Some Adjunctions Associated with Extensions and Restrictions of Ideals in the Context of Commutative Rings

Algunas adjunciones asociadas con extensiones y restricciones de ideales en el contexto de anillos conmutativos

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To Professor Carlos Javier Ruiz Salguero, in memoriam

Abstract. Given a commutative ring R and S one of its ideals, the function $I \mapsto (I : S)$ that transforms ideals of R into ideals of R, is right adjoint of the function $I \mapsto IS$. We define the $S-$maximal ideals of R as those ideals J of R such that $(J : S) = J$. If the ring S is pseudo-regular, then the set of $S-$maximal ideals of R is a complete lattice, isomorphic to the lattice of the ideals of S. In particular, the annihilator of S in R is the minimum of the $S-$maximal ideals of R. So the lattice structure of $S-$maximal ideals of R does not depend on the ring R.
On the other hand, the ideals of S can be extended to ideals of R and the ideals of R can be restricted to ideals of S. These two processes are not adjoint to each other, but if we restrict to appropriated collections of ideals we can obtain adjunctions.

Keywords: Ideal, Prime ideal, Semi-prime ideal, Ordered set, Adjoint functions.

Resumen. Dados un anillo conmutativo R y S uno de sus ideales, la función $I \mapsto (I : S)$, que transforma ideales de R en ideales de R es adjunta a derecha de la función $I \mapsto IS$. Se definen los ideales $S-$maximales de R como aquellos ideales J de R tales que $(J : S) = J$. Si el anillo S es seudo-regular, entonces el conjunto de ideales $S-$maximales de R es un retículo completo, isomorfo al retículo de los ideales de S. En particular, el anulador de S en R es el mínimo de los ideales $S-$maximales de R. La estructura de retículo de los ideales $S-$maximales de R no depende entonces del anillo R.
Por otro lado, los ideales de S se pueden extender a ideales de R y los ideales de R se pueden restringir a ideales de S. Estos dos procesos no son adjuntos entre sí, pero si se restringen a colecciones apropiadas de ideales sí se obtienen sendas adjunciones.

Palabras claves: Ideal, ideal primo, ideal semi-primo, conjunto ordenado, funciones adjuntas.

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1. Introduction

Remember that if $f : X \to Y$ and $g : Y \to X$ are functions between ordered sets, f *is left adjoint of* g (and g *is right adjoint of* f) if for all $x \in X$ and for all $y \in Y$ we have

$$f(x) \leq y \Leftrightarrow x \leq g(y).$$

This is equivalent to f and g are monotone non-decreasing functions such that $f\big(g(y)\big) \leq y$ and $x \leq g\big(f(x)\big)$, for all $x \in X$ and for all $y \in Y$.

This is a particular case of the concept of *adjoint functors* in category theory (see [5]).

The following theorem is well known and we will use it repeatedly along this work.

Theorem 1.1. *If f is left adjoint of g then*

(i) f respects least upper bounds and g respects greatest lower bounds and

(ii) $\operatorname{Im} g$ and $\operatorname{Im} f$ are isomorphic as ordered sets.

For more information about adjoint functions the reader may consult for example [3].

In this paper all rings are **commutative** and not necessarily with identity. The set of ideals of a ring A is denoted by $\mathcal{J}(A)$ and it is considered an ordered set by the inclusion relation. If K is an ideal of the ring A we say that A is an *i-extension* of K. In this case, we denote $r_A(K)$ the radical of the ideal K in the ring A, namely, $r_A(K) = \{x \in A : x^n \in K, \text{ for some } n > 0\}$.

Henceforth S is a fixed ring.

In the first section we see that given an i-extension R of S, the function that maps each ideal I of R to $(I : S)$ is right adjoint of the function that maps each ideal I of R to IS. In other words, we see that "to multiply is left adjoint of to divide". Moreover, we introduce the notion of $S-$maximal ideal.

In Section 2 we define the pseudo-regular rings, we prove some of its properties and we show that if S is pseudo-regular then the collection of the $S-$maximal ideals of an i-extension R of S is a complete lattice whose structure is independent of the i-extension.

In the third section we introduce a mechanism to extend ideals of S to ideals of one of its i-extensions and a mechanism to restrict ideals of the i-extension to ideals of S and we study some properties of these mechanisms. In particular, we prove that these two mechanisms are not adjoint to each other unless that S is pseudo-regular.

In the last section we restrict these mechanisms to appropriate collections of ideals in order to obtain three pairs of adjoint functions.

2. Multiplication and division of ideals

We show that given an i-extension R of S, the processes of multiply and divide by S are adjoints. In this section we introduce the notion of $S-$maximal ideal of an i-extension and present some examples.

The following definition of quotient was taken from [2].

Definition 2.1. Let A be a ring and let I, J be ideals of A. The quotient of I by J is defined by

$$(I : J) = \{x \in A : xJ \subseteq I\}.$$

The following proposition can be deduced immediately from the previous definition.

Proposition 2.2. *If I, J are ideals of the ring A then*

(i) $(I : J)$ *is an ideal of* A.

(ii) $I \subseteq (I : J)$.

(iii) $I \subseteq (IJ : J)$.

(iv) $(I : J)\, J \subseteq I$.

(v) $(I : J) = (I \cap J : J)$.

