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An optimal sixteenth order convergent method to solve nonlinear equations

Un método convergente de orden dieciséis para resolver ecuaciones no lineales

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Abstract. This study presents a new four-step iterative method for solving nonlinear equations. The method is based on Newton's method and has order of convergence sixteen. As this method requires four function evaluations and one derivative evaluation at each step, it is optimal in the sense of the Kung and Traub conjecture. In terms of computational cost, this implies that the efficiency index of our method is $\sqrt[5]{16} = 1.741$. Preliminary numerical results indicate that the algorithm is more efficient and performs better than other existing methods.

Key words and phrases. Nonlinear equations, four-step methods, efficiency index, order of convergence, simple root.

Resumen. Este estudio presenta un nuevo método iterativo para resolver ecuaciones no lineales. El método está basado en método de Newton y su orden de convergencia es dieciséis. Como este método requiere cuatro evaluaciones de funciones y la evaluación de una derivada en cada paso, es óptimo en el sentido de la conjetura de Kung y Traub. En términos de costo computacional esto implica que el índice de eficiencia de nuestro método es $\sqrt[5]{16} = 1.741$. Resultados numéricos preliminares indican que el algoritmo es más eficiente que otros métodos existentes.

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1. **Introduction**

Many complex problems in Science and Engineering involve functions of nonlinear and transcendental nature in equations of the form

$$f(x) = 0, \tag{1.1}$$

in which $f : I \to R$, for an open interval I, is a continuously differentiable real function. The boundary value problems appearing in the kinetic theory of gases, elasticity, and other applied areas are reduced to solving that kind of equations. Many optimization problems also lead to such equations. With the advancements in computer hardware and software, this problem has gained added importance.

Suppose that $\alpha \in I$ is a simple root of f, that is, $f(\alpha) = 0$ and $f'(\alpha) \neq 0$. *Numerical iterative methods* are often used to obtain the approximate solution of such problems because it is not always possible to obtain their exact solution by usual algebraic processes. *Newton's method* is undoubtedly the most famous iterative method to find α, by using the scheme

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}, \tag{1.2}$$

which converges quadratically in some neighborhood of α [5],[8].

A number of ways have been considered by many researchers [1],[9],[2] so to improve the local order of convergence of Newton's method at the expense of additional evaluations of functions, derivatives and changes in the points of iterations. All these modifications are in the direction of increasing the local *order of convergence*, with a view to increasing their *efficiency indices*. Indeed, algorithm cost is another offsetting factor that decides the selection of a method for particular types of problems. The effective cost of an algorithm is directly affected by the following parameters:

(i) **Termination criterion.** Termination is the ending criterion of a process which depends on the level of acceptability of the allowable error. Since a numerical method gives only the approximation of the result, it is a critical step in deciding the accuracy of any method and reliability of the result.

(ii) **Number of iterations used.** Iteration is the repetition of a particular process like a generalized rule that we adopt in the first step and later implement to the succeeding steps. The number of iterations used in obtaining the result of a particular problem is the next factor that decides the length of the solution of a problem. It is preferable to have a process that requires a smaller number of iterations to reach its final solution.

(iii) **Number of function evaluations.** It depends on the number of times the given function and their supporting functions used in the formula have to be recalculated before arriving at the final result. It is always desirable to have a method that requires a smaller number of function evaluations for reaching its final result.

This work concerns a new sixteenth-order method for estimating the simple roots of the nonlinear equation (1.1). We need the following definitions.

Definition 1. [8] *Let f be a real function with a simple root α and $\{x_n\}_{n\geq 0}$ be a sequence of real numbers, converging towards α. We say that the order of convergence of the sequence is p, if there exists a constant real number $C \neq 0$, called the asymptotic error constant, such that*

$$\lim_{n\to\infty} \frac{|x_{n+1}-\alpha|}{|x_n-\alpha|^p} = C.$$

Definition 2. [8] *Let $e_n = x_n - \alpha$ be the error in the n-th iteration. We call the relation*

$$e_{n+1} = Ce_n^p + O(e_n^{p+1})$$

the error equation.

If we can obtain the error equation for any iterative method, then the value of p is its *order of convergence* and C is the *asymptotic error constant.*

Usually, the efficiency of a method is measured using the concept of *efficiency index*, defined as follows.

