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On the nuclear trace of Fourier integral operators

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Abstract. In this paper we characterise the r-nuclearity of Fourier integral operators on Lebesgue spaces. Fourier integral operators will be considered in $\mathbb{R}^n$, the discrete group $\mathbb{Z}^n$, the n-dimensional torus and symmetric spaces (compact homogeneous manifolds). We also give formulae for the nuclear trace of these operators. Explicit examples will be given on $\mathbb{Z}^n$, the torus $\mathbb{T}^n$, the special unitary group SU(2), and the projective complex plane $\mathbb{CP}^2$. Our main theorems will be applied to the characterization of r-nuclear pseudo-differential operators defined by the Weyl quantization procedure.
Keywords: Fourier integral operator, nuclear operator, nuclear trace, spectral trace, compact homogeneous manifold.
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Sobre la traza nuclear de operadores integrales de Fourier

Resumen. En esta investigación se caracteriza la r-nuclearidad de operadores integrales de Fourier en espacios de Lebesgue. Las nociones de traza nuclear y operador nuclear sobre espacios de Banach son conceptos análogos a aquellas de traza espectral y de operador de clase traza en espacios de Hilbert. Operadores integrales de Fourier, por otro lado, surgen para expresar soluciones a problemas de Cauchy hiperbólicos o para estudiar la función espectral asociada a un operador geométrico sobre una variedad diferenciable. Los operadores integrales de Fourier se consideran actuando sobre $\mathbb{R}^n$, el grupo discreto $\mathbb{Z}^n$, el toro de dimensión n y finalmente, espacios simétricos (variedades compactas homogéneas). Se presentan ejemplos explícitos de tales caracterizaciones sobre $\mathbb{Z}^n$, el grupo especial unitario SU(2), y el plano complejo proyectivo $\mathbb{CP}^2$. Los resultados principales de la presente investigación se aplican en la caracterización de operadores pseudo diferenciales nucleares definidos mediante el proceso de cuantificación de Weyl.
Palabras clave: Operador integral de Fourier, operador nuclear, traza nuclear, traza espectral, variedad compacta homogénea.

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1. Introduction

In this paper we characterise the r-nuclearity of Fourier integral operators on Lebesgue spaces. Fourier integral operators will be considered in $\mathbb{R}^n$, the discrete group $\mathbb{Z}^n$, the n-dimensional torus and symmetric spaces (compact homogeneous manifolds). We also give formulae for the nuclear trace of these operators. Explicit examples will be given on $\mathbb{Z}^n$, the torus $\mathbb{T}^n$, the special unitary group SU(2), and the projective complex plane $\mathbb{CP}^2$. Our main theorems will be applied to the characterization of r-nuclear pseudo-differential operators defined by the Weyl quantization procedure.

1.1. Outline of the paper

Let us recall that the Fourier integral operators (FIOs) on $\mathbb{R}^n$, are integral operators of the form

$$Ff(x) := \int_{\mathbb{R}^n} e^{i\phi(x,\xi)} a(x,\xi)(\mathscr{F}f)(\xi)d\xi, \tag{1}$$

where $\mathscr{F}f$ is the Fourier transform of f, or in a more general setting, linear integral operators formally defined by

$$Tf(x) := \int_{\mathbb{R}^{2n}} e^{i\phi(x,\xi)-i2\pi y\cdot\xi} a(x,y,\xi)f(y)dyd\xi. \tag{2}$$

As it is well known, FIOs are used to express solutions to Cauchy problems of hyperbolic equations as well as for obtaining asymptotic formulas for the Weyl eigenvalue function associated to geometric operators (see Hörmander [32], [33], [34], and Duistermaat and Hörmander [25]).

According to the theory of FIOs developed by Hörmander [32], the phase functions ϕ are positively homogeneous of order 1 and they are considered smooth at $\xi \neq 0$, while the symbols are considered satisfying estimates of the form

$$\sup_{(x,y)\in K} |\partial_x^\beta \partial_\xi^\alpha a(x,y,\xi)| \leq C_{\alpha,\beta,K}(1+|\xi|)^{\kappa-|\alpha|}, \tag{3}$$

for every compact subset K of $\mathbb{R}^{2n}$. Let us observe that L^p-properties for FIOs can be found in the references Hörmander [32], Eskin [26], Seeger, Sogge and Stein [51], Tao [52], Miyachi [37], Peral [39], Asada and Fujiwara [2], Fujiwara [28], Kumano-go [35], Coriasco and Ruzhansky [10], [11], Ruzhansky and Sugimoto [44], [45], [46], [47], Ruzhansky [50], and Ruzhansky and Wirth [49].

A fundamental problem in the theory of Fourier integral operators is that of classifying the interplay between the properties of a symbol and the properties of its associated Fourier integral operator.

In this paper our main goal is to give, in terms of symbol criteria and with simple proofs, characterizations for the r-nuclearity of Fourier integral operators on Lebesgue spaces. Let us mention that this problem has been considered in the case of pseudo-differential operators by several authors. However, the obtained results belong to one of two possible approaches. The first ones are sufficient conditions on the symbol trough of summability conditions with the attempt of studying the distribution of the spectrum for the corresponding pseudo-differential operators. The second ones provide, roughly speaking, a decomposition for the symbols associated to nuclear operators, in terms of the Fourier transform, where the spatial variables and the momentum variables can be analyzed separately. Nevertheless, in both cases the results can be applied to obtain Grothendieck-Lidskii's formulae on the summability of eigenvalues when the operators are considered acting in L^p spaces.

Necessary conditions for the r-nuclearity of pseudo-differential operators in the compact setting can be summarized as follows. The nuclearity and the 2/3-nuclearity of pseudo-differential operators on the circle $\mathbb{S}^1$ and on the lattice $\mathbb{Z}$ can be found in Delgado and Wong [14]. Later, the r-nuclearity of pseudo-differential operators was extensively developed on arbitrary compact Lie groups and on (closed) compact manifolds by Delgado and Ruzhansky in the works [16], [17], [18], [19], [21], and by the author in [9]; other conditions can be found in the works [20], [22], [23]. Finally, the subject was treated for compact manifolds with boundary by Delgado, Ruzhansky, and Tokmagambetov in [24].

On the other hand, characterizations for nuclear operators in terms of decomposition of the symbol trough of the Fourier transform were investigated by Ghaemi, Jamalpour Birgani, and Wong in [29], [30], [36] for $\mathbb{S}^1, \mathbb{Z}$, and also for arbitrary compact and Hausdorff groups. Finally the subject has been considered for pseudo-multipliers associated to the harmonic oscillator (which can be qualified as pseudo-differential operators according to the Ruzhansky-Tokmagambetov calculus when the reference operators is the quantum harmonic oscillator) in the works of the author [3], [7], [8].

1.2. Nuclear Fourier integral operators

In order to present our main result we recall the notion of nuclear operators. By following the classical reference Grothendieck [31], we recall that a densely defined linear operator $T : D(T) \subset E \to F$ (where $D(T)$ is the domain of T, and E, F are choose to be Banach spaces) extends to a r-nuclear operator from E into F, if there exist sequences $(e_n')_{n\in\mathbb{N}_0}$ in E' (the dual space of E) and $(y_n)_{n\in\mathbb{N}_0}$ in F such that, the discrete representation

$$Tf = \sum_{n\in\mathbb{N}_0} e_n'(f)y_n, \quad \text{with} \quad \sum_{n\in\mathbb{N}_0} \|e_n'\|_{E'}^r \|y_n\|_F^r < \infty, \tag{4}$$

holds true for all $f \in D(T)$. The class of $r-$nuclear operators is usually endowed with the natural semi-norm

$$n_r(T) := \inf\left\{ \left\{ \sum_n \|e_n'\|_{E'}^r \|y_n\|_F^r \right\}^{\frac{1}{r}} : T = \sum_n e_n' \otimes y_n \right\}, \tag{5}$$

and, if $r = 1$, $n_1(\cdot)$ is a norm and we obtain the ideal of nuclear operators. In addition, when $E = F$ is a Hilbert space and $r = 1$ the definition above agrees with that of trace class operators. For the case of Hilbert spaces H, the set of r-nuclear operators agrees with the Schatten-von Neumann class of order r (see Pietsch [40], [41]). In order to characterise the r-nuclearity of Fourier integral operators on $\mathbb{R}^n$, we will use (same as in the references mentioned above) Delgado's characterization (see [15]), for nuclear integral operators on Lebesgue spaces defined in σ-finite measure spaces, which in this case will be applied to $L^p(\mathbb{R}^n)$-spaces. Consequently, we will prove that r-nuclear Fourier integral operators defined as in (1) have a nuclear trace given by

$$\operatorname{Tr}(F) = \int\limits_{\mathbb{R}^n}\int\limits_{\mathbb{R}^n} e^{i\phi(x,\xi)-i2\pi x\cdot\xi} a(x,\xi)dx\,d\xi. \tag{6}$$

In this paper our main results are the following theorems.

Theorem 1.1. *Let $0 < r \leq 1$. Let $a(\cdot,\cdot)$ be a symbol such that $a(x,\cdot) \in L^1_{loc}(\mathbb{R}^n)$, a.e.w., $x \in \mathbb{R}^n$. Let $2 \leq p_1 < \infty$, $1 \leq p_2 < \infty$, and let F be the Fourier integral operator associated to $a(\cdot,\cdot)$. Then, $F : L^{p_1}(\mathbb{R}^n) \to L^{p_2}(\mathbb{R}^n)$ is r-nuclear if, and only if, the symbol $a(\cdot,\cdot)$ admits a decomposition of the form*

$$a(x,\xi) = e^{-i\phi(x,\xi)} \sum_{k=1}^{\infty} h_k(x)(\mathscr{F}^{-1}g_k)(\xi), \;\; a.e.w., \; (x,\xi), \tag{7}$$

where $\{g_k\}_{k\in\mathbb{N}}$ and $\{h_k\}_{k\in\mathbb{N}}$ are sequences of functions satisfying

$$\mathbb{E}^r(g,h) := \sum_{k=0}^{\infty} \|g_k\|^r_{L^{p_1'}} \|h_k\|^r_{L^{p_2}} < \infty. \tag{8}$$

Theorem 1.2. *Let $0 < r \leq 1$, and let us consider a measurable function $a(\cdot,\cdot)$ on $\mathbb{R}^{2n}$. Let $1 < p_1 \leq 2$, $1 \leq p_2 < \infty$, and F be the Fourier integral operator associated to $a(\cdot,\cdot)$. Then, $F : L^{p_1}(\mathbb{R}^n) \to L^{p_2}(\mathbb{R}^n)$ is r-nuclear if the symbol $a(\cdot,\cdot)$ admits a decomposition of the form*

$$a(x,\xi) = \frac{1}{e^{i\phi(x,\xi)}} \sum_{k=1}^{\infty} h_k(x)g_k(\xi), \;\; a.e.w., \; (x,\xi), \tag{9}$$

where $\{g_k\}_{k\in\mathbb{N}}$ and $\{h_k\}_{k\in\mathbb{N}}$ are sequences of functions satisfying

$$\mathbb{E}^r(g,h) := \sum_{k=0}^{\infty} \|g_k\|^r_{L^{p_1}} \|h_k\|^r_{L^{p_2}} < \infty. \tag{10}$$

This theorem is sharp in the sense that the previous condition is a necessary and sufficient condition for the r-nuclearity of F when $p_1 = 2$.

The previous results are analogues of the main results proved in Ghaemi, Jamalpour Birgani, and Wong [29], [30], Jamalpour Birgani [36], and Cardona and Barraza [3]. Theorem 1.1, can be used for understanding the properties of the corresponding symbols in Lebesgue spaces. Moreover, we obtain the following result as a consequence of Theorem 1.1.

