Algebraic and rational differential invariants

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These notes start with an introduction to differential invariants. They continue with an algebraic treatment of the theory. The algebraic, differential algebraic and differential geometric tools that are necessary to the development of the theory are explained in detail. We expose the recent results on the topic of rational and algebraic differential invariants. Finally we give a new algebraic version of the finiteness theorem of Lie–Tresse for the case of finite dimensional algebraic groups.

Keywords: differential invariants, rational invariants,
jet spaces, geometric invariant theory.

Estas notas comienzan con una introducción a los invariantes diferenciales. Continuan con un tratamiento algebraico de los mismos. Las herramientas de álgebra, álgebra diferencial y geometría diferencial necesarias para el desarrollo de la teoría son explicadas en detalle. Se exponen los resultados recientes sobre el tema de los invariantes diferenciales racionales y algebraicos, con nuevas demostraciones. Finalmente se da una nueva versión algebraica del teorema de Lie–Tresse para el caso de grupos algebraicos de dimensión finita.

Palabras claves: invariantes diferenciales, invariantes racionales,
espacios de jets, teoría geométrica de invariantes.

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1 Introduction

Differential invariants appear in many problems related to differential equations or differential geometry, as the reduction of differential equations, or the equivalence problem. They were introduced by S. Lie [12]. The theorem of the finiteness invariant for actions of finite–dimensional groups was demonstrated by Lie himself, and extended by Tresse [21]. The modern proof is attribuited to Kumpera [11]. A more general result, in the local smooth context, is exposed in [16].

Very frequently, the differential invariants appearing in applications are rational or algebraic functions. Some research has been devoted to the computation of rational and algebraic differential invariants [6, 7]. A recent article [10] presents a global version of Lie–Tresse theorem. However, they deal with a larger class of functions, that are rational in some of the variables and smooth in others.

In this notes we deal with the rational and algebraic differential invariants of an algebraic action. With respect to the computation of the invariants, we mostly follow the results developed by Hubert and Kogan [4, 5]. With respect to the theory of differential invariants we use the language of *nearby point* bundles proposed by Weil [22] and developed by Muñoz, Rodríguez, Kolář, Michor, Shurygin, Slovak and others [8, 15, 19]. Finally we present some new results on the finiteness of the rational differential invariants, namely the Theorems 4.15 and 4.28.

We consider an algebraic group G of transformations of a smooth algebraic variety Z. This means that each element of G is an invertible regular (polynomial) map $\sigma\colon Z \to Z$. The group G does not only act on Z but also in any other structure that we can assign functorially to Z. For instance if $\mathbb{C}(Z)$ is the field of rational functions Z then each $\sigma \in G$ induces $\sigma^*\colon \mathbb{C}(Z) \to \mathbb{C}(Z)$, $f \mapsto \sigma^* f = f \cdot \sigma$. A *rational invariant* is a rational function such that $\sigma^* f = f$ for all $\sigma \in G$. An *algebraic invariant* is a function which is algebraic over the field of rational invariants. In the same way we can define invariant metrics, invariant submanifolds, invariant vector fields and so on.

We can also consider that some of the coordinates in Z are not complex numbers, but functions of some independent variables that do not appear explicitly. In this way the space Z is substituted by a natural bundle. The first example we can give of natural bundle is the tangent bundle of order k, $T^{(k)}Z$. The elements of $T^{(k)}Z$ are order k Taylor series of curves in Z, and therefore if M is coordinated by $x_1, \cdots, x_n$ then

$T^{(k)}Z$ is coordinated by $x_1, x_1', x_1'', \cdots, x_1^{(k)}, x_2, \cdots, x_n, \cdots, x_n^{(k)}$, because the values of these variables determine the Taylor series of a curve,

$$\begin{aligned}\gamma(\varepsilon) \quad = \quad & \Big(x_1 + \cdots + x_1^{(k)}\,\frac{\varepsilon^k}{k!} + o(\varepsilon^{k+1}), \\ & \qquad \cdots, x_n + \cdots + x_n^{(k)}\,\frac{\varepsilon^k}{k!} + o(\varepsilon^{k+1})\Big)\ .\end{aligned}$$

The group G also acts on the space $T^{(k)}Z$ and therefore in the field $\mathbb{C}(T^{(k)}Z)$ of rational functions in the variables $x_i^{(j)}$, $1 \leq i \leq n$, $0 \leq j \leq k$. A rational function $f \in \mathbb{C}(T^{(k)}Z)$ which is invariant under the action of G is called a *rational differential invariant or order* k. For instance, if we consider the group of Moebius tranformations in $\mathbb{P}^1(\mathbb{C})$,

$$x \mapsto \frac{ax+b}{cx+d}\,,$$

with

$$a\,d - b\,c \neq 0\,,$$

then the Schwartzian derivative,

$$S = \frac{2x'x''' - 3x''^2}{x'^2}\,,$$

is a well known rational differential invariant of order 2. An *algebraic differential invariant* is by definition a function which is algebraic over a field of rational differential invariants. For instance, if we consider the action of Euclidean movements in $\mathbb{C}^2$ then the curvature,

$$\kappa = \frac{x'y'' - x''y'}{\sqrt{(x'^2 + y'^2)^3}}\,,$$

is also a well known algebraic differential invariant.

2 Lie algorithm for ordinary differential equations

In this section we review a classical algorithm for order reduction of ordinary differential equations. This algorithm allows us to introduce differential invariants and their computation. We expose the theory in the classical language of differential equations, later we will go back to the language of algebraic varieties.

2.1 Symmetries of a differential equation

Let us consider an arbitrary non linear ordinary differential equation,

$$F(x,\, y,\, y',\, \cdots,\, y^{(r)}) = 0\,, \tag{1}$$

where $F\colon \mathbb{R}^{r+2} \to \mathbb{R}$ is a smooth function.

Definition 2.1. We say that a smooth transformation,

$$\begin{aligned} \sigma \quad &: \quad \mathbb{R}^2 \to \mathbb{R}^2\,, \\ \sigma \quad &: \quad (x,\, y) \,\mapsto\, (x^* = X(x,\, y),\, y^* = Y(x,\, y))\,, \end{aligned}$$

is a symmetry of (1) if for any solution $y = y(x)$ of (1) σ carries its graph to a curve,

$$\sigma(\Gamma_y) = \{(X(x,\, y(x)),\, Y(x,\, y(x)))|\, x \in \mathrm{dom}(y)\}\,,$$

which is *locally* the graph of solutions of (1).

Example 2.1. The following example helps us to understand the meaning of the word *locally* in the above definition. Let us consider the differential equation:

$$y''^2 - y' - 1 = 0\,. \tag{2}$$

It is, in fact, the general equation satisfied by all the circles of radius 1 in the plane.[2] Its general solution depending on three arbitrary constants is:

$$y(x) = \lambda_1 \pm \sqrt{1 - (x - \lambda_2)^2}\,,$$

with $\lambda_2 - 1 \le x \le \lambda_2 + 1$. It is clear that the rotation of $\pi/2$ radians with respect to the origin,

[2] It can be derived easily from the well known curvature expression $\kappa = \frac{y''}{\sqrt{1+y'^2}}$. The equation just fix the curvature to be ± 1.

