Revista de Matemática: Teoría y Aplicaciones 2009 **16**(2): 205–220
cimpa – ucr   issn: 1409-2433

PORTFOLIO OPTIMIZATION USING PARTICLE SWARMS WITH STRIPES

Mario Villalobos–Arias*

Recibido/Received: 2 Oct 2008 — Aceptado/Accepted: 5 Mar 2009

Abstract

In this paper it is consider the Portfolio Optimization Problem developed by Markowitz [11]. The basic assumption is that the investor tries to maximize his/her profit and at the same time, wants to minimize the risk. This problem is usually solved using a scalarization approach (with one objective). Here it is solved it as a bi-objective optimization problem. It uses a new version of the algorithm of Particle Swarm Optimization for Multi-Objective Problems to which it implemented a method of the stripes to improve dispersion.

Keywords: Portfolio optimization, particle swarm optimization, multiobjetive, optimization.

Resumen

En el presente trabajo se considera el problema de optimización de portafolios desarrollado por Markowitz [11]. El supuesto básico es que el inversor intenta maximizar sus beneficios y al mismo tiempo, quiere minimizar el riesgo. Este problema se suele resolver mediante un enfoque de esscalarización (con un objetivo). Aquí se resuelve como un problema de optimización multiobjetivo. Utiliza una nueva versión del algoritmo de optimización por enjambre de partículas para problemas multiobjetivo, a los que se puso en práctica un método de las franjas para mejorar la dispersión.

Palabras clave: Optimización de portafolios, optimización por enjambre de partículas, multiobjetivo, optimización.

Mathematics Subject Classification: 90C27, 90C29, 90C59, 91B28.

*CIMPA, Escuela de Matemática, Universidad de Costa Rica 2060 San José, Costa Rica. E-Mail: mario.villalobos@ucr.ac.cr

1 Introduction

In real-world, there are many problems with several objectives that we aim to optimize simultaneously, an example is the Portfolio Optimization, in wich the investor want to maximize his/her profit and the same time minimize the risk.

These problems are called "multiobjective" or "vector" optimization problems, and have been studied by many authors who have proposed a number of solution techniques [1, 4, 6, 12, 13].

The solution of a multiobjective optimization problem requires a suitable definition of "optimality" (usually called "Pareto optimality"). Such problems normally have not one, but an infinite set of solutions, which represent possible trade-offs among the objectives (such solutions constitute the so-called "Pareto optimal set", defined in Section 2).

In these multiobjective optimization problems (MOPs) one wishes to optimize a vector function, say $F(x) = (f_1(x), \dots, f_n(x))$. A typical way to approach these problems is to transform the MOPs into single-objective (or "scalar") problems (e.g., by using a linear aggregating function). This approach indeed makes sense if the functions $f_1, \dots, f_n$ are of the same type and expressed in the same units, but otherwise (for instance, if f_1 denotes distance, f_2 denotes time, and so on) the scalarized problem might be meaningless.

Diverse metaheuristics have been adopted to solve MOP [1]–[3], [5, 7], hence it is reasonable use a heuristic, such as particle swarm optimization (PSO), to solve the portfolio optimization problem as a multiobjective problem. In this paper we use the Particle Swarm Optimization algorithm for multiobjective (MOPSO) [2], and we use the stripes approach to improve this algorithm [14].

The rest of the paper is organized as follow, the next section MOP is presented, in section 3 an introduction to PSO is presented. The Portfolio Optimization Problem (POP) is presented in the section 4, some classical solution of the POP are in the section 5. The data that we use and the result obtained is presented in sections 6 and 7. Finally in section 8 the conclusions and future work are presented.

2 The multiobjective optimization problem

Let X be a set and $F : X \longrightarrow I\!R^d$ a given vector function with components $f_i : X \longrightarrow I\!R$ for each $i \in \{1, \dots, d\}$. The multiobjective optimization problem (MOP) we are concerned with is to find $x^* \in X$ such that

$$F(x^*) = \min_{x \in X} F(x) = \min_{x \in X} [f_1(x), \dots, f_d(x)], \tag{1}$$

where the minimum is understood in the sense of the standard Pareto order in which two vectors in $I\!R^d$ are compared as follows.

If $\vec{u} = (u_1, \dots, u_d)$ and $\vec{v} = (v_1, \dots, v_d)$ are vectors in $I\!R^d$, then

$$\vec{u} \preceq \vec{v} \iff u_i \le v_i \ \forall\, i \in \{1, \dots, d\}.$$

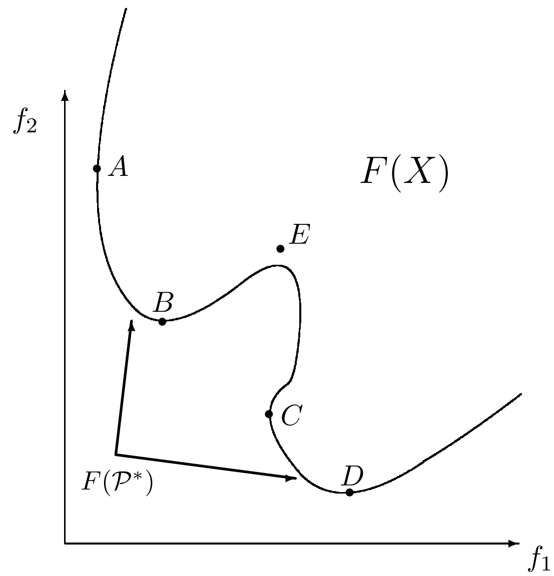

Figure 1: Example of a Pareto front for two objective case.