Theorem 1.3. *Let $a(\cdot,\cdot)$ be a symbol such that $a(x,\cdot) \in L^1_{loc}(\mathbb{R}^n)$, a.e.w., $x \in \mathbb{R}^n$. Let $2 \leq p_1 < \infty$, $1 \leq p_2 < \infty$, and let F be the Fourier integral operator associated to $a(\cdot,\cdot)$. If $F : L^{p_1}(\mathbb{R}^n) \to L^{p_2}(\mathbb{R}^n)$ is nuclear, then $a(x,\xi) \in L^{p_2}_x L^{p_1}_\xi(\mathbb{R}^n \times \mathbb{R}^n) \cap L^{p_1}_\xi L^{p_2}_x(\mathbb{R}^n \times \mathbb{R}^n)$; this means that*

$$\|a(x,\xi)\|_{L^{p_2}_x L^{p_1}_\xi(\mathbb{R}^n\times\mathbb{R}^n)} := \left(\int\limits_{\mathbb{R}^n} \left(\int\limits_{\mathbb{R}^n} |a(x,\xi)|^{p_2} dx \right)^{\frac{p_1}{p_2}} d\xi \right)^{\frac{1}{p_1}} < \infty, \tag{11}$$

and

$$\|a(x,\xi)\|_{L^{p_1}_\xi L^{p_2}_x(\mathbb{R}^n\times\mathbb{R}^n)} := \left(\int\limits_{\mathbb{R}^n}\left(\int\limits_{\mathbb{R}^n}|a(x,\xi)|^{p_1}d\xi\right)^{\frac{p_2}{p_1}}dx\right)^{\frac{1}{p_2}} < \infty. \tag{12}$$

Sufficient conditions in order that pseudo-differential operators in $L^2(\mathbb{R}^n)$ can be extended to (trace class) nuclear operators are well known. Let us recall that the Weyl-quantization of a distribution $\sigma \in \mathscr{S}'(\mathbb{R}^{2n})$ is the pseudo-differential operator defined by

$$Af(x) \equiv \sigma^\omega(x,D_x)f(x) = \int\limits_{\mathbb{R}^n}\int\limits_{\mathbb{R}^n} e^{i2\pi(x-y)\cdot\xi}\sigma\left(\frac{x+y}{2},\xi\right)f(y)dyd\xi. \tag{13}$$

As it is well known $\sigma = \sigma_A(\cdot,\cdot) \in L^1(\mathbb{R}^{2n})$, implies that $A : L^2 \to L^2$ is class trace, and $A : L^2 \to L^2$ is Hilbert-Schmidt if, and only if, $\sigma_A \in L^2(\mathbb{R}^{2n})$. In the framework of the Weyl-Hörmander calculus of operators A associated to symbols σ in the $S(m,g)$-classes (see [34]), there exist two remarkable results. The first one, due to Lars Hörmander, which asserts that $\sigma_A \in S(m,g)$ and $\sigma \in L^1(\mathbb{R}^{2n})$, implies that $A : L^2 \to L^2$ is a trace class operator. The second one, due to L. Rodino and F. Nicola, expresses that $\sigma_A \in S(m,g)$ and $m \in L^1_w$ (the weak-L^1 space), and implies that $A : L^2 \to L^2$ is Dixmier traceable [43]. Moreover, an open conjecture by Rodino and Nicola (see [43]) says that $\sigma_A \in L^1_w(\mathbb{R}^{2n})$ gives an operator A with finite Dixmier trace. General properties for pseudo-differential operators on Schatten-von Neumann classes can be found in Buzano and Toft [6].

As an application of Theorem 1.1 to the Weyl quantization we present the following theorem.

Theorem 1.4. *Let $0 < r \leq 1$. Let $a(\cdot,\cdot)$ be a differentiable symbol. Let $2 \leq p_1 < \infty$, $1 \leq p_2 < \infty$, and let $a^\omega(x,D_x)$ be the Weyl quantization of the symbol $a(\cdot,\cdot)$. Then, $a^\omega(x,D_x) : L^{p_1}(\mathbb{R}^n) \to L^{p_2}(\mathbb{R}^n)$ is r-nuclear if, and only if, the symbol $a(\cdot,\cdot)$ admits a decomposition of the form*

$$a(x,\xi) = \sum_{k=1}^{\infty}\int\limits_{\mathbb{R}^n} e^{-i2\pi z\cdot\xi}h_k\left(x+\frac{z}{2}\right)g_k\left(x-\frac{z}{2}\right)dz, \ \ a.e.w., \ (x,\xi), \tag{14}$$

where $\{g_k\}_{k\in\mathbb{N}}$ and $\{h_k\}_{k\in\mathbb{N}}$ are sequences of functions satisfying

$$\sum_{k=0}^{\infty}\|g_k\|^r_{L^{p_1'}}\|h_k\|^r_{L^{p_2}} < \infty. \tag{15}$$

Remark 1.5. Let us recall that the Wigner transform of two complex functions h, g on $\mathbb{R}^n$, is formally defined as

$$\mathscr{W}(h,g)(x,\xi) := \int\limits_{\mathbb{R}^n} e^{-i2\pi z\cdot\xi}h\left(x+\frac{z}{2}\right)\overline{g}\left(x-\frac{z}{2}\right)dz, \ \ a.e.w., \ (x,\xi). \tag{16}$$

With a such definition in mind, if $2 \leq p_1 < \infty$, $1 \leq p_2 < \infty$, under the hypothesis of Theorem 1.4, $a^\omega(x,D_x) : L^{p_1}(\mathbb{R}^n) \to L^{p_2}(\mathbb{R}^n)$, is r-nuclear if, and only if, the symbol

$a(\cdot,\cdot)$ admits a decomposition (defined trough of the *Wigner transform*) of the type

$$a(x,\xi)=\sum_{k=1}^{\infty}\mathscr{W}(h_k,\overline{g}_k)(x,\xi),\ \ a.e.w.,\ (x,\xi), \tag{17}$$

where $\{g_k\}_{k\in\mathbb{N}}$ and $\{h_k\}_{k\in\mathbb{N}}$ are sequences of functions satisfying

$$\sum_{k=0}^{\infty}\|g_k\|^r_{L^{p_1'}}\|h_k\|^r_{L^{p_2}}<\infty. \tag{18}$$

The proof of our main result (Theorem 1.1) will be presented in Section 2 as well as the proof of Theorem 1.4. The nuclearity of Fourier integral operators on the lattice $\mathbb{Z}^n$ and on compact Lie groups will be discussed in Section 3 as well as some trace formulae for FIOs on the -dimensional torus $\mathbb{T}^n=\mathbb{R}^n/\mathbb{Z}^n$ and the unitary special group SU(2). Finally, in Section 4 we consider the nuclearity of FIOs on arbitrary compact homogeneous manifolds, and we discuss the case of the complex projective space $\mathbb{CP}^2$. In this setting, we will prove analogues for the theorems 1.1 and 1.3 in every context mentioned above.

2. Symbol criteria for nuclear Fourier integral operators

2.1. Characterization of nuclear FIOs

In this section we prove our main result for Fourier integral operators F defined as in (1). Our criteria will be formulated in terms of the symbols a. First, let us observe that every FIO F has a integral representation with kernel $K(x,y)$. In fact, straightforward computation shows us that

$$Ff(x):=\int_{\mathbb{R}^n}K(x,y)f(y)dy, \tag{19}$$

where

$$K(x,y):=\int_{\mathbb{R}^n}e^{i\phi(x,\xi)-i2\pi y\cdot\xi}a(x,\xi)d\xi,$$

for every $f\in\mathscr{D}(\mathbb{R}^n)$. In order to analyze the r-nuclearity of the Fourier integral operator F we will study its kernel K, by using as a fundamental tool the following theorem (see J. Delgado [13], [15]).

Theorem 2.1. *Let us consider $1\leq p_1,p_2<\infty$, $0<r\leq 1$ and let p_i' be such that $\frac{1}{p_i}+\frac{1}{p_i'}=1$. Let (X_1,μ_1) and (X_2,μ_2) be σ-finite measure spaces. An operator $T:L^{p_1}(X_1,\mu_1)\rightarrow L^{p_2}(X_2,\mu_2)$ is r-nuclear if, and only if, there exist sequences $(h_k)_k$ in $L^{p_2}(\mu_2)$, and (g_k) in $L^{p_1'}(\mu_1)$, such that*

$$\sum_k\|h_k\|^r_{L^{p_2}}\|g_k\|^r_{L^{p_1'}}<\infty,\text{ and } Tf(x)=\int\limits_{X_1}(\sum_k h_k(x)g_k(y))f(y)d\mu_1(y),\ \text{a.e.w. } x, \tag{20}$$

for every $f \in L^{p_1}(\mu_1)$. In this case, if $p_1 = p_2$, and $\mu_1 = \mu_2$, (see Section 3 of *[13]*) *the nuclear trace of T is given by*

$$\operatorname{Tr}(T) := \int_{X_1} \sum_k g_k(x) h_k(x) d\mu_1(x). \tag{21}$$

Remark 2.2. Given $f \in L^1(\mathbb{R}^n)$, define its Fourier transform by

$$\mathscr{F} f(\xi) := \int_{\mathbb{R}^n} e^{-i2\pi x\cdot\xi} f(x)dx. \tag{22}$$

If we consider a function f, such that $f \in L^1(\mathbb{R}^n)$ with $\mathscr{F} f \in L^1(\mathbb{R}^n)$, the Fourier inversion formula gives

$$f(x) = \int_{\mathbb{R}^n} e^{i2\pi x\cdot\xi} \mathscr{F} f(\xi)d\xi. \tag{23}$$

Moreover, the Hausdorff-Young inequality $\|\mathscr{F} f\|_{L^{p'}} \leq \|f\|_{L^p}$ (with $\|\mathscr{F} f\|_{L^2} = \|f\|_{L^2}$) shows that the Fourier transform is a well defined operator on L^p, $1 < p \leq 2$.

Proof of Theorem 1.1. Let us assume that F is a Fourier integral operator as in (1) with associated symbol a. Let us assume that $F : L^{p_1}(\mathbb{R}^n) \to L^{p_2}(\mathbb{R}^n)$ is r-nuclear.

Then there exist sequences h_k in L^{p_2} and g_k in $L^{p_1'}$ satisfying

$$Ff(x) = \int_{\mathbb{R}^n} \left(\sum_{k=1}^{\infty} h_k(x) g_k(y) \right) f(y)dy, \ f \in L^{p_1}, \tag{24}$$

with

$$\sum_{k=0}^{\infty} \|g_k\|^r_{L^{p_1'}} \|h_k\|^r_{L^{p_2}} < \infty. \tag{25}$$

For all $z \in \mathbb{R}^n$, let us consider the set $B(z;r)$, i.e., the euclidean ball centered at z with radius $r > 0$. Let us denote by $|B(z;r)|$ the Lebesgue measure of $B(z;r)$. Let us choose $\xi_0 \in \mathbb{R}^n$ and $r > 0$. If we define $\delta^r_{\xi_0} := |B(\xi_0;r)|^{-1} \cdot 1_{B(\xi_0;r)}$, where $1_{B(\xi;r)}$ is the characteristic function of the ball $B(\xi_0;r)$, the condition $2 \leq p_1 < \infty$, together with the Hausdorff-Young inequality gives

$$\|\mathscr{F}^{-1}(\delta^r_{\xi_0})\|_{L^{p_1}} = \|\mathscr{F}^{-1}(\delta^r_{\xi_0})\|_{L^{(p_1')'}} \leq \|\delta^r_{\xi_0}\|_{L^{p_1'}} = 1. \tag{26}$$

So, for every $r > 0$ and $\xi_0 \in \mathbb{R}^n$, the function $\mathscr{F}^{-1}\delta^r_{\xi_0} \in L^{p_1}(\mathbb{R}^n) = \operatorname{Dom}(F)$, and we get,

$$F(\mathscr{F}^{-1}\delta^r_{\xi_0})(x) = \int_{\mathbb{R}^n} \left(\sum_{k=1}^{\infty} h_k(x) g_k(y) \right) \mathscr{F}^{-1}\delta^r_{\xi_0}(y)dy.$$

Taking into account that $K(x,y) = \sum_{k=1}^{\infty} h_k(x)g_k(y) \in L^1(\mathbb{R}^{2n})$ (see, e.g., Lemma 3.1 of [20]), that $\|\mathscr{F}^{-1}\delta^r_{\xi_0}\|_{L^\infty} \leq \|\delta^r_{\xi_0}\|_{L^1} = 1$, and that (in view of the Lebesgue Differentiation Theorem)

$$\lim_{r\to 0^+} \mathscr{F}^{-1}\delta^r_{\xi_0}(x) = \lim_{r\to 0^+} \frac{1}{|B(\xi_0, r)|} \int_{B(\xi_0,r)} e^{i2\pi x\cdot\xi} d\xi = e^{i2\pi x\cdot\xi_0}, \tag{27}$$

an application of the Dominated Convergence Theorem gives

$$\lim_{r\rightarrow 0^+} F(\mathscr{F}^{-1}\delta^r_{\xi_0})(x)=\int_{\mathbb{R}^n}\left(\sum_{k=1}^{\infty}h_k(x)g_k(y)\right)e^{i2\pi y\cdot\xi_0}dy=\sum_{k=1}^{\infty}h_k(x)(\mathscr{F}^{-1}g_k)(\xi_0). \quad (28)$$

In fact, for $a.e.w.$ $x\in\mathbb{R}^n$,

$$\begin{aligned}\left|\left(\sum_{k=1}^{\infty}h_k(x)g_k(y)\right)(\mathscr{F}^{-1}\delta^r_{\xi_0})(y)\right| &= \left|\left(\sum_{k=1}^{\infty}h_k(x)g_k(y)\right)\frac{1}{|B(\xi_0,r)|}\int_{B(\xi_0,r)}e^{i2\pi y\cdot\xi}d\xi\right|\\ &\leq \left|\sum_{k=1}^{\infty}h_k(x)g_k(y)\right| = |K(x,y)|.\end{aligned}$$