Definition 3. [8] *The efficiency index of a method is given by*

$$EI = p^{1/\beta},$$

where p is the order of convergence and β is the whole number of function evaluations per iteration.

Many multi-step higher-order convergent methods have been introduced in the recent past that use *inverse, Hermite, and rational interpolations* [6], [7], [10], [4]. In developing these methods, so far, the *conjecture of Kung and Traub* has remained the focus of attention. It states the following.

Conjecture 1. [3] *An optimal iterative method without memory based on m function evaluations would achieve an optimal convergence order of 2^{m-1}. Hence, the efficiency index is $2^{(m-1)/m}$.*

Recently, Sargolzaei and Soleymani [6] showed that, using Hermite interpolation, one can get a four-step fourteenth-order convergent method from a three-step optimal eighth-order method. Their method is as follows:

$$\begin{cases} y_n = x_n - \frac{f(x_n)}{f'(x_n)}, \\ z_n = y_n - \frac{f(x_n)}{f(x_n)-2f(y_n)} \frac{f(y_n)}{f'(x_n)}, \\ w_n = z_n - \frac{f(x_n)+f(z_n)}{f(x_n)} \frac{f[x_n,y_n]f(z_n)}{f[x_n,z_n]f[y_n,z_n]}, \\ x_{n+1} = w_n - \frac{f(w_n)}{2f[x_n,w_n]+f[z_n,w_n]-2f[x_n,z_n]+(z_n-w_n)f[z_n.x_n,x_n]}, \end{cases} \tag{1.3}$$

in which,

$$f[x_n, y_n] = \frac{f(x_n) - f(y_n)}{x_n - y_n}, \quad f[z_n, x_n, x_n] = \frac{f[z_n, x_n] - f'(x_n)}{z_n - x_n}.$$

They showed the method (1.3) has error equation

$$e_{n+1} = c_2^3 c_4(-4c_2^2 + c_3)^2(4c_2^3 - c_2c_3 + c_4)e_n^{14} + O(e_n^{15}),$$

where $e_n = x_n - \alpha$ and $c_k = \frac{1}{k!}\frac{f^{(k)}(\alpha)}{f'(\alpha)}$, $k \geq 2$. Note that the method (1.3) includes four function evaluations and one first derivative evaluation, which is not optimal in the sense of Kung and Traub.

In the next section, a modification of (1.3) is presented to obtain an optimal four-step sixteenth-order convergent method. To this end, using computed quantities, we add a term to the fourth step of method (1.3) in such a way that coefficients of e_n^{14} and e_n^{15} in the error equation will be zero. The important feature of the new method is that it only adds some arithmetic calculations without any evaluation of the function at another point iterated by (1.3), but its order of convergence increases from fourteen to sixteen. Therefore, this modified method has an efficiency index which equals $\sqrt[5]{16} = 1.741$, and is better than the $\sqrt[5]{14} = 1.695$ of method (1.3). Hence, we provide a new example which agrees with the conjecture of Kung and Traub for $m = 5$.

2. **The new method and its convergence analysis**

In this section, we use the computed quantities in method (1.3) so to modify it and construct a sixteenth-order iterative method. To this end, let us define the following notation:

$$a_n = \frac{f(w_n)}{f(z_n)f(y_n)}, \quad b_n = \frac{f(y_n)^3}{f(x_n)^4}, \quad c_n = \frac{f(z_n)}{f(x_n)^2} - \frac{f(y_n)^3}{f(x_n)^4}, \quad u_n = \frac{f(w_n)}{f(x_n)f(z_n)}$$

$$v_n = \frac{f(y_n)f(z_n)}{f(x_n)^3}, \quad s_n = \left(f(z_n) - \frac{f(y_n)^3}{f(x_n)^2}\right)\frac{f(y_n)}{f(x_n)^3}, \quad t_n = \left(\frac{f(z_n)}{f(y_n)} - \frac{f(y_n)^2}{f(x_n)^2}\right)^2 \frac{1}{f(x_n)}.$$

