned}$$

then $[X_1+f,[Y_1+h,Z_1+g]_c]_c+[Z_1+f,[X_1+h,Y_1+g]_c]_c+[Y_1+f,[Z_1+h,X_1+g]_c]_c=0$. In other words, the Jacobi equality is satisfied.

Also, we see that $\overline{\varphi}(X_2)\in \mathrm{End}(\mathfrak{g}_1\oplus\mathfrak{g}_2^*)$ is a derivation of Lie algebras: let $(X_1,f),(Y_1,h)$ be in $\mathfrak{g}_1\times\mathfrak{g}_2^*$; because $\varphi(X_2)$ and $\pi(X_2)$ are $\mathbb{K}$-endomorphism, by the form as it is defined $\overline{\varphi}(X_2)$ is an endomorphism; on the other hand,

$$\begin{aligned}
\overline{\varphi}(X_2)([X_1+f,Y_1+h]_c) = \overline{\varphi}(X_2)([X_1,Y_1]_1+\psi(X_1,Y_1)) &= \varphi(X_2)([X_1,Y_1]_1)\\
&\quad + \pi(X_2)(\psi(X_1,Y_1)),
\end{aligned}$$

and,

$$\begin{aligned}
[x_1+f,\overline{\varphi}(X_2)(Y_1+h)]_c+[\overline{\varphi}(X_2)(X_1+f),Y_1+h]_c &= [X_1+f,\varphi(X_2)(Y_1+h)\\
&\quad +\pi(X_2)(h)]_c+[\varphi(X_2)(X_1)+\pi(X_2)(f),Y_1+h]_c\\
&= [X_1,\varphi(X_2)(Y_1)]_1+\psi(X_1,\varphi(X_2)(Y_1))+[\varphi(X_2)(X_1),Y_1]_1+\psi(\varphi(X_2)(X_1),Y_1).
\end{aligned}$$

Because $\varphi(X_2)\in \mathrm{der}_a(\mathfrak{g}_1,B_1)$, $\varphi(X_2)([X_1,Y_1]_1) = [X_1,\varphi(X_2)(Y_1)]_1 +[\varphi(X_2)(X_1),(Y_1)]_1$, let us see that $\pi(X_2)(\psi(X_1,Y_1))=\psi(X_1,\varphi(X_2)(Y_1))+\psi(\varphi(X_2)(X_1),Y_1)$; due to $\pi:\mathfrak{g}_2^*\to\mathrm{End}(\mathfrak{g}_2^*)$ is the coadjoint representation, we have that

$$\pi(X_2)(\psi(X_1,Y_1))(Y_2)=\psi(X_1,Y_1)(-[X_2,Y_2]_2).$$

Since we have the equalities

$$\begin{aligned}
\psi(X_1,Y_1)(-[X_2,Y_2]_2) &= B_1(\varphi(-[X_2,Y_2]_2)(X_1),Y_1)\\
&= B_1(-[\varphi(X_2),\varphi(Y_2)](X_1),Y_1)\\
&= B_1([\varphi(Y_2),\varphi(X_2)](X_1),Y_1)\\
&= B_1((\varphi(Y_2)\circ\varphi(X_2)-\varphi(X_2)\circ\varphi(Y_2))(X_1),Y_1)\\
&= -B_1(\varphi(X_2)(\varphi(Y_2)(X_1)),Y_1)+B_1(\varphi(Y_2)(\varphi(X_2)(X_1)),Y_1)\\
&= B_1(\varphi(Y_2)(X_1),\varphi(X_2)(Y_1))+B_1(\varphi(Y_2)(\varphi(X_2)(X_1)),Y_1)\\
&= (\psi(X_1,\varphi(X_2)(Y_1))+\psi(\varphi(X_2)(X_1),Y_1))(Y_2),
\end{aligned}$$

it follows that $\pi(X_2)(\psi(X_1,Y_1)) = \psi(X_1,\varphi(X_2)(Y_1)) + \psi(\varphi(X_2)(X_1),Y_1)$. Furthermore, as $\overline{\varphi}([X_2,Y_2]_2) = [\overline{\varphi}(X_2),\overline{\varphi}(Y_2)]$, due to

