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Compensated compactness for a hyperbolic system of balance laws with three equations

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We consider the problem studied by Lu *et. al.* in [Proc. Am. Math. Soc. **131**, 3511 (2003)] to which we add a source term. This is a system of three balance laws. We use standard techniques, such as viscous approximations and compensated compactness, to prove the existence of solutions.

Keywords: compensated compactness, conservations laws.

Se considera el problema estudiado por Lu *et. al.* en [Proc. Am. Math. Soc. **131**, 3511 (2003)] al cual añadimos un término fuente. Este es un sistema de tres leyes de balance. Se usan técnicas standrad, tal como aproximaciones viscosas y compacidad compensada, para demostrar la existencia de soluciones.

Palabras claves: compacidad compensada, leyes de conservación.

MSC: 35L65

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1 The system of balance laws

In this paper we study the following system of balance laws

$$\begin{aligned} v_t - u_x &= \epsilon\, v_{xx} + g(v,\, u,\, s)\,, \\ u_t - \sigma(v,\, s)_x &= \epsilon\, u_{xx} + f(v,\, u)\,, \\ s_t - c_1\, s_x + \frac{s - c\,v}{\tau} &= \epsilon\, s_{xx}\,, \end{aligned} \tag{1.1}$$

with initial condition

$$(v(x,\,0),\, u(x,\,0),\, s(x,\,0)) = (v_0(x),\, u_0(x),\, s_0(x))\,. \tag{1.2}$$

The Cauchy problem (1.1)–(1.2) was considered by Lu and Klingenberg in [1] without the source terms $g(v, u, s)$ and $f(v, u)$.

The third equation in (1.1) contains a relaxation mechanism with $c\,v$ as the equilibrium value for s. Here, τ is the relaxation time and ϵ is the viscous parameter. The relaxation and dissipation limit of (v, u, s) in (1.1) satisfies $s = c\,v$, where the pair (v, u) is an entropy solution of the equilibrium system

$$\begin{aligned} v_t - u_x &= g(v,\, u,\, c\,v)\,, \\ u_t - \sigma(v,\, c\,v)_x &= f(v,\, u)\,, \end{aligned} \tag{1.3}$$

when $\epsilon = 0$ and $\tau \to 0$.

2 Viscous solution and convergence

In this section we obtain the existence of solutions. We have the following result concerning viscous solutions and their viscous limits.

Theorem. *Part I. If*

C_1**.**

$$\begin{aligned} \|v_0\|_{L^2(\mathbb{R})\cap L^\infty(\mathbb{R})},\, \|u_0\|_{L^2(\mathbb{R})\cap L^\infty(\mathbb{R})},\, \|s_0\|_{L^2(\mathbb{R})\cap L^\infty(\mathbb{R})} &\leq M_1\,, \\ \lim_{|x|\to\infty} \left(\frac{d^i v_0}{dx^{\,i}},\, \frac{d^i u_0}{dx^{\,i}},\, \frac{d^i s_0}{dx^{\,i}} \right) &= (0,\, 0,\, 0)\,. \end{aligned}$$

for $i = 0, 1$*;*

C_2**.**

$$\begin{aligned} |\sigma_s(v,\, s)| &\leq M_2\,, \\ \overline{\sigma}' &\geq d > \max{}^{\text{`}}\left(0,\, c^2 - c + \frac{2c^2 c_1^2}{(M_2+1)^2}\right)\,, \end{aligned}$$

where $\overline{\sigma}(v) = \sigma(v, c\,v)$*.*

C_3.

$$\int_{-\infty}^{\infty}\int_0^{v_0(x)} \overline{\sigma}(r)\,dr\,dx \leq M_3\,.$$

C_4**.**

$$g(v,\, u,\, s) = -(\overline{\sigma}\,'(v))^{-1/2}\,\frac{(\overline{\sigma}(v) + c\,v - c\,s)}{1 + (\overline{\sigma}(v) + c\,v - c\,s)^2}\,\tilde{g}(v,\, u,\, s)\,,$$

and $f(v,u) = -\frac{u}{1+u^2}\tilde{f}(v,u)$*, with* $\tilde{g}(v,u,s) \geq 0$*,* $\tilde{f}(v,u) \geq 0 \in C^1(\mathbb{R}^3)$*, and these functions are* $o(|(v,u,s)|^2)$ *when* $|(v,u,s)| \to \infty$*,* $0 \leq \tilde{g}(v,u,s), 0 \leq \tilde{f}(v,u)$ *and their partial derivatives go to zero at infinity.*