Suppose that $G(a_n, b_n, c_n)$ and $H(u_n, v_n, s_n, t_n)$ are analytic functions in a neighborhood of the origin. Consider the following modification of (1.3):

$$\begin{cases} y_n &= x_n - \frac{f(x_n)}{f'(x_n)}, \\ z_n &= y_n - \frac{f(x_n)}{f(x_n)-2f(y_n)}\,\frac{f(y_n)}{f'(x_n)}, \\ w_n &= z_n - \frac{f(x_n)+f(z_n)}{f(x_n)}\,\frac{f[x_n,y_n]f(z_n)}{f[x_n,z_n]f[y_n,z_n]}, \\ x_{n+1} &= w_n - \frac{f(w_n)}{2f[x_n,w_n]+f[z_n,w_n]-2f[x_n,z_n]+(z_n-w_n)f[z_n.x_n,x_n]} \\ & \quad - \frac{f(w_n)f(z_n)}{f'(x_n)}\,[G(a_n,b_n,c_n)+2H(u_n,v_n,s_n,t_n)]. \end{cases} \tag{2.1}$$

The next theorem shows that method (2.1) is sixteenth-order convergent.

Theorem 1. *Suppose that $G(a,b,c)$ and $H(u,v,s,t)$ are analytic functions in a neighborhood of the origin, satifying*

$$G(\mathbf{0}) = -2H(\mathbf{0}), \quad \nabla G(\mathbf{0}) = \begin{bmatrix} 1 \\ -3 \\ -4 \end{bmatrix}, \quad \nabla H(\mathbf{0}) = \begin{bmatrix} 1 \\ -6 \\ -6 \\ -2 \end{bmatrix}.$$

If $\alpha \in I$ is a simple root of (1.1) and the initial point x_0 is sufficiently close to α, then iterative scheme (2.1) defines a family of sixteenth-order convergent methods with the following error equation:

$$\begin{aligned} e_{n+1} = &-c_2^3(c_2^2-c_3)^2(3c_2^3-4c_2c_3+c_4)\left(9c_2^5-62c_2^3c_3-8c_2^2c_4+c_3c_4\right. \\ &\left.+\, c_2(18c_3^2+c_5)\right)e_n^{16}+O(e_n^{17}), \end{aligned}$$

where, $e_n = x_n - \alpha$ and $c_k = \frac{1}{k!}\frac{f^{(k)}(\alpha)}{f'(\alpha)}$, $k \geq 2$.

Proof. Using the Taylor expansion of the function f, we have

$$f(x_n) = f'(\alpha)[e_n + c_2e_n^2 + c_3e_n^3 + c_4e_n^4 + c_5e_n^5 + c_6e_n^6 + O(e_n^7)], \tag{2.2}$$

and

$$f'(x_n) = f'(\alpha)[1 + 2c_2e_n + 3c_3e_n^2 + 4c_4e_n^3 + 5c_5e_n^4 + 6c_6e_n^5 + O(e_n^6)]. \tag{2.3}$$

Dividing (2.2) by (2.3) gives

$$\frac{f(x_n)}{f'(x_n)} = e_n - c_ne_n^2 + 2(c_2^2-c_3)e_n^3 + (7c_2c_3 - 4c_2^3 - 3c_4)e_n^4 + O(e_n^5). \tag{2.4}$$

So,

$$y_n = \alpha + c_2e_n^2 + 2(-c_2^2+c_3)e_n^3 + (-7c_2c_3 + 4c_2^3 + 3c_4)e_n^4 + O(e_n^5), \tag{2.5}$$

which results in

$$f(y_n) = f'(\alpha)[c_2e_n^2 + 2(-c_2^2+c_3)e_n^3 + (-7c_2c_3 + 4c_2^3 + 3c_4)e_n^4 + O(e_n^5)]. \tag{2.6}$$

In the same way, for the second step of (2.1) we have

$$z_n = \alpha + (c_2^3 - c_2c_3)e_n^4 - 2(2c_2^4 - 4c_2^2c_3 + c_3^2 + c_2c_4)e_n^5 + O(e_n^6), \tag{2.7}$$

which implies

$$f(z_n) = f'(\alpha)[(c_2^3 - c_2c_3)e_n^4 - 2(2c_2^4 - 4c_2^2c_3 + c_3^2 + c_2c_4)e_n^5 + O(e_n^6)]. \tag{2.8}$$

