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Arithmetical functions in two variables. An analogue of a result of Delange

Oscar F. Casas
Universidad de los Andes, Bogotá

Abstract. In this paper, an analogue of a result of Delange [1] for multiplicative functions defined in the Cartesian product $\mathbb{N}\times\mathbb{N}$ is given. Our definition of multiplicative functions differs from that used by Delange.

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Resumen. Se demuestra un análogo de un resultado de Delange [1] para funciones aritméticas multiplicaticas definidas en el monoide $\mathbb{N}\times\mathbb{N}$ is given. Nuestra definición de función aritmética multiplicativa difiere de la usada por Delange.

1. Introduction

In 1968, Halász [2] proved the following proposition:

Proposition. [Halász] *Let $f(n)$ be an arithmetical multiplicative function satisfying the following condition*

$$|f(n)| \leq 1, \qquad \text{for all } n \in \mathbb{N}^*.$$

Then there exists a complex constant C, a real constant a, and a real function $L(u)$, such that

$$|L(u)| = 1; \qquad \frac{L(u_1)}{L(u)} \to 0, \qquad \text{as } u \to \infty \text{ and } u \leq u_1 \leq 2u\ ,$$

and

$$M(x) = \sum_{n\leq x} f(n) = C \cdot L(\log x)x^{1+ia} + o(x)$$

DELANGE, in 1970, generalizes this proposition to the case of arithmetical functions in several variables using a special definition of multiplicative functions (see Section 3). In this paper we define the cartesian product of two arithmetical monoids, and show that again it is an arithmetical monoid (see Section 2). So the usual definition of multiplicative funtions on an arithmetical monoid (as in [3]) is now available for the product of arithmetical monoids. This will allow us to prove in this new setting a result analogue to the one proved by DELANGE in [1] (see Section 3) .

2. Product of arithmetical monoids

For the sake of completeness, we recall the definition of arithmetical monoid given by KNOPFMACHER in [3], and show that it is possible to define in a canonical way the cartesian product of two arithmetical monoids as a new arithmetical monoid, so that we can use all the arithmetical properties known for arithmetical monoids.

Definition 2.1. A monoid M is a non empty set equipped with an associative and commutative operation, denoted multiplicatively. The monoid M is said to be a unitary monoid if there exists $e \in M$ such that $ea = a = ae$ for all $a \in M$. Moreover, M is said to be arithmetical if it is unitary and in addition the following conditions are satisfied:

(1) There is a subset P of M such that every $a \neq e$ has a unique factorization of the form

$$a = p_1^{\alpha_1} \cdots p_k^{\alpha_k}, \quad p_i \in P, \quad \alpha_i \in \mathbb{N}^*, \quad i = 1, \ldots, k, \quad \forall a \neq e \,,$$

where the p_i are distinct elements of P, the α_i are positive integers, and the uniqueness is understood up to the order of the appearing factors.

(2) There exists a function

$$| \quad | : M \to \mathbb{R}_{>0}$$

such that:

(a) $|e| = 1, \quad |p| > 1, \qquad \forall\, p \in P$

(b) $|ab| = |a|\,|b|\,, \qquad \forall\, a, b \in M$

(c) For each $x \in \mathbb{R}$ the total number $N_M(x)$ of elements of the set $\{a \in M \,:\, |a| \leq x\}$ is finite.

The set P is called *set of primes* of M, and the function $| \quad |$ is called a *norm.*

The classical example of arithmetical monoid is $\mathbb{N} = \{1, 2, \ldots\}$, where the norm is the usual absolute value and the primes of the monoid are the positive prime numbers.

Definition 2.2. Let M and M' two arithmetical monoids and denoted by $\mathbb{G}$ their cartesian product. The product of two elements of $\mathbb{G}$ is defined by the expression

$$(a, a')(b, b') = (ab, a'b'), \qquad \text{for all } (a, a'), (b, b') \in \mathbb{G}$$

Proposition 2.3. *With the operation defined above,* $\mathbb{G}$ *is a unitary monoid.*

Proof. It is not difficult to verify that the operation is well defined, associative, commutative and furthermore, if e, e' are the units of M, M' respectively, the element (e, e') is the unit for $\mathbb{G}$. Therefore $\mathbb{G}$ is a unitary monoid. ☑

We say that (a, a') divides (b, b') $((a, a') \mid (b, b'))$, if there exists $(c, c') \in \mathbb{G}$ such that, $(b, b') = (a, a')(c, c')$. Also, an element $(p, p') \neq (e, e')$ is prime if $(a, a') \mid (p, p')$ then $(a, a') = (p, p')$ or $(a, a') = (e, e')$. The proof of the next proposition is an inmediate consequence of the definition of prime.