$$\begin{aligned} x^* &= -y \\ y^* &= x\,, \end{aligned}$$

sends circles of radius 1 to circles of radius 1. Hence, it is a symmetry of (2). Take a particular solution,

$$y(x) = \sqrt{1-(x-1)^2}\,,$$

for $0 \le x \le 2$. The curve $\sigma(\Gamma_f)$ is now the union of two graphs of solutions,

$$y^*_{\pm}(x^*) = 1 \pm \sqrt{1-(x^*)^2}\,,$$

for $-1 \le x^* \le 0$, and the point $(-1, 1)$ in with the implicit function theorem fails and $\sigma(\Gamma_y)$ can not be described as the graph of a function y of x:

$$\Gamma_y = \Gamma_{y^*_+} \cup \Gamma_{y^*_-} \cup \{(-1,\,1)\}\,.$$

2.2 Prolongation of a transformation of the plane

There is something tricky about the above example. We *a priori* know that the rotation transformation is a symmetry of (2) because we actually know the general solution. How can we check that a certain transformation is a symmetry or not of a differential equation? Let us consider a general transformation σ of the plane,

$$\begin{aligned} x^* &= X(x,\,y)\,, \\ y^* &= Y(x,\,y)\,. \end{aligned} \tag{3}$$

Let $y = y(x)$ be a smooth function of x. Along the graph of the function, the transformed coordinates above, become functions of the independent variable x,

$$\begin{aligned} x^* &= X(x,\,y(x))\,, \\ y^* &= Y(x,\,y(x))\,. \end{aligned}$$

Assuming that the function $x^*(x) = X(x, y(x))$ can be considered as a new coordinate function along the curve $\Gamma_y = \{(x, y(x)) | x \in \mathrm{dom}(y)\}$ we can write y^* as a function of x^* defined by $y^*(X(x, y(x))) = Y(x, y(x))$.

Let us show how the r–th derivative of the transformed function $(y^*)^{(r)}(x^*)$ depends on the values $x, y(x), y'(x), \cdots, y^{(r)}(x)$. In order to get a general formula let us consider a function $\phi(x)$ depending of x, the function $y(x)$ and the derivatives of $y(x)$ up to r–th order,

$$\phi(x) = F(x,\, y(x),\, y'(x),\, \cdots,\, y^{(r)}(x))\,.$$

The total derivative is computed using the chain rule:

$$\begin{aligned} \frac{d\phi}{dx}(x) \quad = \quad & F_x(x,\, y(x),\, y'(x),\, \cdots,\, y^{(r)}(x)) \\ & + y'(x)\, F_y(x,\, y(x),\, y'(x),\, \cdots,\, y^{(r)}(x)) \\ & + \cdots + y^{(r+1)}\, F_{y^{(r)}}(x,\, y(x),\, y'(x),\, \cdots,\, y^{(r)}(x))\,. \end{aligned}$$

Therefore we can define a *total derivative* operator that acts on the functions depending on the variables $x, y, y', y'', \cdots$,

$$\frac{\mathfrak{d}}{\mathfrak{d}x} = \frac{\partial}{\partial x} + y'\,\frac{\partial}{\partial y} + y''\,\frac{\partial}{\partial y'} + \cdots + y^{(k+1)}\,\frac{\partial}{\partial y^{(k)}} + \cdots\,,$$

therefore,

$$\frac{d}{dx}F(x,\, y(x),\, \cdots,\, y^{(r)}(x)) = \left(\frac{\mathfrak{d}F}{\mathfrak{d}x}\right)(x,\, y(x),\, \cdots,\, y^{(r+1)}(x))\,.$$

Note that the *total derivative* $\mathfrak{d}F/\mathfrak{d}x$ is a function that depends on the variables $x, y, \cdots, y^{(r+1)}$.

Now, let us consider again the smooth transformation (3) acting upon a smooth function $y(x)$ and yielding $y^*(x^*)$ defined by $x^*(x) = X(y(x), x)$, $y^*(x^*(x)) = Y(x, y(x))$.

By the chain rule we have,

$$\frac{dy^*}{dx}(x) = \frac{dy^*}{dx^*}(x^*(x))\,\frac{dx^*}{dx}(x)\,,$$

and therefore,

$$\frac{dy^*}{dx^*}(x^*) = \left(\frac{dx^*}{dx}(x)\right)^{-1} \frac{dy^*}{dx}(x) = \frac{\frac{\mathfrak{d}X}{\mathfrak{d}x}(x, y(x), y'(x))}{\frac{\mathfrak{d}Y}{\mathfrak{d}x}(x, y(x), y'(x))}\,.$$

Thus, we extend the transformation σ of the plane to a transformation $\sigma^{(1)}$ of the space $\mathbb{R}^3$ with coordinates x, y, y' defined:

$$\begin{aligned} x^* &= X(x,\, y)\,, \\ y^* &= Y(x,\, y)\,, \\ y'^* &= Y'(x,\, y,\, y') = \frac{\frac{\mathfrak{d}X}{\mathfrak{d}x}(x, y, y')}{\frac{\mathfrak{d}Y}{\mathfrak{d}x}(x, y, y')}\,. \end{aligned}$$

The transformation σ maps the graph of any function that on the point x takes the value y and first derivative y' to the graph of a function that of the point x^* takes the value y^* and first derivative y'^*. Note that the denominator of the expression for y'^* above may vanish. Hence, $\sigma^{(1)}$ is not defined in the whole space, but just outside the surface

$$\Delta_X = \left\{(x,\, y,\, y') \in \mathbb{R}^3 \mid \frac{\mathfrak{d}X}{\mathfrak{d}x}(x,\, y,\, y') = 0\right\}\,.$$

This surface corresponds to the functions $y = y(x)$ whose graph does not admit $x^* = X(x, y(x))$ as a local coordinate. Geometrically it means that the graph of the function $y = y(x)$ becomes tangent to the axis OY after transformation by σ.

Indeed, for any order k, by successive differentiation we have that

$$\frac{d}{dx}(y^{*(k)})(x) = y^{*(k+1)}(x^*(x))\,\frac{dx^*}{dx}(x)\,,$$

and therefore,

$$y^{*(k+1)}(x^*) = \left(\frac{dx^*}{dx}(x)\right)^{-1} y^{*(k)}(x)\,.$$

This allows us to write recursively a transformation $\sigma^{(r)}\colon \mathbb{R}^{2+r} \setminus \Delta_X \to \mathbb{R}^{2+r} \setminus \Delta_X$, that we call the r–th prolongation of σ:

$$\begin{aligned}
x^* &= X(x,\, y)\,, \\
y^* &= Y(x,\, y)\,, \\
y'^* &= Y'(x,\, y,\, y') \\
y''^* &= Y''(x,\, y,\, y',\, y'')\,, \\
&\vdots \\
y^{(r)*} &= Y^{(r)}(x,\, y,\, y',\, \cdots,\, y^{(r)})\,.
\end{aligned}$$

where,

$$Y^{(k+1)}(x,\, y,\, y',\, \cdots,\, y^{(k+1)}) = \frac{\frac{\partial Y^{(k)}}{\partial x}(x, y, y', \cdots, y^{(k+1)})}{\frac{\partial X}{\partial x}(x, y, y')}\,.$$

The transformation $\sigma^{(r)}$ says that the graph of a function that takes at the point x the value y and derivatives $y', \cdots, y^{(r)}$ is transformed by σ into the graph of a function that takes at the point x^* the value y^* and derivatives $y'^*. \cdots, y^{(r)*}$. Note that $\sigma^{(r)}$ is defined outside the hypersurface Δ_X whose that does not depend of the order r. The following proposition is a direct implication of the definitions.