$$\begin{aligned}
\overline{\varphi}([X_2,Y_2]_2)(X_1+f) &= \varphi([X_2,Y_2]_2)(X_1) + \pi([X_2,Y_2]_2)(f)\\
&= [\varphi(X_2),\varphi(Y_2)](X_1) + [\pi(X_2),\pi(Y_2)](f)\\
&= (\varphi(X_2)\circ\varphi(Y_2) - \varphi(Y_2)\circ\varphi(X_2))(X_1)\\
&\quad + (\pi(X_2)\circ\pi(Y_2) - \pi(X_2)\circ\pi(Y_2))(f)\\
&= (\varphi(X_2)\circ\varphi(Y_2))(X_1) + (\pi(X_2)\circ\pi(Y_2))(f) - (\varphi(Y_2)\circ\varphi(X_2))(X_1)\\
&\quad - (\pi(Y_2)\circ\pi(X_2))(f)\\
&= \overline{\varphi}(X_2)(\varphi(Y_2)(X_1) + \pi(Y_2)(f)) - \overline{\varphi}(Y_2)(\varphi(X_2)(X_1) + \pi(X_2)(f))\\
&= \overline{\varphi}(X_2)(\overline{\varphi}(Y_2)(X_1+f)) - \overline{\varphi}(Y_2)(\overline{\varphi}(X_2)(X_1+f))\\
&= (\overline{\varphi}(X_2)\circ\overline{\varphi}(Y_2) - \overline{\varphi}(Y_2)\circ\overline{\varphi}(X_2))(X_1+f)\\
&= [\overline{\varphi}(X_2),\overline{\varphi}(Y_2)](X_1+f),
\end{aligned}$$

i.e., $\overline{\varphi}$ is a homomorphism of Lie algebras.

Now, we will see that $(\mathfrak{g},[-,-])$ is a Lie algebra. Let us see that the properties are satisfied:

- The bracket is skew-symmetric: Let X_2+X_1+f, Y_2+Y_1+h be in $\mathfrak{g}$, we remind that $\psi(Y_1,X_1)(Z) = B_1(\varphi(Z)(X_1),Y_1) = -B_1(X_1,\varphi(Z)(Y_1)) = -B_1(\varphi(Z)(Y_1),X_1) = -\psi(Y_1,X_1)(Z)$, with $Z\in\mathfrak{g}_2$, so:

$$\begin{aligned}
[Y_2+Y_1+h, X_2+X_1+f] &= [Y_2,X_2]_2 + ([Y_1,X_1]_1 + \varphi(Y_2)(X_1) - \varphi(X_2)(Y_1))\\
&\quad + (\pi(Y_2)(g) - \pi(X_2)(h) + \psi(Y_1,X_1))\\
&= -([X_2,Y_2]_+ (([X_1,Y_1]) + \varphi(X_2)(Y_1) - \varphi(Y_2)(X_1))\\
&\quad + (\pi(X_2)(h) - \pi(Y_2)(g) - (-\psi(Y_1,X_1))))\\
&= -[X_2+X_1+f, Y_2+Y_1+h].
\end{aligned}$$

- From the previous calculus, we can see without difficulty that $[-,-]$ satisfies the Jacobi identity:

$$\begin{aligned}
&[X_2+X_1+f,[Y_2+Y_1+h,Z_2+Z_1+g]] = [X_2+X_1+f,[Y_2,Z_2]_2+([Y_1,Z_1]_1\\
&\qquad +\varphi(Y_2)(Z_1)-\varphi(Z_2)(Y_1))+(\pi(Y_2)(g)-\pi(Z_2)(h)+\varphi(Y_1,Z_1))]\\
&\qquad = [X_2,[Y_2,Z_2]_2]_2+([X_1,[Y_1,Z_1]_1]_1+[X_1,\varphi(Y_2)(Z_1)]_1-[X_1,\varphi(Z_2)(Y_1)]_1\\
&\qquad +\varphi(X_2)([Y_1,Z_1])+\varphi(X_2)\circ\varphi(Y_2)(Z_1)-\varphi(X_2)\circ\varphi(Z_2)(Y_1)\\
&\qquad -\varphi([Y_2,Z_2]_2)(X_1))\\
&\qquad +(\pi(X_2)\circ\pi(Y_2)(g)-\pi(X_2)\circ\pi(Z_2)(h)+\pi(X_2)(\psi(Y_1,Z_1))\\
&\qquad -\pi([Y_2,Z_2]_2)(f)\\
&\qquad +\psi(X_1,[Y_1,Z_1]_1)+\psi(X_1,\varphi(Y_2)(Z_1))-\psi(X_1,\varphi(Z_2)(Y_1)))\\
&[Z_2+Z_1+g,[X_2+X_1+f,Y_2+Y_1+h]]\\
&\qquad = [Z_2,[X_2,Y_2]_2]_2+([Z_1,[X_1,Y_1]_1]_1+[Z_1,\varphi(X_2)(Y_1)]_1-[Z_1,\varphi(Y_2)(X_1)]_1\\
&\qquad +\varphi(Z_2)([X_1,Y_1])+\varphi(Z_2)\circ\varphi(X_2)(Y_1)-\varphi(Z_2)\circ\varphi(Y_2)(X_1)\\
&\qquad -\varphi([X_2,Y_2]_2)(Z_1))\\
&\qquad +(\pi(Z_2)\circ\pi(X_2)(h)-\pi(Z_2)\circ\pi(Y_2)(f)+\pi(Z_2)(\psi(X_1,Y_1))-\pi([X_2,Y_2]_2)(g)\\
&\qquad +\psi(Z_1,[X_1,Y_1]_1)+\psi(Z_1,\varphi(X_2)(Y_1))-\psi(Z_1,\varphi(Y_2)(X_1)))\\
&[Y_2+Y_1+h,[Z_2+Z_1+g,Z_2+Z_1+f]]\\
&\qquad = [Y_2,[Z_2,X_2]_2]_2+([Y_1,[Z_1,X_1]_1]_1+[Y_1,\varphi(Z_2)(X_1)]_1-[Y_1,\varphi(X_2)(Z_1)]_1\\
&\qquad +\varphi(Y_2)([Z_1,X_1])+\varphi(Y_2)\circ\varphi(Z_2)(X_1)-\varphi(Y_2)\circ\varphi(X_2)(Z_1)\\
&\qquad -\varphi([Z_2,X_2]_2)(Y_1))\\
&\qquad +(\pi(Y_2)\circ\pi(Z_2)(f)-\pi(Y_2)\circ\pi(X_2)(g)+\pi(Y_2)(\psi(Z_1,X_1))-\pi([Z_2,X_2]_2)(h)\\
&\qquad +\psi(Y_1,[Z_1,X_1]_1)+\psi(Y_1,\varphi(Z_2)(X_1))-\psi(Y_1,\varphi(X_2)(Z_1))).
\end{aligned}$$

The sum of the previous terms results by the following reasons: Jacobi identity for the respective algebras; $\varphi(X_2)$ is a derivation with respect to $[-,-]_1$; φ is a Lie homomorphism; π is a homomorphism of Lie algebras; ψ is a 2-cocycle, in other words $\varphi(X_1,[Y_1,Z_1])+\varphi(Z_1,[X_1,Y_1])+\varphi(Y_1,[Z_1,X_1])=0$; previously we show that $\pi(X_2)(\psi(Y_1,Z_1)) = \psi(Y_1,\varphi(X_2)(Z_1)) + \psi(\varphi(X_2)(Y_1),Z_1) = \psi(Y_1,\varphi(X_2)(Z_1)) - \psi(Z_1,\varphi(X_2)(Y_1)$ (we use that $\varphi(X)\in \mathrm{der}_a(\mathfrak{g}_1,B_1)$). As a consequence, we show that $(\mathfrak{g},[-,-])$ is a Lie algebra.

Finally, we need to see that B_γ is an invariant scalar product on $\mathfrak{g}$, so:

- Because each γ and B_1 are symmetric bilinear forms, and moreover the sum is commutative, $B_\gamma(X_2+X_1+f,Y_2+Y_1+h)=B_\gamma(Y_2+Y_1+h,X_2+X_1+f)$. In other words, B_γ is a symmetric bilinear form.

