Ill Posedness of a neural field equation with Heaviside firing rate function

Mala colocación de una ecuación de campo neuronal con tasa de disparo tipo Heaviside

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Abstract. We consider the initial value problem associated to the neural field equation

$$u_t(x,t) = -u(x,t) + \int_{\mathbb{R}^m} w(x,y)[1+\gamma g(u(x,t)-u(y,t))]f(u(y,t))\, dy,$$
$$(x,t) \in \mathbb{R}^m \times (0,\infty),$$

where f is a Heaviside function, then we show that the problem is ill posed in $C_b(\mathbb{R}^m)$. The proof follows from a discontinuity argument apply to the equation's flow.

Keywords: Neural field equation, firing rate function, synaptic and sensitive kernel, well and ill posedness, flow of the equation.

Resumen. En este trabajo probamos que el problema de valor inicial asociado a la ecuación de campo neuronal

$$u_t(x,t) = -u(x,t) + \int_{\mathbb{R}^m} w(x,y)[1+\gamma g(u(x,t)-u(y,t))]f(u(y,t))\, dy,$$
$$(x,t) \in \mathbb{R}^m \times (0,\infty),$$

está mal colocado en $C_b(\mathbb{R}^m)$ si f es una función de tipo Heaviside. La prueba es obtenida por un argumento de discontinuidad aplicado al flujo de la ecuación.

Palabras claves: Ecuación de campo neuronal, tasa de disparo, núcleo sináptico y sensible, buena y mala colocación, flujo de la ecuación.

Mathematics Subject Classification: 35R09.

Recibido: abril de 2015 Aceptado: agosto de 2015

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1. Introduction

Recently Abbassian, Fotouhi and Heidari [1] proposed a neural field model based on the difference in activation at different locations of a neuron population, a kind of behavior called Hebbian learning. That model is

$$u_t(x,t) = -u(x,t) + \int_{\mathbb{R}} w(x-y)[1+\gamma g(u(x,t)-u(y,t))]f(u(y,t))\,dy,$$
$$(x,t) \in \mathbb{R} \times (0,\infty),$$

where $u(x,t)$ is the average membrane potential of a neuron population at position x and time t, w is a kernel of synaptic coupling (usually even), $\gamma > 0$ is a fixed parameter called the Kernel Coefficient Strength, g is proposed to be gaussian $g(u) = e^{-u^2}$ and f is a firing rate function, tipically with the shape of the logistic function, that is

$$f(s) = \frac{1}{1+e^{-(s-\eta)}}.$$

The work of Abbasian et al. [1] is concerned with the existence and stability of solutions as rest state, time-independent solutions (bump type), and traveling waves, for several types of kernels.

The case $\gamma = 0$ is the classical Amari's model or Wilson-Cowan type (see [2, 8])

$$u_t(x,t) = -u(x,t) + \int_{\mathbb{R}} w(x-y)f(u(y,t))\,dy, \quad (x,t) \in \mathbb{R} \times (0,\infty), \tag{1}$$

which describes the dynamics of the spatio-temporal electrical activity in neural tissue in one spatial dimension.

If $f(u(x,t)) > 0$ then the neurons at point x are said to be active.

We refer the reader to [3] or [4] and references therein for more details about this models.

Because of the convolution $w * (f \circ u)$ the equation (1) is a nonlocal model. Some recent work on non-local operators arising in neural field models can be found in [5, 6].

Potthast and Beim Graben [7] provided a rigorous approach to study global existence of solutions to the Wilson-Cowan type of the model with the smooth firing rate function as well as with the unit step function. They demonstrated that the latter case requires more restrictions on the choice of a functional space as well as some extra assumptions on the kernel w.