Proposition 2.1. *The transformation σ is a symmetry of the differential equation* (1) *if and only if the r–th prolongation $\sigma^{(r)}$ transforms the hypersurface $\{F = 0\}$ of $\mathbb{R}^2$ into itself. That is, there exist a function λ defined in a neighbourhood of the hypersurface $\{F = 0\}$ in $\mathbb{R}^{r+2}$ such that,*

$$F(x^*,\, y^*,\, y'^*,\, \cdots,\, y^{(r)*}) = \lambda(x,\, y,\, y',\, \cdots,\, y^{(r)}) \cdot F(x,\, y,\, y',\, \cdots,\, y^{(r)})\,.$$

2.3 One–parameter groups of symmetries

In applications we use continuous families of symmetries of differential equations. A one–parameter group $\{\sigma_t\}_{t\in\mathbb{R}}$ of transformations of the plane is a family of transformation for each $t \in \mathbb{R}$ such that $\sigma_0 = \mathrm{Id}$ and $\sigma_{t+s} = \sigma_t \circ \sigma_s$. We say that a one–parameter group is a group of symmetries of a given differential equation if all its elements are symmetries of the equations.

Example 2.2. The group of scaling transformations,

$$
\begin{aligned}
\sigma_t \colon x^* &= \lambda\, x\,, \\
y^* &= \lambda\, y\,,
\end{aligned}
\tag{4}
$$

where $\lambda = e^t$ is a one–parameter group of transformations of $\mathbb{R}^2$.

Example 2.3. Let us check that the group of scaling transformations (4) is a group of symmetries of the equation

$$
x^2\, y'' + x\, y'^2 = y\, y'\,. \tag{5}
$$

The formula for the second order prolongation gives,

$$
\begin{aligned}
y'^* &= \left(\frac{\partial}{\partial x}\lambda\, x\right)^{-1} \frac{\partial}{\partial x}\lambda\, y = \lambda^{-1}\,\lambda\, y' = y'\,, \\
y''^* &= \left(\frac{\partial}{\partial x}\lambda\, x\right)^{-1} \frac{\partial}{\partial x} y' = \lambda^{-1}\, y''\,.
\end{aligned}
\tag{6}
$$

Replacing the coordinate variables x, y, y', y'' in (5) by the transformed variables x^*, y^*, y'^*, y''^* we obtain,

$$
\begin{aligned}
x^{*2}\, y'' + x^*\, y'^{*2} - y^*\, y'^* &= \lambda^2\, x^2\, \lambda^{-1}\, y'' + \lambda\, x\, y'^2 - \lambda\, y\, y' \\
&= \lambda\, (x^2\, y'' + x\, y'^2 - y\, y')\,.
\end{aligned}
$$

It is then clear that the scale transformation σ_λ is, for all λ, a symmetry of equation (5).

Example 2.4. Let us consider a general polynomial differential equation of r–th order,

$$
\begin{aligned}
P(x,\, y,\, y',\, \cdots,\, y^{(r)}) &= \sum_{j,k_0,k_1,\cdots,k_r=0}^{N} a_{(j,k_0,k_1,\cdots,k_r)}\, x^j\, y^{k_0}\, y'^{k_1}\, \cdots\, y^{(r)k_r} \\
&= 0\,.
\end{aligned}
\tag{7}
$$

Let us compute the necessary and sufficient conditions on the coefficients $a_{(j,k_0,k_1,\cdots,k_r)}$ so that the scale transformations (4) are symmetries of (7).

First, the r–th prolongation of (4) is given by,

$$
\begin{aligned}
\sigma_\lambda^{(r)}\colon \quad x^* &= \lambda x\,,\\
y^* &= \lambda y\,,\\
y'^* &= y'\,,\\
y''^* &= \lambda^{-1} y''\,,\\
&\vdots\\
y^{(r)*} &= \lambda^{-r+1}\, y^{(r)}\,.
\end{aligned}
$$

Substituting in 7:

$$
\begin{aligned}
& P(x^*,\, y^*,\, y'^*,\, \cdots, y^{(r)*})\\
= & \sum_{j,k_s=0}^{N} \lambda^{j+k_0-k_2-2k_3-\cdots-(r-1)k_r}\, a_{(j,k_0,k_1,\cdots,k_r)}\, x^j\, y^{k_0}\, y'^{k_1} \cdots\, y^{(r)k_r}\,.
\end{aligned}
$$

It turns out that each monomial $a_{(j,k_0,k_1,\cdots,k_r)} x^j y^{k_0} y'^{k_1} \cdot \cdots \cdot y^{(r)k_r}$ is scaled by certain power of λ with an exponent which is a fixed linear combination of the exponents of $x, y, y', \cdots, y^{(r)}$. It turns out that σ_λ is a symmetry of (7) if and only if $P(x, y, y', \cdots, y^{(r)})$ is the sum of monomials having the same scaling exponent. Therefore, it should be of the form:

$$
\begin{aligned}
& P(x,\, y,\, y',\, \cdots,\, y^{(r)})\\
= & \sum_{k_s=0}^{N} a_{(k_0,k_1,\cdots,k_r)}\, x^{j-k_0+k_2+2k_3+\cdots+(r-1)k_r}\, y^{k_0}\, y'^{k_1} \cdots\, y^{(r)k_r}\\
= & \; 0\,,
\end{aligned}
\tag{8}
$$

for certain integer j. This is the general form of a *polynomial ordinary differential equation* admitting the scaling transformations (4) as a group of symmetries.

Let $\{\sigma_t\}_{t\in\mathbb{R}}$ be a one–parameter group of transformations of $\mathbb{R}^n$. Given a point $p \in \mathbb{R}^n$ its orbit is the curve $\{\sigma_t(p)\}_{t\in\mathbb{R}}$. A point $p \in \mathbb{R}^n$ is called *regular* if its orbit is not reduced to the point $\{p\}$ itself. A function $u\colon U \to \mathbb{R}$ defined in an open subset $U \subset \mathbb{R}$ is called an invariant if it is constant along the orbits of the action of $\{\sigma_t\}_{t\in\mathbb{R}}$. It is well know that if p is a regular point then there exist a system of coordinates $z_1, \cdots, z_n$ defined in a neighbourhood of p for which the equations of the transformations of the group are written:

$$\begin{aligned} \sigma_t: \quad z_1^* &= z_1\,,\\ &\vdots\\ z_{n-1}^* &= z_{n-1}\,,\\ z_n^* &= z_n + t\,. \end{aligned} \tag{9}$$

Those coordinates are called *canonical coordinates* for the group $\{\sigma_t\}_{t\in\mathbb{R}}$. The orbits correspond to lines, and $z_1 \cdots, z_{n-1}$ form a complete set of functionally independent invariants of the actions. To find an invariant defined in the whole space, or a large portion of it, is not possible in this general setting and depends on some topological considerations about the action. However, we see that we can expect to find, at least locally, $n-1$ functionally independent invariants. If the one–parameter group acts in $\mathbb{R}^2$, then we have a unique invariant up to functional dependence.

Example 2.5. An invariant of the group of scaling transformations is the function $z = y/x$. It is not defined along the OY axis, were $z^{-1} = x/y$ can be chosen as an invariant. They are functionally independent in the intersection of their domains of definition. The point $(0,0)$, around which no invariant is defined, is a *singular* (that is, non regular) point of the action of scaling transformations.

The first application of one–parameter groups of symmetries is in solving first order differential equations. The following theorem is true in a broader sense, but we state it in the particular case in which we can find the canonical variables (9) for the group action. In the general setting it is enough to know an *infinitesimal symmetry* and the integration is done by means of an *integrating factor*. However we state the theorem in this narrower case because the proof use the invariants for reduction.