- Since we have the equalities

$$
\begin{aligned}
&B_\gamma([X_2+X_1+f,Y_2+Y_1+h],Z_2+Z_1+g)\\
&\quad= B_\gamma([X_2,Y_2]_2+([X_1,Y_1]_1+\varphi(X_2)(Y_1)-\varphi(Y_2)(X_1))\\
&\qquad+(\pi(X_2)(h)-\pi(Y_2)(f)+\psi(X_1,Y_1)),Z_2+Z_1+g)\\
&\quad= \gamma([X_2,Y_2]_2,Z_2)+B_1([X_1,Y_1]_1+\varphi(X_2)(Y_1)-\varphi(Y_2)(X_1),Z_1)\\
&\qquad+(\pi(X_2)(h)-\pi(Y_2)(f)+\psi(X_1,Y_1))(Z_2)+g([X_2,Y_2]_2)\\
&\quad= \gamma(X_2,[Y_2,Z_2]_2)+B_1([X_1,Y_1]_1,Z_1)+B_1(\varphi(X_2)(Y_1),Z_1)\\
&\qquad-B_1(\varphi(Y_2)(X_1),Z_1)+(\pi(X_2)(h)(Z_2)-\pi(Y_2)(f)(Z_2)\\
&\qquad+\psi(X_1,Y_1)(Z_2))+g([X_2,Y_2]_2)\\
&\quad= \gamma(X_2,[Y_2,Z_2]_2)+B_1(X_1,[Y_1,Z_1]_1)+\psi(Y_1,Z_1)(X_2)\\
&\qquad+B_1(X_1,\varphi(Y_2)(Z_1))\\
&\qquad-B_1(X_1,\varphi(Z_2)(Y_1))+h(-[X_2,Z_2]_2)-f(-[Y_2,Z_2]_2)\\
&\qquad+B_1(\varphi(Z_2)(X_1),Y_1)+g(-[Y_2,X_2]_2)\\
&\quad= \gamma(X_2,[Y_2,Z_2]_2)+B_1(X_1,[Y_1,Z_1]_1)+\psi(Y_1,Z_1)(X_2)\\
&\qquad+B_1(X_1,\varphi(Y_2)(Z_1))\\
&\qquad-B_1(X_1,\varphi(Z_2)(Y_1))-\pi(Z_2)(h)(X_2)+f([Y_2,Z_2]_2)\\
&\qquad-B_1(X_1,\varphi(Z_2)(Y_1))+\pi(Y_2)(g)(X_2)\\
&\quad= \gamma(X_2,[Y_2,Z_2]_2)+B_1(X_1,[Y_1,Z_1]_1)+B_1(X_1,\varphi(Y_2)(Z_1))\\
&\qquad-B_1(X_1,\varphi(Z_2)(Y_1))\\
&\qquad+f([Y_2,Z_2]_2)+\pi(Y_2)(g)(X_2)-\pi(Z_2)(h)(X_2)+\psi(Y_1,Z_1)(X_2)\\
&\quad= \gamma(X_2,[Y_2,Z_2]_2)+B_1(X_1,[Y_1,Z_1]_1+\varphi(Y_2)(Z_1)-\varphi(Z_2)(Y_1))\\
&\qquad+f([Y_2,Z_2]_2)\\
&\qquad+(\pi(Y_2)(g)-\pi(Z_2)(h)+\psi(Y_1,Z_1))(X_2)\\
&\quad= B_\gamma(X_2+X_1+f,[X_2,Y_2]_2+([X_1,Y_1]_1+\varphi(X_2)(Y_1)-\varphi(Y_2)(X_1))\\
&\qquad+(\pi(X_2)(h)-\pi(Y_2)(f)+\psi(Y_1,Z_1))\\
&\quad= B_\gamma(X_2+X_1+f,[Y_2+Y_1+h,Z_2+Z_1+g]),
\end{aligned}
$$

we conclude that the bilinear form B_γ is invariant with respect to $[-,-]$.

- Suppose that for all $Y_2+Y_1+h\in\mathfrak{g}$, we have that $B_\gamma(X_2+X_1+f,Y_2+Y_1+h)=0$, so $B_\gamma(X_2+X_1+f,Y_2+Y_1+h)=\gamma(X_2,Y_2)+B(X_1,Y_1)+f(Y_2)+h(X_2)=0$. Now, taking an arbitrary element of $\mathfrak{g}$:
  - If $Y_2=0$ and $h=0$ then $\gamma(X_2,0)+B(X_1,Y_1)=0$, because $\gamma(X_2,0)=0$, then $B(X_1,Y_1)=0$. Due to B_1 is a nondegenerate bilinear form and for all $Y_1\in\mathfrak{g}_1$, $B(X_1,Y_1)=0$, we have that $X_1=0$.
  - From the previous fact and now taking $Y_2=0$, we have that $\gamma(X_2,0)+B(0,Y_1)+f(0)+h(X_2)=h(X_2)=0$. Due to h is an arbitrary map, as a result $X_2=0$.