Proposition 2.4. *If* P *and* P' *are the sets of primes of* M *and* M'*, respectively, then the elements of the form* (p, e'), (e, p')*, where* $p \in P$ *and* $p' \in P'$*, are primes in* $\mathbb{G}$.

Since all $a \in M$ and $a' \in M'$, with $e \neq a$ and $e' \neq a'$, can be expressed in an only way as product of elements of P and P' respectively, the following result is evident.

Corollary 2.1. *Let* $a = p_1^{\alpha_1} \cdots p_k^{\alpha_k}$, $p_i \in P$*, and* $a' = {p'}_1^{\beta_1} \cdots {p'}_n^{\beta_n}$, ${p'}_j \in P'$ *be the factorizations of* $a \neq e$ *and* $a' \neq e'$*. Then*

$$(a, a') = (p_1^{\alpha_1}, e') \cdots (p_k^{\alpha_k}, e')(e, {p'}_1^{\beta_1}) \cdots (e, {p'}_n^{\beta_n}),$$

uniquely up to the order of the appearing factors.

Definition 2.5. An element $(d, d') \in \mathbb{G}$ is called the greatest common divisor of two elements $(a, a'), (b, b') \in \mathbb{G}$ (in which case we write $(d, d') = \mathbf{g.c.d.}((a, a'), (b, b')))$ if it satisfies the following conditions:

(1) $(d, d') \mid (a, a')$ and $(d, d') | (b, b')$.
(2) Si $(c, c') | (a, a')$ and $(c, c') | (b, b')$, then $(c, c') | (d, d')$

If the greatest common divisor of two elements $(a, a'), (b, b') \in \mathbb{G}$ is (e, e') then $(a, a'), (b, b') \in \mathbb{G}$ are said to be relatively prime.

Proposition 2.6. $(d, d') = \mathbf{g.c.d.}((a, a'), (b, b'))$ *if, and only if,* $d = \mathbf{g.c.d.}(a, b)$ *and* $d' = \mathbf{g.c.d.}(a', b')$

Proof. $(d, d') = \mathbf{g.c.d.}((a, a'), (b, b'))$ if, and only if, there are (c, c') and (f, f') in $\mathbb{G}$ such that $(a, a') = (d, d')(c, c')$, and $(b, b') = (d, d')(f, f')$. Now if (h, h') is a common divisor of $(a, a'), (b, b')$ then $(h, h') \mid (d, d')$, which amounts to say that $a = dc$, $a' = d'c'$, $b = df$, $b' = d'f'$ and if $a = hk$, $a' = h'k'$, $b = hl$, $b' = h'l'$ then $d = hn$, $d' = h'n'$, which is equivalent to say that $d \mid a$, $d \mid b$,

$d' \mid a',\ d' \mid b'$ and if $h \mid a,\ h \mid b,\ h' \mid a',\ h' \mid b'$ then $h \mid d,\ h' \mid d'$. Clearly this amounts to say that $d = \mathbf{g.c.d.}(a,b)$ and $d' = \mathbf{g.c.d.}(a',b')$. ☑

Proposition 2.7. *The monoid $\mathbb{G}$ is an arithmetical monoid.*

Proof. To verify that actually $\mathbb{G}$ is an arithmetical monoid let us find a norm for $\mathbb{G}$, satisfying the required condictions. To achieve this let us consider

$$|(a,a')| = |a|_M \, |a'|_{M'} \ ,$$

where $|\quad|_M$ is the norm of M and $|\quad|_{M'}$ is the norm of M'. It is not difficult to verify that this map satisfies the conditions a) and b) of the definition of an arithmetical monoid. Let us see that it also satisfies condition c). Indeed, for $x \in \mathbb{R}$ we have that $N_M\left(\dfrac{x}{|a'|_{M'}}\right)$ is finite for each $a' \in M'$. Therefore,

$$N_{\mathbb{G}}(x) = \sum_{a' \in M'} N_M\left(\frac{x}{|a'|_{M'}}\right) \ ,$$

where every term of this sum is finite. Moreover, $N_M\left(\dfrac{x}{|a'|_{M'}}\right) = 0$ if $|a'|_{M'} > x$. Thus,

$$N_{\mathbb{G}}(x) = \sum_{a' \in N_{M'}(x)} N_M\left(\frac{x}{|a'|_{M'}}\right),$$

is a finite sum of finite valued terms, so $N_{\mathbb{G}}(x)$ is finite for each $x \in \mathbb{R}$. ☑

Definition 2.8. A function $F : \mathbb{G} \longrightarrow \mathbb{C}$ is called an arithmetical function.

In what follows the algebra of the arithmetical functions defined on $\mathbb{G}$ will denoted by $Dir(\mathbb{G})$.