This relation is a partial order. We also write $\vec{u} \prec \vec{v}$ if $\vec{u} \preceq \vec{v}$ and $\vec{u} \neq \vec{v}$. In this case we say that u *dominates* v. By example in Figure 1 point B dominates point E.

Definition 1 *A point $x^* \in X$ is called a* Pareto optimal solution *for the MOP (1) if there is no $x \in X$ such that $F(x) \prec F(x^*)$. The set*

$$\mathcal{P}^* = \{x \in X \ : \ x \textit{ is a Pareto optimal solution}\}$$

is called the Pareto optimal set *for the MOP (1), and its image under F, i.e.*

$$F(\mathcal{P}^*) \ := \ \{F(x) \ : \ x \in \mathcal{P}^*\} ,$$

is called the Pareto front*.*

In Figure 1 the Pareto front corresponds to the parts on the boundary of $F(X)$ joining the points A and B, and also the points C and D.
Here we say that x *dominates* y when $F(x) \prec F(y)$. Let $Y \subseteq X$ and $y \in Y$. If there is no $x \in Y$, that dominates y , we say that y is *nondominated* (with respect to Y). Observe that all the elements in the Pareto front are nondominated with respect to X.

3 Particle swarm optimization

Particle swarm optimization algorithm (PSO) was introduced by Kennedy and Eberhart in 1995 [8] is based on the interaction of a set of particles that correspond to possible solutions of an optimization problem, moving each particle in a numerical space looking for the optimal position. A particularity of PSO is that particles communicate and hence —as in a social system— a particle with a good position (measured by its objective function value) *influences* on the other ones, *attracting* them.

In the PSO algorithm a set of M particles is handled in a multidimensional space and it is intended to improve its performance according to its own experience and the experience of its neighbors. Indeed, each particle has three tendencies:

(i) to follow its present direction, following the particle's inertia,

(ii) to go back to its best historical position and

(iii) to imitate its best neighbor.

3.1 Modeling PSO

If $z^m(t)$ represents the m–th particle, then its velocity in iteration $t+1$ is defined as

$$v^m(t+1) = \alpha v^m(t) + r_1(z^{m*} - z^m(t)) + r_2(z^* - z^m(t))$$

where $v^m(t)$ is the direction of the preceding iteration, z^{m*} is the best historial position ever obtained by particle m, z^* is the best particle ever obtained during the algorithm, r_1 and r_2 are random numbers, and α is a parameter. So, we define the new position of particle m as

$$z^m(t+1) = z^m(t) + v^m(t+1).$$

For more details about PSO see [8, 9] and for PSO for multiobjective problems see [1, 2].

PSO with stripes

To solve the portfolio optimization problem as a multiobjective problem we use a new approach presented in [14], call PSO with stripes (MOPSO-ST) that intent to solve the problem of the diversity of the MOP.

4 The portfolio optimization problem: Markowitz model

Here we present the Portfolio Optimization Problem developed by Markowitz [11]. The basic assumption is that the investor tries to maximize his/her profit and, at the same time, wants to minimize the risk.

We consider a market where s different securities (i.e. stocks) are traded. These securities have prices $p_1, p_2, \ldots, p_s$ at the initial time $t = 0$. We restrict ourselves to a one-period model. This means that the investor makes his decisions at the beginning of the period and is not allowed to revise his decisions until the end of the period. Let $P_1(T), P_2(T), \ldots, P_s(T)$ be the prices of the securities at the final time $t = T$, we assume that these final prices are not foreseeable. Therefore, they are modeled as non-negative random variables on a probability space $(\Omega, \mathcal{F}, \mathcal{P})$.

The return of the stocks is given by the variables $r_1, r_2, \ldots, r_s$ given by

$$r_i = \frac{P_i(T) - p_i}{p_i}, \quad i = 1, \ldots, s. \tag{2}$$

Observe that r_i is also a random variable.

We assume that we know (or have estimated) their means, variances and covariances.

$$\begin{array}{rcl} E(r_i) & = & \mu_i \quad \text{for all } i = 1, \ldots, s, \\ Cov(r_i, r_j) & = & \sigma_{ij}^2 \quad \text{for all } i, j = 1, \ldots, s. \end{array} \tag{3}$$

Using the variables x_i for the share of the $i-$th security on the portfolio, we can calculate the return of the portfolio $R_p = R_p(x_1, \ldots, x_s)$ by

$$R_p = \sum_{i=1}^{s} x_i r_i, \tag{4}$$

with the restrictions on the shares

$$\sum_{i=1}^{s} x_i = 1 \ \text{ and } x_i \geq 0 \ \ i = 1, \ldots, s.$$

We have observed that the r_i are random variables with means μ_i and covariances $\sigma_{ij}^2 = E(r_i - E(r_i))(r_j - E(r_j))$. Thus the return of the portfolio R_p is a random variable as well, and its mean μ_p is given by

$$\mu_p \ = \ E(R_p) \ = \ \sum_{i=1}^{s} x_i E(r_i) \ = \ \sum_{i=1}^{s} x_i \mu_i.$$

We measure the risk contained in the portfolio by the variance of its return

$$\sigma_p^2 \ = Var(R_p) = \ E\left[\{R_p - E(R_p)\}^2\right] \ = \ \sum_{j=1}^{s}\sum_{i=1}^{s} x_i \sigma_{ij}^2 x_j \ = \ \sum_{i,j=1}^{s} x_i x_j \sigma_{ij}^2.$$

We will also impose the constraints

$$x_i \leq c_i, \quad \text{for all} \ \ i = 1, \ldots, s,$$

where the c_i are constants.