– In this way $\gamma(0,Y_2)+B(0,Y_1)+f(Y_2)+h(0)=f(Y_2)=0$, inasmuch as $Y_2\in\mathfrak{g}_2$ is arbitrary, we conclude that $f=0$. Therefore, we have that X_2+X_1+f is the zero of $\mathfrak{g}$.

In the following example we construct a quadratic Lie algebra from a Poisson-admissible algebra by means the concept of double extension.

Example 4.5 (See [2]). Let $(\mathcal{A},.,B)$ be a quadratic Poisson-admissible algebra. Then $(\mathcal{A}^-,[-,-],B)$ is a quadratic algebra and $(\mathcal{A}^+,\circ,B)$ is a symmetric commutative algebra. Let us consider the Lie algebra three-dimensional $\mathfrak{sl}(2)$. The vector space $\mathfrak{sl}(2)\otimes\mathcal{A}^+$ with bracket $[-,-]_1$ defined by

$$[x\otimes a,y\otimes b]:=[x,y]\otimes a\circ b,\quad \forall(x,a),(y,b)\in\mathfrak{sl}(2)\times\mathcal{A},$$

is a Lie algebra. We consider the form $B_1:(\mathfrak{sl}(2)\otimes\mathcal{A}^+)\times(\mathfrak{sl}(2)\otimes\mathcal{A}^+)\to\mathbb{K}$ defined by

$$B_1(x\otimes a,y\otimes b):=K(x,y)B(a,b),\quad \forall(x,a),(y,b)\in\mathfrak{sl}(2)\times\mathcal{A},$$

so $(\mathfrak{sl}(2)\otimes\mathcal{A}^+,[-,-]_1,B_1)$ is a quadratic Lie algebra, where K is the Killing form of $\mathfrak{sl}(2)$, that is to say $K(x,y):=trace(\mathrm{ad}_x\mathrm{ad}_y)$. Similarly as in example 3.10, if D is a derivation of $(\mathcal{A}^+,\circ)$, then $\overline{D}:=id_{\mathfrak{sl}(2)}\otimes D$ is a derivation of Lie algebra $(\mathfrak{sl}(2)\otimes\mathcal{A}^+,[-,-]_1)$. Moreover, if D is skew-symmetric with respect to B, it follows that $\overline{D}$ is skew-symmetric with respect to B_1. In fact, let $(x,a),(y,b)$ be in $\mathfrak{sl}(2)\times\mathcal{A}^+$,

$$\begin{aligned}B_1(\overline{D}(x\otimes a),y\otimes b)&=K(x,y)B(D(a),b)=-K(x,y)B(a,D(b))\\&=-B_1(x\otimes a,\overline{D}(y\otimes b)).\end{aligned}$$

In addition, $\overline{D}$ is an inner derivation of the Lie algebra $(\mathfrak{sl}(2)\otimes\mathcal{A}^+,[-,-]_1)$.

Since $(\mathcal{A},.)$ is a Poisson-admissible algebra, then for all $x\in\mathcal{A}$ we have that $\delta_x:=\mathrm{ad}_{\mathcal{A}^-}x$ is derivation of $(\mathcal{A}^+,\circ)$ and furthermore is skew-symmetric with respect to B. So, for all $x\in\mathcal{A}$, $\overline{\delta_x}$ e is a skew-symmetric derivation of $(\mathfrak{sl}(2)\otimes\mathcal{A}^+,[-,-]_1,B_1)$, moreover $\overline{\delta_x}$ can not be inner (see proposition 4.6). Let us consider $x\notin Z(\mathcal{A})$. We have that $\mathrm{ad}_{\mathcal{A}^-}x\neq 0$ (because if we consider $x\in Z(\mathcal{A})$, $\mathrm{ad}_{\mathcal{A}^-}x=0$ and it does not provide any additional information to the algebra, which we want to make). Then we can consider $\mathfrak{g}(\mathcal{A}):=\mathcal{A}^-\oplus(\mathfrak{sl}(2)\otimes\mathcal{A}^+)\otimes(\mathcal{A}^-)^*$ the double extension of $(\mathfrak{sl}(2)\otimes\mathcal{A}^+,[-,-]_1,B_1)$ by the Lie algebra $\mathcal{A}^-$ by means the homomorphism $\varphi:\mathcal{A}^-\to\mathrm{der}_a(\mathfrak{sl}(2)\otimes\mathcal{A}^+,B_1)$ defined as $\varphi(X):=\delta_x$, for all $x\in\mathcal{A}$. Therefore, $(\mathfrak{g}(\mathcal{A}),[-,-],T_0)$ is a quadratic Lie algebra, where $T_0(x+s\otimes a+f,y+s'\otimes b+h):=K(s,s')B(a,b)+f(y)+h(x)$, for all $x,y,a,b\in\mathcal{A}$, $f,h\in\mathcal{A}^*$.