Now, by expanding each existing divided differences in the third step of (2.1) to a Taylor series, we have

$$\begin{aligned} f[x_n,z_n] &= f'(\alpha)[1+c_2e_n+c_3e_n^2+c_4e_n^3+\cdots+O(e_n^9)],\\ f[y_n,z_n] &= f'(\alpha)[1+c_2^2e_n^2+2c_2(-c_2^2+c_3)e_n^3+\cdots+O(e_n^9)],\\ f[x_n,y_n] &= f'(\alpha)[1+c_2e_n+(c_2^2+c_3)e_n^2+(-2c_2^3+3c_2c_3+c_4)e_n^3\\ &\quad+(4c_2^4-8c_2^2c_3+2c_3^2+4c_2c_4+c_5)e_n^4+\cdots+O(e_n^9)]. \end{aligned}$$

According to the above expansions, for the third step of (2.1), one can get

$$w_n=\alpha+c_2^2(c_2^2-c_3)(3c_2^3-4c_2c_3+c_4)e_n^8+O(e_n^9). \tag{2.9}$$

Therefore,

$$f(w_n)=f'(\alpha)[c_2^2(c_2^2-c_3)(3c_2^3-4c_2c_3+c_4)e_n^8+O(e_n^9)]. \tag{2.10}$$

In a similar way, we can write the Taylor series of each existing divided differences in the fourth step of (2.1):

$$\begin{aligned} f[x_n,w_n] &= f'(\alpha)[1+c_2e_n+c_3e_n^2+c_4e_n^3+O(e_n^4)],\\ f[z_n,w_n] &= f'(\alpha)[1+c_2^2(c_2^2-c_3)e_n^4+O(e_n^5)],\\ f[z_n,x_n,x_n] &= f'(\alpha)[c_2+2c_3e_n+3c_4e_n^2+4c_5e_n^3+O(e_n^4)]. \end{aligned}$$

By considering the above mentioned relations, one can deduce the equality

$$\begin{aligned} &\frac{f(w_n)}{2f[x_n,w_n]+f[z_n,w_n]-2f[x_n,z_n]+(z_n-w_n)f[z_n,x_n,x_n]}\\ &\qquad=c_2^2(c_2^2-c_3)(3c_2^3-4c_2c_3+c_4)e_n^8+O(e_n^9). \end{aligned} \tag{2.11}$$

Now, consider the first-order Taylor series of $G(\mathbf{X})$ and $H(\mathbf{Y})$ around $\mathbf{0}$, in which $\mathbf{X}=[a,b,c]^T$ and $\mathbf{Y}=[u,v,s,t]^T$:

$$\begin{aligned} G(\mathbf{X}) &= G(\mathbf{0})+\mathbf{X}^T\nabla G(\mathbf{0})+O(\|\mathbf{X}\|^2)\\ &= G(\mathbf{0})+\frac{1}{f'(\alpha)}\left[((3G_a(\mathbf{0})+G_b(\mathbf{0}))c_2^3-(4G_a(\mathbf{0})+G_c(\mathbf{0}))c_4)e_n^2+O(e_n^3)\right], \end{aligned}$$

$$\begin{aligned} H(\mathbf{Y}) &= H(\mathbf{0})+\mathbf{Y}^T\nabla H(\mathbf{0})+O(\|\mathbf{Y}\|^2)\\ &= H(\mathbf{0})+\frac{1}{f'(\alpha)}\left[((3H_u(\mathbf{0})+H_v(\mathbf{0}))c_2^4-(4H_u(\mathbf{0})+H_v(\mathbf{0})+H_s(\mathbf{0}))c_2^2c_3\right.\\ &\qquad\left.+H_t(\mathbf{0})c_3^2+H_u(\mathbf{0})c_2c_4)e_n^3+O(e_n^4)\right]. \end{aligned}$$

Hence,

$$\begin{aligned}
G(\mathbf{X})+2H(\mathbf{Y}) &= G(\mathbf{0})+2H(\mathbf{0})+\tfrac{1}{f'(\alpha)}\left[((3G_a(\mathbf{0})+G_b(\mathbf{0}))c_2^3\right.\\
&-(4G_a(\mathbf{0})+G_c(\mathbf{0}))c_2c_3+G_a(\mathbf{0})c_4)e_n^2\\
&-2((4G_a(\mathbf{0})+5G_b(\mathbf{0})-2G_c(\mathbf{0})-3H_u(\mathbf{0})-H_v(\mathbf{0}))c_2^4\\
&+(-9G_a(\mathbf{0})-3G_b(\mathbf{0})-2G_c(\mathbf{0})+4H_u(\mathbf{0})+H_v(\mathbf{0})+H_s(\mathbf{0}))c_2^3c_3\\
&+(2G_a(\mathbf{0})+G_c(\mathbf{0})-H_t(\mathbf{0}))c_3^2+(4G_b(\mathbf{0})+G_c(\mathbf{0})-H_u(\mathbf{0}))c_2c_4\\
&\quad\left.-G_a(\mathbf{0})c_5)e_n^3+O(e_n^4)\right].
\end{aligned}$$