Then, for fixed ε *and* τ*, the solutions* (v,u,s) *of the Cauchy problem (1.1)–(1.2) belong to* $C^2(\mathbb{R}\times[0,T))$*; that is, they exist in* $(-\infty,\infty)\times[0,T]$ *for any given* $T > 0$ *and satisfy*

$$\begin{aligned} |v(x,\, t)|,\, |u(x,\, t)|,\, |s(x,\, t)| &\leq M(\varepsilon,\, \tau,\, T)\,, && (2.1) \\ \|v^2(\cdot,\, t)\|_{L^1(\mathbb{R})},\, \|u^2(\cdot,\, t)\|_{L^1(\mathbb{R})}, & && \\ \|s^2(\cdot,\, t)\|_{L^1(\mathbb{R})} &\leq M\,, && (2.2) \\ \left\|\frac{(s-cv)^2}{\tau^{1/2}}\right\|_{L^2(\mathbb{R}\times\mathbb{R}^+)} &\leq M\,, && (2.3) \\ \|(\varepsilon)^{1/2}\,v_x\|_{L^2(\mathbb{R}\times\mathbb{R}^+)},\, \|(\varepsilon)^{1/2}\,u_x\|_{L^2(\mathbb{R}\times\mathbb{R}^+)}, & && \\ \|(\varepsilon)^{1/2}s_x\|_{L^2(\mathbb{R}\times\mathbb{R}^+)} &\leq M\,. && (2.4) \end{aligned}$$

Part II. If

C_5.
- $\sigma(v,\, s) = h(v) - c\, s$;
- $h(v) \in C^3(\mathbb{R})$, $h(0) = 0$, $h' \geq d > 0$, *where* d *is a constant;*
- $h'' \neq 0$, $h'' \in L^1 \cap L^\infty(\mathbb{R})$;
- $h''' \in L^\infty(\mathbb{R})$, $||h'''||_{L^1} \leq M$.

Then, there exist a subsequence $(v^{\varepsilon,\tau}(x,t), u^{\varepsilon,\tau}(x,t), s^{\varepsilon,\tau}(x,t))$ *of solutions of the Cauchy problem (1.1)–(1.2) and there exist* L^2*–bounded functions* $(\overline{v}, \overline{u}, \overline{s})$ *such that*

$$(v^{\varepsilon,\tau}(x,\, t),\, u^{\varepsilon,\tau}(x,\, t),\, s^{\varepsilon,\tau}(x,\, t)) \to (\overline{v},\, \overline{u},\, \overline{s}),$$

a.e. (x,t), *as* $(\varepsilon,\tau) \to (0,0)$, *subject to the condition* $\tau(M_2+1)^2 \leq \varepsilon$ *and* $(\overline{v}, \overline{u})$ *is an entropy solution of the equilibrium system (1.3) with the initial data* (u_0, v_0).

Proof. A unique smooth local solution for the Cauchy problem (1.1)–(1.2), for any fixed ϵ and $\tau > 0$, is obtained with the help of an equivalent integral operator and the Banach fixed point theorem; see [7] and [12]. Furthermore, the solution satisfies

$$\left|\frac{\partial^i v}{\partial x^i}\right| + \left|\frac{\partial^i u}{\partial x^i}\right| + \left|\frac{\partial^i s}{\partial x^i}\right| \leq M(t_1,\, \epsilon,\, \tau) < +\infty\,, \tag{2.5}$$

with $i = 0, 1, 2$, where $M(t_1, \epsilon, \tau)$ is a positive constant that depends only on t_1, ϵ and τ, and t_1 depends on $|v_0|_{L^\infty}$, $|u_0|_{L^\infty}$ and $|s_0|_{L^\infty}$. Moreover

$$\lim_{|x|\to\infty} \left(\frac{\partial^i v}{\partial x^i}, \frac{\partial^i u}{\partial x^i}, \frac{\partial^i s}{\partial x^i}\right) = (0,\, 0,\, 0)\,, \tag{2.6}$$

for $i = 0, 1$, uniformly in $t \in [0, t_1]$.