Since $K\in L^1(\mathbb{R}^{2n})$, and the function $\kappa(x,y):=|K(x,y)|$ is non-negative on the product space $\mathbb{R}^{2n}$, by the Fubinni theorem applied to positive functions, the $L^1(\mathbb{R}^{2n})$-norm of K can be computed from iterated integrals as

$$\int\int|K(x,y)|dy,dx=\int\left(\int|K(x,y)|dy\right)dx=\int\left(\int|K(x,y)|dx\right)dy. \quad (29)$$

By Tonelly theorem, for $a.e.w.$ $x\in\mathbb{R}^n$, the function $\kappa(x,\cdot)=|K(x,\cdot)|\in L^1(\mathbb{R}^n)$. Now, by the dominated convergence theorem, we have

$$\begin{aligned}&\lim_{r\rightarrow 0^+}F(\mathscr{F}^{-1}\delta^r_{\xi_0})(x)\\ &=\lim_{r\rightarrow 0^+}\int_{\mathbb{R}^n}K(x,y)\mathscr{F}^{-1}\delta^r_{\xi_0}(y)dy=\int_{\mathbb{R}^n}K(x,y)\lim_{r\rightarrow 0^+}\mathscr{F}^{-1}\delta^r_{\xi_0}(y)dy\\ &=\int_{\mathbb{R}^n}K(x,y)e^{i2\pi y\xi_0}dy=\lim_{\ell\rightarrow\infty}\int_{|y|\leq\ell}K(x,y)\cdot e^{i2\pi y\xi_0}dy\\ &=\lim_{\ell\rightarrow\infty}\int_{\mathbb{R}^n}\left(\sum_{k=1}^{\infty}h_k(x)g_k(y)\right)\cdot e^{i2\pi y\xi_0}\cdot 1_{\{|y|\leq\ell\}}dy.\end{aligned}$$

Now, from Lemma 3.4-(d) in [20],

$$\begin{aligned}&\lim_{\ell\rightarrow\infty}\int_{\mathbb{R}^n}\left(\sum_{k=1}^{\infty}h_k(x)g_k(y)\right)\cdot e^{i2\pi y\xi_0}\cdot 1_{\{|y|\leq\ell\}}dy\\ &\quad=\lim_{\ell,m\rightarrow\infty}\int_{\mathbb{R}^n}\left(\sum_{k=1}^{m}h_k(x)g_k(y)\right)\cdot e^{i2\pi y\xi_0}\cdot 1_{\{|y|\leq\ell\}}dy\\ &\quad=\lim_{\ell,m\rightarrow\infty}\sum_{k=1}^{m}h_k(x)\int_{\mathbb{R}^n}g_k(y)\cdot e^{i2\pi y\xi_0}\cdot 1_{\{|y|\leq\ell\}}dy\\ &\quad=\lim_{\ell,m\rightarrow\infty}\sum_{k=1}^{m}h_k(x)\int_{|y|\leq\ell}g_k(y)\cdot e^{i2\pi y\xi_0}dy\\ &\quad=\sum_{k=1}^{\infty}h_k(x)(\mathscr{F}^{-1}g_k)(\xi_0).\end{aligned}$$

On the other hand, if we compute $F(\mathscr{F}^{-1}\delta^r_{\xi_0})$ from the definition (1), we have

$$F(\mathscr{F}^{-1}\delta^r_{\xi_0})(x) = \frac{1}{|B(\xi_0,r)|}\int_{B(\xi_0,r)} e^{i\phi(x,\xi)}a(x,\xi)d\xi.$$

From the hypothesis that $a(x,\cdot) \in L^1_{loc}(\mathbb{R}^n)$ for $a.e.w\ x \in \mathbb{R}^n$, the Lebesgue Differentiation theorem gives

$$\lim_{r\to 0^+} F(\mathscr{F}^{-1}\delta^r_{\xi_0}) = e^{i\phi(x,\xi_0)}a(x,\xi_0). \tag{30}$$

Consequently, we deduce the identity

$$e^{i\phi(x,\xi_0)}a(x,\xi_0) = \sum_{k=1}^{\infty} h_k(x)(\mathscr{F}^{-1}g_k)(\xi_0), \tag{31}$$

which in turn is equivalent to

$$a(x,\xi_0) = e^{-i\phi(x,\xi_0)}\sum_{k=1}^{\infty} h_k(x)(\mathscr{F}^{-1}g_k)(\xi_0). \tag{32}$$

So, we have proved the first part of the theorem. Now, if we assume that the symbol a of the FIO F satisfies the decomposition formula (32) for fixed sequences h_k in L^{p_2} and g_k in $L^{p_1'}$ satisfying (52), then from (1) we can write (in the sense of distributions)

$$\begin{aligned} Ff(x) &= \int_{\mathbb{R}^n} e^{\phi(x,\xi)}a(x,\xi)\mathscr{F}f(\xi)d\xi = \int_{\mathbb{R}^n}\sum_{k=1}^{\infty} h_k(x)(\mathscr{F}^{-1}g_k)(\xi)(\mathscr{F}f)(\xi)d\xi \\ &= \int_{\mathbb{R}^n}\sum_{k=1}^{\infty} h_k(x)\int_{\mathbb{R}^n} e^{i2\pi y\xi}g_k(y)dy(\mathscr{F}f)(\xi)d\xi \\ &= \int_{\mathbb{R}^n}\left(\sum_{k=1}^{\infty} h_k(x)g_k(y)\right)\left(\int_{\mathbb{R}^n} e^{i2\pi y\xi}(\mathscr{F}f)(\xi)d\xi\right)dy \\ &= \int_{\mathbb{R}^n}\left(\sum_{k=1}^{\infty} h_k(x)g_k(y)\right)f(y)dy, \end{aligned}$$

where in the last line we have used the Fourier inversion formula. So, by Delgado Theorem (Theorem 2.1) we finish the proof. ☑

Proof of Theorem 1.2. Let us consider the Fourier integral operator F,

$$Ff(x) := \int_{\mathbb{R}^n} e^{i\phi(x,\xi)}a(x,\xi)\widehat{f}(\xi)d\xi, \tag{33}$$

associated with the symbol a. The main strategy in the proof will be to analyze the natural factorization of F in terms of the Fourier transform,

$$(\mathscr{F}f)(\xi) := \int_{\mathbb{R}^n} e^{-i2\pi x\cdot\xi}f(x)dx. \tag{34}$$

Clearly, if we define the operator with kernel (associated to $\sigma = (\phi, a)$),

$$K_\sigma g(x) = \int_{\mathbb{R}^n} e^{i\phi(x,\xi)} a(x,\xi) g(\xi),\ g \in \mathscr{S}(\mathbb{R}^n),\ K_\sigma(x,\xi) = e^{i\phi(x,\xi)} a(x,\xi), \tag{35}$$

then $F = K_\sigma \circ \mathscr{F}$. Taking into account the Hausdorff-Young inequality

$$\|\mathscr{F} f\|_{L^{p_1'}(\mathbb{R}^n)} \leq \|f\|_{L^{p_1}(\mathbb{R}^n)}, \tag{36}$$

the Fourier transform extends to a bounded operator from $L^{p_1}(\mathbb{R}^n)$ into $L^{p_1'}(\mathbb{R}^n)$. So, if we prove that the condition (10) assures the r-nuclearity of K_σ from $L^{p_1'}(\mathbb{R}^n)$ into $L^{p_2}(\mathbb{R}^n)$, we can deduce the r-nuclearity of F from $L^{p_1}(\mathbb{R}^n)$ into $L^{p_2}(\mathbb{R}^n)$. Here, we will be using the fact that the class of r-nuclear operators is a bilateral ideal on the set of bounded operators between Banach spaces.

Now, $K_\sigma : L^{p_1'}(\mathbb{R}^n) \to L^{p_2}(\mathbb{R}^n)$ is r-nuclear if, and only if, there exist sequences $\{h_k\}, \{g_k\}$ satisfying

$$K_\sigma(x,\xi) = e^{i\phi(x,\xi)} a(x,\xi) = \sum_k h_k(x) g_k(\xi), \tag{37}$$

where

$$\sum_k \|h_k\|_{L^{p_2}}^r \|g_k\|_{L^{p_1}}^r < \infty, \text{ and } K_\sigma f(x) = \int_{\mathbb{R}^n} (\sum_k h_k(x) g_k(\xi)) g(\xi) d\xi, \text{ a.e.w. } x, \tag{38}$$

for every $g \in L^{p_1'}(\mathbb{R}^n)$. Here, we have used the fact that for $1 < p_1 \leq 2$, $L^{p_1''}(\mathbb{R}^n) = L^{p_1}(\mathbb{R}^n)$. We end the proof by observing that (37) is in turns equivalent to (9). ☑

Proof of Theorem 1.3. Let $a(\cdot,\cdot)$ be a symbol such that $a(x,\cdot) \in L^1_{loc}(\mathbb{R}^n)$, *a.e.w.*, $x \in \mathbb{R}^n$. Let $2 \leq p_1 < \infty$, $1 \leq p_2 < \infty$, and let F be the Fourier integral operator associated to $a(\cdot,\cdot)$. If $F : L^{p_1}(\mathbb{R}^n) \to L^{p_2}(\mathbb{R}^n)$ is nuclear, then Theorem 1.1 guarantees the decomposition

$$a(x,\xi) = e^{-i\phi(x,\xi)} \sum_{k=1}^{\infty} h_k(x) (\mathscr{F}^{-1} g_k)(\xi),\ \ a.e.w.,\ (x,\xi),$$

where $\{g_k\}_{k\in\mathbb{N}}$ and $\{h_k\}_{k\in\mathbb{N}}$ are sequences of functions satisfying

$$\sum_{k=0}^{\infty} \|g_k\|_{L^{p_1'}} \|h_k\|_{L^{p_2}} < \infty. \tag{39}$$

So, if we take the $L^{p_2}_x$-norm, we have,

$$\begin{aligned} \|a(x,\xi)\|_{L^{p_2}_x} &= \left\| e^{-i\phi(x,\xi)} \sum_{k=1}^{\infty} h_k(x)(\mathscr{F}^{-1} g_k)(\xi) \right\|_{L^{p_2}_x} \\ &= \left\| \sum_{k=1}^{\infty} h_k(x)(\mathscr{F}^{-1} g_k)(\xi) \right\|_{L^{p_2}_x} \\ &\leq \sum_{k=1}^{\infty} \|h_k\|_{L^{p_2}} |(\mathscr{F}^{-1} g_k)(\xi)|. \end{aligned}$$

Now, if we use the Hausdorff-Young inequality, we deduce that $\|\mathscr{F}^{-1}g_k\|_{L^{p_1}} \leq \|g_k\|_{L^{p_1'}}$. Consequently,

$$\begin{aligned}\|a(x,\xi)\|_{L^{p_2}_x L^{p_1}_\xi(\mathbb{R}^n\times\mathbb{R}^n),} &= \left(\int\limits_{\mathbb{R}^n}\left(\int\limits_{\mathbb{R}^n}|a(x,\xi)|^{p_2}dx\right)^{\frac{p_1}{p_2}}d\xi\right)^{\frac{1}{p_1}}\\ &\leq \left\|\sum_{k=1}^{\infty}\|h_k\|_{L^{p_2}}|(\mathscr{F}^{-1}g_k)(\xi)|\right\|_{L^{p_1}_\xi}\\ &\leq \sum_{k=1}^{\infty}\|h_k\|_{L^{p_2}}\|\mathscr{F}^{-1}g_k\|_{L^{p_1}}\\ &\leq \sum_{k=1}^{\infty}\|h_k\|_{L^{p_2}}\|g_k\|_{L^{p_1'}}<\infty.\end{aligned}$$

In an analogous way we can prove that

$$\|a(x,\xi)\|_{L^{p_1}_\xi L^{p_2}_x(\mathbb{R}^n\times\mathbb{R}^n)} \leq \sum_{k=1}^{\infty}\|h_k\|_{L^{p_2}}\|g_k\|_{L^{p_1'}}<\infty.$$