Theorem 2.3. *For each i-extension R of S, the function*

$$\eta_R : \mathcal{J}(R) \to \mathcal{J}(R) : I \mapsto (I : S)$$

is right adjoint of the function

$$\lambda_R : \mathcal{J}(R) \to \mathcal{J}(R) : J \mapsto JS.$$

Proof. It is clear that these two functions are monotone. Further, by the previous proposition we have that for all $I, J \in \mathcal{J}(R)$

$$\eta_R(\lambda_R(J)) = (JS : S) \supseteq J \text{ and}$$
$$\lambda_R(\eta_R(I)) = (I : S)\, S \subseteq I.$$

□

Corollary 2.4. *Given an i-extension R of S, for every collection $\{J_l\}_{l \in L}$ of ideals of R we have that*

(i) $\left\langle \bigcup_{l \in L} J_l \right\rangle S = \left\langle \bigcup_{l \in L} J_l S \right\rangle$,

(ii) $\left(\bigcap_{l \in L} J_l : S \right) = \bigcap_{l \in L} (J_l : S)$.

Proof. It is enough to remember that left adjoint functions respect least upper bounds and right adjoint functions respect greatest lower bounds. □

Definition 2.5. Let R be an i-extension of S. An ideal J of R is $S-$maximal if $J = \eta_R(J)$.

Example 2.6. If R is an i-extension of S and S is contained in the annihilator of R, namely $S \subseteq (0 : R)$, then $\eta_R(I) = R$ for all $I \in \mathcal{J}(R)$ and therefore, R is the only $S-$maximal ideal of R.

Example 2.7. Let R be an i-extension of S. If I is a prime ideal of R and $S - I \neq \phi$ then I is $S-$maximal.

Example 2.8. Let R be an i-extension of S. If $S \subseteq I \subseteq R$ then I is $S-$maximal if and only if $I = R$, since $\eta_R(I) = R$.

Example 2.9. If $R = \mathbb{Z}$ and $S = 2\mathbb{Z}$ then $n\mathbb{Z}$ is a $S-$maximal ideal of $\mathbb{Z}$ if and only if n is odd.

If R is an i-extension of S it is natural to ask which are the $S-$maximal ideals of the ring R. In the following section we give an answer in the case of the pseudo-regular rings.

3. Pseudo-regular rings

In this section we consider a particular kind of commutative rings that we have named *pseudo-regular rings*, because its definition is a weak version of von Neumann regular rings (see [7]). In [4], Gilmer studies eleven conditions that are consequence of the existence of identity in a ring and which are not equivalent when the ring has not identity. We call pseudo-regularity one of these conditions.

When the ring S is pseudo-regular, each ideal of S is an ideal in each i-extension of S and moreover, the $S-$restriction of each ideal J of an i-extension of S coincides with the product ideal JS. Using these facts we characterize the $S-$maximal ideals of any i-extension of S.

The following theorem was taken from [4].

Theorem 3.1. *Let A be a commutative ring. The following statements are equivalent:*

(i) *For each $b \in A$, $b \in bA$.*

(ii) *For each ideal I of A, $AI = I$.*

(iii) *If $\{x_1, ..., x_n\}$ is a finite set of elements of A, there exists $y \in A$ such that $x_i y = x_i$, for each i.*

(iv) *If I and J are co-maximal ideals of A, then $I \cap J = IJ$.*

Definition 3.2. A commutative ring A is **pseudo-regular** if it satisfies some of the conditions of the previous theorem.

Example 3.3. Every ring with identity is pseudo-regular.

Example 3.4. Every von Neumann regular ring is pseudo-regular. In particular, the Boolean rings are pseudo-regular.

Example 3.5. $2\mathbb{Z}$ is not a pseudo-regular ring.

The following proposition is evident.

Proposition 3.6. *The collection of pseudo-regular rings is closed for products and quotients and it is not closed for sub-rings nor ideals.*

Lemma 3.7. *If S is pseudo-regular and R is an i-extension of S, then*

(i) $\mathcal{J}(S) \subseteq \mathcal{J}(R)$.

(ii) For all $J \in \mathcal{J}(R)$, $JS = J \cap S$.

(iii) For all $I \in \mathcal{J}(S)$, $\lambda_R(\eta_R(I)) = I$.

(iv) Let $J \in \mathcal{J}(R)$. *J is a $S-$maximal ideal of R if and only if* $J \in \eta_R(\mathcal{J}(S))$.

Proof. (i) If $I \in \mathcal{J}(S)$ then clearly I is a sub-group of $(R,+)$. Now, if $x \in R$ and $z \in I$ then $xz \in S$, thus there exists $s \in S$ such that $xz = (xz)s$. Consequently $xz = (xs)z \in I$ since $xs \in S$. Therefore $I \in \mathcal{J}(R)$.

(ii) Let $J \in \mathcal{J}(R)$. It is enough to prove that $J \cap S \subseteq JS$. If $x \in J \cap S$, there exists $s \in S$ such that $x = xs$ and then $x \in JS$.

(iii) Let $I \in \mathcal{J}(S)$. It is enough to see that $I \subseteq \lambda_R(\eta_R(I))$. If $x \in I$, there exists $s \in S$ such that $x = xs$ and as $x \in (I : S)$ then $x \in (I : S)S = \lambda_R(\eta_R(I))$.

(iv) Let $J \in \mathcal{J}(R)$. We have

$$\eta_R(J) = (J : S) = (J \cap S : S) = \eta_R(J \cap S),$$

therefore, if J is $S-$maximal then $J = \eta_R(J) = \eta_R(J \cap S) \in \eta_R(\mathcal{J}(S))$. On the other hand, if $J = \eta_R(I)$ then $\lambda_R(J) = \lambda_R(\eta_R(I)) = I$ and thus

$$\begin{aligned} J &= \eta_R(\lambda_R(J)) \\ &= (SJ : S) \\ &= (S \cap J : S) \\ &= (J : S) \\ &= \eta_R(J). \end{aligned}$$

□

The proof of the following proposition is a simple routine exercise.

