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CONVERGENCE IN NONLINEAR SYSTEMS WITH A FORCING TERM

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Abstract

The problem of convergence, as $t \longrightarrow +\infty$, of solutions of the system (1) is considered. It is assumed that the functions α, f and g are of class C^1 for all values of their arguments, furthermore $g'(x) > 0$, $f'(x) \geq r > 0$, $0 < n \leq \alpha'(y) \leq N < +\infty$ and the functions $a(t)$ and $p(t)$ are continuous on $[0, +\infty)$ with $0 < \underline{a} \leq a(t) \leq A < +\infty$ and $p(t) \geq 0$.

Resumen

Se considera el problema de la convergencia, cuando $t \longrightarrow +\infty$, de las soluciones del sistema (1). Se asume que las funciones α, f y g son de clase C^1 para todos los valores de sus argumentos y que además $g'(x) > 0$, $f'(x) \geq r > 0$, $0 < n \leq \alpha'(y) \leq N < +\infty$ y las funciones $a(t)$ y $p(t)$ son continuas sobre $[0, +\infty)$ con $0 < \underline{a} \leq a(t) \leq A < +\infty$ y $p(t) \geq 0$.

1 Preliminars

We consider the system:

$$\begin{aligned} x' &= \alpha(y) - f(x), \\ y' &= p(t) - a(t)g(x), \end{aligned} \tag{1}$$

where the functions involved are of class C^1 for all values of their arguments, we also assume that $g'(x) > 0$, $f'(x) \geq r > 0$, for all $x \in \mathbb{R}$ and $0 < n \leq \alpha'(y) \leq N < +\infty$, for all $y \in \mathbb{R}$ and the functions $a(t)$ and $p(t)$ are continuous functions on $[0, +\infty)$ with $0 < \underline{a} \leq a(t) \leq A < +\infty$ and $p(t) \geq 0$.

In this paper we study the problem of convergence, when $t \longrightarrow +\infty$, of all solutions of the system (1). If the functions involved in this system are such that $\alpha(y) = y$ and $a(t) \equiv 1$, the system reduces to the well known Liénard's equation with forcing term:

$$x'' + f'(x)x' + g(x) = p(t). \tag{2}$$

The purpose of this note is to generalize, to the system (1), the results of [2] concerning to the convergence of all solutions of (2) to a bounded solution, under the earlier conditions.

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2 Results

Let $(\varphi(t), \psi(t))$ be a bounded solution of (1). If we set $v(t) = x(t) - \varphi(t)$, $w(t) = y(t) - \psi(t)$ we obtain the system:

$$\begin{aligned} v' &= \alpha(w+\psi) - \alpha(\psi) - [f(v+\varphi) - f(\varphi)], \\ w' &= -a(t)[g(v+\varphi) - g(\varphi)]. \end{aligned} \tag{3}$$

In the half plane $v > 0$ we consider lines $w + kv = \text{const.}$, where $k \geq 0$ for $nw > av$ and $k < 0$ for $nw \leq av$. We find conditions under which trajectories of (3) intersect these lines at points where $w = lv$, in the direction of the origin; this means that $w + k$ is decreasing on the solutions of system (3), i.e. there exist positive numbers P and Q such that, all the solutions of (3) cannot leave the domain $|v| < P$, $|w| < Q$ by crossing the boundary $v = \pm P, w = \pm Q$ when $t \to +\infty$. Differentiating $w + kv$ with (3) we obtain:

$$k \int_{\underline{z}}^{\overline{z}} \alpha'(r)dr - \int_{\underline{u}}^{\overline{u}} \left[a(t)g'(s) - kf'(s)\right] ds = v\left\{-a(t)g'(u) - kf'(u) + kl\alpha'(z)\right\}, \tag{4}$$

where $\varphi = \underline{u} \leq u \leq \overline{u} = \varphi + v$, $\psi = \underline{z} \leq z \leq \overline{z} = \psi + w$. It follows from this that the conditions for intersection in the desired direction takes the form:

$$k < \frac{a(t)g'(u)}{l\alpha'(z) - f'(u)} := R(l, t, u, z). \tag{5}$$

Let:

$$H(l) = \inf_{t,u,z} R(l, t, u, z)$$

we have that the infimum take place when $\alpha_1 l > r$, for u such that $f(u) < l\alpha_1$ and $\alpha_1 = \inf_z \alpha'(z)$. Hence we have the following:

Lemma 1 *The function $R(l, t, u, z)$ reaches the infimum in the absolute extremes of the function $\alpha'(z)$.*

Let $U(l) \leq H(l)$ be any piecewise smooth function satisfying $U(l) \geq 0$ for $\alpha_1 l > r$, $U(l) < 0$ for $\alpha_1 l \leq r$ and

$$\lim U_{l \longrightarrow (r/N)^-} U(l) = -\infty.$$

Consider the homogeneous differential equation:

$$\frac{dw}{dv} + U(w/v) = 0. \tag{6}$$

Let the integral curve of equation (6) which passes through the point $v = 0, w = w_0 > 0$ and which lies in the right half-plane intersect the w-axis again at $-w_1 < 0$ (if there is no such intersection, take $w_1 = +\infty$).