On the other hand, from (2.3), (2.8), and (2.10) obtain

$$\frac{f(w_n)f(z_n)}{f'(x_n)}=f'(\alpha)\left[c_2^3(c_2^2-c_3)^2(3c_2^3-4c_2c_3+c_4)e_n^{12}+O(e_n^{13})\right].$$

This implies that

$$\begin{aligned}
&\frac{f(w_n)f(z_n)}{f'(x_n)}[G(\mathbf{X})+2H(\mathbf{Y})]=\\
&\quad f'(\alpha)[G(\mathbf{0})+2H(\mathbf{0})][c_2^3(c_2^2-c_3)^2(3c_2^3-4c_2c_3+c_4)]e_n^{12}\\
&\quad -2f'(\alpha)[G(\mathbf{0})+2H(\mathbf{0})]c_2^2(c_2^2-c_3)\left(22c_2^7-70c_2^5c_3+16c_2^4c_4\right.\\
&\quad \left.-23c_2^2c_3c_4+3c_3^2c_4+c_2^3(64c_3^2-c_5)+c_2(-14c_3^3+2c_4^2+c_3c_5)\right)e_n^{13}+O(e_n^{14}).
\end{aligned}$$

So, if $G(\mathbf{0})=-2H(\mathbf{0})$ we have

$$\begin{aligned}
&\frac{f(w_n)f(z_n)}{f'(x_n)}[G(\mathbf{X})+2H(\mathbf{Y})]=\\
&c_2^3(c_2^2-c_3)^2(3c_2^3-4c_2c_3+c_4)((3G_a(\mathbf{0})+G_b(\mathbf{0}))c_2^3-(4G_a(\mathbf{0})+G_c(\mathbf{0}))c_2c_3+G_a(\mathbf{0})c_4)e_n^{14}\\
&-(c_2^2(c_2^2-c_3)((156G_a(\mathbf{0})+74G_b(\mathbf{0})-12G_c(\mathbf{0})-18H_u(\mathbf{0})-6H_v(\mathbf{0}))c_2^{10}\\
&+(66H_u(\mathbf{0})-706G_a(\mathbf{0})-228G_b(\mathbf{0})-28G_c(\mathbf{0})+20H_v(\mathbf{0})+6H_s(\mathbf{0}))c_2^8c_3\\
&+(172G_a(\mathbf{0})+42G_b(\mathbf{0})+2G_c(\mathbf{0})-12H_u(\mathbf{0})-2H_v(\mathbf{0}))c_2^7c_4-(16G_a(\mathbf{0})+2G_c(\mathbf{0}))c_2^2c_3^2c_5\\
&+(28H_u(\mathbf{0})-488G_a(\mathbf{0})-62G_b(\mathbf{0})-46G_c(\mathbf{0})+4H_v(\mathbf{0})+2H_s(\mathbf{0}))c_2^5c_3c_4+6G_a(\mathbf{0})c_3^2c_4^2\\
&+(384G_a(\mathbf{0})+12G_b(\mathbf{0})+60G_c(\mathbf{0})-16H_u(\mathbf{0})-2H_v(\mathbf{0})-2H_s(\mathbf{0})-2H_t(\mathbf{0}))c_2^3c_3^2c_4\\
&+(1114G_a(\mathbf{0})+210G_b(\mathbf{0})+158G_c(\mathbf{0})-80H_u(\mathbf{0})-22H_v(\mathbf{0})-14H_s(\mathbf{0})-6H_t(\mathbf{0}))c_2^6c_3^2\\
&-(12G_a(\mathbf{0})+2G_b(\mathbf{0}))c_2^6c_5+((2H_t(\mathbf{0})-56G_a(\mathbf{0})-8G_c(\mathbf{0}))c_3^3+4G_a(\mathbf{0})c_4^2+4G_a(\mathbf{0})c_2c_3c_4c_5\\
&+((128G_a(\mathbf{0})+36G_c(\mathbf{0})+8H_t(\mathbf{0}))c_3^3+(2H_u(\mathbf{0})-70G_a(\mathbf{0})-6G_c(\mathbf{0}))c_4^2-4G_a(\mathbf{0})c_2^3c_4c_5\\
&+((32H_u(\mathbf{0})-696G_a(\mathbf{0})-52G_b(\mathbf{0})-158G_c(\mathbf{0})+8H_v(\mathbf{0})+8H_s(\mathbf{0})+14H_t(\mathbf{0}))c_3^3\\
&+(52G_a(\mathbf{0})+4G_b(\mathbf{0})+2G_c(\mathbf{0})-2H_u(\mathbf{0}))c_4^2+(28G_a(\mathbf{0})+2G_b(\mathbf{0})+2G_c(\mathbf{0}))c_3c_5c_2^4))e_n^{15}\\
&+O(e_n^{16}).
\end{aligned}$$

