Weber's problem on the Riemannian Manifolds: Some upper bounds for the minimun Weber's function

El problema de Weber sobre variedades Riemannianas: Algunas cotas Superiores para el mínimo de la función de Weber

Franco Rubio López [*] , Patricia Edith Alvarez Rodriguez [**] , Heyssen Dueñes Chávez [***]

Abstract

In this paper we obtain some upper bounds for the minimum of the Weber function on a strongly convex ball in a Riemannian manifold with positive sectional curvature; where the minimum is reached on the weighted geometric median of "m" given points in the strongly convex.

Keywords: The Weber problem, Weighted Geometric Median, Riemannian manifold, Strongly convex set.

Resumen

En este artículo se obtiene algunas cotas superiores para el mínimo de la función de Weber sobre una bola fuertemente convexa en una variedad Riemanniana con curvartura seccional positiva; dicho mínimo se alcanza sobre la mediana geométrica pesada de "m" puntos dados en la bola fuertemente convexa.

Palabras Clave: Problema de Weber, Mediana Geométrica Pesada, Variedad Riemanniana, Conjunto fuertemente convexo.

1. Introduction.

Mathematically, Weber's problem in the plane consists in finding a point that minimizes the sum of the weighed distances to "m" fixed points. Weiszfeld [7] was the first to formulate an iterative method to approximate the solution to Weber's problem.

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[*] **ID ORCID**: https://orcid.org/0000-0002-0168-3806 , Departamento de Matemáticas, Universidad Nacional de Trujillo, Av. Juan Pablo II s/n, Ciudad Universitaria, Trujillo-Perú.(frubio@unitru.edu.pe)

[**] Departamento de Matemáticas, Universidad Nacional de Trujillo, Av. Juan Pablo II s/n, Ciudad Universitaria, Trujillo-Perú (palvarezr@upao.edu.pe)

[***] **ID ORCID**: https://orcid.org/0000-0001-5083-7800 , Departamento de Matemáticas, Universidad Nacional de Trujillo, Av. Juan Pablo II s/n, Ciudad Universitaria, Trujillo-Perú.(ricardo18b@gmail.com)

If the fixed points are located in large regions on the surface of the earth, the approximation through a plane, and the use of the Euclidean distance, will provide unsatisfactory results; and since the sphere is used to model the planet earth, Drezner and Wesolowsky [2], Hansen [5], generalized Weber's problem to the sphere, and used the shortest arc distance to estimate the distance between two points.

Since both the plane and the sphere are regular surfaces; and since regular surfaces are a particular case of Riemannian manifolds, Fletcher [4], Aftab Khurrum [1], generalized the problem to Riemannian manifolds.

In addition, Fletcher [4] establishes conditions to guarantee the existence and uniqueness of the weighted geometric median. In this paper, upper bounds are established for the minimum of the Weber function on a strongly convex ball in a Riemannian manifolds; which is reached in the weighted geometric median.

2. The Weber Problem on the Plane

Consider "m" different points in the plane $a_1, a_2, \ldots, a_m \in \mathbb{R}^2$, $a_i = (x_i, \ y_i), \forall \ i = 1, 2, \ldots, m$, with weights $w_i > 0$, $\forall \ i = 1, 2, \ldots, m$.

Weber's problem is to find a point of $\mathbb{R}^2$ that minimizes the function $F: \mathbb{R}^2 \to \mathbb{R}$, defined by:

(1)
$$F(x, y) = \sum_{i=1}^m w_i \sqrt{(x - x_i)^2 + (y - y_i)^2} \ ;$$

ie:

(2)
$$\min_{X \in \mathbb{R}^2} F(X) = \min_{X \in \mathbb{R}^2} \sum_{i=1}^m w_i d_i(X) \ ,$$

where $d_i(X) = \sqrt{(x - x_i)^2 + (y - y_i)^2}$ is the Euclidean distance between the point $X = (x, y)$ and the point $a_i = (x_i, \ y_i)$. The points $a_i \in \mathbb{R}^2$ are called vértices and the function $F(x)$ is called Weber function.