Proposition 3.8. *Let R be an i-extension of S.*

(i) If I is an ideal of S then the set $\{x \in R : xS \subseteq I\}$ is an ideal of R which contains I.

(ii) If J is an ideal of R then the set $J \cap S$ is an ideal of S contained in J.

As a consequence, for each i-extension R of S we can define the functions

$$\begin{aligned} \psi_R &: \mathcal{J}(S) \to \mathcal{J}(R) : I \mapsto \{x \in R : xS \subseteq I\} \text{ and} \\ \varphi_R &: \mathcal{J}(R) \to \mathcal{J}(S) : J \mapsto J \cap S. \end{aligned}$$

If I is an ideal of S, we say that $\psi_R(I)$ is the $S-$extension[1] of I to the ring R. If J is an ideal of R, we say that $\varphi_R(J)$ is the $S-$restriction of J to the ring S.

[1]The S-extension should not be confused with the notion of extension presented in [2], that corresponds to the ideal of R generated by I.

Theorem 3.9. *If S is pseudo-regular and R is an i-extension of S, then the function*

$$\psi_R : \mathcal{J}(S) \to \mathcal{J}(R)$$

is right adjoint to the function

$$\varphi_R : \mathcal{J}(R) \to \mathcal{J}(S).$$

Proof. By Lemma 3.7, ψ_R is the restriction of η_R to the set of ideals of S and φ_R is the same function λ_R co-restricted to the set of ideals of S. It is clear that ψ_R and φ_R are monotone functions and moreover

$$\begin{aligned} \varphi_R\left(\psi_R(I)\right) &= I, \text{ for all } I \in \mathcal{J}(S) \text{ and} \\ \psi_R\left(\varphi_R(J)\right) &\supseteq J, \text{ for all } J \in \mathcal{J}(R). \end{aligned}$$

□

Corollary 3.10. *If S is pseudo-regular and R is an i-extension of S, then the set of $S-$maximal ideals of R is a complete lattice, which is isomorphic to the lattice of ideals of S. The minimum of the $S-$maximal ideals of R is the annihilator of S in R.*

Proof. By the adjunction of the functions φ_R and ψ_R it is obtained that $\operatorname{Im}\varphi_R$ and $\operatorname{Im}\psi_R$ are isomorphic as ordered sets, where $\operatorname{Im}\varphi_R$ is $\mathcal{J}(S)$ and $\operatorname{Im}\psi_R$ is the set of $S-$maximal ideals of R. Also $\psi_R(0) = (0 : S)$ is the annihilator of S in R. □

Remark 3.11. Note that when S is pseudo-regular, the structure of the collection of $S-$maximal ideals of R does not depend on R. In other words, this lattice is the same **for each i-extension of** S.

Remark 3.12. The function ψ_R not always coincides with the restriction of η_R. Indeed, there may be ideals of S that are not ideals of R. Similarly, the function φ_R not always coincides with the co-restriction of λ_R, because there may be ideals S, J of R such that $JS \neq J \cap S$.

Example 3.13.

1. Let $R = \mathbb{R}[x]$ be the polynomial ring over $\mathbb{R}$ in the indeterminate x and let $S = \langle x \rangle$ be the ideal of R generated by the polynomial x. Consider $p(x) = x^2 + x$ which clearly is an element of S. Call I the ideal of the ring S generated by $p(x)$, namely

$$\begin{aligned} I &= \langle p(x) \rangle_S \\ &= \{p(x)k(x) + zp(x) : k(x) \in S,\ z \in \mathbb{Z}\} \\ &= \{p(x)[k(x) + z] : k(x) \in S,\ z \in \mathbb{Z}\} \\ &= \{p(x)q(x) : q(x) \in R,\ q_0 \in \mathbb{Z}\} \\ &\subseteq \{m(x) : m(x) \in R,\ m_1 \in \mathbb{Z},\ m_0 = 0\}. \end{aligned}$$

   Therefore, we conclude that I is an ideal of S that is not an ideal of R because, $p(x) \in I$ and taking $q(x) = \frac{1}{2} \in R$ we see that $p(x)q(x) \notin I$.

2. Consider $R = 2\mathbb{Z}$ and two of its ideals $S = 6\mathbb{Z}$, $J = 10\mathbb{Z}$ such that $JS = 60\mathbb{Z} \neq 30\mathbb{Z} = J \cap S$.

4. Extension and restriction of ideals

As we saw in the previous section, if R is an i-extension of S, the ideals of S can be extended to ideals of R and the ideals of R can be restricted to ideals of S. In this section we show that the functions φ_R and ψ_R are morphisms of ordered sets that, in general, are not adjoint to each other and we present some of its properties.

The following proposition is evident:

Proposition 4.1. *φ_R and ψ_R are monotone functions.*

Proposition 4.2. *(i) If $\{I_l\}_{l\in L}$ is a collection of ideals of S then $\psi_R\left(\bigcap_{l\in L} I_l\right) = \bigcap_{l\in L}\psi_R(I_l)$.*

(ii) If I, K are ideals of S then $\psi_R(I+K) \supseteq \psi_R(I) + \psi_R(K)$.

(iii) If I is an ideal of S then $r_R(\psi_R(I)) \subseteq \psi_R(r_S(I))$.