Thus, we finish the proof. ☑

2.2. The nuclear trace for FIOs on $\mathbb{R}^n$

If we choose a r-nuclear operator $T : E \to E$, $0 < r \leq 1$, with the Banach space E satisfying the Grothendieck approximation property (see Grothendieck [31]), then there exist (a nuclear decomposition) sequences $(e_n')_{n\in\mathbb{N}_0}$ in E' (the dual space of E) and $(y_n)_{n\in\mathbb{N}_0}$ in E satisfying

$$Tf = \sum_{n\in\mathbb{N}_0} e_n'(f)y_n, \quad f\in E, \tag{40}$$

and

$$\sum_{n\in\mathbb{N}_0}\|e_n'\|^r_{E'}\|y_n\|^r_F<\infty. \tag{41}$$

In this case the nuclear trace of T is (a well-defined functional) given by $\mathrm{Tr}(T) = \sum_{n\in\mathbb{N}_0^n} e_n'(f_n)$, because L^p-spaces have the Grothendieck approximation property and, as consequence, we can compute the nuclear trace of every r-nuclear pseudo-multipliers. We will compute it from Delgado Theorem (Theorem 2.1). For doing so, let us consider a $r-$nuclear Fourier integral operator $F : L^p(\mathbb{R}^n) \to L^p(\mathbb{R}^n)$, $2 \leq p < \infty$. If a is the symbol associated to F, in view of (9), we have (in the sense of distributions)

$$\int\limits_{\mathbb{R}^{2n}} e^{i\phi(x,\xi)-2\pi i x\cdot\xi}a(x,\xi)d\xi\,dx = \int\limits_{\mathbb{R}^n}\int\limits_{\mathbb{R}^n} e^{i\phi(x,\xi)-2\pi i x\cdot\xi}e^{-i\phi(x,\xi)}\sum_{k=1}^{\infty}h_k(x)(\mathscr{F}g_k)(-\xi)d\xi\,dx$$
$$= \int\limits_{\mathbb{R}^n}\sum_{k=1}^{\infty}h_k(x)\int\limits_{\mathbb{R}^n}e^{-2\pi i x\cdot\xi}(\mathscr{F}^{-1}g_k)(\xi)d\xi\,dx$$
$$= \int\limits_{\mathbb{R}^n}\sum_{k=1}^{\infty}h_k(x)g_k(x)dx = \mathrm{Tr}(F).$$

So, we obtain the trace formula

$$\mathrm{Tr}(F) = \int\limits_{\mathbb{R}^n}\int\limits_{\mathbb{R}^n} e^{i\phi(x,\xi)-2\pi i x\cdot\xi}a(x,\xi)d\xi\,dx. \tag{42}$$

Now, in order to determinate a relation with the eigenvalues of F, we recall that the nuclear trace of an r-nuclear operator on a Banach space coincides with the spectral trace, provided that $0 < r \leq \frac{2}{3}$. For $\frac{2}{3} \leq r \leq 1$. We recall the following result (see [42]).

Theorem 2.3. *Let $T : L^p(\mu) \to L^p(\mu)$ be a r-nuclear operator as in (40). If $\frac{1}{r} = 1 + |\frac{1}{p} - \frac{1}{2}|$, then*

$$\mathrm{Tr}(T) := \sum_{n\in\mathbb{N}_0^n} e_n'(f_n) = \sum_n \lambda_n(T), \tag{43}$$

where $\lambda_n(T)$, $n \in \mathbb{N}$ is the sequence of eigenvalues of T with multiplicities taken into account.

As an immediate consequence of the preceding theorem, if the FIO $F : L^p(\mathbb{R}^n) \to L^p(\mathbb{R}^n)$ is r-nuclear, the relation $\frac{1}{r} = 1 + |\frac{1}{p} - \frac{1}{2}|$ implies

$$\mathrm{Tr}(F) = \int\limits_{\mathbb{R}^n}\int\limits_{\mathbb{R}^n} e^{i\phi(x,\xi)-2\pi i x\cdot\xi}a(x,\xi)d\xi\,dx = \sum_n \lambda_n(T), \tag{44}$$

where $\lambda_n(T)$, $n \in \mathbb{N}$ is the sequence of eigenvalues of F with multiplicities taken into account.

2.3. Characterization of nuclear pseudo-differential operators defined by the Weyl quantization

As it was mentioned in the introduction, the Weyl-quantization of a distribution $\sigma \in \mathscr{S}'(\mathbb{R}^{2n})$ is the pseudo-differential operator defined by

$$\sigma^{\omega}(x, D_x)f(x) = \int\limits_{\mathbb{R}^n}\int\limits_{\mathbb{R}^n} e^{i2\pi(x-y)\cdot\xi}\sigma\left(\frac{x+y}{2},\xi\right)f(y)dyd\xi. \tag{45}$$

There exist relations between pseudo-differential operators associated to the classical quantization

$$\sigma(x, D_x)f(x) = \int\limits_{\mathbb{R}^n}\int\limits_{\mathbb{R}^n} e^{i2\pi(x-y)\cdot\xi}\sigma(x,\xi) f(y)dyd\xi = \int\limits_{\mathbb{R}^n} e^{i2\pi x\cdot\xi}\sigma(x,\xi)(\mathscr{F}f)(\xi)d\xi, \quad (46)$$

or in a more general setting, τ-quantizations defined for every $0<\tau\leq 1$, by the integral expression

$$\sigma^{\tau}(x, D_x)f(x) = \int\limits_{\mathbb{R}^n}\int\limits_{\mathbb{R}^n} e^{i2\pi(x-y)\cdot\xi}\sigma(\tau x + (1-\tau)y, \xi) f(y)dyd\xi \quad (47)$$

(with $\tau = \frac{1}{2}$ corresponding to the Hörmander quantization), as it can be viewed in the following proposition (see Delgado [12]).

Proposition 2.4. *Let $a, b \in \mathscr{S}'(\mathbb{R}^{2n})$. Then, $a^{\tau}(x, D_x) = b^{\tau'}(x, D_x)$ if, and only if,*

$$a(x,\xi) = \int\limits_{\mathbb{R}^n}\int\limits_{\mathbb{R}^n} e^{-i2\pi(\xi-\eta)z} b(x + (\tau'-\tau)z, \eta)dzd\eta, \quad (48)$$

provided that $0 < \tau, \tau' \leq 1$.

Theorem 2.5. *Let $0 < r \leq 1$. Let $a(\cdot,\cdot)$ be a differentiable symbol. Let $2 \leq p_1 < \infty$, $1 \leq p_2 < \infty$, and let $a^{\omega}(x, D_x)$ be the Weyl quantization of the symbol $a(\cdot,\cdot)$. Then, $a^{\omega}(x, D_x) : L^{p_1}(\mathbb{R}^n) \to L^{p_2}(\mathbb{R}^n)$ is r-nuclear if, and only if, the symbol $a(\cdot,\cdot)$ admits a decomposition of the form*

$$a(x,\xi) = \sum_{k=1}^{\infty}\int\limits_{\mathbb{R}^n} e^{-i2\pi z\cdot\xi} h_k(x + (1-\tau)z) g_k(x - \tau z)dz, \;\; a.e.w., \;\; (x,\xi), \quad (49)$$

where $\{g_k\}_{k\in\mathbb{N}}$ and $\{h_k\}_{k\in\mathbb{N}}$ are sequences of functions satisfying

$$\sum_{k=0}^{\infty} \|g_k\|^r_{L^{p_1'}} \|h_k\|^r_{L^{p_2}} < \infty. \quad (50)$$

Proof. Let us assume that $a^{\tau}(x, D_x)$ is r-nuclear from $L^{p_1}(\mathbb{R}^n)$ into $L^{p_2}(\mathbb{R}^n)$. By Proposition 2.4, $a^{\tau}(x, D_x) = b(x, D_x)$, where

$$a(x,\xi) = \int\limits_{\mathbb{R}^n}\int\limits_{\mathbb{R}^n} e^{-i2\pi(\xi-\eta)z} b(x + (1-\tau)z, \eta)dzd\eta.$$

By Theorem 1.1 applied to $\phi(x,\xi) = 2\pi x\cdot\xi$, and taking into account that $b(x, D_x)$ is r-nuclear, there exist sequences h_k in L^{p_2} and g_k in $L^{p_1'}$ satisfying

$$b(x,\xi) = e^{-i2\pi x\cdot\xi}\sum_{k=1}^{\infty} h_k(x)(\mathscr{F}^{-1}g_k)(\xi), \;\; a.e.w., \;\; (x,\xi), \quad (51)$$

with

$$\sum_{k=0}^{\infty} \|g_k\|^r_{L^{p_1'}} \|h_k\|^r_{L^{p_2}} < \infty. \tag{52}$$

So, we have

$$a(x,\xi) = \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} e^{-i2\pi(\xi-\eta)z - i2\pi(x+(1-\tau)z)\eta} \left(\sum_{k=1}^{\infty} h_k(x+(1-\tau)z)(\mathscr{F}^{-1}g_k)(\eta)\right) dz d\eta.$$

Since

$$\begin{aligned} -i2\pi(\xi-\eta)z - i2\pi(x+(1-\tau)z)\cdot\eta \\ &= -i2\pi\xi\cdot z + i2\pi\eta\cdot z - i2\pi x\cdot\eta - i2\pi(1-\tau)z\cdot\eta \\ &= -i2\pi\xi\cdot z - i2\pi x\cdot\eta + i2\pi\tau z\cdot\eta, \end{aligned}$$

we have (in the sense of distributions)

$$\begin{aligned} a(x,\xi) &= \sum_{k=1}^{\infty} \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} e^{-i2\pi\xi\cdot z - i2\pi x\cdot\eta + i2\pi\tau z\cdot\eta} h_k(x+(1-\tau)z)(\mathscr{F}^{-1}g_k)(\eta) dz d\eta \\ &= \sum_{k=1}^{\infty} \int_{\mathbb{R}^n} e^{-i2\pi\xi\cdot z} h_k(x+(1-\tau)z) \int_{\mathbb{R}^n} e^{-i2\pi(x-\tau z)\cdot\eta}(\mathscr{F}^{-1}g_k)(\eta) d\eta dz \\ &= \sum_{k=1}^{\infty} \int_{\mathbb{R}^n} e^{-i2\pi z\cdot\xi} h_k(x+(1-\tau)z) g_k(x-\tau z) dz. \end{aligned}$$

So, we have proved the first part of the characterization. On the other hand, if we assume (49), then

$$a(x,\xi) = \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} e^{-i2\pi(\xi-\eta)z} b(x+(1-\tau)z,\eta) dz d\eta,$$

where $b(x,\xi)$ is defined as in (51). So, from Theorem 1.1 we deduce that $b(x,D_x)$ is r-nuclear, and from the equality $a^\tau(x,D_x) = b(x,D_x)$ we deduce the r-nuclearity of $a^\tau(x,D_x)$. The proof is complete. ☑

Remark 2.6. Let us observe that from Theorem 2.5 with $\tau = 1/2$, we deduce the Theorem 1.4 mentioned in the introduction.

3. Characterizations of Fourier integral operators on $\mathbb{Z}^n$ and arbitrary compact Lie groups

3.1. FIOs on $\mathbb{Z}^n$

In this subsection we characterize those Fourier integral operators on $\mathbb{Z}^n$ (the set of points in $\mathbb{R}^n$ with integral coordinates) admitting nuclear extensions on Lebesgue spaces. Now we define pseudo-differential operators and discrete Fourier integral operators on $\mathbb{Z}^n$. The discrete Fourier transform of $f \in \ell^1(\mathbb{Z})$ is defined by

$$(\mathscr{F}_{\mathbb{Z}^n} f)(\xi) = \sum_{m \in \mathbb{Z}^n} e^{-i2\pi m\cdot\xi} f(m), \ \xi \in [0,1]^n. \tag{53}$$

The Fourier inversion formula gives

$$f(m) = \int_{[0,1]^n} e^{i2\pi m\cdot\xi} (\mathscr{F}_{\mathbb{Z}^n} f)(\xi) d\xi, \ \ m \in \mathbb{Z}^n. \tag{54}$$

In this setting pseudo-differential operators on $\mathbb{Z}^n$ are defined by the integral form

$$t_m f(n') := \int_{[0,1]^n} e^{i2\pi n'\cdot\xi} m(n',\xi)(\mathscr{F}_{\mathbb{Z}^n} f)(\xi) d\xi, \ \ f \in \ell^1(\mathbb{Z}^n), \, n' \in \mathbb{Z}^n. \tag{55}$$

These operators were introduced by Molahajloo in [38]. However, the fundamental work of Botchway L., Kibiti G., Ruzhansky M. [5] provides a symbolic calculus and other properties for these operators on ℓ^p-spaces. In particular, Fourier integral operators on $\mathbb{Z}^n$ were defined in such reference as integral operators of the form

$$\mathfrak{f}_{a,\phi} f(n') := \int_{[0,1]^n} e^{i\phi(n',\xi)} a(n',\xi)(\mathscr{F}_{\mathbb{Z}^n} f)(\xi) d\xi, \ \ f \in \ell^1(\mathbb{Z}^n), \, n' \in \mathbb{Z}^n. \tag{56}$$

Our main tool in the characterization of nuclear FIOs on $\mathbb{Z}^n$ is the following result, due to Jamalpour Birgani [36].