Finally,

$$
\begin{aligned}
x_{n+1}-\alpha = & -c_2^3(c_2^2-c_3)^2(3c_2^3-4c_2c_3+c_4)((3G_a(\mathbf{0})+G_b(\mathbf{0}))c_2^3 \\
& -(4G_a(\mathbf{0})+G_c(\mathbf{0}))c_2c_3+(G_a(\mathbf{0})-1)c_4)e_n^{14} \\
& +c_2^2(c_2^2-c_3)((156G_a(\mathbf{0})+74G_b(\mathbf{0})-12G_c(\mathbf{0})-18H_u(\mathbf{0})-6H_v(\mathbf{0}))c_2^{10} \\
& +(66H_u(\mathbf{0})-706G_a(\mathbf{0})-128G_b(\mathbf{0})-28G_c(\mathbf{0})+20H_v(\mathbf{0})+6H_s(\mathbf{0}))c_2^8c_3 \\
& +(192G_a(\mathbf{0})+42G_b(\mathbf{0})+2G_c(\mathbf{0})-12H_u(\mathbf{0})-2H_v(\mathbf{0})-38)c_2^7c_4+(6G_a(\mathbf{0})-6)c_3^2c_4^2 \\
& +(126-488G_a(\mathbf{0})-62G_b(\mathbf{0})-46G_c(\mathbf{0})+28H_u(\mathbf{0})+4H_v(\mathbf{0})+2H_s(\mathbf{0}))c_2^5c_3c_4 \\
& +(384G_a(\mathbf{0})+12G_b(\mathbf{0})+60G_c(\mathbf{0})-16H_u(\mathbf{0})-2H_v(\mathbf{0})-2H_s(\mathbf{0})-2H_t(\mathbf{0})-120)c_2^3c_3^2c_4 \\
& -(4G_a(\mathbf{0})-4)c_2^3c_4c_5-(12G_a(\mathbf{0})+2G_b(\mathbf{0})-6)c_2^6c_5+(4G_a(\mathbf{0})-4)c_2c_4^3 \\
& +(4G_a(\mathbf{0})-4)c_2c_3c_4c_5 \\
& +(1114G_a(\mathbf{0})+210G_b(\mathbf{0})+158G_c(\mathbf{0})-80H_u(\mathbf{0})-22H_v(\mathbf{0})-14H_s(\mathbf{0})-6H_t(\mathbf{0}))c_2^6c_3^2 \\
& +(28-56G_a(\mathbf{0})-8G_c(\mathbf{0})+2H_t(\mathbf{0}))c_2c_3^3c_4+(128G_a(\mathbf{0})+36G_c(\mathbf{0})-8H_t(\mathbf{0}))c_2^2c_3^4 \\
& +(44-70G_a(\mathbf{0})-6G_c(\mathbf{0})+2H_u(\mathbf{0}))c_2^2c_3c_4^2-(16G_a(\mathbf{0})+2G_c(\mathbf{0})-8)c_2^2c_3^2c_5 \\
& +(14H_t(\mathbf{0})-696G_a(\mathbf{0})-52G_b(\mathbf{0})-158G_c(\mathbf{0})+32H_u(\mathbf{0})+8H_v(\mathbf{0})+8H_s(\mathbf{0}))c_2^4c_3^3 \\
& +(52G_a(\mathbf{0})+4G_b(\mathbf{0})+2G_c(\mathbf{0})-2H_u(\mathbf{0})-30)c_2^4c_4^2 \\
& +(28G_a(\mathbf{0})+2G_b(\mathbf{0})+2G_c(\mathbf{0})-14)c_2^4c_3c_5)e_n^{15}+\cdots+O(e_n^{17}).
\end{aligned}
$$