Proof. (i)

$$\begin{aligned} a \in \psi_R\left(\bigcap_{l\in L} I_l\right) &\Leftrightarrow aS \subseteq \bigcap_{l\in L} I_l \\ &\Leftrightarrow a \in \psi_R(I_l) \text{ for each } l \in L \\ &\Leftrightarrow a \in \bigcap_{l\in L}\psi_R(I_l). \end{aligned}$$

(ii) $I \subseteq I+K$ and $K \subseteq I+K$ thus, by the monotony of ψ_R we have $\psi_R(I) \subseteq \psi_R(I+K)$ and $\psi_R(K) \subseteq \psi_R(I+K)$. Therefore, $\psi_R(I) + \psi_R(K) \subseteq \psi_R(I+K)$.

(iii) If $a \in r_R(\psi_R(I))$ then $a^k \in \psi_R(I)$, for some $k > 0$. Thus, $a^k S \subseteq I$, for some $k > 0$, namely, $(as)^k = a^k s^k \in I$, for all $s \in S$ and for some $k > 0$. Therefore, $as \in r_S(I)$ for all $s \in S$, namely, $aS \subseteq r_S(I)$ and $a \in \psi_R(r_S(I))$.

□

Proposition 4.3. *Let R_i be an i-extension of S_i and let I_i be an ideal of S_i, for each $i \in L$. If $R = \prod_{i\in L} R_i$ and $S = \prod_{i\in L} S_i$ then R is an i-extension of S and $\psi_R\left(\prod_{i\in L} I_i\right) = \prod_{i\in L}\psi_{R_i}(I_i)$.*

Proof. Since $(r_i) \in \psi_R\left(\prod_{i\in L} I_i\right)$ is equivalent to $(r_i)S \subseteq \prod_{i\in L} I_i$, namely, $r_i S_i \subseteq I_i$ for each $i \in L$, we have $r_i \in \psi_{R_i}(I_i)$ for each i, namely, $(r_i) \in \prod_{i\in L}\psi_{R_i}(I_i)$. □

Example 4.4. Note that in general $\psi_R(IK) \not\supseteq \psi_R(I)\psi_R(K)$. Consider $R = \mathbb{Z}$ and $S = 2\mathbb{Z}$.
$\psi_R(6\mathbb{Z}) = \{z \in \mathbb{Z} : z2\mathbb{Z} \subseteq 6\mathbb{Z}\} = 3\mathbb{Z}$.

$\psi_R(8\mathbb{Z}) = \{z \in \mathbb{Z} : z2\mathbb{Z} \subseteq 8\mathbb{Z}\} = 4\mathbb{Z}$.
$\psi_R(48\mathbb{Z}) = \{z \in \mathbb{Z} : z2\mathbb{Z} \subseteq 48\mathbb{Z}\} = 24\mathbb{Z}$.
$\psi_R(6\mathbb{Z}.8\mathbb{Z}) = \psi_R(48\mathbb{Z}) = 24\mathbb{Z} \subsetneqq 12\mathbb{Z} = 3\mathbb{Z}.4\mathbb{Z} = \psi_R(6\mathbb{Z})\,\psi_R(8\mathbb{Z})$.

Now we study the relationship between these two morphisms of ordered sets.

Proposition 4.5. *Let R be an i-extension of S. $I \subseteq \varphi_R(\psi_R(I))$ for each ideal I of S.*

Proof. Consider $a \in I$. We have that $aS \subseteq I$ and $a \in S$ then, $a \in \psi_R(I)$ and $a \in S$. Therefore, $a \in \psi_R(I) \cap S$, namely, $a \in \varphi_R(\psi_R(I))$. □

The following example shows us that, in general, this inclusion is strict.

Example 4.6. Consider $R = \mathbb{Z}$, $S = 2\mathbb{Z}$ and $I = 4\mathbb{Z}$.
Then $4\mathbb{Z} \subsetneqq \varphi_R(\psi_R(4\mathbb{Z})) = \varphi_R(\{z \in \mathbb{Z} : z2\mathbb{Z} \subseteq 4\mathbb{Z}\}) = \varphi_R(2\mathbb{Z}) = 2\mathbb{Z}$.

Proposition 4.7. *Let R be an i-extension of S. $J \subseteq \psi_R(\varphi_R(J))$ for each ideal J of R.*

Proof. Consider $b \in J$. We have that $bS \subseteq S$ and $bS \subseteq J$ thus, $bS \subseteq J \cap S = \varphi_R(J)$. Therefore, $b \in \psi_R(\varphi_R(J))$. □

In general, this inclusion also is strict, as is shown below.

Example 4.8. Consider $R = \mathbb{Z}$, $S = 4\mathbb{Z}$ and $J = 2\mathbb{Z}$. We see that $2\mathbb{Z} \subsetneqq \psi_R(\varphi_R(2\mathbb{Z})) = \psi_R(2\mathbb{Z} \cap 4\mathbb{Z}) = \psi_R(4\mathbb{Z}) = \{z \in \mathbb{Z} : z4\mathbb{Z} \subseteq 4\mathbb{Z}\} = \mathbb{Z}$.

The situation presented in this example is a particular case of the following proposition.

Proposition 4.9. *Let R be an i-extension of S. If J is a proper ideal of R containing S then $J \neq \psi_R(\varphi_R(J))$.*

Proof. It is enough to see that $\psi_R(\varphi_R(J)) = R$. $a \in \psi_R(\varphi_R(J))$ if and only if $aS \subseteq \varphi_R(J) = J \cap S$, namely, $aS \subseteq S$. Equivalently, $a \in R$. □

From the results above we see that in general, the functions φ_R and ψ_R are not adjoint to each other. The following theorem establishes a necessary and sufficient condition on the ring S in order to obtain an adjunction between these functions.

Theorem 4.10. *The following statements are equivalent:*

(i) The ring S is pseudo-regular.