Theorem 3.1. *Let $0 < r \leq 1$, $1 \leq p_1 < \infty$, $1 \leq p_2 < \infty$, and let t_m be the pseudo-differential operator associated to the symbol $m(\cdot,\cdot)$. Then, $t_m : \ell^{p_1}(\mathbb{Z}^n) \to \ell^{p_2}(\mathbb{Z}^n)$ is r-nuclear if, and only if, the symbol $m(\cdot,\cdot)$ admits a decomposition of the form*

$$m(n',\xi) = e^{-i2\pi n'\xi} \sum_{k=1}^{\infty} h_k(n')(\mathscr{F}_{\mathbb{Z}^n} g_k)(-\xi), \ \ a.e.w., \ (n',\xi), \tag{57}$$

where $\{g_k\}_{k\in\mathbb{N}}$ and $\{h_k\}_{k\in\mathbb{N}}$ are sequences of functions satisfying

$$\sum_{k=0}^{\infty} \|g_k\|^r_{\ell^{p_1'}} \|h_k\|^r_{\ell^{p_2}} < \infty. \tag{58}$$

As a consequence of the previous result, we give a simple proof for our characterization.

Theorem 3.2. *Let $0 < r \leq 1$, $1 \leq p_1 < \infty$, $1 \leq p_2 < \infty$, and let $\mathfrak{f}_{a,\phi}$ be the Fourier integral operator associated to the phase function ϕ and to the symbol $a(\cdot,\cdot)$. Then, $\mathfrak{f}_{a,\phi} : \ell^{p_1}(\mathbb{Z}^n) \to \ell^{p_2}(\mathbb{Z}^n)$ is r-nuclear if, and only if, the symbol $a(\cdot,\cdot)$ admits a decomposition of the form*

$$a(n',\xi) = e^{-i\phi(n',\xi)} \sum_{k=1}^{\infty} h_k(x)(\mathscr{F}_{\mathbb{Z}^n} g_k)(-\xi), \ \ a.e.w., \ (n',\xi), \tag{59}$$

where $\{g_k\}_{k\in\mathbb{N}}$ and $\{h_k\}_{k\in\mathbb{N}}$ are sequences of functions satisfying

$$\sum_{k=0}^{\infty} \|g_k\|^r_{\ell^{p_1'}} \|h_k\|^r_{\ell^{p_2}} < \infty. \tag{60}$$

Proof. Let us write the operator $\mathfrak{f}_{a,\phi}$ as

$$\mathfrak{f}_{a,\phi}f(x)=\int_{[0,1]^n}e^{i\phi(n',\xi)}a(n',\xi)(\mathscr{F}_{\mathbb{Z}^n}f)(\xi)d\xi=\int_{[0,1]^n}e^{i2\pi n'\cdot\xi}m(n',\xi)(\mathscr{F}_{\mathbb{Z}^n}f)(\xi)d\xi, \tag{61}$$

where $m(n',\xi)=e^{i\phi(n',\xi)-i2\pi n'\cdot\xi}a(n',\xi)$. So, the discrete Fourier integral operator $\mathfrak{f}_{a,\phi}$ coincides with the discrete pseudo-differential operator t_m with symbol m. By using Theorem 3.5, the operator $\mathfrak{f}_{a,\phi}=t_m:\ell^{p_1}(\mathbb{Z}^n)\to\ell^{p_2}(\mathbb{Z}^n)$ is r-nuclear if, and only if, the symbol $m(\cdot,\cdot)$ admits a decomposition of the form

$$m(n',\xi)=e^{-i2\pi n'\xi}\sum_{k=1}^{\infty}h_k(n')(\mathscr{F}_{\mathbb{Z}^n}g_k)(-\xi),\ \ a.e.w.,\ (n',\xi), \tag{62}$$

where $\{g_k\}_{k\in\mathbb{N}}$ and $\{h_k\}_{k\in\mathbb{N}}$ are sequences of functions satisfying

$$\sum_{k=0}^{\infty}\|g_k\|^r_{\ell^{p_1'}}\|h_k\|^r_{\ell^{p_2}}<\infty. \tag{63}$$

Let us note that from the definition of m we have

$$e^{i\phi(n',\xi)-i2\pi n'\cdot\xi}a(n',\xi)=e^{-i2\pi n'\cdot\xi}\sum_{k=1}^{\infty}h_k(n')(\mathscr{F}_{\mathbb{Z}^n}g_k)(-\xi),\ \ a.e.w.,\ (n',\xi),$$

which, in turn, is equivalent to

$$a(n',\xi)=e^{-i\phi(n',\xi)}\sum_{k=1}^{\infty}h_k(n')(\mathscr{F}_{\mathbb{Z}^n}g_k)(-\xi). \tag{64}$$

Thus, the proof is complete. ☑

Remark 3.3. The nuclear trace of a nuclear discrete pseudo-differential operator on $\mathbb{Z}^n$, $t_m:\ell^p(\mathbb{Z}^n)\to\ell^p(\mathbb{Z}^n)$, $1\leq p<\infty$, can be computed according to the formula

$$\mathrm{Tr}(t_m)=\sum_{n'\in\mathbb{Z}^n}\int_{[0,1]^n}m(n',\xi)d\xi. \tag{65}$$

From the proof of the previous criterion, we have that $\mathfrak{f}_{a,\phi}=t_m$, where $m(n',\xi)=e^{\phi(n',\xi)-i2\pi n'\xi}a(n',\xi)$ and, consequently, if $\mathfrak{f}_{a,\phi}:\ell^p(\mathbb{Z}^n)\to\ell^p(\mathbb{Z}^n)$, $1\leq p<\infty$, is r-nuclear, its nuclear trace is given by

$$\mathrm{Tr}(\mathfrak{f}_{a,\phi})=\sum_{n'\in\mathbb{Z}^n}\int_{[0,1]^n}e^{\phi(n',\xi)-i2\pi n'\xi}a(n',\xi)d\xi. \tag{66}$$

Now, we present an application of the previous result.

Theorem 3.4. *Let $2\leq p_1<\infty$, and $1\leq p_2<\infty$. If $\mathfrak{f}_{a,\phi}:\ell^{p_1}(\mathbb{Z}^n)\to\ell^{p_2}(\mathbb{Z}^n)$ is nuclear, then $a(n',\xi)\in\ell^{p_2}_{n'}L^{p_1}_{\xi}(\mathbb{Z}^n\times\mathbb{T}^n)\cap L^{p_1}_{\xi}\ell^{p_2}_{n'}(\mathbb{Z}^n\times\mathbb{T}^n)$; this means that*

$$\|a(n',\xi)\|_{\ell^{p_2}_{n'}L^{p_1}_{\xi}(\mathbb{Z}^n\times\mathbb{T}^n),}:=\left(\int_{\mathbb{T}^n}\left(\sum_{n'\in\mathbb{Z}^n}|a(n',\xi)|^{p_2}\right)^{\frac{p_1}{p_2}}d\xi\right)^{\frac{1}{p_1}}<\infty, \tag{67}$$

and

$$\|a(n',\xi)\|_{L^{p_1}_{\xi}\ell^{p_2}_{n'}(\mathbb{Z}^n\times\mathbb{T}^n),} := \left(\sum_{n'\in\mathbb{Z}^n}\left(\int\limits_{\mathbb{T}^n}|a(x,\xi)|^{p_1}d\xi\right)^{\frac{p_2}{p_1}}\right)^{\frac{1}{p_2}} < \infty. \tag{68}$$

The proof is only an adaptation of the proof that we have done for Theorem 1.3. We only need to use a discrete Hausdorff-Young inequality. In this case, we use

$$\|\mathscr{F}_{\mathbb{Z}^n}g_k\|_{L^{p_1}(\mathbb{T}^n)} \leq \|g_k\|_{\ell^{p_1'}(\mathbb{Z}^n)}. \tag{69}$$

3.2. FIOs on compact Lie groups

In this subsection we characterize nuclear Fourier integral operators on compact Lie groups. Although the results presented are valid for arbitrary Hausdorff and compact groups, we restrict our attention to Lie groups taking into account their differentiable structure, which in our case could give potential applications of our results to the understanding on the spectrum of certain operators associated to differential problems.

Let us consider a compact Lie group G with Lie algebra $\mathfrak{g}$. We will equip G with the Haar measure μ_G. The following identities follow from the Fourier transform on G:

$$(\mathscr{F}_G\varphi)(\xi) \equiv \widehat{\varphi}(\xi) = \int_G \varphi(x)\xi(x)^*dx, \qquad \varphi(x) = \sum_{[\xi]\in\widehat{G}} d_\xi \mathrm{Tr}(\xi(x)\widehat{\varphi}(\xi));$$

and the Peter-Weyl Theorem on G implies the Plancherel identity on $L^2(G)$,

$$\|\varphi\|_{L^2(G)} = \left(\sum_{[\xi]\in\widehat{G}} d_\xi \mathrm{Tr}(\widehat{\varphi}(\xi)\widehat{\varphi}(\xi)^*)\right)^{\frac{1}{2}} = \|\widehat{\varphi}\|_{L^2(\widehat{G})}.$$

Notice that, since $\|A\|^2_{HS} = \mathrm{Tr}(AA^*)$, the term within the sum is the Hilbert-Schmidt norm of the matrix $\widehat{\varphi}(\xi)$. Any linear operator A on G mapping $C^\infty(G)$ into $\mathcal{D}'(G)$ gives rise to a *matrix-valued global (or full) symbol* $\sigma_A(x,\xi) \in \mathbb{C}^{d_\xi\times d_\xi}$ given by

$$\sigma_A(x,\xi) = \xi(x)^*(A\xi)(x), \tag{70}$$

which can be understood from the distributional viewpoint. Then it can be shown that the operator A can be expressed in terms of such a symbol as [48],

$$Af(x) = \sum_{[\xi]\in\widehat{G}} d_\xi \mathrm{Tr}[\xi(x)\sigma_A(x,\xi)\widehat{f}(\xi)]. \tag{71}$$

So, if $\Phi : G\times\widehat{G} \to \cup_{[\xi]\in\widehat{G}}\mathrm{GL}(d_\xi)$ is a measurable function (the phase function), and $a : G\times\widehat{G} \to \cup_{[\xi]\in\widehat{G}}\mathbb{C}^{d_\xi\times d_\xi}$ is a distribution on $G\times\widehat{G}$, the Fourier integral operator $F = F_{\Phi,a}$ associated to the symbol $a(\cdot,\cdot)$ and to the phase function Φ is defined by the Fourier series operator

$$Ff(x) = \sum_{[\xi]\in\widehat{G}} d_\xi \mathrm{Tr}[\Phi(x,\xi)a(x,\xi)\widehat{f}(\xi)]. \tag{72}$$

In order to present our main result for Fourier integral operators, we recall the following criterion (see Ghaemi, Jamalpour Birgani, Wong [30]).

Theorem 3.5. *Let $0 < r \leq 1$, $1 \leq p_1 < \infty$, $1 \leq p_2 < \infty$, and let A be the pseudo-differential operator associated to the symbol $\sigma_A(\cdot,\cdot)$. Then, $A : L^{p_1}(G) \to L^{p_2}(G)$ is r-nuclear if, and only if, the symbol $\sigma_A(\cdot,\cdot)$ admits a decomposition of the form*

$$\sigma_A(x,\xi) = \xi(x)^* \sum_{k=1}^{\infty} h_k(x)(\mathscr{F}_G \overline{g}_k)(\xi)^*, \quad a.e.w., \ (x,\xi), \tag{73}$$

where $\{g_k\}_{k\in\mathbb{N}}$ and $\{h_k\}_{k\in\mathbb{N}}$ are sequences of functions satisfying

$$\sum_{k=0}^{\infty} \|g_k\|^r_{L^{p_1'}} \|h_k\|^r_{L^{p_2}} < \infty. \tag{74}$$

As a consequence of the previous criterion, we give a simple proof for our characterization.

Theorem 3.6. *Let $0 < r \leq 1$, $1 \leq p_1 < \infty$, $1 \leq p_2 < \infty$, and let F be the Fourier integral operator associated to the phase function Φ and to the symbol $a(\cdot,\cdot)$. Then, $F : L^{p_1}(G) \to L^{p_2}(G)$ is r-nuclear if, and only if, the symbol $a(\cdot,\cdot)$ admits a decomposition of the form*

$$a(x,\xi) = \Phi(x,\xi)^{-1} \sum_{k=1}^{\infty} h_k(x)(\mathscr{F}_G \overline{g}_k)(\xi)^*, \quad a.e.w., \ (x,\xi), \tag{75}$$

where $\{g_k\}_{k\in\mathbb{N}}$ and $\{h_k\}_{k\in\mathbb{N}}$ are sequences of functions satisfying

$$\sum_{k=0}^{\infty} \|g_k\|^r_{L^{p_1'}} \|h_k\|^r_{L^{p_2}} < \infty. \tag{76}$$

Remark 3.7. For the proof we use use the characterization of r-nuclear pseudo-differential operators mentioned above. However, this result will be generalized in the next section to arbitrary compact homogeneous manifolds.