Theorem 1 *If $w_1 < w_0$, then $(x(t), y(t)) \longrightarrow (\varphi(t), \psi(t))$ as $t \longrightarrow +\infty$, where $(x(t), y(t))$ is an arbitrary solution of (1).*

PROOF. Let $\gamma : (v(t), w(t))$ be a trajectory of system (1) which goes through a point P of the half-plane $v > 0$, by (5) and the definition of the function $U(l)$, the trajectories of the system (3) either intersect the integral curves of equation (6) in the direction of the origin or intersect the half-axis $w < 0$. Assuming the opposite, suppose that the trajectory γ lies completely inside the bounded region limited by axis w and some trajectory L_1 of (6).

We write the system (3) in the form:

$$\begin{array}{rcl} v' & = & -\alpha'(\varepsilon y + (1-\varepsilon)\psi)w - f'(\theta x + (1-\theta)\varphi)v, \\ w' & = & -a(t)g'(\nu x + (1-\nu)\varphi)v \end{array} \tag{7}$$

where $0 \leq \varepsilon,\ \theta, \nu \leq 1$. $x(t) = v(t) + \varphi(t)$ and $y(t) = w(t) + \psi(t)$ is a bounded solution, from this we have:

$$f'(\theta x + (1-\theta)\varphi) \leq A < +\infty, \quad g'(\nu x + (1-\nu)\varphi) \geq \delta > 0,$$

for some positive constants A and δ.

Since $\alpha'(y)$ is bounded we obtain $v' \geq -Av + Dw$ where: $D = \begin{cases} n \text{ if } w > 0 \\ N \text{ if } w < 0 \end{cases}$

Then if $v(t) \geq v_0 > 0$ for all $t \geq t_0$ we have that $w'(t) \leq -\delta \underline{a} v_0$ and $w(t) \longrightarrow -\infty$, as $t \longrightarrow +\infty$. But this is a contradiction. So:

$$w_1 < w_0 \tag{8}$$

From this it follows that the origin is globally stable for the system (3). ∎

3 Examples and related remarks

Example 1. Let:

$$0 < \gamma \leq g'(x) \leq \Gamma < +\infty \tag{9}$$

For $U(l)$ we can take:

$$U(l) = \begin{cases} (\underline{a}\gamma)/(nl - r) & \text{if } l > a/n \\ (A\Gamma)/(nl - r) & \text{if } l \leq a/n \end{cases},$$

when $a(t) \equiv 1$ and $\alpha(y) = y$, the condition (8) for the integral curves of equation (6) coincides with the convergence condition in (6).

Example 2. When $f'(x) \not\equiv r$, the convergence criterion for system (1) obtained in Example 1 can be improved if, in the choice of $U(l)$, we take into account not only the greatest and least values of $g'(x)$ and $f'(x)$ but also relations among them. Apparently the first criterion of this type was obtained in [5]. If we consider the case where an estimate:

$$0 < \gamma \leq g'(x) \leq s + tf'(x) < +\infty; \quad s, t \in \mathbb{R}_+,$$

holds. Then we have:

$$U(l) = \begin{cases} (\underline{a}\gamma)(ln - r) & \text{if } l > r/n \\ ((s + tr)A)(nl - r) & \text{if } r/n > l > -s/tn \\ -t & \text{if } l \leq -s/tn \end{cases}$$

If we set $\alpha(y) = y, a(t) \equiv 1$ and $\gamma = 0$, we obtain:

$$U(l) = \begin{cases} 0 & \text{if } l > r \\ (s+t)/(l-r) & \text{if } r > l > -s/t, \\ -t & \text{if } l < -s/t \end{cases}$$

which coincides with the results given in [2].

Example 3. We take now:

$$0 < \gamma \leq g'(x) \leq s + t\sqrt{f'(x)} < +\infty, \quad t, s \in \mathbb{R}_+,$$

thus we derive that:

$$U(l) = \begin{cases} \underline{a}\gamma/(nl-r) & \text{if } l > r/n \\ (s+t\sqrt{r})A/(ln-r) & \text{if } r/n > l > -r/n - 2s\sqrt{r}/tn \\ t^2/2(s-\sqrt{s^2-t^2nl}) & \text{if } l \leq -r/n - 2s\sqrt{r}/tn \end{cases}$$

When $a(t) \equiv 1$, this result coincides with the one obtained in [8].

Example 4. We consider a forced oscillator with diferential equation:

$$x'' + cx' + g(x) = p(t), \tag{10}$$

where $c > 0$ is a fixed constant, $g : \mathbb{R} \longrightarrow \mathbb{R}$ is sufficiently smooth and has limit at infinity, satisfying the inequality $g(-\infty) < g(x) < g(+\infty)$, forall $x \in \mathbb{R}$.

The function $p \in L^\infty(\mathbb{R})$ is bounded but not necessarily periodic. We remark that system (1) is the equation (10) under suitable conditions.

Assume that $0 < \gamma \leq g'(x) \leq \Gamma < c^2/4$, our Theorem 1 coincides with Theorem 1.2 in [1].

Remark. If we take $g'(x)$ and $f'(x)$ bounded functions, our results are consistent with those obtained in [7].

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