(ii) For each i-extension R of S, $\varphi_R(\psi_R(I)) = I$ for all ideal I of S.

(iii) For each i-extension R of S, φ_R is left adjoint of ψ_R.

Proof. (i)$\Rightarrow$ (ii). By Proposition 4.5 it is enough to prove that $\varphi_R(\psi_R(I)) \subseteq I$. Take $b \in \varphi_R(\psi_R(I))$, namely, $b \in \psi_R(I) \cap S$. Thus, $bS \subseteq I$ and $b \in S$, then $b \in bS \subseteq I$. Therefore, $b \in I$.
(ii)$\Rightarrow$ (i). Let us suppose that S is not pseudo-regular, namely, there exists $a \in S$ such that $a \notin aS$. Clearly aS is an ideal of S. As $aS \subseteq aS$ and $a \in S$, then $a \in \psi_R(aS) \cap S$, so $a \in \varphi_R(\psi_R(aS))$. Hence, $\varphi_R(\psi_R(aS)) \neq aS$.
(ii)$\Rightarrow$ (iii). As φ_R and ψ_R are morphisms of ordered sets such that $\varphi_R(\psi_R(I)) = I$, for all ideal I of S and $J \subseteq \psi(\varphi(J))$, for all ideal J of R then, φ_R is left adjoint of ψ_R.
(iii)$\Rightarrow$ (ii). As φ_R is left adjoint of ψ_R then $\varphi_R(\psi_R(I)) \subseteq I$, for all ideal I of S. By Proposition 4.5, $\varphi_R(\psi_R(I)) \supseteq I$, for all ideal I of S. Then, $\varphi_R(\psi_R(I)) = I$, for all ideal I of S. □

Note that we can consider the ring S as an i-extension of itself, which allows us to establish an additional characterization of pseudo-regular rings.

Theorem 4.11. *The following statements are equivalent:*

(i) S *is pseudo-regular.*

(ii) $\psi_S(I) = I$, *for each ideal* I *of* S.

Proof. (i)$\Rightarrow$ (ii). By Theorem 4.10, if I is an ideal of S then $I = \varphi_S(\psi_S(I)) = \psi_S(I) \cap S = \psi_S(I)$.
(ii)$\Rightarrow$(i). Consider $a \in S$. As aS is an ideal of S, then $\psi_S(aS) = aS$. On the other hand, it is clear that $a \in \psi_S(aS) = \{x \in S : xS \subseteq aS\}$, so $a \in aS$ and S is pseudo-regular. □

Example 4.12. Let us consider $S = 4\mathbb{Z}$.
Note that $\psi_S(24\mathbb{Z}) = \{a \in 4\mathbb{Z} : a4\mathbb{Z} \subseteq 24\mathbb{Z}\} = 12\mathbb{Z}$ and then $4\mathbb{Z}$ is not pseudo-regular.

The Cohen-Seidenberg theorems (the “going-up” and “going-down” theorems) about prime ideals in integral extensions are proved in [2]. We can note that in the context of i-extensions similar properties are satisfied. For i-extensions of pseudo-regular rings we have a similar version of the “going-up” property, that in this case can be extended to the complete collection of ideals.

Proposition 4.13. *Let* S *be a pseudo regular ring and let* R *be an i-extension of* S. *If* I_1, I_2 *are ideals of* S *such that* $I_1 \subseteq I_2$ *and* J_1 *is an ideal of* R *such that* $\varphi_R(J_1) = I_1$, *then there exists* J_2, *ideal of* R, *such that* $J_1 \subseteq J_2$ *and* $\varphi_R(J_2) = I_2$.

Proof. As $I_1 \subseteq I_2$ then, by Proposition 4.1, $\psi_R(I_1) \subseteq \psi_R(I_2)$. By Proposition 4.7, $J_1 \subseteq \psi_R(\varphi_R(J_1)) = \psi_R(I_1) \subseteq \psi_R(I_2)$. $\psi_R(I_2)$ is an ideal of R and as S is pseudo-regular then $\varphi_R(\psi_R(I_2)) = I_2$; it is enough to take $J_2 = \psi_R(I_2)$. □

5. Some adjunctions associated with $S-$extensions and $S-$restrictions of ideals.

In this section we study the behavior of the extension and restriction morphisms, no longer imposing conditions on the ring S, but on its ideals. Hereinafter R is a fixed i-extension of S and functions ψ_R and φ_R will be denoted just by ψ and φ, respectively.

5.1. $S-$extension and $S-$restriction of prime ideals

Proposition 5.1. *If J is a prime ideal of R that does not contain S, then*

(i) $\varphi(J)$ *is a prime ideal of* S.

(ii) $\psi(\varphi(J)) = J$.

Proof. As J does not contain S then $\varphi(J)$ is a proper ideal of S.

(i) Consider $a, b \in S$. If $ab \in \varphi(J)$ then $ab \in J$, but as J is a prime ideal of R, it is concluded that $a \in J$ or $b \in J$ and hence, $a \in \varphi(J)$ or $b \in \varphi(J)$.

(ii) By Proposition 4.7, it is enough to prove that $\psi(\varphi(J)) \subseteq J$. Let us consider $y \in S - \varphi(J)$, then $y \notin J$. If $a \in \psi(\varphi(J))$ then $aS \subseteq \varphi(J)$. In particular, $ay \in \varphi(J)$, then $ay \in J$. Hence, $a \in J$.

□

This proposition shows us that if we restrict φ to the set of prime ideals of R which do not contain S, its image is a subset of the set of the prime ideals of S.