To obtain the estimates in (2.1) we multiply the first equation in (1.1) by $\overline{\sigma}(v) + c\,v - c\,s$, the second equations by u, the third equation by $s - c\,v$ and add the results; this is the same procedure used by Lu and Klingenber in [1]. Then we get

$$
\begin{aligned}
& \left(\int_0^v (\overline{\sigma}(r)+c\,r)\,dr+\frac{u^2}{2}-c\,s\,v+\frac{s^2}{2}\right)_t \\
& +\left(c\,u\,s-u\,(\overline{\sigma}(v)+c\,v)+\frac{c_1 s^2}{2}\right)_x \\
& -c\,c_1\,v\,s_x-u\,\left(\sigma(v,\,s)+s-(\sigma(v,\,c\,v)+c\,v)\right)_x+\frac{(s-c\,v)^2}{\tau} \\
= \;& \varepsilon\left(\int_0^v (\overline{\sigma}(r)+c\,r)\,dr+\frac{u^2}{2}-c\,s\,v+\frac{s^2}{2}\right)_{xx} \\
& -\varepsilon\,\left(\overline{\sigma}'(v)+c\right)\,v_x^2-\varepsilon\,u_x^2-\varepsilon\,s_x^2+2\,\varepsilon\,c\,s_x\,v_x \\
& +\overline{\sigma}(v)\,g(v,\,u)+c\,v\,g(v,\,u)-c\,s\,g(v,\,u)+u\,f(v,\,u)\,. \qquad (2.7)
\end{aligned}
$$

By condition C_4, $\overline{\sigma}(v)\,g(v,u)+c\,v\,g(v,u)-c\,s\,g(v,u)+u\,f(v,u)\leq 0$, and therefore the inequality (2.7) becomes

$$
\begin{aligned}
& \int_{-\infty}^{\infty}\frac{1}{2}\,(c\,v-s)^2\,dx+\int_{-\infty}^{\infty}\frac{u^2}{2}\,dx+\int_0^T\int_{-\infty}^{\infty}\frac{\varepsilon}{2}\,(u_x)^2\,dx\,dt \\
& +\int_0^T\int_{-\infty}^{\infty}\varepsilon\,(c\,v_x-s_x)^2\,dx\,dt+\int_0^T\int_{-\infty}^{\infty}\frac{(s-c\,v)^2}{4\tau}\,dx\,dt \;\leq\; M\,. \qquad (2.8)
\end{aligned}
$$

This proves the estimates (2.2), (2.3) and (2.4). Again, as in [1], differentiating the first equation in (1.1) with respect to x, multiplying by v_x and integrating the result on $\mathbb{R}\times[0,T]$, we obtain

$$
\begin{aligned}
& \int_{-\infty}^{\infty}\left(\frac{v_x(x,T)}{2}\right)^2 dx\,dt \\
\leq \;& \int_{-\infty}^{\infty}\left(\frac{v_x(x,0)}{2}\right)^2 dx\,dt+\int_0^T\int_{-\infty}^{\infty}\frac{(u_x)^2}{\epsilon}\,dx\,dt \\
& +\int_0^T\int_{-\infty}^{\infty}\left(\frac{\partial g(v,u,s)}{\partial v}\,(v_x)^2+\frac{\partial g(v,u,s)}{\partial u}\,u_x\,v_x\right. \\
& \qquad \left.+\frac{\partial g(v,u,s)}{\partial s}\,s_x\,v_x\right)\,dx\,dt \\
\leq \;& M(\epsilon)\,,
\end{aligned}
$$

since $\frac{\partial g(v,u,s)}{\partial v}$, $\frac{\partial g(v,u,s)}{\partial u}$, $\frac{\partial g(v,u,s)}{\partial s}$ $\in C(\mathbb{R}^3)$ and they go to zero at infinity. Then, by (2.4) and the Hölder inequality, we get the above bound. Therefore,

$$\begin{aligned} v^2 &= \left| \int_{-\infty}^{x} 2\, v\, v_x\, dx \right| \leq \int_{-\infty}^{x} 2\, |v|\, |v_x|\, dx \\ &\leq \int_{-\infty}^{\infty} v^2\, dx + \int_{-\infty}^{\infty} (v_x)^2\, dx \\ &\leq M(\epsilon)\,. \end{aligned}$$

Similarly for u and s. We get the estimates in (2.1) and the proof of part 1 in Theorem 2 is complete.