Proposition 4.6. *Let $(A,.,B)$ a Poisson-admissible algebra, $(\mathfrak{sl}(2),[-,-])$ the Lie algebra whose basis is $\{H,E,F\}$ such that $[H,E]=E$, $[H,F]=-F$, $[E,F]=2H$ and $D\in\mathrm{der}(\mathcal{A})$. $\overline{D}$ is an inner derivation of the Lie algebra $(\mathfrak{sl}(2)\otimes\mathcal{A}^+,[-,-]_1)$ if and only if $D=0$.*

Proof. If $D = 0$, because $\overline{D} : \mathfrak{sl}(2) \otimes \mathcal{A}^+ \to \mathfrak{sl}(2) \otimes \mathcal{A}^+$, $x \otimes D(a) = x \otimes 0$ with $x \in \mathfrak{sl}(2)$, and $a \in A^+$, it follows that $\overline{D} = 0_{\mathfrak{sl}(2)\otimes\mathcal{A}^+}$.

On the other hand, if $\overline{D}$ is an inner derivation, then

$$\overline{D} = \mathrm{ad}(H \otimes a_1) + \mathrm{ad}(E \otimes a_2) + (F \otimes a_3),$$

with $a_1, a_2, a_3 \in \mathcal{A}$. Sea $a \in \mathcal{A}$ y $H \in \mathfrak{sl}(2)$ an element of the basis, so

$$\begin{aligned}
\overline{D}(H \otimes a) &= (\mathrm{ad}(H \otimes a_1) + \mathrm{ad}(E \otimes a_2) + \mathrm{ad}(F \otimes a_3))(H \otimes a) \\
H \otimes D(a) &= [H \otimes a_1, H \otimes a] + [E \otimes a_2, H \otimes a] + [F \otimes a_3, H \otimes a] \\
&= [H, H] \otimes a_1 \circ a + [E, H] \otimes a_2 \circ a + [F, H] \otimes a_3 \circ a \\
&= -E \otimes a_2 \circ a + F \otimes a_3 \circ a,
\end{aligned}$$

because H, E, F are different elements in the basis, $H \otimes D(a)$ can not generate from elements with form $E \otimes b$, $F \otimes c$ con $b, c \in \mathcal{A}$. Moreover, as $H \neq 0$ we have that $D(a) = 0$, due to a is arbitrary. Therefore, $D = 0$. □

Lemma 4.7. *If D is a derivation of a quadratic Poisson-admissible algebra $(\mathcal{A}, ., B)$ additionally skew-symmetric with respect to B, then the endomorphism $\widetilde{D}$ over $\mathfrak{g}(\mathcal{A})$ defined by*

$$\widetilde{D}(x) := D(x); \widetilde{D}(f) = -f \circ D; \widetilde{D}(s \otimes a) := s \otimes D(a), \forall a, x \in \mathcal{A},\ f \in \mathcal{A}^*,\ s \in \mathfrak{sl}(2),$$

is a derivation of Lie algebra $\mathfrak{g}(\mathcal{A})$, skew-symmetric with respect to T, that is the invariant scalar product over $g(\mathcal{A})$. Even more if D is invertible, then $\widetilde{D}$ is invertible.