Proposition 5.2. *If I is a prime ideal of S, then*

(i) $\psi(I)$ *is a prime ideal of* R *which does not contain* S.

(ii) $\varphi(\psi(I)) = I$.

Proof. (i) Consider $c \in S - I$, then $cS \nsubseteq I$, thus $c \notin \psi(I)$ and $\psi(I)$ is a proper subset of R, moreover, does not contain S.

Let $a, b \in R$, such that $ab \in \psi(I)$. Let us suppose that $a \notin \psi(I)$, then there exists $y \in S$ such that $ay \notin I$.

As $(ay)b = (ab)y \in abS \subseteq I$, then $b \in I$ and hence, $bS \subseteq I$. Therefore, $b \in \psi(I)$.

(ii) By Proposition 4.5, it is enough to see that $\varphi(\psi(I)) \subseteq I$. Take $y \in S - I$ and $a \in \varphi(\psi(I))$. Thus $a \in \psi(I)$, namely, $aS \subseteq I$. In particular, $ay \in I$ which implies that $a \in I$.

□

As a consequence of the above results we can establish the following theorem, where $\Pr(S)$ is the set of prime ideals of S[2] and $\Pr_S(R)$ is the set of prime ideals of R which do not contain S.

Theorem 5.3. *The function* $\varphi : \Pr_S(R) \to \Pr(S)$ *is an isomorphism of ordered sets with inverse* ψ.

A study about the isomorphism presented in the previous theorem can be found in [1].

Example 5.4. Let A be a commutative ring. We call $U(A)$ the set $A \times \mathbb{Z}$ endowed with the operations:

$$\begin{aligned} (a,\alpha) + (b,\beta) &= (a+b, \alpha+\beta) \text{ and} \\ (a,\alpha)(b,\beta) &= (ab + \beta a + \alpha b, \alpha\beta). \end{aligned}$$

$U(A)$ is a commutative ring of characteristic 0, with identity $(0,1)$ and which naturally contains the ring A, when we identify it with $A_0 = A \times \{0\}$, through the homomorphism $i_A : A \to U_0(A) : i_A(a) = (a, 0)$. This is the process used in standard way to adjoint identity to the ring A. It is easily verified that A_0 is an ideal of $U(A)$, so that $U(A)$ is an i-extension of A_0.

Note that I is an ideal (prime ideal) of A if and only if $I \times \{0\}$ is an ideal (prime ideal) of A_0. So, $\mathcal{J}(A) \approx \mathcal{J}(A_0)$ and $\Pr(A) \approx \Pr(A_0)$. On the other hand, if J is an ideal of $U_0(A)$, in order to $\varphi(J)$ be a proper ideal of A it is necessary that J does not contain A_0. In $U_0(A)$ there are ideals that contain A_0, and others that do not contain it. For example, if I is a proper ideal of A then $I \times \{0\}$ is an ideal of $U_0(A)$ that does not contain A_0. Now, if p is a prime number, then $A \times \langle p \rangle$ is a prime ideal of $U_0(A)$ that clearly contains A_0. In fact, $A \times \langle p \rangle$ is the kernel of the surjective homomorphism $\varphi_p : U_0(A) \to \mathbb{Z}_p : \varphi_p(a,\alpha) = \overline{\alpha}$, where $\overline{\alpha}$ represents the equivalence class of α modulo p.

The results of this section allow us to affirm that the sets $\Pr_{A_0}(U(A))$ and $\Pr(A)$, ordered by inclusion, are isomorphic.

We note that in the context of prime ideals and i-extensions, we have analogous properties to "going-up" and "going-down". We mention the corresponding version to "going-up". For "going down" it is enough to invert the inclusions in the statement of the following proposition.

Proposition 5.5. ***(Going-up property.)*** *If* I_1, I_2 *are prime ideals of* S *such that* $I_1 \subseteq I_2$ *and* J_1 *is a prime ideal of* R *such that* $\varphi(J_1) = I_1$, *then there exists* J_2, *prime ideal of* R, *such that* $J_1 \subseteq J_2$ *and* $\varphi(J_2) = I_2$.

Proof. As $I_1 \subseteq I_2$ then, by Proposition 4.1, $\psi(I_1) \subseteq \psi(I_2)$. Thus, by the mentioned isomorphism in the previous theorem, $J_1 = \psi(\varphi(J_1)) = \psi(I_1) \subseteq \psi(I_2)$. It is enough to consider that $J_2 = \psi(I_2)$, because $\psi(I_2)$ is a prime ideal of R and $\varphi(J_2) = \varphi(\psi(I_2)) = I_2$. □

In the remainder of this section we intend to make other restrictions on the considered collections of ideals, to achieve an adjunction between the morphisms ψ and φ.

[2] We do not use the notation $Spec(S)$, because here we are not considering the topology.

5.2. $S-$extension and $S-$restriction of semi-prime ideals

The notion of *semi-prime ideal* is found in the literature. The interested reader may consult for example [6], to extend the information about these ideals.

Definition 5.6. An ideal I of the ring S is called **semi-prime** if each element of S with some power on I, is also an element of I.

From the previous definition we observe that I is a semi-prime ideal of S if and only if $I = r(I)$, where $r(I) = \{x \in S : x^n \in I, \text{ for some } n \in \mathbb{Z}^+\}$, the radical of I. Clearly the prime ideals of S are semi-prime, but the converse is not true. One can easily see that $\langle 6 \rangle$ is an semi-prime ideal of $\mathbb{Z}_{12}$ that is not prime, as well as 0 is a semi-prime ideal of $\mathbb{Z}_6$ that is not a prime ideal.