Therefore, the investor wants to find a vector $\vec{x} = (x_1, x_2, \ldots, x_s)$ that maximizes the mean return

$$\mu_p \ = \ \sum_{i=1}^{s} x_i \mu_i =: -f_1(\vec{x})$$

and at the same time minimizes the risk

$$\sigma_p^2 \ = \ \sum_{i,j=1}^{s} x_i x_j \sigma_{ij}^2 =: f_2(\vec{x}),$$

subject to the constraints

$$\sum_{i=1}^{s} x_i = 1 \;\text{ and } 0 \leq x_i \leq c_i \;\; \forall\, i = 1, \ldots, s.$$

Thus, we have the next definition.

Definition 2 *The classical* portfolio optimization problem *(POP) with two objective functions is to find the vector* $\vec{x}^* = (x_1^*, x_2^*, \ldots, x_s^*)$ *such that*

$$\begin{aligned} (f_1(\vec{x}^*), f_2(\vec{x}^*)) &= \min_{\vec{x}} \left(-\sum_{i=1}^{s} x_i\mu_i \;,\; \sum_{i,j=1}^{s} x_i x_j \sigma_{ij}^2 \right) \\ \textit{subject to} \quad & \sum_{i=1}^{s} x_i = 1, \\ & 0 \leq x_i \leq c_i \;\; \forall\, i = 1, \ldots, s. \end{aligned} \tag{5}$$

5 Classical solution

The classical way to solve this problem is by solving a single–objective (or scalar) problem, (see for example, [10]). One can also consider several variants of (5).
For instance we may require a lower bound (R_c) on the mean return, and then choose the portfolio with minimal variance, that is

$$\begin{aligned} \min_{\vec{x}} \sigma_p^2 &= \min_{\vec{x}} \sum_{i,j=1}^{s} x_i x_j \sigma_{ij}^2 \\ \text{subject to} \quad & \mu_p \geq R_c \\ & \sum_{i=1}^{s} x_i = 1, \\ & 0 \leq x_i \leq c_i \;\; \forall\, i = 1, \ldots, s \end{aligned} \tag{6}$$

Alternatively, one can consider the dual problem of setting up an upper bound (σ_c) on the portfolio variance, and then maximize the mean return.

$$\begin{aligned} \max_{\vec{x}} \mu_p &= \max_{\vec{x}} \sum_{i=1}^{s} x_i\mu_i \\ \text{subject to} \quad & \sigma_p^2 \leq \sigma_c \\ & \sum_{i=1}^{s} x_i = 1, \\ & 0 \leq x_i \leq c_i \;\; \forall\, i = 1, \ldots, s. \end{aligned} \tag{7}$$

In any of these two forms of the POP, we usually find a single point of the Pareto front. (see Figure 2).

Still another variant of the POP is

$$\begin{aligned} \min_{\vec{x}}(\sigma_p^2 - \mu_p) &= \min_{\vec{x}} \left(\sum_{i,j=1}^{s} x_i x_j \sigma_{ij}^2 - \sum_{i=1}^{s} x_i \mu_i \right) \\ \text{subject to} \quad & \sum_{i=1}^{s} x_i = 1, \\ & 0 \leq x_i \leq c_i \quad \forall\, i = 1, \ldots, s. \end{aligned} \tag{8}$$

Again, the solution of this single–objective problem gives only one point of the Pareto front, and the investor does not have the option to select another portfolio with a similar risk and/or a better return.

This situation is illustrated in Figure 2, which shows a classical Pareto front for the POP. If the value of σ_c is close to 0, we can see that a small increase in the risk can give a much higher return. In contrast, if σ_p^2 is large, then to obtain a small increase in the return requires a large increase in the risk.

In the single–objective formulation of the POP, the investor cannot appreciate these subtleties.

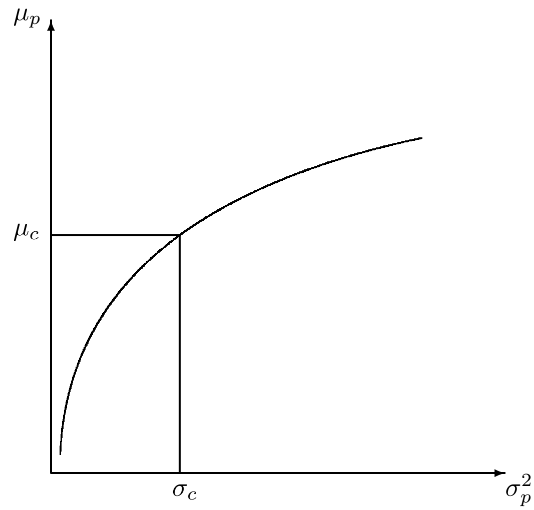

Figure 2: Graphical illustration of the Pareto front for the POP.