Proof. Let $x + s \otimes a + f, y + r \otimes b + h$ be in $\mathfrak{g}(\mathcal{A})$, with $\mathfrak{g}(\mathcal{A})$ defined as in Example 4.5. Let us see that $\widetilde{D}$ is a derivation:

$$\begin{aligned}
\widetilde{D}([x + s \otimes a + f, y + r \otimes b + h]) &= \widetilde{D}([x, y] + ([s \otimes a, r \otimes b] + \varphi(x)(r \otimes b) \\
&\quad - \varphi(y)(s \otimes a)) \\
&\quad + (\pi(x)(h) - \pi(y)(f) + \psi(s \otimes a, r \otimes b)) \\
&= \widetilde{D}([x, y]) + \widetilde{D}([s, r] \otimes (a \circ b) + r \otimes [x, b] - s \otimes [y, a]) \\
&\quad + \widetilde{D}(\pi(x)(h) - \pi(y)(f) + \psi(s \otimes a, r \otimes b)) \circ D \\
&= D([x, y]) + [s, r] \otimes D(a \circ b) + r \otimes D([x, b]) - s \otimes D([y, a]) - \pi(x)(h) \circ D \\
&\quad + \pi(y)(f) \circ D - \psi(s \otimes a, r \otimes b) \circ D,
\end{aligned}$$

and,

$$\begin{aligned}
&[\widetilde{D}(x+s\otimes a+f), y+r\otimes b+h]+[x+s\otimes a+f, \widetilde{D}(y+r\otimes b+h)]\\
&\quad=[D(x)+s\otimes D(a)-f\circ D, y+r\otimes b+h]\\
&\quad+[x+s\otimes a+f, D(y)+s\otimes D(b)-h\circ D]\\
&\quad=[D(x),y]+([s\otimes D(a), r\otimes b]+\varphi(D(x))(y\otimes b)-\varphi(y)(s\otimes D(a)))\\
&\quad+(\pi(D(x))(h)-\pi(y)(-f\circ D)+\psi(s\otimes D(a), r\otimes b))+[x,D(y)]\\
&\quad+([s\otimes a, r\otimes D(b)]+\varphi(x)(r\otimes D(b))-\varphi(D(y))(s\otimes a))\\
&\quad+(\pi(x)(-h\circ D)-\pi(D(y))(f)+\psi(s\otimes a, r\otimes D(b))).
\end{aligned}$$

The two previous elements in $\mathfrak{g}(\mathcal{A})$ are equal, since if we take an arbitrary $z\in\mathcal{A}^-$, then we have

$$\begin{aligned}
[s\otimes D(a), r\otimes b]+\varphi(D(x))(y\otimes b)-\varphi(y)(s\otimes D(a))+[s\otimes a, r\otimes D(b)]&\\
+\varphi(x)(r\otimes D(b))&\\
-\varphi(D(y))(s\otimes a)&=[s\otimes D(a), r\otimes b]+r\otimes[D(x),b]-s\otimes[y,D(a)]\\
&\quad+[s\otimes a, r\otimes D(b)]+r\otimes[x,D(b)]-s\otimes[D(y),a]\\
&=[s\otimes D(a), r\otimes b]+[s\otimes a, r\otimes D(b)]\\
&\quad+r\otimes([D(x),b]+[x,D(b)])\\
&\quad-s\otimes([y,D(a)]+[D(y),a])\\
&=[s,r]\otimes D(a)\circ b+[s,r]\otimes a\circ D(b)+r\otimes D([x,b])\\
&\quad-s\otimes D([y,a])\\
&=[s,r]\otimes D(a\circ b)+r\otimes D([x,b])-s\otimes D([y,a]),
\end{aligned}$$

and,

$$\begin{aligned}
(\pi(D(x))(h)+\pi(x)(-h\circ D))(z)&=h(-[D(x),z])-h\circ D(-[x,z])\\
&=h(-[D(x),z])+h(D([x,z]))\\
&=h([x,z]-[D(x),z])\\
&=h([x,D(z)])\\
&=(-\pi(x)(h)\circ D)(z),\\
(\pi(D(y))(f)+\pi(y)(-f\circ D))(z)&=(-\pi(y)(f)\circ D)(z),
\end{aligned}$$