Proposition 5.7. *I is a semi-prime ideal of the ring S if and only if $a^2 \in I$ implies $a \in I$.*

Proof. It is evident that if I is a semi-prime ideal of S then $a^2 \in I$ implies that $a \in I$. Now, let us suppose that $a^2 \in I$ implies $a \in I$. Let $n \geq 2$ such that $a^n \in I$. Let us choose a positive integer m such that $2^m > n$. We have that $a^{2^m} = a^{2^m - n} a^n \in I$ and as $a^{2^m} = \left(a^{2^{m-1}}\right)^2$, then $a^{2^{m-1}} \in I$. Similarly we obtain $a^{2^{m-2}} \in I, ..., a^2 \in I$, hence $a \in I$. □

As a consequence of the previous proposition we easily observe that the intersection is a closed operation in the collection of semi-prime ideals of S.

Proposition 5.8. *For each semi-prime ideal I of S, $\varphi(\psi(I)) = I$.*

Proof. It is enough to see that $\varphi(\psi(I)) \subseteq I$. Consider $a \in \varphi(\psi(I))$, namely, $a \in \psi(I) \cap S$, then $a \in S$ and $aS \subseteq I$. Thus, $a^2 \in I$ then $a \in r(I)$ and as I is semi-prime, we conclude that $a \in I$. □

Proposition 5.9. *If I is a semi-prime ideal of S, then $\psi(I)$ is a semi-prime ideal of R.*

Proof. Let $a \in r(\psi(I))$ namely, there exists $n \in \mathbb{Z}^+$ such that $a^n \in \psi(I)$. As $a^n S \subseteq I$ and for each $s \in S$ we have that $s^n \in S$, then $a^n s^n = (as)^n \in I$, for each $s \in S$. Hence, $as \in r(I) = I$ for each $s \in S$, thus $aS \subseteq I$ and $a \in \psi(I)$. □

Proposition 5.10. *If J is a semi-prime ideal of R then $\varphi(J)$ is a semi-prime ideal of S.*

Proof. Consider $a \in r(\varphi(J))$, namely, $a \in S$ and there exists $n \in \mathbb{Z}^+$ such that $a^n \in J \cap S$. Thus, $a \in S$ and $a \in r(J) = J$, then $a \in J \cap S = \varphi(J)$. □

We denote by $Smpr(A)$ the set of semi-prime ideals of A. From the above propositions we obtain the following result.

Proposition 5.11. *The function $\varphi : Smpr(R) \rightarrow Smpr(S)$ is left adjoint of $\psi : Smpr(S) \rightarrow Smpr(R)$.*

Remark 5.12. $\varphi : \operatorname{Im}\psi \rightarrow \operatorname{Im}\varphi$ is an isomorphism of ordered sets, where $\operatorname{Im}\varphi = Smpr(S)$ and $J \in \operatorname{Im}\psi$ if and only if it is satisfied that $a \in R$ and $aS \subseteq J$ imply $a \in J$.

We also obtain a version of the property "going-up" restricting ourselves now to semi-prime ideals. This property is mentioned in the following proposition and its proof is similar to the proof of Proposition 4.13.

Proposition 5.13. *Let S be a ring and let R be an i-extension of S. If I_1, I_2 are semi-prime ideals of S such that $I_1 \subseteq I_2$ and J_1 is a semi-prime ideal of R such that $\varphi(J_1) = I_1$, then there exists J_2, semi-prime ideal of R, such that $J_1 \subseteq J_2$ and $\varphi(J_2) = I_2$.*

5.3. $S-$extension and $S-$restriction of pseudo-prime ideals.

The above work suggests the notion of pseudo-prime ideals.

Definition 5.14. An ideal I of the ring S is **pseudo-prime** if I is proper and for each $a \in S$, $aS \subseteq I$ implies $a \in I$.

It is evident that if a ring has identity, then all its proper ideals are pseudo-prime. We denote by $Spr(S)$ the collection of pseudo-prime ideals of S.

Proposition 5.15. *If I is a proper and semi-prime ideal of S, then I is pseudo-prime.*

Proof. Consider $a \in S$, such that $aS \subseteq I$. If we suppose that $a \in S - I$, then $a^2 \in I$, but it contradicts that I is semi-prime. Hence, $a \in I$. □

In a Boolean ring each proper ideal is semi-prime, then by the previous proposition, it is also a pseudo-prime ideal. The following example shows us that the converse of previous proposition is false.

Example 5.16. Consider the ring $\mathbb{R}[x]$. As this ring is unitary, all its proper ideals are pseudo-prime. In particular, $\langle x^2 \rangle$ is pseudo-prime but it is not semi-prime. It is enough to see that $x \notin \langle x^2 \rangle$, but $x^2 \in \langle x^2 \rangle$.

From the previous result we have that each prime ideal of S is pseudo-prime. However, not every pseudo-prime ideal of S is prime, for example 0 is a pseudo-prime ideal of $\mathbb{Z}_6$, that is not prime. On the other hand, $8\mathbb{Z}$ is an ideal of $2\mathbb{Z}$ that is not pseudo-prime because $4 \in 2\mathbb{Z} - 8\mathbb{Z}$ and $4(2\mathbb{Z}) \subseteq 8\mathbb{Z}$.

The following propositions characterize the pseudo-prime and semi-prime ideals of $2\mathbb{Z}$.