Proof. Let us observe that the Fourier integral operator F, can be written as

$$Ff(x) = \sum_{[\xi]\in\widehat{G}} d_\xi \mathrm{Tr}[\Phi(x,\xi)a(x,\xi)\widehat{f}(\xi)] = \sum_{[\xi]\in\widehat{G}} d_\xi \mathrm{Tr}[\xi(x)\sigma_A(x,\xi)\widehat{f}(\xi)], \tag{77}$$

where $\sigma_A(x,\xi) = \xi(x)^*\Phi(x,\xi)a(x,\xi)$. So, the Fourier integral operator F coincides with the pseudo-differential operator A with symbol σ_A. In view of Theorem 3.5, the operator $F = A : L^{p_1}(G) \to L^{p_2}(G)$ is r-nuclear if, and only if, the symbol $\sigma_A(\cdot,\cdot)$ admits a decomposition of the form

$$\sigma_A(x,\xi) = \xi(x)^* \sum_{k=1}^{\infty} h_k(x)(\mathscr{F}_G \overline{g}_k)(\xi)^*, \quad a.e.w., \ (x,\xi), \tag{78}$$

where $\{g_k\}_{k\in\mathbb{N}}$ and $\{h_k\}_{k\in\mathbb{N}}$ are sequences of functions satisfying

$$\sum_{k=0}^{\infty} \|g_k\|^r_{L^{p_1'}} \|h_k\|^r_{L^{p_2}} < \infty. \tag{79}$$

Let us note that from the definition of σ_A we have

$$\xi(x)^*\Phi(x,\xi)a(x,\xi) = \xi(x)^*\sum_{k=1}^{\infty} h_k(x)(\mathscr{F}_G\overline{g}_k)(\xi)^*, \ \ a.e.w., \ (x,\xi),$$

which is equivalent to

$$a(x,\xi) = \Phi(x,\xi)^{-1}\sum_{k=1}^{\infty} h_k(x)(\mathscr{F}_G\overline{g}_k)(\xi)^*, \ \ a.e.w., \ (x,\xi). \tag{80}$$

Thus, we finish the proof. ☑

Remark 3.8. The nuclear trace of a r-nuclear pseudo-differential operator on G, $A : L^p(G) \to L^p(G)$, $1 \leq p < \infty$, can be computed according to the formula

$$\mathrm{Tr}(A) = \int\limits_G \sum_{[\xi]\in\widehat{G}} d_\xi \mathrm{Tr}[\sigma_A(x,\xi)]dx. \tag{81}$$

From the proof of the previous theorem, we have that $F = A$, where $\sigma_A(x,\xi) = \xi(x)^*\Phi(x,\xi)a(x,\xi)$ and, consequently, if $F : L^p(G) \to L^p(G)$, $1 \leq p < \infty$, is r-nuclear, its nuclear trace is given by

$$\mathrm{Tr}(F) = \int\limits_G \sum_{[\xi]\in\widehat{G}} d_\xi \mathrm{Tr}[\xi(x)^*\Phi(x,\xi)a(x,\xi)]dx. \tag{82}$$

Now, we illustrate the results above with some examples.

Example 3.9. (The torus). *Let us consider the n-dimensional torus $G = \mathbb{T}^n := \mathbb{R}^n/\mathbb{Z}^n$ and its unitary dual $\widehat{\mathbb{T}}^n := \{e_\ell : \ell \in \mathbb{Z}^n\}$, $e_\ell(x) := e^{i2\pi\ell\cdot x}$, $x \in \mathbb{T}^n$. By following Ruzhansky and Turunen [48], a Fourier integral operator F associated to the phase function $\phi : \mathbb{T}^n \times \widehat{\mathbb{T}}^n \to \mathbb{C}$, and to the symbol $a : \mathbb{T}^n \times \widehat{\mathbb{T}}^n \to \mathbb{C}$, is defined according to the rule*

$$F\varphi(x) = \sum_{e_\ell\in\widehat{\mathbb{T}}^n} e^{i\phi(x,e_\ell)}a(x,e_\ell)\widehat{\varphi}(e_\ell), \ x \in \mathbb{T}^n, \tag{83}$$

where $\widehat{\varphi}(e_\ell) = \int_{\mathbb{T}^n} f(x)e_\ell(x)dx$ is the Fourier transform of f at e_ℓ. If we identify $\widehat{\mathbb{T}}^n$ with $\mathbb{Z}^n$, and we define $a(x,\ell) := a(x,e_\ell)$, and $\widehat{\varphi}(\ell) := \widehat{\varphi}(e_\ell)$, we give the more familiar expression for F,

$$F\varphi(x) = \sum_{\ell\in\mathbb{Z}^n} e^{i\phi(x,\ell)}a(x,\ell)\widehat{\varphi}(\ell), \ x \in \mathbb{T}^n. \tag{84}$$

Now, by using Theorem 3.6, F is r-nuclear, $0 < r \leq 1$ if, and only if, the symbol $a(\cdot,\cdot)$ admits a decomposition of the form

$$a(x,\ell) = e^{-i\phi(x,\ell)}\sum_{k=1}^{\infty} h_k(x)\overline{(\widehat{\overline{g}}_k)(\ell)}, \ \ a.e.w., \ (x,\ell), \tag{85}$$

where $\{g_k\}_{k\in\mathbb{N}}$ *and* $\{h_k\}_{k\in\mathbb{N}}$ *are sequences of functions satisfying*

$$\sum_{k=0}^{\infty}\|g_k\|^r_{L^{p_1'}}\|h_k\|^r_{L^{p_2}}<\infty. \tag{86}$$

The last condition have been proved for pseudo-differential operators in [29]. In this case, the nuclear trace of F *can be written as*

$$\mathrm{Tr}(F)=\int_{\mathbb{T}^n}\sum_{\ell\in\mathbb{Z}^n}e^{\phi(x,\ell)-i2\pi x\cdot\ell}a(x,\ell)dx. \tag{87}$$

Example 3.10. (The group SU(2)). *Let us consider the group* $\mathrm{SU}(2)\cong\mathbb{S}^3$ *consinting of those orthogonal matrices* A *in* $\mathbb{C}^{2\times 2}$, *with* $\det(A)=1$. *We recall that the unitary dual of* $\mathrm{SU}(2)$ *(see [48]) can be identified as*

$$\widehat{\mathrm{SU}}(2)\equiv\{[t_l]:2l\in\mathbb{N},d_l:=\dim t_l=(2l+1)\}. \tag{88}$$

There are explicit formulae for t_l *as functions of Euler angles in terms of the so-called Legendre-Jacobi polynomials, see [48]. A Fourier integral operator* F *associated to the phase function* $\Phi:\mathrm{SU}(2)\times\widehat{\mathrm{SU}}(2)\to\cup_{\ell\in\frac{1}{2}\mathbb{N}_0}\mathrm{GL}(2\ell+1)$, *and to the symbol* $a:\mathrm{SU}(2)\times\widehat{\mathrm{SU}}(2)\to\cup_{\ell\in\frac{1}{2}\mathbb{N}_0}\mathbb{C}^{(2\ell+1)\times(2\ell+1)}$, *is defined as*

$$F\varphi(x)=\sum_{[t_\ell]\in\widehat{\mathrm{SU}}(2)}(2\ell+1)\mathrm{Tr}[\Phi(x,t_\ell)a(x,t_\ell)\widehat{\varphi}(t_\ell)],\ x\in\mathrm{SU}(2), \tag{89}$$

where

$$\widehat{\varphi}(e_\ell)=\int_{\mathrm{SU}(2)}f(x)t_\ell(x)dx\in\mathbb{C}^{(2\ell+1)\times(2\ell+1)},\ \ell\in\frac{1}{2}\mathbb{N}_0$$

is the Fourier transform of f *at* t_ℓ. *As in the case of the* n*-dimensional torus, if we identify* $\widehat{\mathrm{SU}}(2)$ *with* $\frac{1}{2}\mathbb{N}_0$, *and we define* $a(x,\ell):=a(x,t_\ell)$, *and* $\widehat{\varphi}(\ell):=\widehat{\varphi}(t_\ell)$, *we can write*

$$F\varphi(x)=\sum_{\ell\in\frac{1}{2}\mathbb{N}_0}(2\ell+1)\mathrm{Tr}[\Phi(x,\ell)a(x,\ell)\widehat{\varphi}(\ell)],\ x\in\mathrm{SU}(2). \tag{90}$$

Now, by using Theorem 3.6, F *is* r*-nuclear,* $0<r\leq 1$ *if, and only if, the symbol* $a(\cdot,\cdot)$ *admits a decomposition of the form*

$$a(x,\ell)=\Phi(x,\ell)^{-1}\sum_{k=1}^{\infty}h_k(x)\overline{(\widehat{\overline{g}}_k)(\ell)},\ \ a.e.w.,\ (x,\ell), \tag{91}$$

where $\{g_k\}_{k\in\mathbb{N}}$ *and* $\{h_k\}_{k\in\mathbb{N}}$ *are sequences of functions satisfying*

$$\sum_{k=0}^{\infty}\|g_k\|^r_{L^{p_1'}}\|h_k\|^r_{L^{p_2}}<\infty. \tag{92}$$

The last condition have been proved for pseudo-differential operators in [30] on arbitrary Hausdorff and compact groups. In this case, in an analogous expression to the one presented above for $\mathbb{R}^n$, $\mathbb{Z}^n$, *and* $\mathbb{T}^n$, *the nuclear trace of* F *can be written as*

$$\mathrm{Tr}(F)=\int_{\mathrm{SU}(2)}\sum_{\ell\in\frac{1}{2}\mathbb{N}_0}(2\ell+1)\mathrm{Tr}[t_\ell(A)^*\Phi(A,\ell)a(A,\ell)]dA. \tag{93}$$

By using the diffeomorphism $\varrho : \mathrm{SU}(2) \to \mathbb{S}^3$, *defined by*

$$\varrho(A) = x := (x_1, x_2, x_3, x_4), \text{ for } A = \begin{bmatrix} x_1 + ix_2 & x_3 + ix_4 \\ -x_3 + ix_4 & x_1 - ix_2 \end{bmatrix}, \tag{94}$$

we have

$$\begin{aligned} \mathrm{Tr}(F) &= \int\limits_{\mathrm{SU}(2)} \sum_{\ell \in \frac{1}{2}\mathbb{N}_0} (2\ell+1)\mathrm{Tr}[t_\ell(A)^* \Phi(A,\ell) a(A,\ell)] dA \\ &= \int\limits_{\mathbb{S}^3} \sum_{\ell \in \frac{1}{2}\mathbb{N}_0} (2\ell+1)\mathrm{Tr}[t_\ell(\varrho^{-1}(x))^* \Phi(\varrho^{-1}(x),\ell) a(\varrho^{-1}(x),\ell)] d\sigma(x) \\ &= \int\limits_{\mathbb{S}^3} \sum_{\ell \in \frac{1}{2}\mathbb{N}_0} (2\ell+1)\mathrm{Tr}[t_\ell(x)^* \Phi(x,\ell) a(x,\ell)] d\sigma(x), \end{aligned}$$

where $t_\ell(\varrho^{-1}(x)) =: t_\ell(x)$, $\Phi(\varrho^{-1}(x),\ell) =: \Phi(x,\ell)$, $a(\varrho^{-1}(x),\ell) =: a(x,\ell)$, *and* $d\sigma(x)$ *denotes the surface measure on* $\mathbb{S}^3$. *If we consider the parametrization of* $\mathbb{S}^3$ *defined by* $x_1 := \cos(\frac{t}{2})$, $x_2 := \nu$, $x_3 := (\sin^2(\frac{t}{2}) - \nu^2)^{\frac{1}{2}} \cos(s)$, $x_4 := (\sin^2(\frac{t}{2}) - \nu^2)^{\frac{1}{2}} \sin(s)$, *where*

$$(t,\nu,s) \in D := \{(t,\nu,s) \in \mathbb{R}^3 : |\nu| \leq \sin(\frac{t}{2}),\ 0 \leq t, s \leq 2\pi\},$$

then $d\sigma(x) = \sin(\frac{t}{2}) dt d\nu ds$, and

$$\mathrm{Tr}(F) = \int_0^{2\pi}\int_0^{2\pi}\int_{-\sin(t/2)}^{\sin(t/2)} \sum_{\ell \in \frac{1}{2}\mathbb{N}_0} (2\ell+1)\mathrm{Tr}[t_\ell(t,\nu,s)^* \Phi((t,\nu,s),\ell) a((t,\nu,s),\ell)] \sin(\frac{t}{2}) d\nu dt ds.$$

4. Nuclear Fourier integral operators on compact homogeneous manifolds

The main goal in this section is to provide a characterization for the nuclearity of Fourier integral operators on compact homogeneous manifolds $M \cong G/K$. Taking into account that the Peter-Weyl decompositions of $L^2(M)$ and $L^2(\mathbb{G})$ (where $\mathbb{G}$ is a Hausdorff and compact group) have an analogue structure, we classify the nuclearity of FIOs on compact homogeneous manifolds by adapting to our case the proof of Theorem 2.2 in [30], where those nuclear pseudo-differential operators in $L^p(\mathbb{G})$-spaces were classified.