In the case with smooth firing rate f and a more general synaptic kernel function w, they demostrated well posedness in the continuous bounded functions space by a standard fixed point argument. Since the model can be write as $u' = Fu$ ($u' = u_t$), where

$$(Fu)(x,t) = -u(x,t) + (Ju)(x,t), \tag{2}$$

and

$$(Ju)(x,t) = \int_{\mathbb{R}^n} w(x,y) f(u(y,t))\, dy, \tag{3}$$

the operator Φ defined as

$$\Phi(u) := u_0 + A(u), \quad \text{with} \quad (Au)(x,t) = \int_0^t (Fu)(x,s) ds \quad \text{(Volterra operator)} \tag{4}$$

is a contraction on $C_b(\mathbb{R}^m \times [0,T))$, for a T suficently small. Actually, the live time of the solution is infinity because the solution is bounded, so the well posedness is global.

In the case with Heaviside firing rate function they demostrated that the operator Φ is not continuous, so the same argument does not apply. They used compactness arguments to obtain global results in a weaker space.

In this paper we deal with the Cauchy problem asociated to the Abbassian-Fotouhi-Heidari equation

$$u_t(x,t) = -u(x,t) + \int_\Omega w(x,y)[1 + \gamma g(u(x,t) - u(y,t))] f(u(y,t))\, dy, \quad (x,t) \in \Omega \times (0,\infty), \tag{5}$$

$$u(x,0) = u_0(x), \qquad x \in \Omega \subset \mathbb{R}^m \tag{6}$$

for a Heaviside-type activation function f.

It was proved in [3, 4] that this initial value problem is global well posed on the space $C_b(\mathbb{R}^m)$ and $L^1(\Omega)$ if Ω is compact and f is a smooth function. Their result was based on the contraction mapping principle. Also it was showed that the solutions tends (uniformely) to the Amari' s model solutions when $\gamma \to 0$.

In our case, with Heaviside f (the most common simplification of the neural field models), we can use the same ideas in [7] to show that the approach by a contraction argument on the flow fails, so one say that the Initial Value Problem (IVP) is ill posed. As far as we know this is the only analytical work considering f as Heaviside function.

Our result can be summarizedas as follow: *if the firing rate is a Heaviside function then the flow genered by the equation 5 is not continuous, so it is imposible to obtain solutions to IVP by a contraction argument.*

The rest of the paper is organized as follows. In Section 2 we transform the Abbasian-Fotouhi-Heidari model in a Volterra integral equation, so we define the flow associated to the equation and next we present the functional spaces involved in the well posedness results as well as a useful way of splitting the nonlinear part of the equation. In Section 3 we prove that the flow is not continuous. Finally, in Section 4 we give some idea of the future work on existence of solutions of the Abassian-Fotouhi-Heidari model with Heaviside firing rate function.

2. Preliminary

The equation in (5) suggest (see [7] and [4]) the use of the following operators

$$(Ju)(x,t) := \int_{\mathbb{R}^m} w(x,y)[1+\gamma g(u(x,t)-u(y,t))]f(u(y,t))\, dy, \tag{7}$$

$$(Fu)(x,t) := -u(x,t) + (Ju)(x,t), \tag{8}$$

and

$$(Au)(x,t) := \int_0^t (Fu)(x,s)ds, \tag{9}$$

J is the nonlinear part of the Neural Field Equation (NFE) in (5).

Then we can write the initial value problem (5)-(6) as

$$\begin{cases} u' = Fu, \\ u(x,0) = u_0(x), \end{cases} \tag{10}$$

and, by integration

$$u = u_0 + Au, \tag{11}$$

which is a Volterra integro-differential equation.

Then it is clear that a solution of the IVP (11) is a fixed point of the operator defined by $\varphi \to u_0 + A\varphi$.