Theorem 2.2 (S. Lie)**.** *Assume that a first order differential equation,*

$$F(x, y, y') = 0\,, \tag{10}$$

admits a one–parameter group of symmetries $\{\sigma_t\}_{t\in\mathbb{R}}$ *and that can be written in canonical variables* (9)*. Then the equation* (10) *is integrable* *by means of a change of variables and a quadrature.*

Proof. Let $z(x,y)$, $w(x,y)$ be the coordinates in which the group is written in the canonical form (9), being z the invariant. We restrict our considerations to the the region in which the transformation

$$\begin{aligned} \tau : z &= z(x, y)\,, \\ w &= w(x, y)\,, \end{aligned}$$

is invertible. By means of the first prolongation of the inverse transformation τ^{-1} we compute the differential equation for the variables z, w,

$$G(z,\, w,\, w') = F(x(z,\, w),\, y(z,\, w),\, y'(z,\, w,\, w'))\,.$$

But now, we know that if $w = w(z)$ is a solution then $w = w(z) + t$ is another solution. It follows that,

$$G(z,\, w,\, w') = 0 \Leftrightarrow \forall t \;\; G(z,\, w + t,\, w') = 0\,.$$

Therefore, the variable w does not have a role in the differential equation, and up to a multiplicative constant $\lambda(z, w, w')$ that does not vanish in the hypersurface $\{G = 0\}$ we have:

$$G(z,\, w,\, w') = \lambda(z,\, w,\, w') \cdot H(z,\, w')\,.$$

We solve this last equation with respect to w' obtaining,

$$\begin{aligned} w' &= h(z)\,, \\ w(z;\, k) &= \int h(z)\, dz + k\,. \end{aligned}$$

Finally we find the solutions of the original equation applying the transformation τ^{-1} to the graphs of the functions $w = w(z; k)$. □

Example 2.6. Let us assume that the equation (10) admits the group of scale transformations (4). Applying to the point of coordinates (x, y, y') a suitable scaling transformation it follows that the equation (10) is equivalent to

$$F\left(\frac{x}{y},\, 1,\, y'\right) = 0\,.$$

On the other hand, the following z, w are a set of canonical coordinates for the group,

$$\begin{aligned} z &= \frac{x}{y}\,, \\ w &= \ln(y)\,, \end{aligned}$$

$$\begin{aligned} x &= z\,e^{w}\,, \\ y &= e^{w}\,. \end{aligned}$$

We obtain

$$y' = \frac{\frac{\partial y}{\partial z}(z, w, w')}{\frac{\partial x}{\partial z}(z, w, w')} = \frac{w'}{1 + w'z}\,,$$

and therefore the differential equation (10) is equivalent to

$$G(z,\, w') = F\left(z,\, 1,\, \frac{w'}{1 + w'z}\right) = 0\,,$$

in the coordinates z, w.

2.4 Differential invariants

Let $\{\sigma_t\}_{t\in\mathbb{R}}$ be a one–parameter group of transformations of $\mathbb{R}^2$. The prolongations of order r form a one–parameter group of transformations of $\mathbb{R}^{2+r}$ $\{\sigma_t^{(r)}\}_{t\in\mathbb{R}}$, called the r–th prolongation of the group $\{\sigma_t\}_{t\in\mathbb{R}}$.

Definition 2.2. A *differential invariant* of order r of the action of the one–parameter group $\{\sigma_t\}_{t\in\mathbb{R}}$ is a function $f(x, y, y', \cdots, y^{(r)})$ which is an invariant of the action of the r–th prolongation $\{\sigma_t^{(r)}\}_{t\in\mathbb{R}}$. Invariants defined in open subsets of $\mathbb{R}^2$ are called invariants of order zero.

In order to make sure that f is of order r and not of order $r-1$ we also require the derivative of f with respect $y^{(r)}$ to be different from zero. Since we expect to have a family of $r+1$ independent invariants from order zero to $r-1$, we expect that there exist only one functionally independent differential invariant of order r.

Example 2.7. In the case of scaling transformations, $z = y/x$ and $w = y'$ form a set of functionally independents differential invariants up to order 1.

Let us consider $z = z(x, y, y', \cdots, y^{(k)})$ a function defined in $\mathbb{R}^{2+k}$, including the case k equal to zero. Let $y = y(x)$ be a function of x. We can consider z as a local coordinate along the curve

$$\{(x,\, y(x),\, y'(x),\, \cdots,\, y^{(k)}(x)) |\, x \in \mathrm{dom}(y)\}\,.$$

For any $w(x, y, y' \cdots, y^{(r)})$ defined in $\mathbb{R}^{2+r}$ and,

$$\phi(x) = w(x,\, y(x),\, y'(x),\, \cdots,\, y^{(r)}(x))\,,$$

we can compute the derivatize of ϕ with respect to z by using the total derivative operators,

$$\begin{aligned} &\frac{d\phi}{dz}(x(z)) \\ = &\left(\frac{\partial z}{\partial x}(x,\, y'(x),\, \cdots,\, y^{(k+1)}(x))\right)^{-1} \frac{\partial w}{\partial x}(x,\, y'(x),\, \cdots,\, y^{(r+1)}(x))\,. \end{aligned}$$

Therefore, we write:

$$\frac{\partial}{\partial z} = \left(\frac{\partial z}{\partial x}\right)^{-1} \frac{\partial}{\partial x}\,,$$

as a differential operator that sends a function w defined in $\mathbb{R}^{2+r}$ to $\partial w/\partial z$ defined in $\mathbb{R}^{2+r+1}$, provided that $r > k$. In the particular case in which z and w are differential invariants, then it is clear that $\partial w/\partial z$ is also a differential invariant and the following results hold.

Proposition 2.3. *Let z, w be differential invariants of $\{\sigma_t\}_{t\in\mathbb{R}}$ of order $k < r$ respectively. Then $\frac{\partial w}{\partial z}$ is a differential invariant of $\{\sigma_t\}_{t\in\mathbb{R}}$ of order $r+1$ and thus, functionally independent from z and w.*

Theorem 2.4. *Let z, w be differential invariants of $\{\sigma_t\}_{t\in\mathbb{R}}$ of order 0 and 1 respectively,*

$$
\begin{aligned}
z &= z(x,\, y)\,,\\
w &= w(x,\, y,\, y')\,,\\
w' &= \frac{\mathfrak{d} w}{\mathfrak{d} z}\,,\\
w'' &= \frac{\mathfrak{d}^2 w}{\mathfrak{d} z^2}(x,\, y,\, y',\, y'')\,,\\
&\ \vdots\\
w^{(r-1)} &= \frac{\mathfrak{d}^{(r-1)} w}{\mathfrak{d} z^{(r-1)}}(x,\, y,\, y',\, \cdots,\, y^{(r)})\,,
\end{aligned}
$$

is a complete set of functionally independent differential invariants of order up to r.

Example 2.8. In the case of scaling transformations,

$$
\begin{aligned}
z &= \frac{y}{x}\,,\\
w &= y'\,,\\
w' &= \frac{\mathfrak{d} w}{\mathfrak{d} z} = \left(\frac{\partial}{\partial x}\frac{y}{x} + y'\,\frac{\partial}{\partial y}\frac{y}{x}\right)^{-1} y'' = \frac{x^2 y''}{x y' - y}\,,
\end{aligned}
\tag{11}
$$

is a complete system of differential invariants up to order 2.

2.5 Lie algorithm

Let us now assume that $\{\sigma_t\}_{t\in\mathbb{R}}$ is a one–parameter group of symmetries of the equation (1) and that $z, w, w', \cdots, w^{(r-1)}$ is a complete system of differential invariants up to order r computed using the Theorem 2.4. By hypothesis, the hypersurface $\{F = 0\}$ is invariant under the action of $\{\sigma_t^{(r)}\}_{t\in\mathbb{R}}$ and therefore its equations can be given in terms of the invariants. There exist a function λ non vanishing on $\{F = 0\}$ and a function G depending only on $z, w, \cdots, w^{(r-1)}$ such that,

$$F(x,\, y,\, y',\, \cdots,\, y^{(r)}) = \lambda \cdot G(z,\, w,\, w',\, \cdots,\, w^{(r-1)})\,.$$

For each solution $w = w(z)$ of the above equation we obtain a first order equation,

$$w(x,\, y,\, y') - z(x,\, y) = 0\,,$$

that admits $\{\sigma\}_t$ as a group of symmetries and therefore it can be integrated by means of Theorem 2.2.