4.1. Global FIOs on compact homogeneous manifolds

In order to present our definition for Fourier integral operators on compact homogeneous spaces, we recall some definitions on the subject. Compact homogeneous manifolds can be obtained if we consider the quotient space of a compact Lie groups G with one of its closed subgroups K (there exists an unique differential structure for the quotient $M := G/K$). Examples of compact homogeneous spaces are spheres $\mathbb{S}^n \cong \mathrm{SO}(n+1)/\mathrm{SO}(n)$, real projective spaces $\mathbb{RP}^n \cong \mathrm{SO}(n+1)/\mathrm{O}(n)$, complex projective spaces $\mathbb{CP}^n \cong \mathrm{SU}(n+1)/\mathrm{SU}(1)\times\mathrm{SU}(n)$, and, more generally, Grassmannians $\mathrm{Gr}(r,n) \cong \mathrm{O}(n)/\mathrm{O}(n-r)\times\mathrm{O}(r)$.

Let us denote by $\widehat{G}_0$ the subset of $\widehat{G}$ of representations in G that are of class I with respect to the subgroup K. This means that $\pi \in \widehat{G}_0$ if there exists at least one non trivial invariant vector a with respect to K, i.e., $\pi(h)a = a$ for every $h \in K$. Let us denote by B_π to the vector space of these invariant vectors, and $k_\pi = \dim B_\pi$. Now we follow the notion of Multipliers as in [1]. Let us consider the class of symbols $\Sigma(M)$, for $M = G/K$, consisting of those matrix-valued functions

$$\sigma : \widehat{G}_0 \to \bigcup_{n=1}^{\infty} \mathbb{C}^{n\times n} \ \text{ such that } \ \sigma(\pi)_{ij} = 0 \text{ for all } i,j > k_\pi. \tag{95}$$

Following [1], a Fourier multiplier A on M is a bounded operator on $L^2(M)$ such that for some $\sigma_A \in \Sigma(M)$ it satisfies

$$Af(x) = \sum_{\pi\in\widehat{G}_0} d_\pi \mathrm{Tr}(\pi(x)\sigma_A(\pi)\widehat{f}(\pi)), \quad \text{for } f \in C^\infty(M), \tag{96}$$

where $\widehat{f}$ denotes the Fourier transform of the lifting $\dot{f} \in C^\infty(G)$ of f to G, given by $\dot{f}(x) := f(xK)$, $x \in G$.

Remark 4.1. For every symbol of a Fourier multiplier A on M, only the upper-left block in $\sigma_A(\pi)$ of the size $k_\pi \times k_\pi$ cannot be the trivial matrix zero.

Now, if we consider a phase function $\Phi : M \times \widehat{G}_0 \to \cup_{[\pi]\in\widehat{G}_0}\mathrm{GL}(d_\pi)$, and a distribution $a : M \times \widehat{G}_0 \to \cup_{[\pi]\in\widehat{G}_0}\mathrm{GL}(d_\pi)$, the Fourier integral operator associated to Φ and to $a(\cdot,\cdot)$ is given by

$$F\varphi(x) = \sum_{\pi\in\widehat{G}_0} d_\pi \mathrm{Tr}(\Phi(x,\pi)a(x,\pi)\widehat{\varphi}(\pi)), \quad \text{for } \varphi \in C^\infty(M). \tag{97}$$

We additionally require the condition $\sigma(x,\pi)_{ij} = 0$ for $i,j > k_\pi$ for the distributional symbols considered above. Now, if we want to characterize those r-nuclear FIOs we only need to follow the proof of Theorem 2.2 in [30], where the nuclearity of pseudo-differential operators was characterized on compact and Hausdorff groups. Since the set

$$\{\sqrt[2]{d_\pi}\pi_{ij} : 1 \leq i,j \leq k_\pi\}$$

provides an orthonormal basis of $L^2(M)$, we have the relation

$$\int_M \pi_{nm}(x)\overline{\varkappa_{ij}(x)}dx = \frac{1}{d_\pi}\delta_{\pi\varkappa}\delta_{ni}\delta_{mj}, \ \ [\pi],[\varkappa] \in \widehat{G}_0. \tag{98}$$

If we assume that $F : L^{p_1}(M) \to L^{p_2}(M)$ is r-nuclear, then we have a nuclear decomposition for its kernel, i.e., there exist sequences h_k in L^{p_2} and g_k in $L^{p_1'}$ satisfying

$$Ff(x) = \int_M \left(\sum_{k=1}^{\infty} h_k(x)g_k(y)\right) f(y)dy, \ f \in L^{p_1}(M), \tag{99}$$

with

$$\sum_{k=0}^{\infty}\|g_k\|^r_{L^{p_1'}}\|h_k\|^r_{L^{p_2}}<\infty. \tag{100}$$

So, we have with $1\leq n,m\leq k_\pi$,

$$F\pi_{nm}(x)$$
$$=(\Phi(x,\pi)a(x,\pi))_{nm}=\int_M\left(\sum_{k=1}^{\infty}h_k(x)g_k(y)\right)\pi_{nm}(x)dx=\sum_{k=1}^{\infty}h_k(x)\overline{\widehat{\overline{g_k}}(\pi)}_{mn}.$$

Consequently, if B^t denotes the transpose of a matrix B, we obtain

$$\Phi(x,\pi)a(x,\pi)=\sum_{k=1}^{\infty}h_k(x)\overline{\widehat{\overline{g_k}}(\pi)}^t=\sum_{k=1}^{\infty}h_k(x)\widehat{\overline{g_k}}(\pi)^*, \tag{101}$$

and by considering that $\Phi(x,\pi)\in\mathrm{GL}(d_\pi)$ for every $x\in M$, we deduce the equivalent condition,

$$a(x,\pi)=\Phi(x,\pi)^{-1}\sum_{k=1}^{\infty}h_k(x)\widehat{\overline{g_k}}(\pi)^*. \tag{102}$$

On the other hand, if we assume that the symbol $a(\cdot,\cdot)$ satisfies the condition (102) with $\sum_{k=0}^{\infty}\|g_k\|^r_{L^{p_1'}}\|h_k\|^r_{L^{p_2}}<\infty$, from the definition of Fourier integral operator we can write, for $\varphi\in L^{p_1}(M)$,

$$\begin{aligned}F\varphi(x)&=\sum_{\pi\in\widehat{G}_0}d_\pi\mathrm{Tr}(\Phi(x,\pi)a(x,\pi)\widehat{\varphi}(\pi))=\sum_{\pi\in\widehat{G}_0}d_\pi\mathrm{Tr}(\sum_{k=1}^{\infty}h_k(x)\widehat{\overline{g_k}}(\pi)^*\widehat{\varphi}(\pi))\\&=\sum_{\pi\in\widehat{G}_0}d_\pi\mathrm{Tr}(\sum_{k=1}^{\infty}h_k(x)\int_M g_k(y)\pi(y)dy\widehat{\varphi}(\pi))\\&=\int_M\sum_{k=1}^{\infty}h_k(x)g_k(y)\sum_{\pi\in\widehat{G}_0}d_\pi\mathrm{Tr}(\pi(y)\widehat{\varphi}(\pi))dy\\&=\int_M\sum_{k=1}^{\infty}h_k(x)g_k(y)\varphi(y)dy.\end{aligned}$$

Again, by Delgado's Theorem we obtain the r-nuclearity of F. So, our adaptation of the proof of Theorem 2.2 in [30] to our case of FIOs on compact manifolds leads to the following result.

Theorem 4.2. *Let us assume $M\cong G/K$ be a homogeneous manifold, $0<r\leq 1$, $1\leq p_1,p_2<\infty$, and let F be a Fourier integral operator as in (97). Then, $F:L^{p_1}(M)\to L^{p_2}(M)$ is r-nuclear if, and only if, there exist sequences h_k in L^{p_2} and g_k in $L^{p_1'}$ satisfying*

$$a(x,\pi)=\Phi(x,\pi)^{-1}\sum_{k=1}^{\infty}h_k(x)\widehat{\overline{g_k}}(\pi)^*,\ x\in G,[\pi]\in\widehat{G}_0, \tag{103}$$

with

$$\sum_{k=0}^{\infty}\|g_k\|^r_{L^{p_1'}}\|h_k\|^r_{L^{p_2}}<\infty. \tag{104}$$

Now, we will prove that the previous (abstract) characterization can be applied in order to measure the decaying of symbols in the momentum variables. So, we will use the following formulation of Lebesgue spaces on $\widehat{G}_0$:

$$\mathcal{M}\in\ell^p(\widehat{G}_0)\Longleftrightarrow\|\mathcal{M}\|_{\ell^p(\widehat{G}_0)}=\left(\sum_{[\pi]\in\widehat{G}_0}d_\pi k_\pi^{p(\frac{1}{p}-\frac{1}{2})}\|\mathcal{M}(\pi)\|^p_{\text{HS}}\right)^{\frac{1}{p}}<\infty, \tag{105}$$

for $1\leq p<\infty$.

Theorem 4.3. *Let us assume $M\cong G/K$ be a homogeneous manifold, $2\leq p_1<\infty$, $1\leq p_2<\infty$, and let F be a Fourier integral operator as in* (97). *If $F:L^{p_1}(M)\rightarrow L^{p_2}(M)$ is nuclear, then $a(x,\pi)\in\ell^{p_1}_\pi L^{p_2}_x(M\times\widehat{G}_0)$; this means that*

$$\|a(x,\pi)\|_{\ell^{p_1}_\pi L^{p_2}_x(M\times\widehat{G}_0)}:=\left(\int\limits_M\left(\sum_{[\pi]\in\widehat{G}_0}d_\pi k_\pi^{p_1(\frac{1}{p_1}-\frac{1}{2})}\|a(x,\pi)\|^{p_1}_{\text{HS}}\right)^{\frac{p_2}{p_1}}dx\right)^{\frac{1}{p_2}}<\infty, \tag{106}$$

provided that

$$\|\Phi^{-1}\|_\infty:=\sup_{(x,[\pi])\in M\times\widehat{G}_0}\|\Phi(x,\pi)^{-1}\|_{op}<\infty. \tag{107}$$

Proof. Let $2\leq p_1<\infty$, $1\leq p_2<\infty$, and let F be the Fourier integral operator associated to $a(\cdot,\cdot)$. If $F:L^{p_1}(M)\rightarrow L^{p_2}(M)$ is nuclear, then Theorem 4.2 guarantees the decomposition

$$a(x,\pi)=\Phi(x,\pi)^{-1}\sum_{k=1}^{\infty}h_k(x)\widehat{\overline{g_k}}(\pi)^*,\ x\in G,[\pi]\in\widehat{G}_0, \tag{108}$$

with

$$\sum_{k=0}^{\infty}\|g_k\|^r_{L^{p_1'}}\|h_k\|^r_{L^{p_2}}<\infty. \tag{109}$$

So, if we take the $\ell^{p_1}_\pi$-norm, we have,

$$\begin{aligned}\|a(x,\pi)\|_{\ell^{p_1}_\pi}&=\left\|\Phi(x,\pi)^{-1}\sum_{k=1}^{\infty}h_k(x)\widehat{\overline{g_k}}(\pi)^*\right\|_{\ell^{p_1}_\pi}\\&=\left\|\sum_{k=1}^{\infty}h_k(x)\Phi(x,\pi)^{-1}\widehat{\overline{g_k}}(\pi)^*\right\|_{L^{p_1}_\pi}\\&\leq\sum_{k=1}^{\infty}|h_k(x)|\|\Phi(x,\pi)^{-1}\widehat{\overline{g_k}}(\pi)^*\|_{\ell^{p_1}_\pi}.\end{aligned}$$

By the definition of $\ell^{p_1}_\pi$-norm, we have

$$\begin{aligned}\|\Phi(x,\pi)^{-1}\overline{g_k}(\pi)^*\|_{\ell^{p_1}(\widehat{G}_0)} &= \left(\sum_{[\pi]\in\widehat{G}_0} d_\pi k_\pi^{p_1(\frac{1}{p_1}-\frac{1}{2})}\|\Phi(x,\pi)^{-1}\widehat{\overline{g_k}}(\pi)^*\|_{\mathrm{HS}}^{p_1}\right)^{\frac{1}{p_1}}\\ &\leq \left(\sum_{[\pi]\in\widehat{G}_0} d_\pi k_\pi^{p_1(\frac{1}{p_1}-\frac{1}{2})}\|\Phi(x,\pi)^{-1}\|_{op}\|\widehat{\overline{g_k}}(\pi)^*\|_{\mathrm{HS}}^{p_1}\right)^{\frac{1}{p_1}}\\ &\leq \|\Phi^{-1}\|_\infty\left(\sum_{[\pi]\in\widehat{G}_0} d_\pi k_\pi^{p_1(\frac{1}{p_1}-\frac{1}{2})}\|\widehat{\overline{g_k}}(\pi)^*\|_{\mathrm{HS}}^{p_1}\right)^{\frac{1}{p_1}}\\ &= \|\Phi^{-1}\|_\infty\left(\sum_{[\pi]\in\widehat{G}_0} d_\pi k_\pi^{p_1(\frac{1}{p_1}-\frac{1}{2})}\|\widehat{\overline{g_k}}(\pi)\|_{\mathrm{HS}}^{p_1}\right)^{\frac{1}{p_1}}.\end{aligned}$$