6 The data

To test our algorithm we took 20 securities (i.e. $s = 20$) from the "Mexican Stock Market" (BMV = Bolsa Mexicana de Valores). These securities appear in the "Index of Prices and Quotations" (IPyC = Indice de Precios y Cotizaciones).

date	AlfaA	AmTelA1	Amxl	BImboA	Cemex CPO	Elektra	Femsaubd	gcarsoa1	···
28/09/2004	42.090	23.800	22.027	24.752	63.800	76.200	50.200	51.843	···
29/09/2004	42.880	24.420	22.226	24.655	64.720	76.790	50.620	52.787	···
30/09/2004	43.060	24.600	22.206	24.439	64.090	76.480	50.300	51.992	···
01/10/2004	43.480	24.890	22.756	25.203	64.800	76.750	50.830	52.250	···
04/10/2004	43.280	25.250	23.185	25.350	65.760	76.400	50.870	52.558	···
05/10/2004	43.100	24.600	22.956	25.340	65.470	76.610	51.040	52.648	···
06/10/2004	42.860	24.300	22.526	25.144	67.140	76.690	51.140	52.518	···
07/10/2004	42.990	24.310	22.506	25.291	66.580	78.000	51.010	52.379	···
08/10/2004	42.150	23.810	22.007	24.214	65.120	79.500	50.920	52.131	···
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋱

Table 1: Example of table of prices.

Fecha	AlfaA	AmTelA1	Amxl	BImboA	Cemex CPO	Elektra	Femsaubd	gcarsoa1	···
29/09/2004	1.877	2.605	0.907	-0.395	1.442	0.774	0.837	1.821	···
30/09/2004	0.420	0.737	-0.090	-0.873	-0.973	-0.404	-0.632	-1.506	···
01/10/2004	0.975	1.179	2.474	3.124	1.108	0.353	1.054	0.497	···
04/10/2004	-0.460	1.446	1.888	0.583	1.481	-0.456	0.079	0.590	···
05/10/2004	-0.416	-2.574	-0.991	-0.039	-0.441	0.275	0.334	0.170	···
06/10/2004	-0.557	-1.220	-1.871	-0.772	2.551	0.104	0.196	-0.245	···
07/10/2004	0.303	0.041	-0.089	0.584	-0.834	1.708	-0.254	-0.265	···
08/10/2004	-1.954	-2.057	-2.219	-4.257	-2.193	1.923	-0.176	-0.474	···
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋱

Table 2: Example of table of returns.

We took the prices of the 20 stocks for 100 days, see Table 1; the whole data is in [14]. Then we calculated the return of each security, for each day, using equation (2). See Table 2, in [14] are the whole data.

To compute estimates of the mean returns and the covariances in (3) we used 5−day moving averages, that is, using the data from day $n-4$ to day n with $n = 5, 6, \ldots, 100$. This procedure gave us 95 matrices of order 21×20 whose first row are the mean returns. An example of these matrices appears in Table 3.

Then to these data we applied the ST-MOPSO algorithm, see [14], to obtain a Pareto front for each of the 95 matrices. In each case we used the constraints $c_i = 0.2$ for all $i = 1, \ldots, s$.

7 The results

To apply the results obtained in the previous section, the idea was to use the 5-day data to decide the portfolio for the sixth day. Hence for each of the 95 matrices we tried to obtain a Pareto front with 100 points. The Table 4 is an example of the resulting solution, and Figure 3 shows the corresponding graph. (Appendix B contains all the graphs). The solutions tell us, according to the POP, the fraction (or the share (in percent)) of our