Example 2.9. Let us reduce the equation (5) by means of the group of scale transformations (4). As computed in the Example 2.8 above, $z = \frac{y}{x}$, $w = y'$, $w'' = \frac{x^2 y''}{xy'-y}$ form a complete system of differential invariants up to order 2. A direct substitution gives us,

$$x^2\, y'' + x\, {y'}^2 - y\, y' = x(w - z)\, (w' + w)\,.$$

The reduced equation is then,

$$(w - z)\, (w' + w) = 0\,,$$

that has a singular solution $w(z) = z$ and a family of solutions $w(z) = ke^{-z}$. Reverting them back to the original equation we obtain,

$$\begin{aligned} y' &= \frac{y}{x}\,, \\ y' &= k\, e^{-y/x}\,; \end{aligned}$$

the first one is solved by $y = kx$, the second one is solved implicitly by

$$\int \frac{d(y/x)}{ke^{-(y/x)} - (y/x)} + \ln(x) = k_2\,.$$

3 Rational and algebraic invariants

3.1 Notation and conventions on algebraic varieties

In what follows we consider algebraic groups and varieties defined over an *algebraically closed field* $\mathbb{C}$ of zero characteristic. In our notation an *affine* $\mathbb{C}$*–variety* is an algebraic subset Z of the affine space $\mathbb{C}^m$. That is, Z is the space of solutions of a system of polynomial equations,

$$Z:\ \{P_1(x_1,\, \cdots,\, x_m) = 0,\, \cdots,\, P_r(x_1,\, \cdots,\, x_m) = 0\}\,.$$

By definition a *regular function* $f\colon Z \to \mathbb{C}$ is a polynomial function in the coordinates of $\mathbb{C}^m$. Let us denote by $\mathbb{C}[Z]$ the ring of regular functions in V. Let $\mathfrak{I}(Z)$ be the ideal of polynomials in the variables $x_1, \cdots, x_m$ vanishing on Z. By the Hilbert *Nullstellensatz* $\mathfrak{I}(Z)$ is the radical ideal spanned by $P_1, \cdots, P_r$. Two polynomials define the same regular functions if and only if its difference lie in $\mathfrak{I}(Z)$. Therefore:

$$\mathbb{C}[Z] = \mathbb{C}[x_1, \cdots, x_m]/\mathfrak{I}(Z).$$

In Z we consider the Zariski topology. A basis of open subsets is given by the *affine open subsets.* Let f be a regular function, then

$$U_f = \{x \in Z \mid f(x) \neq 0\}.$$

Note that U_f is identified with an algebraic subset of $\mathbb{C}^{m+1}$ by setting:

$$\begin{aligned} U_f: \qquad & \{P_1(x_1, \cdots, x_m) = 0, \cdots, P_r(x_1, \cdots, x_m) = 0, \\ & x_{m+1}\, f(x_1, \cdots, x_m) - 1 = 0\}. \end{aligned}$$

Therefore we have,

$$\mathbb{C}[U_f] = K[Z]\,[f^{-1}] \supset \mathbb{C}[Z].$$

This allows us to define the *sheaf of regular functions* through localization. For each open subset $U \subset V$ we have,

$$\mathbb{C}[U] = S^{-1}\,\mathbb{C}[Z] = \left\{ \frac{f}{g} \;\middle|\; f \in \mathbb{C}[Z], g \in S \right\},$$

where

$$S = \{g \in \mathbb{C}[Z] \mid \forall x \in U\; g(x) \neq 0\}.$$

Let us recall that affine $\mathbb{C}$–varieties are quasicompact and T_1–separated. In general Z is the union of a finite number of irreducible components, $Z = Z_1 \cup \cdots \cup Z_c$. We define the *ring of rational functions* on Z as the ring of total fractions,

$$\mathbb{C}(Z) = \varinjlim_{U \text{dense}} \mathbb{C}[U] = \left\{ \frac{f}{g} \;\middle|\; f,\, g \in \mathbb{C}[Z],\, \forall i = 1, \cdots, c,\, g|_{Z_i} \neq 0 \right\}.$$

An affine $\mathbb{C}$–variety Z is irreducible if and only if $\mathfrak{I}(Z)$ is a prime ideal. In such a case we have a *field of rational functions*

$$\mathbb{C}(Z) = \lim_{\substack{\to \\ U \text{open}}} \mathbb{C}[U] = \left\{ \frac{f}{g} \,\middle|\, f, g \in \mathbb{C}[Z] \; g \neq 0 \right\}.$$

We have spectral representation for affine algebraic varieties. Let us denote $\mathrm{Spec}_{\mathbb{C}}(\mathbb{C}[Z])$ the set of $\mathbb{C}$–algebra homomorphisms from $\mathbb{C}[Z]$ on $\mathbb{C}$, endowed with the Zariski topology. Then there is a canonical homeomorphism $Z \simeq \mathrm{Spec}_{\mathbb{C}}(\mathbb{C}[Z])$.

Given a point $p \in Z$ the ring of *germs of regular functions* is,

$$\mathcal{O}_{Z,p} = \lim_{\substack{\to \\ p \in U}} \mathbb{C}[U] = \left\{ \frac{f}{g} \,\middle|\, f, g \in \mathbb{C}[Z] \; g(p) \neq 0 \right\} \subset \mathbb{C}(Z),$$

which is a local ring with maximal ideal $\mathfrak{m}_p \subset \mathcal{O}_{Z,p}$. The point p is said to be *regular* if the *Krull dimension* of $\mathcal{O}_{Z,p}$ coincides with the minimal number of generators of its maximal $\mathfrak{m}_p$. Otherwise we say that p is a *singular* point. A numerical criterium for regularity is the following: assume that Z is irreducible and let $\{P_1, \cdots, P_r\}$ be a minimal system of generators of $\mathfrak{I}(Z)$. Then, p is regular if and only if the set $\{d_p P_1, \cdots, d_p P_r\} \subset T^*_x(\mathbb{C}^m)$ consists of $\mathbb{C}$–linearly independent 1–forms. We say that Z is *smooth* if all points of Z are regular.

The Cartesian product of two affine algebraic varieties $Y \subset \mathbb{C}^m$ and $Z \subset \mathbb{C}^n$ is an affine algebraic variety $Y \times Z \subset \mathbb{C}^{n+1}$. The Zariski topology in $Y \times Z$ is tamer than the product topology. In the spectral representation we have $Y \times Z \simeq \mathrm{Spec}_{\mathbb{C}}(\mathbb{C}[Y] \otimes_{\mathbb{C}} \mathbb{C}[Z])$.

Definition 3.1. A *pre*-$\mathbb{C}$–variety is a quasicompact topological space Z endowed with a sheaf of regular functions $U \rightsquigarrow \mathbb{C}[U]$ induced by an atlas $\{(U_i, \phi_i)\}$ verifying:

- **a.** The coordinate maps $\phi_i \colon U_i \to V_i \subset \mathbb{C}^{m_i}$ identify U_i with an algebraic subset of $V_i \subset \mathbb{C}^m_i$ endowed with the Zariski topology.
- **b.** Gluing maps $\phi_{ij} \colon \phi_i(U_i \cap U_j) \subset \mathbb{C}^{m_i} \to \phi_j(U_i \cap U_j) \subset \mathbb{C}^{m_j}$ are given coordinate–wise by regular functions.