If the vertices $a_i \in \mathbb{R}^2$ are not collinear, the Weber function is strictly convex, so the problem (2) has a unique solution; Wendel and Hurter [8], proved that this solution is in the convex capsule of the vertices. Weiszfeld [7], proposed an iterative method to approximate the solution of the problem (2), by means of a sequence defined by:

(3)
$$x^{k+1} = \frac{\sum_{i=1}^m \frac{w_i x_i}{d_i(x^k, \ y^k)}}{\sum_{i=1}^m \frac{w_i}{d_i(x^k, \ y^k)}} \quad , \quad y^{k+1} = \frac{\sum_{i=1}^m \frac{w_i y_i}{d_i(x^k, \ y^k)}}{\sum_{i=1}^m \frac{w_i}{d_i(x^k, \ y^k)}} \ ,$$

with starting point:

$$x^0 = \frac{\sum_{i=1}^m w_i x_i}{\sum_{i=1}^m w_i} \quad , \quad y^0 = \frac{\sum_{i=1}^m w_i y_i}{\sum_{i=1}^m w_i} \ .$$

3. The Weber Problem on Riemannian Manifold

Let M be a Riemannian manifold n-dimensional of class C^∞ , and d_M , ∇ , $\mathcal{R}$ are the intrinsic metric of M, Riemannian Connection and Curvature Operator respectively; where:

$$\mathcal{R}(X, Y)Z = \nabla_Y \nabla_X Z - \nabla_X \nabla_Y Z + \nabla_{[X,Y]} Z \ , \quad \forall \ X, Y, Z \in \Gamma(M),$$

$$\Gamma(M) = \{X: M \to TM / \quad X \in C^\infty(M)\} \ , \text{ TM is the tangent bundle of M.}$$

Definition 1. Let $S \subset M$. S is called a Strongly Convex set if for each pair of points q_1, q_2 of the closed set $\bar{S}$ of S, there is a unique minimizing geodesic $\alpha: [0,1] \to S$, $\alpha(0) = q_1$, $\alpha(1) = q_2$, such that $\alpha([0,1]) \subset S$.

Also, for each $p \in M$, there is $\delta > 0$, such that the geodesic ball $B(p, \delta)$ is strongly convex [6]. Let $A \subset M$ be a strongly convex set in M. The set $T_x A$ of tangent vectors to A at point x is a convex cone in tangent space to M at x: $T_x M$.

Given X_x, $Y_x \in T_x A$, then: $tX_x + tY_x \in T_x A$, $\forall\, t \geq 0$; and there are curves:

$$\alpha,\ \beta: [0,1] \to A,\ \alpha(0) = \beta(0) = x,\ \alpha'(0) = X_x\ \ ,\ \beta'(0) = Y_x.$$

Also, we consider a deformation of class C^∞ of curves in A:

(4)
$$\tau: [0,1]x[0,t_0] \to A$$

$$(s,t) \to \tau(s,t)$$

such that

a) $\tau(s,0) = x$.

b) $\tau(0,t) = \alpha(t) = exp_x(tX_x)$,

    $\tau(1,t) = \beta(t) = exp_x(tY_x)$.

c) For each $t \in [0,t_0]$, the curve:

$$\gamma_t: [0,1] \to A \text{ , defined by } \gamma_t(s) = \tau(s,t),$$

is a parameterized geodesic with a parameter proportional to the arc length. This allows to consider the parameterized geodesic:

$$\lambda_s: [0,t_0] \to A,\ \forall\, s \in [0,1], \text{ such that } \lambda_s(t) = \gamma_t(s) = \tau(s,t).$$

Theorem 1. The vector field $J(s) = \frac{\partial \tau(s,t)}{\partial t}$, $\forall\ s \in [0,1]$, defined along the curve $\gamma_t(s) = \tau(s,t)$, is a field of Jacobi; ie:

$$\frac{D^2 J(s)}{\partial s^2} +\ \mathcal{R}\left(\frac{d\gamma_t(s)}{ds}, J(s)\right) \frac{d\gamma_t(s)}{ds} = 0$$

<div align="center">Proof</div>

Using the curvature operator $\mathcal{R}$, we have:

$$\mathcal{R}\left(\frac{d\gamma_t(s)}{ds}, J(s)\right)\frac{d\gamma_t(s)}{ds} = \nabla_{J(s)}\nabla_{\frac{d\gamma_t(s)}{ds}}\frac{d\gamma_t(s)}{ds} - \nabla_{\frac{d\gamma_t(s)}{ds}}\nabla_{J(s)}\frac{d\gamma_t(s)}{ds} + \nabla_{\left[\frac{d\gamma_t(s)}{ds}, J(s)\right]}\frac{d\gamma_t(s)}{ds}$$

$$= -\nabla_{\frac{d\gamma_t(s)}{ds}}\nabla_{\frac{d\gamma_t(s)}{dt}}\frac{d\lambda_s(t)}{dt} = -\frac{D^2 J(s)}{\partial s^2}\ .$$