The above relation shows that by choosing

$$
G(\mathbf{0})=-2H(\mathbf{0}), \quad \nabla G(\mathbf{0})=\begin{bmatrix} 1 \\ -3 \\ -4 \end{bmatrix}, \quad \nabla H(\mathbf{0})=\begin{bmatrix} 1 \\ -6 \\ -6 \\ -2 \end{bmatrix},
$$

we have

$$
\begin{aligned}
e_{n+1} = & -c_2^3(c_2^2-c_3)^2(3c_2^3-4c_2c_3+c_4)(9c_2^5-62c_2^3c_3-8c_2^2c_4+c_3c_4 \\
& +c_2(18c_3^2+c_5))e_n^{16}+O(e_n^{17}),
\end{aligned}
$$

which means the method (2.1) is sixteenth order convergent. □

3. Numerical results

There are some other sixteen-order methods in the literature. The authors of [10] study a sixteen-order method (named as ZHFK method) defined by

$$
\begin{cases}
y_n = x_n - \frac{f(x_n)}{f'(x_n)}, \\
z_n = y_n - \frac{f(x_n)}{f(x_n)-2f(y_n)} \frac{f(y_n)}{f'(x_n)}, \\
w_n = z_n - \frac{f(x_n)+f(z_n)}{f(x_n)} \frac{f[x_n,y_n]f(z_n)}{f[x_n,z_n]f[y_n,z_n]}, \\
x_{n+1} = w_n - \frac{f(w_n)}{h(w_n)},
\end{cases} \tag{3.1}
$$

where

$$h(w_n) = f[w_n, z_n] + (w_n - z_n)f[w_n, z_n, y_n] + (w_n - z_n)(w_n - y_n)f[w_n, z_n, y_n, x_n] \\ +(w_n - z_n)(w_n - y_n)(w_n - x_n)f[w_n, z_n, y_n, x_n, 2],$$

and

$$\begin{aligned} f[w_n, z_n, y_n] &= \tfrac{f[w_n,z_n]-f[w_n,y_n]}{w_n-y_n} \\ f[w_n, z_n, y_n, x_n] &= \tfrac{f[w_n,z_n]-f[z_n,y_n]}{(w_n-x_n)(w_n-y_n)} - \tfrac{f[z_n,y_n]-f[y_n,x_n]}{(w_n-z_n)(z_n-x_n)} \\ f[w_n, z_n, y_n, x_n, 2] &= \tfrac{f[w_n,z_n]-f[z_n,y_n]}{(w_n-x_n)^2(w_n-y_n)} - \tfrac{f[z_n,y_n]-f[y_n,x_n]}{(w_n-x_n)^2(z_n-x_n)} \\ &\quad - \tfrac{f[z_n,y_n]-f[y_n,x_n]}{(z_n-x_n)^2(w_n-x_n)} + \tfrac{f[y_n,x_n]-f'(x_n)}{(w_n-x_n)(z_n-x_n)(y_n-x_n)}. \end{aligned}$$

Also, in [4] following sixteenth-order method (LMMW method) was proposed:

$$\begin{cases} y_n = x_n - \frac{f(x_n)}{f'(x_n)}, \\ z_n = y_n - \frac{2f(x_n)-f(y_n)}{2f(x_n)-5f(y_n)} \frac{f(y_n)}{f'(x_n)}, \\ w_n = z_n - \frac{f(z_n)}{f'(z_n)} \\ x_{n+1} = w_n - \frac{2f(z_n)-f(w_n)}{2f(z_n)-5f(w_n)} \frac{f(w_n)}{f'(z_n)}. \end{cases} \tag{3.2}$$

It is obvious that the efficiency indices of (3.1) and (3.2) are $\sqrt[5]{16} = 1.741$ and $\sqrt[6]{16} = 1.584$, respectively.