An arithmetical function F is called multiplicative if F is not identically zero and if it satisfies the following condition:

$$F((a,a')(b,b')) = F(a,a')F(b,b') \text{ whenever } \mathbf{g.c.d.}((a,a'),(b,b')) = (e,e').$$

Proposition 2.9. *If $F \in \mathrm{Dir}(\mathbb{G})$ is multiplicative, then the functions $G \in \mathrm{Dir}(M)$ and $H \in \mathrm{Dir}(M')$ defined by $G(a) = F(a,e')$ and $H(a') = F(e,a')$ are multiplicative.*

Proof. Let $F \in \mathrm{Dir}(\mathbb{G})$ multiplicative and $(a,a'),(b,b')$ two elements of $\mathbb{G}$ relatively prime. We have that

$$F(ab,e') = F((a,e')(b,e')) = F(a,e')F(b,e')$$
$$F(e,a'b') = F((e,a')(e,b')) = F(e,a')F(e,b'). \quad ☑$$

3. Analogue of a theorem of Delange

In [1] DELANGE generalizes the fore mentioned result of HALÁSZ to arithmetical functions in two variables, in the following way:

First, he says that a complex valued function f of q positive integers is said to be multiplicative if $f(1,1,\ldots,1)=1$ and

$$f(m_1n_1, m_2n_2, \ldots, m_qn_q) = f(m_1, m_2, \ldots, m_q)f(n_1, n_2, \ldots, n_q)$$

whenever $(m_1m_2\cdots m_q, n_1n_2\cdots n_q)=1$. Clearly, this definition differs from ours as given in Section 2. Finally, he proves thus the following result:

Proposition 3.1. *If f is a multiplicative function, and*

$$|F(m_1, m_2, \ldots, m_q)| \le 1$$

for all positive integers $m_1, m_2, \ldots, m_q$, then, as $x_1, x_2, \ldots, x_q \to \infty$ independently, either:

1. *f has zero mean value, i.e.,*

$$Q \equiv \frac{1}{x_1x_2\cdots x_q} \sum_{m_1\le x_1,\ldots,m_q\le x_q} f(m_1,\ldots,m_q) \to 0$$

or

2. $$Q = C\dot{x}_1^{ia_1}\cdots x_q^{ia_q} L_1(\log x_1)\cdots L_q(\log x_q) + o(1)$$

where C is a non-zero complex constant, $a_1, \ldots, a_q$ are real constants, and $L_1, \ldots, L_q$ are complex functions defined on $\mathbb{R}^+$ satisfying for $j=1$, $\ldots$, q

$$|L_j(t)| = 1, \qquad \textit{for all } t \in \mathbb{R}^+$$

and

$$\lim_{t\to\infty} \frac{L_j(\lambda t)}{L_j(t)} = 1, \qquad \textit{for all } \lambda > 0,$$

the limits being uniform on all closed subintervals of $(0,\infty)$

Here we prove the following analogue for the product monoid $\mathbb{G}$ using our definition of multiplicative arithmetical functions given in Section 2.

Proposition 3.2. *If F is an arithmetical multiplicative function that satisfies the following condition:*

$$|F(n,m)| \le 1, \qquad \textit{for all } (n,m) \in \mathbb{G},$$

then there exists a complex constant C, two real constants a_1 and a_2 and two complex functions L_1 and L_2 defined on $\mathbb{R}^+$ satisfying for $j=1,2$

$$|L_j(t)| = 1, \qquad \textit{for all } t \in \mathbb{R}^+$$

and

$$\lim_{t\to\infty} \frac{L_j(\lambda t)}{L_j(t)} = 1, \quad \textit{for all } \lambda > 0, \textit{ uniform on all closed subintervals of } (0,\infty),$$

such that when $z \to \infty$ $(z = (x, y))$

$$\frac{1}{xy} \sum_{\substack{n \le x \\ m \le y}} F(n, m) = C \cdot x^{ia_1} y^{ia_2} L_1(\log x) L_2(\log y) + o(1)$$

Proof. Since $|F(n, m)| \le 1$ for all $n, m \in \mathbb{N}$, we have that

$$|F(n, 1)| \le 1, \qquad \text{for all } n \in \mathbb{N},$$

and

$$|F(1, m)| \le 1, \qquad \text{for all } m \in \mathbb{N}.$$

Moreover,

$$\begin{aligned} \frac{1}{xy} \sum_{\substack{n \le x \\ m \le y}} F(n, m) &= \frac{1}{xy} \sum_{\substack{n \le x \\ m \le y}} F(n, 1) F(1, m) \\ &= \left(\frac{1}{x} \sum_{n \le x} F(n, 1) \right) \left(\frac{1}{y} \sum_{m \le y} F(1, m) \right) \\ &= \left(\frac{1}{x} \sum_{n \le x} h_1(n) \right) \left(\frac{1}{y} \sum_{m \le y} h_2(m) \right), \end{aligned}$$