Therefore, $J(s)$ is a field of Jacobi.   ∎

Theorem 2. Let $\tau(s,t)$ be a deformation given by (4), and be the geodesic:

$$\lambda_s \colon [0,t_0] \to A, \quad \forall\, s \in [0,1], \text{ such that } \lambda_s(t) = \gamma_t(s) = \tau(s,t).$$

Then:

(5)
$$\frac{d\lambda_s(0)}{dt} = (1-s)X_{x_0} + sY_{x_0}, \quad \forall s \in [0,1].$$

$$\text{Proof}$$

By theorem (1), the vector field $J(s) = \frac{\partial \tau(s,t)}{\partial t}$, $\forall\ s \in [0,1]$, is a field of Jacobi along the geodesic $\gamma_t(s) = \tau(s,t)$.

In particular for $t = 0$, and by (a) of the deformation (4), the Jacobi equation is reduced:

$$\frac{D^2 J(s)}{\partial s^2} = \frac{D}{ds}\frac{D}{ds}\big(J(s)\big) = \frac{D}{ds}\frac{D}{ds}\left(\frac{\partial \tau(s,0)}{\partial t}\right) = 0.$$

By the conditions:

$$\frac{\partial \tau(0,0)}{\partial t} = \frac{d\alpha(0)}{dt} = X_{x_0}, \qquad \frac{\partial \tau(1,0)}{\partial t} = \frac{d\beta(0)}{dt} = Y_{x_0},$$

we have:

$$\frac{d\lambda_s(0)}{dt} = (1-s)X_{x_0} + sY_{x_0}, \quad \forall s \in [0,1]. \qquad \blacksquare$$

Definition 2. Given points $x_1, x_2, \ldots, x_m \in M$, and positive real number $w_1, w_2, \ldots, w_m$, the function $F \colon M \to \mathbb{R}$, defined by:

(6)
$$F(x) = \sum_{i=1}^{m} w_i d_M(x,\ x_i)$$

is called Weber's function.

Theorem 3. Let $p_0 \in M$, $B(p_0, r)$ a strongly convex ball. For each $x_0 \in B(p_0, r)$, the function $F_{x_0} \colon B(p_0, r) \to \mathbb{R}$ defined by $F_{x_0}(x) = d_M(x_0, x)$ is a convex function.

$$\text{Proof}$$

Let $x_1, x_2 \in B(p_0, r)$ be two arbitrary points. Then there are minimal geodesics

$$\alpha \colon [0,1] \to B(p_0, r), \ \alpha(0) = x_0, \ \alpha(1) = x_1, \ \frac{d\alpha(0)}{dt} = X_{x_0}$$

$$\beta \colon [0,1] \to B(p_0, r), \ \beta(0) = x_0, \ \beta(1) = x_2, \ \frac{d\beta(0)}{dt} = Y_{x_0},$$

such that:

$$l(\alpha) = \int_0^1 \left\|\frac{d\alpha(t)}{dt}\right\| dt = \|X_{x_0}\|,$$

$$l(\beta) = \int_0^1 \left\|\frac{d\beta(t)}{dt}\right\| dt = \|Y_{x_0}\|.$$

By deformation $\tau(s,t)$ given by (4), there is a minimal geodesic: $\gamma_1 \colon [0,1] \to B(p_0, r)$, such that:

$$\gamma_1(0) = \tau(0,1) = \alpha(1) = x_1,$$
$$\gamma_1(1) = \tau(1,1) = \beta(1) = x_2.$$

For each "s" consider the geodesic:

$$\lambda_s : [0,1] \to B(p_0, r), \quad \lambda_s(t) = \tau(s, t);$$

what satisfies: $\lambda_s(0) = \tau(s, 0) = x_0$, $\lambda_s(1) = \tau(s, 1) = \gamma_1(s)$.