0.48	0.68	0.84	0.48	0.52	0.11	0.33	0.31	0.78	1.08	0.52	1.02	0.26	1.36	0.81	0.27	0.16	0.60	-0.12	0.19
3.89	5.13	1.61	0.54	1.63	1.46	1.36	2.18	4.89	9.07	0.44	-0.05	-0.58	4.25	-0.14	2.89	1.30	1.30	10.88	1.23
5.13	15.14	7.65	1.32	5.85	0.40	1.08	3.57	12.53	11.03	5.41	0.43	1.90	11.16	0.25	8.03	2.89	1.42	9.91	4.39
1.61	7.65	7.99	6.58	5.18	0.02	1.84	2.64	6.23	1.75	6.97	3.74	2.84	-0.23	4.65	3.19	1.81	2.99	-2.37	2.19
0.54	1.32	6.58	9.87	3.36	0.61	2.74	1.73	1.61	-1.75	6.48	5.70	2.07	-8.08	7.23	-1.10	0.79	4.17	-6.42	-0.15
1.63	5.85	5.18	3.36	5.27	0.82	2.08	4.62	4.13	5.17	3.44	1.70	2.41	0.77	2.36	4.66	2.39	2.92	4.95	2.79
1.46	0.40	0.02	0.61	0.82	1.11	1.15	1.80	0.50	4.35	-0.62	0.13	-0.29	-0.18	0.30	0.95	0.71	1.22	6.10	0.35
1.36	1.08	1.84	2.74	2.08	1.15	1.77	2.58	0.94	3.80	1.03	1.37	0.55	-1.82	1.90	1.23	1.09	2.22	4.46	0.66
2.18	3.57	2.64	1.73	4.62	1.80	2.58	5.72	2.26	8.35	0.57	0.58	1.45	0.43	1.13	4.71	2.57	3.15	11.17	2.56
4.89	12.53	6.23	1.61	4.13	0.50	0.94	2.26	10.86	9.65	4.60	0.64	0.98	9.20	0.46	5.86	2.10	1.15	8.47	3.11
9.07	11.03	1.75	-1.75	5.17	4.35	3.80	8.35	9.65	25.47	-2.17	-2.10	-1.11	11.01	-2.40	9.46	4.23	3.52	33.95	4.24
0.44	5.41	6.97	6.48	3.44	-0.62	1.03	0.57	4.60	-2.17	6.81	3.86	2.40	-1.71	4.68	0.98	0.78	2.06	-7.50	1.03
-0.05	0.43	3.74	5.70	1.70	0.13	1.37	0.58	0.64	-2.10	3.86	3.34	1.23	-4.95	4.22	-1.01	0.26	2.20	-5.20	-0.25
-0.58	1.90	2.84	2.07	2.41	-0.29	0.55	1.45	0.98	-1.11	2.40	1.23	1.71	-1.02	1.65	1.45	0.79	1.08	-2.60	1.11
4.25	11.16	-0.23	-8.08	0.77	-0.18	-1.82	0.43	9.20	11.01	-1.71	-4.95	-1.02	17.07	-6.62	6.85	1.27	-3.02	14.28	3.22
-0.14	0.25	4.65	7.23	2.36	0.30	1.90	1.13	0.46	-2.40	4.68	4.22	1.65	-6.62	5.37	-1.12	0.46	2.98	-6.00	-0.23
2.89	8.03	3.19	-1.10	4.66	0.95	1.23	4.71	5.86	9.46	0.98	-1.01	1.45	6.85	-1.12	6.72	2.64	1.40	11.85	3.62
1.30	2.89	1.81	0.79	2.39	0.71	1.09	2.57	2.10	4.23	0.78	0.26	0.79	1.27	0.46	2.64	1.28	1.35	5.18	1.45
1.30	1.42	2.99	4.17	2.92	1.22	2.22	3.15	1.15	3.52	2.06	2.20	1.08	-3.02	2.98	1.40	1.35	2.93	3.55	0.86
10.88	9.91	-2.37	-6.42	4.95	6.10	4.46	11.17	8.47	33.95	-7.50	-5.20	-2.60	14.28	-6.00	11.85	5.18	3.55	48.78	5.02
1.23	4.39	2.19	-0.15	2.79	0.35	0.66	2.56	3.11	4.24	1.03	-0.25	1.11	3.22	-0.23	3.62	1.45	0.86	5.02	2.03

Table 3: Example of a table of mean return μ_i (first row) and covariances σ_{ij} (rows 2 to 21).

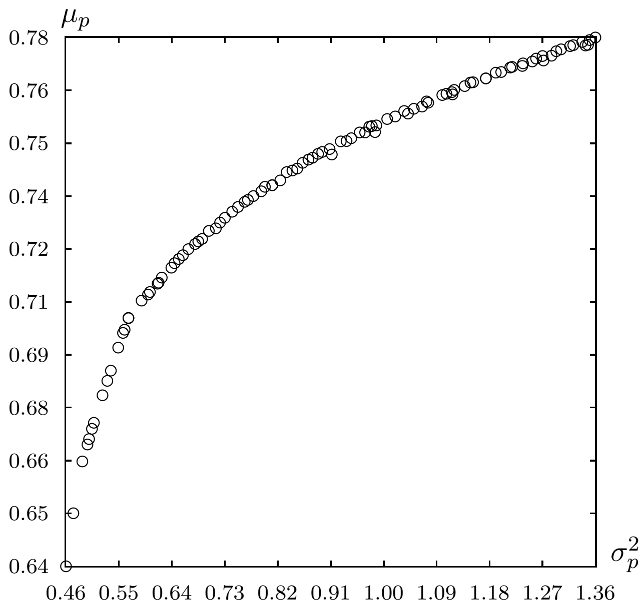

Figure 3: Example of the graph of a solution of POP.

wealth that we should invest in each of the 20 securities.

Since each of the 100 points in the Pareto front is a possible portfolio, handling this information turns out to be quite complicated. Therefore, we sorted the solutions according to their risks and took only three solutions per day: the solution with the minimal risk, the solution with the maximal risk, and a solution with a medium risk. Then we computed the return of for day 6 of these 3 solutions using equation 4, and we compared the return of these three solutions with the return of the IPyC of the BMV. The results are shown in Table 5.

To see how this would work in a real situation, we did an experiment beginning with "one unit" of investment (say, one peso) and following the corresponding wealth day-per-day; that is, every day we multiply the current value by $1 + r$ to obtain the value of our investment the day after. The results are shown in the Table 6 and their graphical representation appears in Figure 4. It can be seen that each of our three solutions gives a better return than the IPyC–in some cases the return is up to 8% above the IPyC return (day 72 of Table 6). Only in the last few days our solutions were similar to the IPyC–perhaps because the IPyC was behaving "optimally". For instance, from the Table 6.4 we can see that the IPyC return of 21.1% is very close to our solutions with minimal and maximum risks, 20.7% and 21.0%, respectively, but below the 24.6% given by our medium risk solution.