Definition 2.1. A solution of the IVP (11) is a function $u \in C(\mathbb{R}^m \times [0,\rho))$, for some ρ, such that u satisfies (10) or (11), and the flow of the equation $u' = Fu$ is the map

$$(t, u_0) \to \varphi(t, u_0)$$

defined as

$$\varphi(t,u_0) = u_0 + \int_0^t (F\varphi(\cdot,u_0))(\cdot_x, s)ds, \tag{12}$$

where

$$\varphi(\cdot,u_0)(x,s) := \varphi(s,u_0)(x). \tag{13}$$

Now we present the appropriate spaces for the well posedness result, and we define the kernel considered by Potthas and Bein Graben [7], that significantly extend the others one.

Definition 2.2. If M is a metric space and $\Omega \subset \mathbb{R}^n$ then

i) $C_b(M) := \{h : M \longrightarrow \mathbb{R} \mid h \text{ is continuous and bounded}\}$, endowed with the norm

$$\|h\| := \sup_{x\in M} |h(x)|.$$

ii) $C_b^1(\Omega) := \{h : \Omega \longrightarrow \mathbb{R} \mid h \in C^1(\Omega) \,\text{and}\, h' \in C_b(\Omega)\}$, with the norm

$$\|h\| := \sup_{x\in\Omega}(|h(x)| + |h'(x)|).$$

iii) $C_b^{0,1}(\mathbb{R}^m \times [0,\infty)) := \{h : \mathbb{R}^m \times [0,\infty) \longrightarrow \mathbb{R} \mid \forall t > 0 \;\; h(\cdot,t) \in C_b(\mathbb{R}^m),\; \forall x \in \mathbb{R}^m \; h(x,\cdot) \in C_b^1[0,\infty)\}$, and norm

$$\|h\| := \sup_{x\in\mathbb{R}^m, t\geq 0}\Big(|h(x,t)| + \Big|\frac{dh}{dt}(x,t)\Big|\Big).$$

Definition 2.3. (Synaptic kernel) The synaptic integral kernel is a function $w : \mathbb{R}^m \times \mathbb{R}^m \to \mathbb{R}$ such that

$$w \in L^\infty(\mathbb{R}^m \times \mathbb{R}^m), \quad \|w\|_{L^\infty} = \sup_{x,y\in\mathbb{R}^m} |w(x,y)| \leq C_\infty, \tag{14}$$

and

$$\|w\|_{L_x^\infty L_y^1} := \sup_{x\in\mathbb{R}^m} \|w(x,\cdot)\|_{L_y^1} \leq C_w, \tag{15}$$

that is

$$w \in L_x^\infty L_y^1(\mathbb{R}^m \times \mathbb{R}^m), \tag{16}$$

as a function $x \mapsto w(x,\cdot)$ of $L_x^\infty(\mathbb{R}^m)$ with $w(x,\cdot) \in L_y^1(\mathbb{R}^m)$ for all x, and for some constants $C_\infty, C_w > 0$.

Moreover, w satisfy the Lipschitz condition

$$\|w(x,\cdot) - w(\widetilde{x},\cdot)\|_{L^1} \leq K_w|x - \widetilde{x}| \quad x,\; \widetilde{x} \in \mathbb{R}^m \tag{17}$$

for some constant $K_w > 0$, and w is *sensitive* in the sense that

$$\sup_{x\in\mathbb{R}^m}\left|\int_G w(x,y)dy\right| > 0 \quad \text{for all open set } G \subset \mathbb{R}^m. \tag{18}$$

An inmediate consequence of properties of w is that

$$|(Ju)(x,t)| \leq (1+\gamma)C_w \tag{19}$$

for all $0 \leq f, g \leq 1$.

For the sake of completeness we present some results in the cases $f, g \in C_b^1(\mathbb{R})$ and $0 \leq f, g \leq 1$, which can be seen in [4].