Then, the function $F_{x_0} \diamond \gamma_1 : [0,1] \to \mathbb{R}$ verifies:

$$(F_{x_0} \diamond \gamma_1)(s) = F_{x_0}(\gamma_1(s)) = F_{x_0}(\lambda_s(1)) = d_M(x_0, \lambda_s(1)) =$$

$$= \int_0^1 \left\| \frac{d\lambda_s(t)}{dt} \right\| dt = \left\| \frac{d\lambda_s(0)}{dt} \right\| = \left\| (1-s)X_{x_0} + sY_{x_0} \right\|$$

$$\leq (1-s)\left\| X_{x_0} \right\| + s\left\| Y_{x_0} \right\|$$

$$= (1-s)d_M\big(x_0, \alpha(1)\big) + s d_M\big(x_0, \beta(1)\big)$$

$$= (1-s)d_M(x_0, x_1) + s d_M(x_0, x_2)$$

$$= (1-s)F_{x_0}(x_1) + s F_{x_0}(x_2), \ \forall s \in [0,1]$$

Therefore, F_{x_0} is a convex function. ∎

Theorem 4. Let $x_1, x_2, \dots, x_m \in B(p_0, r)$. Then the function $F : B(p_0, r) \to \mathbb{R}$ defined by:

(7) $$F(x) = \sum_{i=1}^m w_i d_M(x, x_i) \ , \ w_i > 0 , \ \forall \, i = 1, \dots, m$$

is convex function.

<div align="center">Proof</div>

By theorem (3), the Weber's function (6):

$$F(x) = \sum_{i=1}^m w_i d_M(x, x_i)$$

$$= \sum_{i=1}^m w_i F_{x_i}(x) ,$$

where the functions F_{x_i} , $\forall \, i = 1, \dots, m$, are convex functions. Also,

$$w_i F_{x_i} \ , \ \forall \, i = 1, \dots, m \ , \text{ are convex functions.}$$

Therefore: $F(x) = \sum_{i=1}^m w_i d_M(x, x_i)$ is a convex function. ∎

4. Some bound for the Weber function

In this section we introduce the weighted geometric median (Fletcher, 2009).

Definition 3. Let $U \subset M$ be a strongly convex set, $x_1, x_2, \dots, x_m \in U$. The Weighted Geometric Median of $F(x) = \sum_{i=1}^m w_i d_M(x, x_i)$, $w_i > 0 , \forall \, i = 1, \dots, m, \ \sum_{i=1}^m w_i = 1$, is:

(8) $$x^* = arg \min_{x \in U} \sum_{i=1}^m w_i d_M(x, x_i).$$

If $w_i = \frac{1}{m}$, $\forall \, i = 1, \dots, m$, x^* is call simply the Geometric Median [4].

Let $p_0 \in M$, $B(p_0, r) = exp_{p_0}(B(0, r))$, $B(0, r) \subset T_{p_0}M$ is the ball open in tangent space $T_{p_0}M$, where the exponential map exp_{p_0} is a diffeomorphism.

Now, let's consider the points $x_1, x_2, \ldots, x_m \in B(p_0, r)$, and be

$$v_i = exp_{p_0}^{-1}(x_i) \in B(0, r), \quad \forall i = 1, \ldots, m.$$

Theorem 5. Let M be a Riemannian manifold with sectional curvature $k(x) > 0$, $\forall x \in M$. Then:

(9) $$F(x) = \sum_{i=1}^m w_i d_M(x, x_i) < 2r.$$

<div align="center">Proof</div>

As $k(x) > 0, \forall x \in M$, and by Toponogov's theorem, we have:

$$d_M(x, x_i) \leq \|v - v_i\|, \quad \forall i = 1, \ldots, m$$

$$d_M(x, x_i) \leq \|v - v_i\| < 2r, \quad \forall i = 1, \ldots, m$$

Also: $$w_i d_M(x, x_i) \leq w_i \|v - v_i\| < 2rw_i, \quad \forall i = 1, \ldots, m$$

$$\sum_{i=1}^m w_i d_M(x, x_i) \leq \sum_{i=1}^m w_i \|v - v_i\| < 2r \sum_{i=}^m w_i = 2r$$

Therefore:

$$F(x) = \sum_{i=1}^m w_i d_M(x, x_i) < 2r.$$ ∎

Theorem 6. Let M be a Riemannian manifold with sectional curvature $0 < k(x) \leq \Delta$, $B(p_0, r) \subset M$ a strongly convex ball, $r < \frac{\pi}{4\sqrt{\Delta}}$, and $x_1, x_2, \ldots, x_m \in B(p_0, r)$, then exists and is unique the weighted gemetric median defined by (8). Also:

(10) $$F(x) = \sum_{i=1}^m w_i d_M(x, x_i) < \frac{\pi}{2\sqrt{\Delta}}.$$

<div align="center">Proof</div>

By theorem (4) the function $F(x) = \sum_{i=1}^m w_i d_M(x, x_i)$ is convex; and how the sectional curvature of M is bounded, and $r < \frac{\pi}{4\sqrt{\Delta}}$, then by Fletcher [4], exists and is unique the weighted gemetric median defined by (8).

By theorem (5):

$$F(x) = \sum_{i=1}^m w_i d_M(x, x_i) < 2r < \frac{\pi}{2\sqrt{\Delta}}.$$

Therefore:

$$F(x) = \sum_{i=1}^m w_i d_M(x, x_i) < \frac{\pi}{2\sqrt{\Delta}}.$$ ∎

Corollary 1. Let M be a Riemannian manifold with sectional curvature $0 < k(x) \leq \Delta$, $B(p_0, r) \subset M$ a strongly convex ball, $r < \frac{\pi}{4\sqrt{\Delta}}$, and $x_1, x_2, \ldots, x_m \in B(p_0, r)$, then

(11) $$F^* = \min_{x \in B(p_0, r)} \sum_{i=1}^m w_i d_M(x, x_i) < \frac{\pi}{2\sqrt{\Delta}}.$$

<div align="center">Proof</div>

By theorem (6), exists and is unique the weighted geometric median $x^* \in B(p_0, r)$; and be

$F^* = F(x^*)$. Then:

$$F^* = F(x^*) \leq F(x) < \frac{\pi}{2\sqrt{\Delta}} , \quad \forall \, x \in B(p_0, r). \qquad \blacksquare$$

Let $x_1, x_2, \ldots, x_m \in B(p_0, r)$, $v_i = exp_{p_0}^{-1}(x_i) \in B(0, r)$, $\forall \, i = 1, \ldots, m$,

$$v = exp_{p_0}^{-1}(x) \in B(0, r), \quad \forall \, x \in B(p_0, r) .$$

This allows to consider the function $f: B(0, r) \subset T_{p_0} M \to \mathbb{R}$, defined by:

$$f(v) = \sum_{i=1}^m w_i \|v - v_i\|.$$

Theorem 7. Let M be a Riemannian manifold with sectional curvature $k(x) > 0$, $\forall \, x \in M$. Then:

(12) $$F(x) \leq f(v), \qquad \forall \, x \in B(p_0, r).$$

<div align="center">Proof</div>

As $k(x) > 0$, $\forall \, x \in M$, and by Toponogov theorem we have:

$$d_M(x, \, x_i) \leq \|v - v_i\| , \quad \forall \, i = 1, \ldots, m.$$

Also: $\qquad w_i d_M(x, \, x_i) \leq w_i \|v - v_i\|, \quad w_i > 0 , \, \forall \, i = 1, \ldots, m.$

Then $\qquad F(x) = \sum_{i=1}^m w_i d_M(x, \, x_i) \leq \sum_{i=1}^m w_i \|v - v_i\| = f(v).$

Therefore:

$$F(x) \leq f(v), \quad \forall \, x \in B(p_0, r). \qquad \blacksquare$$

Theorem 8. Let M be a Riemannian manifold with sectional curvature $0 < k(x) \leq \Delta$, $B(p_0, r) \subset M$ a strongly convex ball, $r < \frac{\pi}{4\sqrt{\Delta}}$, and be a $x_1, x_2, \ldots, x_m \in B(p_0, r)$,

$$\sum_{i=1}^m w_i = 1, \qquad F^* = \min_{x \in B(p_0, r)}(F(x)), \qquad f^* = \min_{v \in B(0, r)}(f(v)).$$

Then:

(12) $$F^* \leq f^* < \frac{\pi}{2\sqrt{\Delta}} .$$

<div align="center">Proof</div>

Let $x^* \in B(p_0, r)$ such that: $F^* = F(x^*)$, and $v^* \in B(0, r)$ such that: $f^* = f(v^*)$. We have

Case 1. Let $v^* = exp_{p_0}^{-1}(x^*)$, then:

$$F^* = F(x^*) \leq f(v^*) = f^* < \frac{\pi}{2\sqrt{\Delta}}.$$

Case 2. Let $v_{x^*} = exp_{p_0}^{-1}(x^*)$, such that $v_{x^*} \neq v^*$, then $f(v^*) < f(v_{x^*})$.