together with

$$
\begin{aligned}
(\psi(s\otimes D(a), r\otimes b) + \psi(s\otimes a, r\otimes D(b)))(z) \\
&= B_1(\varphi(z)(s\otimes D(a)), r\otimes b) + B_1(\varphi(z)(s\otimes a), r\otimes D(b)) \\
&= B_1(s\otimes [z, D(a)], r\otimes b) + B_1(s\otimes [z,a], r\otimes D(b)) \\
&= K(s,r)B([z, D(a)], b) + K(s,r)B([z,a], D(b)) \\
&= K(s,r)B([z, D(a)], b) - K(s,r)B(D([z,a]), b) \\
&= K(s,r)B([z, D(a)] - D([z,a]), b) \\
&= K(s,r)B(-[D(z), a], b) \\
&= -B_1(s\otimes [D(z), a], r\otimes b) \\
&= -B_1(\varphi(D(z))(s\otimes a), r\otimes b) \\
&= (-\psi(s\otimes a, r\otimes b)\circ D)(z).
\end{aligned}
$$

Furthermore, $\widetilde{D}$ is skew-symmetric with respect to T, since

$$
\begin{aligned}
T(\widetilde{D}(x+s\otimes a+f), y+r\otimes b+h) &= T(D(x)+s\otimes D(a) - f\circ D, y + r\otimes b + h) \\
&= B_1(s\otimes D(a), r\otimes b) - (f\circ D)(y) + h(D(x)) \\
&= K(s,r)B(D(a), b) - (f\circ D)(y) + h(D(x)) \\
&= -(K(s,r)B(a, D(b)) - (h\circ D)(x) + f(D(y))) \\
&= -T(x+s\otimes a+f, D(y) + r\otimes D(b) - h\circ D) \\
&= -T(x+s\otimes a+f, \widetilde{D}(y+r\otimes b+h)).
\end{aligned}
$$

If D is invertible, then there exists D^{-1}, so we consider $\widetilde{D}^{-1}$, where

$$
\begin{aligned}
&\widetilde{D}^{-1}(x) := D^{-1}(x); \widetilde{D}^{-1}(f) = -f\circ D^{-1}; \\
&\widetilde{D}^{-1}(s\otimes a) := s \ \otimes \ D^{-1}(a), \forall a, x\in\mathcal{A}, \ f\in\mathcal{A}^*, \ s\in\mathfrak{sl}(2),
\end{aligned}
$$

$$
\begin{aligned}
(\widetilde{D}\circ\widetilde{D}^{-1})(x+s\otimes a+f) &= \widetilde{D}(D^{-1}(x) + s\otimes D^{-1}(a) - f\circ D^{-1}) \\
&= D(D^{-1}(x)) + s\otimes D(D^{-1}(a)) - (-f\circ D^{-1})\circ D \\
&= x + s\otimes a + f\circ D^{-1}\circ D \\
&= x + s\otimes a + f.
\end{aligned}
$$

Similarly we have that $(\widetilde{D}^{-1}\circ\widetilde{D})(x+s\otimes a+f) = x+s\otimes a+f$. Thus, $\widetilde{D}\circ\widetilde{D}^{-1} = \widetilde{D}^{-1}\circ\widetilde{D} = \mathrm{id}_{\mathfrak{g}(\mathcal{A})}$, therefore $\widetilde{D}$ is invertible. □

Theorem 4.8 (see [2])**.** *If $(\mathcal{A}, B, \omega)$ is a quadratic Poisson-admisible algebra and D a skew-symmetric invertible derivation (with respect to B) of $\mathcal{A}$ such that $\omega(x,y) = B(D(x), y)$, for all $x, y\in\mathcal{A}$, then $(g(\mathcal{A}), T, \Omega)$ is a symplectic* *quadratic Lie algebra where* Ω *is the symplectic structure over the Lie algebra* $\mathfrak{g}(\mathcal{A})$ *defined by:*

$$\Omega(X,Y) := T(\widetilde{D}(X),Y),\ \forall X,Y\in\mathfrak{g}(\mathcal{A}).$$

Proof. Note that $(g(\mathcal{A}),T)$ is a quadratic Lie algebra given by construction. Since D is an invertible skew-symmetric (with respect to B) derivation of $\mathcal{A}$, Lemma 4.7 guarantees that $\widetilde{D}$ is an invertible skew-symmetric (with respect to T) derivation defined on $\mathfrak{g}(\mathcal{A})$. Now, from Proposition 3.8 we conclude that Ω define a symplectic structure on $\mathfrak{g}(\mathcal{A})$. $\square$

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