Consequently,

$$\|a(x,\pi)\|_{\ell^{p_1}_\pi} \leq \|\Phi^{-1}\|_\infty\sum_{k=1}^{\infty}|h_k(x)|\|\widehat{\overline{g_k}}(\pi)\|_{\ell^{p_1}_\pi}. \tag{110}$$

Now, if we use the Hausdorff-Young inequality, we deduce, $\|\widehat{\overline{g_k}}(\pi)\|_{\ell^{p_1}_\pi} \leq \|g_k\|_{L^{p_1'}}$. Consequently,

$$\begin{aligned}\|a(x,\pi)\|_{\ell^{p_1}_\pi L^{p_2}_x(M\times\widehat{G}_0)} &= \left(\int_M\left(\sum_{[\pi]\in\widehat{G}_0} d_\pi k_\pi^{p_1(\frac{1}{p_1}-\frac{1}{2})}\|a(x,\pi)\|_{\mathrm{HS}}^{p_1}\right)^{\frac{p_2}{p_1}}dx\right)^{\frac{1}{p_2}}\\ &\leq \|\Phi^{-1}\|_\infty\left\|\sum_{k=1}^{\infty}|h_k(x)|\|\widehat{\overline{g_k}}(\pi)\|_{\ell^{p_1}_\pi}\right\|_{L^{p_2}_x}\\ &\leq \sum_{k=1}^{\infty}\|h_k\|_{L^{p_2}}\|g_k\|_{L^{p_1'}}<\infty.\end{aligned}$$

Thus, we finish the proof. ☑

Remark 4.4. If $K=\{e_G\}$ and $M=G$ is a compact Lie group, the condition

$$\|\Phi^{-1}\|_\infty := \sup_{(x,[\pi])\in M\times\widehat{G}_0}\|\Phi(x,\pi)^{-1}\|_{op} = \sup_{(x,[\xi])\in G\times\widehat{G}}\|\Phi(x,\xi)^{-1}\|_{op}<\infty \tag{111}$$

arises naturally in the context of pseudo-differential operators. Indeed, if we take $\Phi(x,\xi)=\xi(x)$, then $\Phi(x,\xi)^{-1}=\xi(x)^*$, and

$$\sup_{(x,[\xi])\in G\times\widehat{G}}\|\xi(x)^*\|_{op}=1. \tag{112}$$

Remark 4.5. As a consequence of Delgado's theorem, if $F : L^p(M) \to L^p(M)$, $M \cong G/K$, $1 \le p < \infty$, is r-nuclear, its nuclear trace is given by

$$\mathrm{Tr}(F) = \int_M \sum_{[\pi]\in \widehat{G}_0} d_\pi \mathrm{Tr}[\pi(x)^*\Phi(x,\pi)a(x,\pi)]dx. \tag{113}$$

Example 4.6 (The complex projective plane $\mathbb{CP}^2$). *A point $\ell \in \mathbb{CP}^n$ (the n-dimensional complex projective space) is a complex line through the origin in $\mathbb{C}^{n+1}$. For every n, $\mathbb{CP}^n \cong \mathrm{SU}(n+1)/\mathrm{SU}(1)\times\mathrm{SU}(n)$. We will use the representation theory of $\mathrm{SU}(3)$ in order to describe the nuclear trace of Fourier integral operators on $\mathbb{CP}^2 \cong \mathrm{SU}(3)/\mathrm{SU}(1)\times\mathrm{SU}(2)$. The Lie group $\mathrm{SU}(3)$ (see [27]) has dimension 8, and 3 positive square roots α, β and ρ with the property*

$$\rho = \frac{1}{2}(\alpha+\beta+\rho). \tag{114}$$

We define the weights

$$\sigma = \frac{2}{2}\alpha + \frac{1}{3}\beta, \ \ \tau = \frac{1}{3}\alpha + \frac{2}{3}\beta. \tag{115}$$

With the notations above the unitary dual of $\mathrm{SU}(3)$ can be identified with

$$\widehat{\mathrm{SU}}(3) \cong \{\lambda := \lambda(a,b) = a\sigma + b\tau : a,b \in \mathbb{N}_0,\}. \tag{116}$$

In fact, every representation $\pi = \pi_{\lambda(a,b)}$ has highest weight $\lambda = \lambda(a,b)$ for some $(a,b) \in \mathbb{N}_0^2$. In this case $d_{\lambda(a,b)} := d_{\pi_{\lambda(a,b)}} = \frac{1}{2}(a+1)(b+1)(a+b+2)$. For $G = \mathrm{SU}(3)$ and $K = \mathrm{SU}(1)\times\mathrm{SU}(2)$, let us define

$$\mathbb{N}_{00}^2 := \{(a,b) \in \mathbb{N}_0\times\mathbb{N}_0 : \pi_{\lambda(a,b)} \in \widehat{G}_0 = \widehat{\mathrm{SU}}(3)_0\}. \tag{117}$$

Now, let us consider a phase function $\Phi : \mathbb{CP}^2 \times \mathbb{N}_{00}^2 \to \cup_{(a,b)\in\mathbb{N}_0\times\mathbb{N}_0}\mathrm{GL}(d_{\lambda(a,b)})$, a distribution $a : \mathbb{CP}^2\times\mathbb{N}_{00}^2 \to \cup_{(a,b)\in\mathbb{N}\times\mathbb{N}}\mathrm{GL}(d_{\lambda(a,b)})$, and the Fourier integral operator F associated to Φ and to $a(\cdot,\cdot)$:

$$F\varphi(\ell) = \sum_{(a,b)\in\mathbb{N}_{00}^2} \frac{1}{2}(a+1)(b+1)(a+b+2)\mathrm{Tr}(\Phi(\ell,(a,b))a(\ell,(a,b))\widehat{\varphi}(\pi_{\lambda(a,b)})), \tag{118}$$

where $\varphi \in C^\infty(\mathbb{CP}^2)$. We additionally require the condition $\sigma(\ell,(a,b))_{ij} = 0$ for $\ell \in \mathbb{CP}^2$, and $i,j > k_{\pi_{\lambda(a,b)}}$, for those distributional symbols considered above. As a consequence of Remark 4.5, if $F : L^p(\mathbb{CP}^2) \to L^p(\mathbb{CP}^2)$, $1 \le p < \infty$, is r-nuclear, its nuclear trace is given by

$$\mathrm{Tr}(F) = \int_{\mathbb{CP}^2} \sum_{(a,b)\in\mathbb{N}_{00}^2} (a+1)(b+1)(a+b+2)\mathrm{Tr}[\pi_{\lambda(a,b)}(\ell)^*\Phi(\ell,(a,b))a(\ell,(a,b))]\frac{d\ell}{2}. \tag{119}$$

If $\vartheta : \mathbb{CP}^2 \to \mathrm{SU}(3)/\mathrm{SU}(1)\times\mathrm{SU}(2)$ *is a diffeomorphism and* $K = \mathrm{SU}(1)\times\mathrm{SU}(2)$, *then*

$$\int\limits_{\mathbb{CP}^2} \sum_{(a,b)\in\mathbb{N}_{00}^2} (a+1)(b+1)(a+b+2)\mathrm{Tr}[\pi_{\lambda(a,b)}(\ell)^*\Phi(\ell,(a,b))a(\ell,(a,b))]\frac{d\ell}{2}$$

$$= \int\limits_{\mathrm{SU}(3)/\mathrm{SU}(1)\times\mathrm{SU}(2)} \sum_{(a,b)\in\mathbb{N}_{00}^2} (a+1)(b+1)(a+b+2)$$
$$\mathrm{Tr}[\pi_{\lambda(a,b)}(\vartheta^{-1}(g))^*\Phi(\vartheta^{-1}(g),(a,b))a(\vartheta^{-1}(g),(a,b))]\frac{dg}{2}$$

$$= \int\limits_{\mathrm{SU}(3)/\mathrm{SU}(1)\times\mathrm{SU}(2)} \sum_{(a,b)\in\mathbb{N}_{00}^2} (a+1)(b+1)(a+b+2)$$
$$\mathrm{Tr}[\pi_{\lambda(a,b)}(g)^*\Phi(g,(a,b))a(g,(a,b))]\frac{dg}{2}$$

$$= \int\limits_{\mathrm{SU}(3)} \sum_{(a,b)\in\mathbb{N}_{00}^2} (a+1)(b+1)(a+b+2)$$
$$\mathrm{Tr}[\pi_{\lambda(a,b)}(gK)^*\Phi(gK,(a,b))a(gK,(a,b))]\frac{d\mu_{\mathrm{SU}(3)}(g)}{2},$$

where we have denoted $\pi_{\lambda(a,b)}(g) := \pi_{\lambda(a,b)}(\vartheta^{-1}(g))$, $\Phi(g,(a,b)) := \Phi(\vartheta^{-1}(g),(a,b))$ *and* $a(g,(a,b)) := a(\vartheta^{-1}(g),(a,b))$. *If we consider the parametrization of* $\mathrm{SU}(3)$ *(see, e.g., Bronzan [4]),*

$$g \equiv g(\theta_1,\theta_2,\theta_3,\phi_1,\phi_2,\phi_3,\phi_4,\phi_5) := (u_{ij})_{i,j=1,2,3},$$

where $0 \le \theta_i \le \frac{\pi}{2}$, $0 \le \phi_i \le 2\pi$, *and*

- $u_{11} = \cos\theta_1\cos\theta_2 e^{i\phi_1}$;
- $u_{12} = \sin\theta_1 e^{i\phi_3}$;
- $u_{13} = \cos\theta_1\sin\theta_2 e^{i\phi_4}$;
- $u_{21} = \sin\theta_1\sin\theta_3 e^{-i\phi_4-i\phi_5} - \sin\theta_1\cos\theta_2\cos\theta_3 e^{i\phi_1+i\phi_2-i\phi_3}$;
- $u_{22} = \cos\theta_1\cos\theta_3 e^{i\phi_2}$;
- $u_{23} = -\cos\theta_1\sin\theta_3 e^{-i\phi_1-i\phi_5} - \sin\theta_1\sin\theta_2\cos\theta_3 e^{i\phi_2-i\phi_3+i\phi_4}$;
- $u_{31} = -\sin\theta_1\cos\theta_2\sin\theta_3 e^{i\phi_1-i\phi_3+i\phi_5} - \sin\theta_2\cos\theta_3 e^{-i\phi_2-i\phi_4}$;
- $u_{32} = \cos\theta_1\sin\theta_3 e^{i\phi_5}$;
- $u_{33} = \cos\theta_2\cos\theta_3 e^{-i\phi_1-i\phi_2} - \sin\theta_1\sin\theta_2\sin\theta_3 e^{-i\phi_3+i\phi_4+i\phi_5}$,

then, the group measure is the determinant given by

$$d\mu_{\mathrm{SU}(3)}(g) = \frac{1}{2\pi^5}\sin\theta_1\cos^3\theta_1\sin\theta_2\cos\theta_2\sin\theta_3\cos\theta_3 d\theta_1 d\theta_2 d\theta_3 d\phi_1 d\phi_2 d\phi_3 d\phi_4 d\phi_5, \tag{120}$$

and we have the trace formula

$$\operatorname{Tr}(F) = \int_0^{\pi/2}\int_0^{\pi/2}\int_0^{\pi/2}\int_0^{2\pi}\int_0^{2\pi}\int_0^{2\pi}\int_0^{2\pi}\int_0^{2\pi} \sum_{(a,b)\in\mathbb{N}_{00}^2} (a+b+ab+1)(a+b+2)\operatorname{Tr}[\pi_{\lambda(a,b)}(g(\theta,\phi)K)^* \Phi(g(\theta,\phi)K,(a,b))a(g(\theta,\phi)K,(a,b))] \times \frac{1}{4\pi^5}\sin\theta_1\cos^3\theta_1\sin\theta_2\cos\theta_2\sin\theta_3\cos\theta_3 d\theta_1 d\theta_2 d\theta_3 d\phi_1 d\phi_2 d\phi_3 d\phi_4 d\phi_5,$$

with $(\theta,\phi) = (\theta_1,\theta_2,\theta_3,\phi_1,\phi_2,\phi_3,\phi_4,\phi_5)$.

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