As $F(x) \leq f(v)$, y $v = \varphi(x)$, where $\varphi = exp_{p_0}^{-1}$, we have:

$$F(x) \le f\big(\varphi(x)\big) = G(x), \text{ where } G = f \circ \varphi \colon B(p_0, r) \to \mathbb{R}$$

then:

$$\min_{x \in B(p_0, r)} (F(x)) \le \min_{x \in B(p_0, r)} (G(x))$$

$$= \min_{x \in B(p_0, r)} (f \circ \varphi(x))$$

$$= f^* = \min_{x \in B(p_0, r)} (f(v)) < \frac{\pi}{2\sqrt{\Delta}}.$$

Therefore:

$$F^* \le f^* < \frac{\pi}{2\sqrt{\Delta}}. \qquad\qquad\qquad\blacksquare$$

4. Example

In this section we present results on the unit sphere.

Let $M = S^2 = \{(x, y, z) \in \mathbb{R}^3 \ / \ x^2 + y^2 + z^2 = 1 \}$ the unit sphere, whose sectional curvature $k(x) = 1, \ \forall x \in S^2$; therefore the bounded above for the sectional curvature is $\Delta = 1$.

Now, consider the ball strongly convex $B(p_0, r)$, where $p_0 = (0, 0, 1)$, and $r = \frac{\pi}{4}$, Drezner and Wesolowsky [2], and using spherical Coordinates:

$$x = \cos(\varphi)\cos(\theta), \quad y = \cos(\varphi)\sin(\theta), \quad z = \sin(\varphi), \quad \frac{\pi}{4} < \varphi < \frac{\pi}{2}, \ 0 < \theta < 2\pi,$$

a PC with Intel Core i3 microprocessor, Windows 7 operating system, 2.20 GHz, 4GB of RAM,

we have the following cases:

Case 1. We consider $m = 5$ points chosen randomly

Coordinates and weights			
	θ_i	φ_i	w_i
1	5.1191	0.8620	0.2000
2	5.6913	1.0041	0.2000
3	0.7979	1.2149	0.2000
4	5.7389	1.5374	0.2000
5	3.9732	1.5432	0.2000

Table 1. Spherical coordinates of the points and their weights.

Using the Weiszfeld algorithm in the sphere was obtained:

Results		
Optimal Value: F^*	0.3222718	
Geometric Median: x^*	(0.0436559, -0.0280022, 0.9986541)	

Table 2. The Geometric Median and the Optimal value on the Sphere.

Using the Weiszfeld algorithm in the tangent space $T_{p_0} S^2$ was obtained:

Results		
Optimal Value: f^*	0.3365514	
Geometric Median: v^*	(0.0445683, -0.0286065, 1.0000000)	

Table 3. The Geometric Median and the Optimal value on the Tangent space $T_{p_0} S^2$.
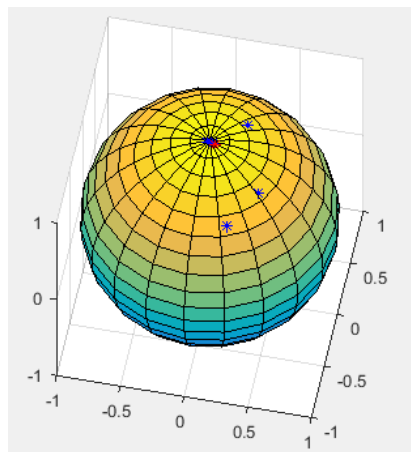
Table 2 and Table 3 show that: $F^* < f^*$.

Figure 1. The red point is the geometric median for 5 points.

Case 2. We consider $m = 10$ points chosen randomly

Coordinates and weights			
	θ_i	φ_i	w_i
1	4.4362	1.1300	0.1000
2	0.2000	1.0851	0.1000
3	1.7400	1.3866	0.1000
4	0.2901	1.4099	0.1000
5	0.6103	0.9322	0.1000
6	5.1739	1.1701	0.1000
7	4.3657	1.1354	0.1000
8	1.9924	1.2930	0.1000
9	5.9704	1.3425	0.1000
10	0.2164	1.3781	0.1000

Table 4. Spherical coordinates of the points and their weights.