Lemma 2.4. *The operator F is well defined as a map from $C_b^{0,1}(\mathbb{R}^m \times [0,\infty))$ on itself. Therefore, If $u(x,\cdot)$ is continous in t, then $(Ju)(x,\cdot)$ is continous in t too.*

Corollary 2.5. *$u(x,t)$ is a solution of (10) if and only if it is solution to (11) on $(x,t) \in \mathbb{R}^m \times (0,\rho)$ for some $\rho > 0$.*

Lemma 2.6. *Let* $u_0 \in C_b(\mathbb{R}^m)$ *and* u *be a solution to (10), then, there is* $C > 0$ *such that*

$$|u(x,t)| \leq C \text{ for all } (x,t) \in \mathbb{R}^m \times [0,\infty), \tag{20}$$

with $C = \max(\|u_0\|_\infty, |(1+\gamma)C_w|)$.

As usual in fixed-point arguments, the greater difficulty arises in handling the nonlinear term, in this case A, which can be divided in three differents operators, $A = A_1 + A_2 + A_3$, where

$$(A_1 v)(x,t) := -\int_0^t v(x,s)\, ds, \tag{21}$$

and A_2 and A_3, are the nonlinear integral operators

$$(A_2 v)(x,t) := \int_0^t \int_{\mathbb{R}^m} w(x,y) f(v(y,s))\, dy\, ds, \tag{22}$$

$$(A_3 v)(x,t) := \gamma \int_0^t \int_{\mathbb{R}^m} w(x,y) g(v(x,s) - v(y,s)) f(v(y,s))\, dy\, ds, \tag{23}$$

for all $0 \leq t \leq \rho$.

But the definition of synaptic kernel and the properties of f and g (Lipschitz functions) made possible the following results:

Lemma 2.7. *The operator* A *is well defined as an map on* $C_b(\mathbb{R}^m \times [0,\rho))$, *and it is a contraction for* $\gamma = \gamma_0$ *fixed and* $\rho > 0$ *sufficiently small.*

Theorem 2.8 (Global well posedness)**.** *The initial value problem (5)-(6) with* $u_0 \in C_b(\mathbb{R}^m)$ *has a unique solution in* $C_b(\mathbb{R}^m \times [0,+\infty))$ *and solutions depend continuously on the initial data.*

3. Flow's discontinuity

In this section we prove that the flow of the differential equation (5) is not continuous in the Heaviside case, so we are considering the integral formulation $u = u_0 + Au$ of the IVP (5)-(6) with activation function

$$f(s) = \begin{cases} 0, & s < \eta \\ 1, & s \geq \eta, \end{cases} \tag{24}$$

where $\eta \in \mathbb{R}$ is an *activation threshold.*

We have

Lemma 3.1. *The operator* Fu *does not depend continuously on* $u \in X = C_b(\mathbb{R}^m \times [0,+\infty))$.

Proof. Consider the sequence $(u_n)_{n\in\mathbb{N}}$ of functions $u_n \in X$ with

$$u_n(x,t) := \begin{cases} 0, & x \leq -2 \\ \left(\eta - \frac{1}{n}\right)(2+x), & x \in (-2,-1) \\ \eta - \frac{1}{n}, & x \in [-1,1] \\ \left(\eta - \frac{1}{n}\right)(2-x), & x \in (1,2) \\ 0, & x \geq 2 \end{cases} \tag{25}$$

for $x \in \mathbb{R}$ and $t \geq 0$.

If we define

$$u(x,t) := \begin{cases} 0, & x \leq -2 \\ \eta(2+x), & x \in (-2,-1) \\ \eta, & x \in [-1,1] \\ \eta(2-x), & x \in (1,2) \\ 0, & x \geq 2 \end{cases} \tag{26}$$

then

$$|u(x,t) - u_n(x,t)| = \begin{cases} 0, & x \leq -2 \\ \frac{1}{n}(2+x), & x \in (-2,-1) \\ \frac{1}{n}, & x \in [-1,1] \\ \frac{1}{n}(2-x), & x \in (1,2) \\ 0, & x \geq 2, \end{cases}$$

so

$$\|u(x,t) - u_n(x,t)\| = \sup_{x\in\mathbb{R},\ t>0} |u(x,t) - u_n(x,t)| = \frac{1}{n}$$

and therefore $u_n \to u$ in X (uniformely).