day\\x_i	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	var.	return
5	0	0	0	0	0	0	0	0	0	1.3	0	20	15.1	20	20	0	0	20	0	0	0.962696	0.788671
5	0	0	0	0	0	0	0	0	0	3.1	0	20	12.0	20	20	0	0	20	0	0	0.994871	0.799627
5	0	0	0	0	0	0	0	0	0	3.6	0	20	11.8	20	20	0	0	20	0	0	1.017500	0.804341
5	0	0	1.7	0	0	0	0	0	0	3.0	0	20	12.0	20	20	0	0	20	0	0	1.078990	0.812383
5	0	0	0	0	0	0	0	0	0	4.8	0	20	13.7	20	20	0	0	20	0	0	1.111420	0.821429
5	0	0	0	0	0	0	0	0	0	5.4	0	20	12.3	20	20	0	0	20	0	0	1.123690	0.823776
5	0	0	0	0	0	0	0	0	0	6.1	0	20	11.5	20	20	0	0	20	0	0	1.151900	0.828414
5	0	0	0	0	0	0	0	0	0	5.6	0	20	13.6	20	20	0	0	20	0	0	1.155620	0.829170
5	0	0	0	0	0	0	0	0	0	7.1	0	20	9.0	20	20	0	0	20	0	0	1.185270	0.832727
5	0	0	0	0	0	0	0	0	0	6.7	0	20	12.1	20	20	0	0	20	0	0	1.200290	0.836255
5	0	0	0	0	0	0	0	0	0	6.6	0	20	13.4	20	20	0	0	20	0	0	1.215770	0.838955
5	0	0	3.4	0	0	0	0	0	0	5.6	0	20	6.3	20	20	0	0	20	0	0	1.230710	0.838862
5	0	0	0	0	0	0	0	0	0	7.8	0	20	8.6	20	20	0	0	20	0	0	1.232530	0.839415
5	0	0	0	0	0	0	0	0	0	7.5	0	20	12.5	20	20	0	0	20	0	0	1.260490	0.845516
5	0	0	1.6	0	0	0	0	0	0	6.6	0	20	11.6	20	20	0	0	20	0	0	1.271040	0.846919
5	0	0	0	0	0	0	0	0	0	8.4	0	20	10.4	20	20	0	0	20	0	0	1.296070	0.849498
5	0	0	0.5	0	0	0	0	0	0	8.0	0	20	11.4	20	20	0	0	20	0	0	1.308080	0.852146
5	0	0	5.9	0	0	0	0	0	0	4.6	0	20	8.7	20	20	0	0	20	0	0	1.342850	0.854090
5	0	0	2.8	0	0	0	0	0	0	7.2	0	20	9.8	20	20	0	0	20	0	0	1.348560	0.858057
5	0	0	2.9	0	0	0	0	0	0	8.1	0	20	6.8	20	20	0	0	20	0	0	1.374120	0.860402
5	0	0	5.6	0	0	0	0	0	0	5.9	0	20	8.0	20	20	0	0	20	0	0	1.393930	0.863270
5	0	0	4.5	0	0	0	0	0	0	6.9	0	20	8.6	20	20	0	0	20	0	0	1.407580	0.866164
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮

Table 4: Example of a solution (the values are percent).