The ring of rational functions $\mathbb{C}(Z)$ of a pre–$\mathbb{C}$–variety is defined in the same way. Again $\mathbb{C}(Z)$ is a field if and only if Z is irreducible. The notion of regular and singular points was defined locally, therefore it is defined in the same way for points of pre–$\mathbb{C}$–varieties. In the same way, we say that a pre–$\mathbb{C}$–variety is *smooth* if all their points are regular.

Definition 3.2. We say that Z is a *separated* if the diagonal embedding,

$$\begin{aligned} \Delta \quad : \quad & Z \to Z \times Z\,, \\ & x \mapsto (x,\, x)\,, \end{aligned}$$

is closed.

Affine $\mathbb{C}$–varieties are separated. Therefore any pre–$\mathbb{C}$–variety is always locally separated. It follows that the non–separatedness is always a consequence of the gluing maps.

Definition 3.3. A $\mathbb{C}$*–variety* Z is a separated pre–$\mathbb{C}$–variety.

Example 3.1. The projective space of dimension n, $\mathbb{P}(n, \mathbb{C})$ is a $\mathbb{C}$–variety.

Definition 3.4. A $\mathbb{C}$*–regular map* between pre–$\mathbb{C}$–varieties is a map, $\varphi\colon Y \to Z$, which is given coordinate–wise by regular functions. We say that φ is *separated* if the diagonal map

$$\Delta_Z \colon Y \to Y \times_Z Y = \{(p, q) \in Y \times Y \mid \varphi(p) = \varphi(q)\}\,,$$

is closed.

If Y and Z are affine $\mathbb{C}$–varieties then there is a one–to–one correspondence between regular maps $\varphi\colon Y \to Z$ and $\mathbb{C}$–algebra morphisms $\varphi^*\colon \mathbb{C}[Z] \to \mathbb{C}[Y]$. We have $\varphi^*(f) = f \circ \varphi$.

We say that two $\mathbb{C}$–varieties are *isomorphic* if there is an invertible regular map between them. A $\mathbb{C}$–variety is said to be *affine* if it is isomorphic to an affine $\mathbb{C}$–variety. A sub–$\mathbb{C}$–variety Z' of Z is the intersection of a closed and an open subset of Z. There is a unique structure of $\mathbb{C}$–variety in Z' that makes the identity $\mathrm{id}\colon Z' \to Z$ a regular map. Closed sub–$\mathbb{C}$–varieties of affine $\mathbb{C}$–varieties are affine $\mathbb{C}$–varieties. A closed sub–$\mathbb{C}$–variety of the projective $\mathbb{P}(n, \mathbb{C})$ is called a *projective variety.* The following classical result holds. It is essential to many applications.

Theorem 3.1 (Chevalley)**.** *Let $\varphi\colon Y \to Z$ be a regular map. Then the image of Y is a finite union of algebraic sub–$\mathbb{C}$–varieties of Z, not necessarily closed.*

Recall that an open subset $U \subset V$ is called *affine* if there it is $\mathbb{C}$–isomorphic to an algebraic subset of the affine space $\mathbb{C}^m$. If U is affine, and $f \in \mathbb{C}[U]$ then $U_f = \{x \in U \mid f(x) \neq 0\}$ is also affine and $\mathbb{C}[U_f] = \mathbb{C}[U][f^{-1}]$.

Algebraic groups

Definition 3.5. An algebraic $\mathbb{C}$–group G is a group object in the category of $\mathbb{C}$–varieties.

The definition implies that each connected component of G is an irreducible $\mathbb{C}$–variety and that composition and inversion morphisms are regular maps.

Algebraic subgroups are closed. If $H \lhd G$ is a normal algebraic subgroup, then the quotient G/H is an algebraic group. The *connected component of the identity* of G is the minimal normal algebraic subgroup of finite index G_0. Therefore, the group G/G_0 is finite.

Exercise 3.2. Prove that algebraic $\mathbb{C}$–groups are *smooth*.

If G is an *affine algebraic group* then it is isomorphic to an algebraic subgroup of $\mathrm{GL}(n, \mathbb{C})$ for some $n > 0$. Therefore, affine algebraic groups are often referred as *linear algebraic groups*.

An Abelian variety is an algebraic group which is a projective variety. One dimensional Abelian varieties are elliptic curves. In general, Abelian varieties are Abelian groups, in particular Abelian varieties over the complex numbers are compact complex torii. The following result exhibits the structure of algebraic groups.

Theorem 3.3 (Chevalley-Barsotti)**.** *Let G be an algebraic group. There exist a normal linear algebraic group $H \lhd G$ such that G/H is an Abelian variety.*

3.2 Algebraic actions

Definition 3.6. An algebraic action of an algebraic group G on $\mathbb{C}$–variety Z is a group action,

$$\begin{aligned} \alpha \quad : \quad & G \times Z \to Z\,, \\ & (\sigma, p) \to a(\sigma, p) = \sigma \cdot p\,, \end{aligned}$$

which is a regular map. A $\mathbb{C}$–variety endowed with an action of G is called a G–variety.

An action of G on an irreducible variety Z induces actions of G by the right side on the ring of regular functions $\mathbb{C}[Z]$ and the field of rational functions $\mathbb{C}(Z)$. The set of invariants are the so called *ring of regular invariants* $\mathbb{C}[Z]^G$ and *field of rational invariants* $\mathbb{C}(Z)^G$.

The construction of quotients by algebraic actions, known as *geometric invariant theory*, is an important research topic, which is beyond the scope of this article. For the general theory, we mostly follow the classical book of Mumford *et al.* [14] and the notes of Brion [3].

Theorem 3.4. *Let Z be a G–variety. The following statements hold:*

- **a.** *Orbits $G \cdot p$ are locally closed sub–$\mathbb{C}$–varieties of Z.*
- **b.** *Stabilizers G^p are $\mathbb{C}$–algebraic subgroups of G.*
- **c.** *Every component of $G \cdot p$ has dimension $\dim_{\mathbb{C}}(G) - \dim_{\mathbb{C}}(G^p)$.*
- **d.** *The closure $\overline{G \cdot p}$ is union of $G \cdot p$ and orbits of smaller dimension than $G \cdot p$.*
- **e.** *Any orbit of minimal dimension is closed.*
- **f.** *For any $p \in Z$, $\dim_{\mathbb{C}}(G \cdot p) + \dim_{\mathbb{C}}(G^p) = \dim_{\mathbb{C}}(G)$.*
- **g.** *The dimension of orbits is lower–continuous. For each n the set $\{p \in Z \,|\, \dim_{\mathbb{C}}(G \cdot p) \leq n\}$ is closed.*
- **h.** *The dimension of stabilizers is upper–continuous. For each n the set $\{p \in Z \,|\, \dim_{\mathbb{C}}(G^p) \geq n\}$ is closed.*

Let Z be a G–variety with action α, the map

$$\begin{aligned} \Psi_\alpha \quad : \quad & G \times Z \to Z \times Z\,, \\ & (\sigma, p) \mapsto (\sigma \cdot p, p)\,, \end{aligned}$$

is called the *the graph map* of α. In virtue of Chevalley's theorem the image of Ψ_α is a finite union of locally closed subsets of $Z \times Z$. Let us denote by Γ_α this image, called *the graph of alpha*,

$$\Gamma_\alpha = \{(p,\, q) \in Z \times Z \,|\, G \cdot p = G \cdot q\}\,.$$

Definition 3.7. An algebraic action α of G in Z is said to be *algebro–geometrically...*

- **a.** *...closed* if for all $p \in Z$ the orbit $G \cdot x \subset Z$ is closed in Z.

b. *...separated* if the graph Γ_α is closed.

c. *...proper* it the graph map Ψ_α is proper.

d. *...free* if the map Ψ_α above is a closed immersion.

e. *...regular* if all orbits have the same dimension.