For all $n \in \mathbb{N}$ we have

$$(Fu_n)(x,t) = -u_n(x,t) + \int_{\mathbb{R}} w(x,y)[1 + \gamma g(u_n(x,t) - u_n(y,t))] f(u_n(y,t)) dy$$

and

$$\int_{\mathbb{R}} w(x,y)[1 + \gamma g(u_n(x,t) - u_n(y,t))] f(u_n(y,t)) dy = 0$$

because u_n does not reach the activation treshold, that is, $0 \leq u_n(x,t) < \eta$ for all $n \in \mathbb{N}$, $x \in \mathbb{R}$ and $t \geq 0$.

Then

$$(Fu_n)(x,t) = -u_n(x,t), \qquad \forall n \in \mathbb{N},\ \ x \in \mathbb{R}\ \ t \geq 0. \tag{27}$$

and

$$(Fu)(x,t) = -u(x,t) + [1 + \gamma g(u(x,t) - \eta)] \int_{[-1,1]} w(x,y) dy,$$

since

$$(fu)(x,t) = \begin{cases} 1, & x \in [-1,1] \\ 0, & |x| > 1 \end{cases} \quad \text{and} \quad u(y,t) = \eta \quad \text{if } y \in [-1,1].$$

So we have

$$(Fu)(x,t) - (Fu_n)(x,t) = -u(x,t) + u_n(x,t) + J_\eta(x), \quad x \in \mathbb{R}$$

with

$$J_\eta := [1 + \gamma g(u(x,t) - \eta)] \int_{[-1,1]} w(x,y)dy, \quad x \in \mathbb{R}, \tag{28}$$

and

$$\lim_{n\to\infty} ((Fu)(x,t) - (Fu_n)(x,t)) = J_\eta(x), \quad x \in \mathbb{R}, \ t \geq 0.$$

This convergence is uniform because $u_n \to u$ in X.

As $g(u) = e^{-u^2} > 0$ and $w(x,t)$ is sensitive then $J_\eta \neq 0$, so F is not continous. □

Theorem 3.2. *The operator A is not continuous from X to X, so the flow is also not continuous.*

Proof. This result is a consequence of the limit

$$|Au_n(x,t) - Au(x,t)| = \left| \int_0^t [(Fu_n)(x,s) - (Fu)(x,s)]ds \right| \tag{29}$$

$$\to |J_\eta(x)|t, \quad \text{if } n \to \infty. \tag{30}$$

□

4. Future work

The next step will be to implement the ideas of Potthast and Grabem [7] on existence of solutions in the non continuous case. The idea is to use a sequence of smooth functions f_n that approximate the Heaviside function f, so we will have a sequence of solutions satisfying

$$u_n - Au_n = u_0, \quad n \in \mathbb{N},$$

and a operators sequence A_n for which one hopes can obtain a subsequence

$$u_k = u_0 + A_k u_k,$$

converging to a $u \in C_b(\mathbb{R}^n \times [0,\rho))$ (in some sense) and

$$u = u_0 + Au \quad \text{or} \quad u - Au = u_0.$$

Note that the operator $(I - A)^{-1}$ does not necessarily exist because A is not a continuous linear operator (or equivalently bounded), but the linear part A_1 is continuous.

The typical tools to deal with this are the compactness arguments and the use of Neumann series for bounded linear operators. For that we need to consider another spaces (Holder types) and other considerations on the kernel, as well as the measure of the set on which a function equals the treshold. We postpone the argument for a future work.

Acknowledgment. This work was partially supported by the Convocatoria del programa nacional de proyectos para el fortalecimiento de la investigación, la creación y la innovación de posgrados de la universidad Nacional de Colombia 2013 - 2015, proyecto Análisis teŕrico - experimental de un modelo de campo Neural usando técnicas de la mecánica cuántica, código Hermes 19375.

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