where $h_1(n) = F(n, 1)$ y $h_2(m) = F(1, m)$. Now if we apply Haláz's result to the functions h_1 and h_2 we have that there are complex constants not zero C_1, C_2, real constants a_1, a_2 and complex functions L_1, L_2 defined on $\mathbb{R}^+$ satisfying

$$|L_j(t)| = 1, \quad \text{for all } t \in \mathbb{R}^+$$

and

$$\lim_{t \to \infty} \frac{L_j(\lambda t)}{L_j(t)} = 1, \quad \text{for all } \lambda > 0 \quad (j = 1, 2),$$

such that

$$\sum_{n \le x} h_1(n) = C_1 x^{1 + ia_1} L_1(\log x) + o(x), \quad \text{when } x \to \infty, \tag{3.1}$$

and

$$\sum_{m \le y} h_2(m) = C_2 y^{1 + ia_2} L_2(\log y) + o(y), \qquad \text{when } y \to \infty; \tag{3.2}$$

therefore,

$$\begin{aligned} & M(x, y) - C_1 C_2 x^{ia_1} y^{ia_2} L_1(\log x) L_2(\log y) \\ =& \frac{1}{xy} \sum_{\substack{n \le x \\ m \le y}} F(n, m) - C_1 C_2 x^{ia_1} y^{ia_2} L_1(\log x) L_2(\log y) \\ =& \frac{1}{xy} \sum_{n \le x} h_1(n) \sum_{m \le y} h_2(m) - C_1 C_2 x^{ia_1} y^{ia_2} L_1(\log x) L_2(\log y). \end{aligned}$$

Now summing adequatedly, we obtain

$$
\begin{aligned}
& M(x,y) - C_1C_2x^{ia_1}y^{ia_2}L_1(\log x)L_2(\log y) \\
=& \frac{1}{y}\sum_{m\le y} h_2(m)\left(\frac{1}{x}\sum_{n\le x} h_1(n) - C_1x^{ia_1}L_1(\log x)\right) + \\
& C_1x^{ia_1}L_1(\log x)\left(\frac{1}{y}\sum_{m\le y} h_2(m) - C_2y^{ia_2}L_2(\log y)\right) + \\
& C_2y^{ia_2}L_2(\log y)\left(\frac{1}{x}\sum_{n\le x} h_1(n) - C_1x^{ia_1}L_1(\log x)\right) - \\
& C_2y^{ia_2}L_2(\log y)\left(\frac{1}{x}\sum_{n\le x} h_1(n) - C_1x^{ia_1}L_1(\log x)\right) \\
=& \left(\frac{1}{x}\sum_{n\le x} h_1(n) - C_1x^{ia_1}L_1(\log x)\right)\left(\frac{1}{y}\sum_{m\le y} h_2(m) - C_2y^{ia_2}L_2(\log y)\right) + \\
& C_1x^{ia_1}L_1(\log x)\left(\frac{1}{y}\sum_{m\le y} h_2(m) - C_2y^{ia_2}L_2(\log y)\right) + \\
& C_2y^{ia_2}L_2(\log y)\left(\frac{1}{x}\sum_{n\le x} h_1(n) - C_1x^{ia_1}L_1(\log x)\right).
\end{aligned}
$$

The first term in the last equality is o(1), by hypothesis; for the last two terms we observe that

$$
\left|C_1x^{ia_1}L_1(\log x)\left(\frac{1}{y}\sum_{m\le y} h_2(m) - C_2y^{ia_2}L_2(\log y)\right)\right| = C_1\,\mathrm{o}(1)\,.
$$

Thus

$$
M(x,y) = C_1C_2x^{ia_1}y^{ia_2}L_1(\log x)L_2(\log y) + \mathrm{o}(1)
$$

completing the proof. ☑

Recurrently, a similar result is obtained for the k variables case, i.e. we will have the existence of a complex constant C, real constans a_1, a_2, …, a_k and complex functions L_1, L_2, …, L_k defined on $\mathbb{R}^+$ such that:

$$
\begin{aligned}
& \frac{1}{x_1x_2\cdots x_k}\sum_{\substack{m_j\le x_j\\ j=1,\ldots,k}} F(m_1,m_2,\ldots,m_k) \\
& \qquad = C\cdot x_1^{ia_1}x_2^{ia_2}\cdots x_k^{ia_k}L_1(\log x_1)L_2(\log x_2)\cdots L_k(\log x_k) + \mathrm{o}(1)\ .
\end{aligned}
$$

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DEPARTMENTO DE MATEMÁTICAS
UNIVERSIDAD DE LOS ANDES
BOGOTÁ, COLOMBIA
e-mail: oscasas@uniandes.edu.co