Proposition 5.17. *I is a pseudo-prime ideal of $2\mathbb{Z}$ if and only if $I = 2k\mathbb{Z}$, where $k = 0$, or k is odd, $k > 1$.*

Proof. $\Leftarrow$) Clearly 0 is a pseudo-prime ideal of $2\mathbb{Z}$. Let k be an odd integer greater than 1. $2k\mathbb{Z}$ is a proper ideal of $2\mathbb{Z}$. Take $a \in 2\mathbb{Z}$ such that $a2\mathbb{Z} \subseteq 2k\mathbb{Z}$, then should be that a is a multiple of 2 and of the odd number k, so $a \in 2k\mathbb{Z}$ and hence $2k\mathbb{Z}$ is a pseudo-prime ideal of $2\mathbb{Z}$.
$\Rightarrow$) It is enough to see that $I = 2k\mathbb{Z}$, where k is an even integer different from zero, is not a pseudo-prime ideal of $2\mathbb{Z}$. If we take $a \in k\mathbb{Z} - 2k\mathbb{Z}$ then $a \in 2\mathbb{Z}$, $a2\mathbb{Z} \subseteq 2k\mathbb{Z}$, but $a \notin 2k\mathbb{Z}$; hence I is not a pseudo-prime ideal of $2\mathbb{Z}$. □

Proposition 5.18. *If I is a pseudo-prime ideal of $2\mathbb{Z}$, then I is a semi-prime ideal of $2\mathbb{Z}$.*

Proof. Clearly 0 is a semi-prime ideal of $2\mathbb{Z}$. Suppose that $I = 2k\mathbb{Z}$, where k is odd, $k > 1$.

Consider $a \in 2\mathbb{Z} - I$, namely, $a = 2t + i$, where $i \in 2k\mathbb{Z}$ and $1 \leq t \leq k - 1$. Hence, $a^2 = 4t^2 + 4ti + i^2$, where k divides $4ti + i^2$ but does not to $4t^2$, then $2k$ does not divide a^2. Thus, $a^2 \in 2\mathbb{Z} - I$ and I is a semi-prime ideal of $2\mathbb{Z}$. □

Corollary 5.19. *The proper pseudo-prime and semi-prime ideals of $2\mathbb{Z}$ coincide.*

The collection of pseudo-prime ideals is closed for intersections, as shown in the following proposition.

Proposition 5.20. *If I_t is a pseudo-prime ideal of S for each $t \in T$, then $\bigcap_{t\in T} I_t$ is a pseudo-prime ideal of S.*

Proof. Clearly $\bigcap_{t\in T} I_t$ is a proper ideal of S. Consider $a \in S$ such that $aS \subseteq \bigcap_{t\in T} I_t$, then $aS \subseteq I_t$ for each $t \in T$, thus $a \in I_t$ for each $t \in T$, namely, $a \in \bigcap_{t\in T} I_t$. □

From the definition of pseudo-prime ideal and Proposition 4.5 we obtain the following result.

Proposition 5.21. $I = \varphi(\psi(I))$, *for each* $I \in Spr(S)$.

Moreover, the collection of pseudo-prime ideals of S is the largest subcollection of ideals of S for which we have the result of the previous proposition.

Proposition 5.22. *If I is a pseudo-prime ideal of S then $\psi(I)$ is a pseudo-prime ideal of R.*

Proof. As $I = \varphi(\psi(I))$, then $\psi(I)$ does not contain S and it is a proper ideal of R. Consider $a \in R$ such that $aR \subseteq \psi(I)$, namely, $aRS \subseteq I$. As S is an ideal of R then $aS \subseteq I$, thus $a \in \psi(I)$. □

We can note from the proof of the previous proposition that, if $I \in Spr(S)$ then $\psi(I)$ is a pseudo-prime ideal of R which does not contain S. The following example shows that if J is a pseudo-prime ideal of R which does not contain S, then $\varphi(J)$ is not necessarily a pseudo-prime ideal of S.

Example 5.23. Consider $R = \mathbb{Z}$ and $S = 2\mathbb{Z}$. $J = 4\mathbb{Z}$ is a pseudo-prime ideal of $\mathbb{Z}$ which does not contain $2\mathbb{Z}$, but $\varphi(4\mathbb{Z}) = 4\mathbb{Z} \cap 2\mathbb{Z} = 4\mathbb{Z}$ is not a pseudo-prime ideal of $2\mathbb{Z}$ because 2 is an element of $2\mathbb{Z}$ such that $2.2\mathbb{Z} \subseteq 4\mathbb{Z}$ and $2 \notin 4\mathbb{Z}$.

The previous example shows that, in general, φ is not properly restricted to the collection of pseudo-prime ideals of R which do not contain S. In order to find a pseudo-prime ideal J of R which does not contain S such that $\varphi(J)$ is a pseudo-prime ideal of S, it is required that for each $a \in S$ such that $aS \subseteq J$ then $a \in J$.

Definition 5.24. Let R be an i-extension of S. We say that an ideal J of R is **pseudo-prime with respect to** S if for each $a \in S$ such that $aS \subseteq J$ then $a \in J$.

We denote $Spr_S(R)$ the collection of ideals of R that are pseudo-prime with respect to S. Thus $\psi(Spr(S)) \subseteq Spr_S(R)$ and $Spr_S(R)$ is a subset, generally proper, of the collection of pseudo-prime ideals of R which do not contain S. Hence, from Proposition 5.21 and as $J \subseteq \psi(\varphi(J))$ for each ideal J of R, we obtain the following corollary.

Corollary 5.25. *The function* $\varphi : Spr_S(R) \to Spr(S)$ *is left adjoint of the function* $\psi : Spr(S) \to Spr_S(R)$.

Finally, we can mention the following result.

Corollary 5.26. *Every proper ideal of a pseudo-regular ring is pseudo-prime.*

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