If (η, q) is the entropy–entropy flux pair associated with (1.3) constructed in [13], then this pair satisfies the following estimates:

(I) $\eta = a^{-1/2} O(1)$, $q = a^{1/2} O(1)$;

(II) $\eta_u = a^{-1/2} O(1)$, $\eta_v = a^{1/2} O(1)$;

(III) $\eta_{uu} = a^{-1/2} O(1)$, $\eta_{uv} = a^{1/2} O(1)$, $\eta_{vv} = a^{3/2} O(1)$;

where $a = \overline{\sigma}'(v)$ and $O(1)$ denotes a L^∞ function. From (1.1) we get

$$\begin{aligned} \eta_t + q_x &= \varepsilon\,((\eta_v\, v_x + \eta_u\, u_x)_x + (\eta_u\, \sigma_s\,(v,\, \alpha(v,\, s))\,(s - c\,v))_x \\ &\quad -\varepsilon\,(\eta_{vv}\,(v_x)^2 + 2\,\eta_{uv}\, u_x\, v_x + \eta_{uu}\,(u_x)^2) \\ &\quad -(\eta_{uv}\, v_x + \eta_{uu}\, u_x)\,(\sigma_s\,(v, \alpha(v,\, s))\,(s - c\,v) \\ &\quad +\eta_v\, g(v,\, u,\, s) + \eta_u\, f(v,\, u) \\ &= I_1 + I_2\,, \end{aligned} \tag{2.9}$$

with $\alpha(v, s)$ between s and cv, and

$$\begin{aligned} I_1 &= (\eta_v\, v_x + \eta_u\, u_x)_x + (\eta_u\, \sigma_s\,(v,\, \alpha(v,\, s))\,(s - c\,v))_x\,, \\ I_2 &= -(\eta_{uv}\, v_x + \eta_{uu}\, u_x)\,(\sigma_s\,(v,\, \alpha(v,\, s))\,(s - c\,v) \\ &\quad -\epsilon\,(\eta_{vv}\,(v_x)^2 + 2\,\eta_{uv}\, u_x\, v_x + \eta_{uu}\,(u_x)^2) \\ &\quad +\eta_v\, g(v,\, u,\, s) + \eta_u\, f(v,\, u)\,. \end{aligned}$$

From the estimates (2.2), (2.3), (2.4) and the previous equality, we get:

1. $\eta(v^{\varepsilon,\tau}, u^{\varepsilon,\tau}) + q(v^{\varepsilon,\tau}, u^{\varepsilon,\tau})$ is bounded in $W^{-1,p}_{Loc}(\mathbb{R} \times \mathbb{R}^+)$ for some $p > 2$ by using estimation (I).

2. Since $\sigma_s(v, s)| \leq M$ due to (II) and (2.3), (2.4), then $(\eta_v v_x + \eta_u u_x)_x + (\eta_u \sigma_s(v, \alpha(v, s))(s - c\,v))_x$ is compact in $H^{-1}_{Loc}(\mathbb{R} \times \mathbb{R}^+)$.

3. From (II), (III), (2.3), (2.4), we have that

$$\begin{aligned}&(\eta_{uv}\, v_x + \eta_{uu}\, u_x)\, \sigma_s(v,\, \alpha(v,\, s))\, (s - c\, v)\\ &+\varepsilon\, (\eta_{vv}\, (v_x)^2 + 2\, \eta_{uv}\, u_x\, v_x + \eta_{uu}\, (u_x)^2)\end{aligned}$$

is bounded in $L^1_{Loc}(\,\mathbb{R} \times \mathbb{R}^+)$.

4. From condition (C_4) of the Theorem 2 we get $|g(v, u, s)| \leq \frac{M}{(\overline{\sigma}(v))^{1/2}}$ and $|f(v, u)| \leq M$. Then we obtain that $\eta_v g(v, u, s) + \eta_u f(v, u)$ is in $L^1_{Loc}(\mathbb{R} \times \mathbb{R}^+)$.

By Murrat's lemma, see [4], $\eta(v^{\varepsilon,\tau}, u^{\varepsilon,\tau}) + q(v^{\varepsilon,\tau}, u^{\varepsilon,\tau}) \in H^{-1}_{Loc}(\mathbb{R} \times \mathbb{R}^+)$. We can associate a Young measure family to the solutions $(v^{\epsilon,\tau}, u^{\epsilon,\tau})$ found in the *Part I* of this theorem (see [10]). Sharer in [13] proved that the support of this family reduces to one point. Thus, the convergence of $(v^{\epsilon,\tau}, u^{\epsilon,\tau})$ can be obtained. From the estimate in (2.3) we obtain the convergence $s^{\epsilon,\tau} \to cv$. This finishes the proof.

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