Now we will compare the accuracy of algorithm (1.3) (named SS me-thod), the modified algorithm (2.1) (named MSS method), and the sixteenth-order methods (3.1) and (3.2). In the MSS method (2.1), we use the simple choice of

$$G(a, b, c) = a - 3b - 4c, \quad H(u, v, s, t) = u - 6v - 6s - 2t,$$

satisfying Theorem 2.1. For numerical experiments, we use the following test functions:

$$\begin{aligned} &f_1(x) = e^{x^2+7x-30} - 1, && \alpha = 3, \\ &f_2(x) = x^2 - e^x - 3x + 2, && \alpha \approx 0.257530285439861, \\ &f_3(x) = \sqrt{x^2 + 2x + 5} - 2\sin x - x^2 + 3, && \alpha \approx 2.331969765588396, \\ &f_4(x) = \sin\tfrac{1}{x} - x, && \alpha \approx 0.897539461280487, \\ &f_5(x) = 2\sin x + 1 - x, && \alpha \approx 2.380061273139339, \\ &f_6(x) = e^{-x} + \cos x, && \alpha \approx 1.746139530408013, \\ &f_7(x) = \cos^2 x - \tfrac{x}{5}, && \alpha \approx 2.320204274495726. \end{aligned}$$

Using two different initial guesses x_0, we implement only three full iterations of the above various sixteen-order methods and compute $|f(x_3)|$ and the time together. All computations were done by MATLAB 10. Numerical results are summarized in Table 1. In this table, we use the notation 0.180E-922/7.276 to show that, for SS method, $|f_1(x_3)| < 0.180 \times 10^{-922}$, when $x_0 = 3.1$, while the time for computing x_3 is 7.276 seconds. From Table 1, it is clear that the weakest accuracy is for LMMW method although it is rather fast, while the strongest accuracy is for MSS method. The new MSS algorithm is comparable with the other methods, and gives better results in all cases.

TABLE 1. Numerical results of various iterative methods

F	x_0	SS method	MSS method	ZHFK method	LMMW method
f_1	3.1	0.180E-922/7.276	0.136E-1176/9.411	0.984E-973/9.590	0.322E-154/6.783
f_1	3.2	0.319E-303/7.442	0.395E-463/9.607	0.222E-402/9.965	0.503E-114/6.624
f_2	0.9	0.156E-3463/8.335	0.347E-5009/10.821	0.241E-3546/10.761	0.252E-1281/6.528
f_2	1	0.976E-2517/8.128	0.109E-3603/10.781	0.839E-2599/10.737	0.540E-917/6.544
f_3	2.5	0.130E-3845/15.330	0.165E-5491/17.885	0.322E-3932/18.094	0.106E-840/16.007
f_3	1.4	0.110E-2576/15.294	0.161E-3687 /17.854	0.127E-2745 /18.255	0.789E-709/15.961
f_4	2	0.368E-1382/11.492	0.276E-1926/14.413	0.242E-1482/14.113	0.514E-472/11.408
f_4	1.5	0.938E-1781/11.818	0.410E-2714 /14.000	0.715E-2009/14.626	0.123E-545/11.453
f_5	4.1	0.663E-935/12.435	0.451E-1014/15.225	0.318E-811/14.383	0.170E-312/11.284
f_5	3.5	0.652E-2350/11.493	0.353E-3013/14.315	0.277E-2249/14.719	0.222E-772/11.327
f_6	1	0.325E-3105/14.912	0.299E-3830/18.183	0.506E-2700/17.530	0.108E-752/15.426
f_6	1.6	0.610E-4633/14.256	0.456E-5377/17.223	0.351E-3867/17.668	0.338E-851/15.816
f_7	2.1	0.102E-2251/11.672	0.815E-2701/15.083	0.120E-2283/15.067	0.347E-756/15.060
f_7	2.6	0.393E-2138/11.680	0.215E-2508/14.666	0.363E-2082/14.373	0.824E-513/14.458

4. Conclusions

In this paper, a general four-step iterative method has been given for solving nonlinear equations. An analytic proof of convergence order of this method was given which demonstrates that the method has an optimal convergence order sixteen. For this method, the number of function evaluations is five per full step, so the efficiency index is $\sqrt[5]{16} = 1.741$, which is optimal in the sense of Kung and Traub.

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