We have the following sequence of implications:

$$\text{free} \implies \text{proper} \implies \text{separated} \implies \text{closed} \impliedby \text{regular}.$$

Lemma 3.5 ([17] Lemma 2.1)**.** *Let Z be an irreducible G–manifold. There is a G–invariant open subset $U \subset Z$ such that the induced action of G in U is separated and regular.*

Algebraic G–modules

Note that a finite dimensional $\mathbb{C}$–vector space E is in a natural way an affine irreducible $\mathbb{C}$–variety. The group of linear transformations $\mathrm{GL}(E, \mathbb{C})$ is an algebraic group acting regularly on E. Let G be an algebraic group, an algebraic *linear representation* of G is a linear representation given by a regular group morphism $\Phi\colon G \to \mathrm{GL}(n, \mathbb{C})$.

Let G act on a $\mathbb{C}$–vector space E (not necessarily of finite dimension). We say that E is a *G–module* if E is the union of finite linear representations of finite dimension. Equivalently, for each $\vec{v} \in E$ the orbit $G \cdot \vec{v}$ is contained in a finite dimensional G–invariant vector subspace in which G acts algebraically.

Theorem 3.6 (Chevalley)**.** *Let G be a linear algebraic group and H a closed subgroup. Then there exist a finite dimensional G–module E and a line $l \subset E$ such that H is the stabilizer of the line,*

$$H = \{\sigma \in G \,|\, G \cdot l \subseteq l\}.$$

The above theorem can be rewritten in terms of projective actions. If we take the projective space $\mathbb{P}(E)$ and the p the point defined by l, we have that $H = G^p$. Therefore, any closed algebraic group is the stabilizer of a point in the projectivization of a finite dimensional G–module.

Categorical and geometric quotients

Definition 3.8. A categorical quotient of Z is a pre–$\mathbb{C}$–variety Q together with a G–invariant regular map $\pi\colon V \to Q$ such that for any G–invariant regular map $\varphi\colon V \to W$ there exist a unique $\bar{\varphi}\colon Y \to Q$ such that

$$\begin{tikzcd} Z \arrow[r, "\pi"] \arrow[d, "\varphi"'] & Q \arrow[dl, "\bar{\varphi}"] \\ W & \end{tikzcd}$$

with $\bar{\varphi} \circ \pi = \varphi$.

From its categorical definition, it follows that categorical quotients are unique up to regular $\mathbb{C}$–isomorphisms. However, the points of a categorical quotient needs not to be in one–to–one correspondence with the orbits of Z. We have the tamer notion of geometric quotient.

Definition 3.9. A geometric quotient of Z is a pre–$\mathbb{C}$–variety Q together with a G–invariant regular map $\pi\colon Z \to Q$ satisfying:

a. For each $x \in V$, $\pi^{-1}(\pi(x)) = G \cdot x$.

b. π is submersive, *i.e.* $U \subset Q$ is open if and only if $\pi^{-1}U$ is open.

c. For each $U \subset Q$, π induces an isomorphism between $\mathbb{C}[U]$ and $\mathbb{C}[\pi^{-1}U]^G$.

Example 3.2 (in [14], page 11)**.** A closed and set–theoretically free action which admits a smooth geometric quotient and is not algebro–geometrically proper nor free. Let us consider $\mathbb{C}[x, y]_n$ the space of homogeneous polynomials of degree n in the variables x, y. The group $\mathrm{SL}(2, \mathbb{C})$ acts algebraically by linear substitutions[3]:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot P(x,\, y) = P(d\,x - b\,y,\, -c\,x + a\,y)\,.$$

We consider the subset $Z \subset (\mathbb{C}[x, y]_1 \setminus \{0\}) \times \mathbb{C}[x, y]_4$ whose elements are pairs $(P_1(x, y), P_2(x, y)^2)$ consisting of a non zero homogeneous polynomial of degree 1 and the square of a quadratic form with discriminant 1:

[3] Note that we have to invert the matrix in order to have a left action

$$\begin{aligned} Z &= \Big\{ \big(\alpha\, x + \beta\, y,\, (a\, x^2 + b\, x\, y + c\, y^2)^2\big) \mid \alpha\, x + \beta\, y \neq 0, \\ &\qquad \Delta(a\, x^2 + b\, x\, y + c\, y^2) = a\, c - \frac{b^2}{4} = 1 \Big\}\,. \end{aligned}$$

Z is a non–singular 4–dimensional $\mathrm{SL}(2,\mathbb{C})$–variety. We also check that for all $q = (P_1(x,y), P_2(x,y)^2) \in Z$ the stabilizer $\mathrm{SL}(2,\mathbb{C})^q$ is reduced to the identity. We define,

$$\begin{aligned} \pi \quad : \quad & Z \to \mathbb{C}\,, \\ & \big(\alpha x + \beta y, (a x^2 + b x y + c y^2)^2\big) \to \big(a\beta^2 - b\beta\alpha + c\alpha^2\big)^2\,. \end{aligned}$$

We have that π is an open surjective regular map, and that the preimage $\pi^{-1}(\lambda)$ for $\lambda \in \mathbb{C}$ consists in one orbit. In fact it is a geometric quotient (see Definition 3.9) $Z/G \simeq \mathbb{C}$ and $\mathbb{C}[Z]^G = \mathbb{C}\left[\big(a\beta^2 - b\beta\alpha + c\alpha^2\big)^2\right]$. However, the graph map Ψ is not even closed. Let $Y \subset \mathrm{SL}(2,\mathbb{C}) \times Z$ be the closed subvariety,

$$Y = \left\{ \left(\begin{pmatrix} 0 & -\lambda^{-1} \\ \lambda & 0 \end{pmatrix}, (\lambda x + y,\, x^2 y^2) \right) \middle|\, \lambda \in \mathbb{C}^* \right\}.$$

Then,

$$\Psi(Y) = \left\{ \left((-\lambda x + y,\, x^2y^2),\, (\lambda x + y,\, x^2 y^2)\right) \middle|\, \lambda \in \mathbb{C}^* \right\}.$$

But $\Psi(Y)$ is not Zariski closed, its closure is:

$$\overline{\Psi(Y)} = \Psi(Y) \cup \{((y, x^2y^2), (y, x^2y^2))\}\,.$$

Therefore, the action of $\mathrm{SL}(2,\mathbb{C})$ on Z is not separated, not proper, not algebro–geometrically free.

Let us recall that a regular map $\varphi\colon Y \to Z$ is an *affine morphism* if for any affine open subset $U \subseteq Z$ the fibre $\varphi^{-1}(U)$ is an affine open subset of U. The following facts can be consulted in [14] chapter 0.

Proposition 3.7. *Le Z be a G–variety. Assume that a geometrical quotient $\pi\colon Z \to Q$ exists.*

a. *The action of G in Z is closed.*

b. *Q is a categorical quotient and therefore unique up to isomorphism.*

c. *The action of G in Z is separated if and only if Q is a $\mathbb{C}$–variety.*

d. *If G and Z are affine then π is an affine morphism.*

The following result (due to Roselicht [18]) guarantees the existence of a geometric quotient of a dense Zariski G–invariant open subset U of Z. Note that if Q is a geometric quotient of U, then $\mathbb{C}(Z)^G \simeq \mathbb{C}(Q)$.