day	IPyC	min. risk	med. risk	max. risk	day	IPyC	min. risk	med. risk	max. risk
1	-0.24%	0.48%	0.23%	-0.21%	48	1.11%	1.24%	2.01%	2.45%
2	0.06%	0.70%	0.37%	0.25%	49	0.24%	-0.29%	-0.20%	-0.53%
3	-1.62%	-0.81%	-1.04%	-1.39%	50	1.13%	0.81%	0.52%	0.25%
4	0.49%	0.52%	0.42%	0.36%	51	0.50%	0.38%	0.80%	0.98%
5	0.43%	0.09%	0.09%	-0.05%	52	0.09%	-0.02%	-0.22%	-0.51%
6	-0.74%	-1.45%	-1.47%	-1.56%	53	0.11%	-0.23%	-0.22%	-0.13%
7	-0.61%	-0.58%	-0.75%	-0.41%	54	-0.05%	0.19%	-0.17%	-0.41%
8	1.05%	0.73%	0.83%	0.56%	55	1.09%	0.79%	1.09%	0.62%
9	0.58%	0.30%	0.39%	0.77%	56	0.48%	0.55%	1.21%	1.26%
10	-0.29%	-0.13%	0.04%	0.04%	57	0.28%	0.21%	0.49%	0.67%
11	0.48%	1.12%	1.16%	1.34%	58	0.42%	0.26%	0.22%	0.07%
12	0.81%	1.00%	1.12%	1.15%	59	0.13%	0.04%	0.43%	0.82%
13	0.53%	0.32%	0.59%	0.33%	60	0.72%	0.80%	0.53%	0.51%
14	-0.43%	0.28%	-0.05%	0.19%	61	0.91%	0.65%	0.88%	0.85%
15	1.44%	1.29%	2.00%	1.94%	62	-0.48%	-0.60%	-0.58%	-0.68%
16	1.53%	0.53%	0.17%	0.37%	63	-0.39%	-0.22%	-0.22%	-0.18%
17	-0.66%	-0.11%	-0.69%	-1.38%	64	0.81%	-0.41%	0.35%	0.21%
18	1.05%	0.77%	0.90%	1.38%	65	-1.92%	-1.66%	-1.65%	-2.47%
19	0.50%	0.41%	0.43%	-0.09%	66	-1.48%	-1.01%	-1.47%	-1.57%
20	-0.10%	-0.01%	-0.10%	-0.05%	67	0.88%	0.05%	0.28%	1.28%
21	1.32%	1.06%	1.61%	1.43%	68	-2.01%	-1.16%	-1.10%	-2.67%
22	0.83%	0.72%	0.85%	0.96%	69	-0.06%	-0.45%	-0.39%	-0.49%
23	-0.60%	-0.50%	-0.29%	-0.61%	70	-1.88%	-0.58%	-0.44%	-0.56%
24	-0.29%	0.00%	0.05%	-0.80%	71	0.91%	0.57%	0.76%	0.83%
25	0.23%	1.05%	0.11%	0.47%	72	1.07%	1.38%	1.22%	1.19%
26	0.06%	0.67%	0.92%	0.40%	73	1.83%	1.27%	1.45%	1.56%
27	1.49%	0.19%	0.64%	1.14%	74	0.96%	0.70%	0.61%	0.61%
28	-0.12%	-0.22%	-0.52%	-0.10%	75	1.63%	1.86%	1.83%	1.84%
29	-0.02%	0.21%	0.29%	0.46%	76	0.05%	0.47%	-0.01%	0.38%
30	-0.04%	0.05%	0.12%	0.05%	77	-2.08%	-1.65%	-2.05%	-2.00%
31	0.58%	0.89%	0.45%	0.56%	78	-0.76%	-1.04%	-1.20%	-1.11%
32	0.15%	0.82%	1.01%	0.96%	79	0.55%	0.57%	0.66%	0.82%
33	-1.69%	-1.07%	-1.44%	-1.75%	80	0.96%	0.48%	0.99%	0.33%
34	0.34%	0.32%	0.42%	0.46%	81	1.37%	0.98%	0.75%	1.26%
35	-0.03%	-0.40%	-0.10%	0.90%	82	-0.45%	-0.43%	-0.77%	-0.75%
36	0.26%	0.17%	0.12%	1.09%	83	0.42%	-0.14%	0.23%	0.25%
37	0.75%	0.83%	0.95%	1.09%	84	0.43%	0.45%	-0.03%	-0.35%
38	0.65%	1.07%	1.15%	1.21%	85	1.82%	1.12%	1.31%	1.30%
39	0.99%	1.71%	1.98%	1.37%	86	-0.01%	-0.31%	-0.02%	0.02%
40	-0.78%	-0.18%	-0.71%	-0.22%	87	0.75%	0.80%	0.91%	0.46%
41	1.07%	1.30%	1.57%	1.33%	88	0.05%	-1.11%	-0.88%	-0.77%
42	-0.97%	-0.66%	-0.70%	-0.69%	89	0.22%	1.17%	1.06%	0.86%
43	-0.05%	-0.65%	-0.29%	-0.27%	90	0.23%	-0.08%	-0.26%	-0.33%
44	0.66%	-0.31%	0.28%	0.19%	91	1.13%	0.46%	0.53%	0.72%
45	-0.59%	-0.46%	-0.20%	-0.01%	92	0.34%	0.47%	0.15%	-0.14%
46	-0.04%	0.65%	0.54%	0.15%	93	0.04%	-0.17%	0.04%	0.05%
47	0.09%	0.27%	0.59%	0.52%	94	-1.10%	-1.11%	-1.16%	-1.12%

Table 5: Table of comparison of return of IPyC, the solution with minimal risk, a medium risk and maximal risk.