Using the Weiszfeld algorithm in the sphere was obtained:

Results	
Optimal Value: F^*	0.3029679
Geometric Median: x^*	(0.1638406, 0.0266638, 0.9861264)

Table 5. The Geometric Median and the Optimal value on the Sphere

Using the Weiszfeld algorithm in the tangent space $T_{p_0}S^2$ was obtained:

Results	
Optimal Value: f^*	0.3134207
Geometric Median: v^*	(0.1633452, 0.0262530, 1.0000000)

Table 6. The Geometric Median and the Optimal value on the Tangent space $T_{p_0}S^2$.
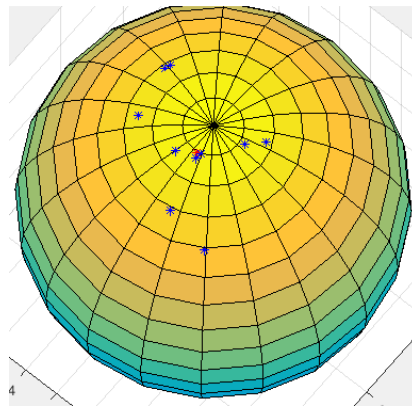
Table 5 and Table 6 show that: $F^* < f^*$.

Figure 2. The red point is the geometric median for 10 points.

Case 3. We consider $m = 20$ points chosen randomly

	Coordinates and weights		
	θ_i	φ_i	w_i
1	1.3053	1.2526	0.0500
2	1.8928	0.9913	0.0500
3	2.9589	1.2589	0.0500
4	1.4482	1.3440	0.0500
5	5.3049	0.9596	0.0500
6	1.2237	0.8776	0.0500
7	1.4195	1.0184	0.0500
8	1.0726	1.0358	0.0500
9	1.4305	1.1185	0.0500
10	2.7376	1.1843	0.0500
11	1.9547	0.8526	0.0500
12	5.8018	0.9916	0.0500
13	2.7031	1.4145	0.0500
14	1.1612	0.8083	0.0500
15	5.6855	1.5149	0.0500
16	6.1559	1.3590	0.0500
17	2.7575	1.1692	0.0500
18	0.6982	1.2398	0.0500
19	1.6215	0.9718	0.0500
20	2.5681	1.1458	0.0500

Table 7. Spherical coordinates of the points and their weights.

Using the Weiszfeld algorithm in the sphere was obtained:

Results	
Optimal Value: F^*	0.3630473
Geometric Median: x^*	(0.0300179, 0.2946810, 0.9551241)

Table 8. The Geometric Median and the Optimal value on the Sphere.

Using the Weiszfeld algorithm in the tangent space $T_{p_0}S^2$ was obtained:

Results	
Optimal Value: f^*	0.3653496
Geometric Median: v^*	(0.0280437, 0.2905243, 1.0000000)

Table 9. The Geometric Median and the Optimal value on the Tangent space $T_{p_0}S^2$.
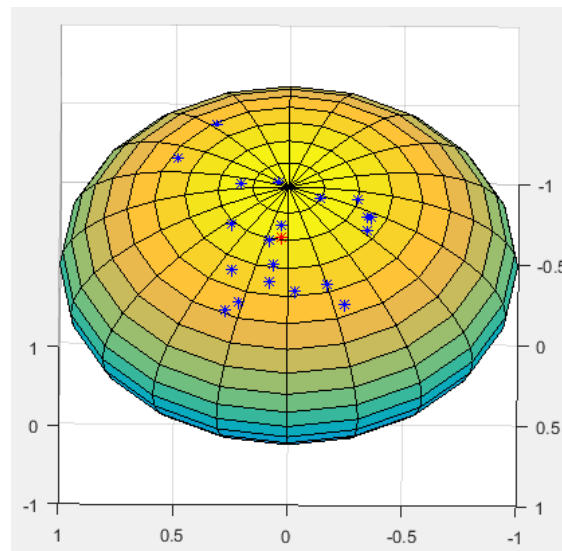
Table 8 and Table 9 show that: $F^* < f^*$.

Figure 3. The red point is the geometric median for 20 points.

7. References

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