day	IPyC	min. risk	med. risk	max. risk	day	IPyC	min. risk	med. risk	max. risk
0	1.000	1.000	1.000	1.000					
1	0.998	1.005	1.002	0.998	48	1.100	1.152	1.172	1.174
2	0.998	1.012	1.006	1.000	49	1.102	1.148	1.170	1.168
3	0.982	1.004	0.996	0.987	50	1.115	1.158	1.176	1.171
4	0.987	1.009	1.000	0.990	51	1.120	1.162	1.185	1.182
5	0.991	1.010	1.001	0.990	52	1.121	1.162	1.183	1.176
6	0.984	0.995	0.986	0.974	53	1.123	1.159	1.180	1.175
7	0.978	0.989	0.978	0.970	54	1.122	1.162	1.178	1.170
8	0.988	0.997	0.987	0.975	55	1.134	1.171	1.191	1.177
9	0.994	1.000	0.991	0.983	56	1.140	1.177	1.205	1.192
10	0.991	0.998	0.991	0.983	57	1.143	1.180	1.211	1.200
11	0.996	1.010	1.002	0.997	58	1.148	1.183	1.214	1.201
12	1.004	1.020	1.014	1.008	59	1.149	1.183	1.219	1.211
13	1.009	1.023	1.020	1.011	60	1.158	1.193	1.225	1.217
14	1.005	1.026	1.019	1.013	61	1.168	1.200	1.236	1.227
15	1.019	1.039	1.040	1.033	62	1.163	1.193	1.229	1.219
16	1.035	1.045	1.041	1.037	63	1.158	1.190	1.226	1.217
17	1.028	1.044	1.034	1.023	64	1.167	1.186	1.231	1.219
18	1.039	1.052	1.043	1.037	65	1.145	1.166	1.210	1.189
19	1.044	1.056	1.048	1.036	66	1.128	1.154	1.193	1.170
20	1.043	1.056	1.047	1.035	67	1.138	1.155	1.196	1.185
21	1.057	1.067	1.064	1.050	68	1.115	1.141	1.183	1.154
22	1.066	1.075	1.073	1.060	69	1.115	1.136	1.178	1.148
23	1.059	1.069	1.070	1.054	70	1.094	1.130	1.173	1.142
24	1.056	1.069	1.070	1.045	71	1.104	1.136	1.182	1.151
25	1.059	1.081	1.071	1.050	**72**	**1.115**	**1.152**	**1.196**	**1.165**
26	1.059	1.088	1.081	1.054	73	1.136	1.166	1.214	1.183
27	1.075	1.090	1.088	1.066	74	1.147	1.175	1.221	1.190
28	1.074	1.087	1.082	1.065	75	1.165	1.196	1.244	1.212
29	1.073	1.090	1.086	1.070	76	1.166	1.202	1.243	1.217
30	1.073	1.090	1.087	1.071	77	1.142	1.182	1.218	1.192
31	1.079	1.100	1.092	1.077	78	1.133	1.170	1.203	1.179
32	1.081	1.109	1.103	1.087	79	1.139	1.177	1.211	1.189
33	1.063	1.097	1.087	1.068	80	1.150	1.182	1.223	1.193
34	1.066	1.101	1.092	1.073	81	1.166	1.194	1.232	1.208
35	1.066	1.096	1.090	1.082	82	1.161	1.189	1.223	1.199
36	1.069	1.098	1.092	1.094	83	1.165	1.187	1.226	1.202
37	1.077	1.107	1.102	1.106	84	1.170	1.192	1.225	1.198
38	1.084	1.119	1.115	1.120	85	1.192	1.206	1.241	1.213
39	1.094	1.138	1.137	1.135	86	1.192	1.202	1.241	1.214
40	1.086	1.136	1.129	1.132	87	1.201	1.212	1.252	1.219
41	1.098	1.151	1.147	1.147	88	1.201	1.198	1.241	1.210
42	1.087	1.143	1.138	1.139	89	1.204	1.212	1.255	1.220
43	1.086	1.136	1.135	1.136	90	1.207	1.211	1.251	1.216
44	1.094	1.132	1.138	1.139	91	1.220	1.217	1.258	1.225
45	1.087	1.127	1.136	1.138	92	1.224	1.222	1.260	1.223
46	1.087	1.135	1.142	1.140	93	1.225	1.220	1.260	1.224
47	1.088	1.138	1.149	1.146	94	1.211	1.207	1.246	1.210

Table 6: Table of comparison of investment of IPyC, the solution with minimal risk, a medium risk and maximal risk.

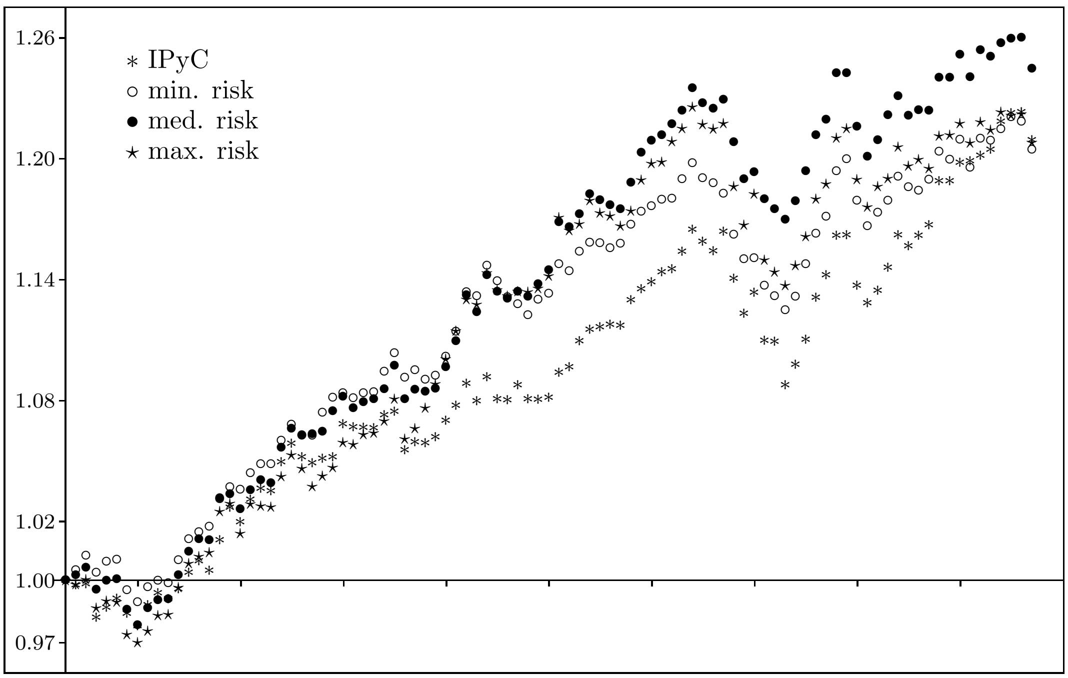

Figure 4: Graphic comparison of the IPyC and the solutions with minimal risk, a medium risk and maximal risk.

8 Conclusions and future work

In this paper we applied the ST-MOPSO algorithm to the Markowitz' portfolio selection problem. As shown in section 7 our results seem to be quite good. But of course before reaching any conclusions we need to do more experimental work. For instance, we do not really know how good are our 5-day moving averages. It would be interesting (and important!) to determine how sensitive our results are to the length of the moving averages.

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