Theorem 3.8 (Roselicht, 1963)**.** *Let Z be a G–variety. There exist a dense Zariski open subset $U \subset Z$ invariant under the action of G such that there exist a geometric quotient Q of U.*

A corollary to Rosenlicht theorem is that if H is a subgroup of G, then the quotient

$$G/H = \{[\sigma] = \sigma H \mid \sigma \in G\},$$

is a geometrical quotient. The dense open subset U the theorem must be invariant by left translations —which are H–equivariant maps of G on itself— and therefore U is G.

Quotients by reductive groups

In the case of actions of reductive groups we have the existence of a categorical quotient, and a geometric quotient if the action is closed. Let us recall that the *radical* of an algebraic group is its maximal normal solvable algebraic subgroup.

Definition 3.10. A linear algebraic group G is called reductive if its radical is a torus, *i.e.* a direct product of multiplicative groups, $\mathbb{T}_m = (\mathbb{C}^*)^m$.

Theorem 3.9. *Let G be a linear algebraic group . The following assertions are equivalent:*

a. *G is reductive.*

b. *G contains no closed subgroups isomorphic to the additive $\mathbb{C}^n$.*

c. *Every finite dimensional G–module is semi–simple.*

d. *Every G–module is semi–simple.*

Let us assume that G is a linear algebraic group over the field $\mathbb{C}$ of complex numbers. The above statements are equivalent to:

e. *G has a compact Lie subgroup $K \subset G$ dense in the Zariski topology.*

Proposition 3.10. *Let G be a reductive algebraic group and Z and affine G–variety. The $\mathbb{C}$–algebra $\mathbb{C}[Z]^G$ is finitely generated.*

Let G be reductive and Z an affine G–variety. Let us denote by Q the spectrum $\mathrm{Spec}_{\mathbb{C}}(\mathbb{C}[Z]^G)$. Since $\mathbb{C}[Z]^G$ is finitely generated, it is an affine $\mathbb{C}$–variety. One way of seeing Q constructively is to take $y_1, \cdots, y_m$ a system of generators of $\mathbb{C}[Z]^G$. Then, the image of the map,

$$\begin{aligned} \pi \quad : \quad & Z \to \mathbb{Q} \subset \mathbb{C}^m \,, \\ & p \mapsto (y_1(p), \cdots, y_n(p)) \,, \end{aligned}$$

is Q, realized as a Zariski closed subset of $\mathbb{C}^m$.

Theorem 3.11. *Let G be reductive algebraic group, Z an affine G–variety, and $Q = \mathrm{Spec}_{\mathbb{C}}(\mathbb{C}[Z]^G)$. Consider the canonical map $\pi\colon Z \to Q$. The following statements hold:*

a. $\mathrm{Spec}_{\mathbb{C}}(\mathbb{C}[Z]^G)$ *is a categorical quotient.*

b. *Each fibre of π contains a unique closed orbit.*

d. *If Z is irreducible so is* $\mathrm{Spec}_{\mathbb{C}}(\mathbb{C}[Z]^G)$.

e. *If Z is smooth so is* $\mathrm{Spec}_{\mathbb{C}}(\mathbb{C}[Z]^G)$.

Then, the affine $\mathbb{C}$–manifold Q can be seen as the space of closed orbits, the action of G on Z. In virtue of Theorem 3.8 there must be a G–invariant Zariski open subset $U \subset Z$ such that a geometric quotient U/G. For reductive groups we have the following construction.

Definition 3.11. A point $p \in Z$ is called *stable* if the $G \cdot p$ is closed and G^p is finite. The set Z^{s} of stable points is a possibly empty G–invariant open subset.

Proposition 3.12. *The map $\pi|_{Z^{\mathrm{s}}}\colon Z^{\mathrm{s}} \to \pi(Z^{\mathrm{s}})$ is a geometric quotient.*

Theorem 3.13 ([14] Th. 1.1 and Am. 1.3, case of ch. 0)**.** *Let G be a reductive algebraic group, Z an affine G–variety, and $Q = \mathrm{Spec}_{\mathbb{C}}(\mathbb{C}[Z]^G)$. Consider the canonical map $\pi\colon Z \to Q$. Then Q is a geometric quotient if and only if the action of G in Z is closed.*

3.3 Rational actions

Birrational geometry

Definition 3.12. Let Y and Z be $\mathbb{C}$–varieties, $\phi\colon U \to Z$, $\varphi\colon U' \to Z$ regular maps defined in some dense open subsets $U, U' \subset Y$. We say that ϕ and φ define the same rational map if $\phi|_{U\cap U'} = \varphi|_{U\cap U'}$.

We denote by $\varphi\colon Y \dashrightarrow Z$ the rational map defined by φ.

A birrational map φ has a *domain of definition* $\mathrm{Dom}(\varphi)$ which is the union of the domains of definitions of all its representatives. By gluing those representatives, it is clear that φ is defined as a regular map in its whole domain $\varphi\colon \mathrm{Dom}(\varphi) \to Z$.

A regular map $Y \times Z$ is said to be *dominant* if its image contains an open dense subset. A rational map is dominant if any of its representatives is dominant. If $\varphi\colon Y \dashrightarrow Z$ is a dominant rational map then it induces a $\mathbb{C}$–algebra morphism $\varphi^*\colon \mathbb{C}(Z) \hookrightarrow \mathbb{C}(Y)$.

A rational dominant map $\varphi\colon Y \dashrightarrow Z$ is said to be *birrational* if it is dominant and admits a rational inverse map. Then, φ^* is a $\mathbb{C}$–algebra isomorphism $\mathbb{C}(Z) \simeq \mathbb{C}(Y)$. *Birrational geometry* leads with the classification of algebraic varieties up to birrational equivalence. For any irreducible algebraic variety Z we consider the group $\mathrm{Birr}_{\mathbb{C}}(Z)$ of birrational automorphisms, which is naturally anti–isomorphic to the group $\mathrm{Aut}_{\mathbb{C}}(\mathbb{C}(Z))$ of $\mathbb{C}$–algebra automorphisms of $\mathbb{C}(Z)$.

Rational actions and their regularization

Definition 3.13. A *rational* action of G in Z is a rational map,

$$a\colon G \times Z \dashrightarrow Z\,,$$

that induces a group morphism,

$$\Phi_a\colon G \to \mathrm{Birr}_{\mathbb{C}}(Z)\,.$$

We say that the action of G in V is *good* if for all $x \in V$ we have $(x, e) \in \mathrm{Dom}(a)$.

The subset of points x in Z such that $(e, x) \in \mathrm{Dom}(a)$ is a open dense subset Z' of Z. Therefore, replacing Z by Z' we can assume that any rational action is good. The following results relate rational and algebraic actions.

Theorem 3.14 (Weil group chunk theorem 1955). *Supose that G acts rationally on Z. The variety Z can be G–equivariantly embedded as a dense open sub–$\mathbb{C}$–variety of some $\mathbb{C}$–variety Y where G acts regularly if and only if the action is good.*

Corollary 3.15. *Suppose that G acts rationally on Z. Then there exist Y birrationally isomorphic to Z where G acts regularly.*

Rational quotients

Let Z be an irreducible G–manifold. We say that a rational invariant $y \in \mathbb{C}(Z)^G$ separates the orbits $G \cdot p$ and $G \cdot q$ if $y(p) \neq y(q)$. We say that a set $\{y_1, \cdots, y_m\}$ *separates orbits in general position* if there exist a non empty open G–invariant subset $U \subset Z$ such that the orbits in U are separated by at least one function in $\{y_1, \cdots, y_m\}$.

Theorem 3.16. *Let Z be an irreducible G–variety. There is a finite set of rational invariants $\{y_1, \cdots, y_m\}$ that separates orbits in general position. Conversely, if $\{y_1, \cdots, y_m\}$ separates orbits in general position then $\mathbb{C}(Z)^G = \mathbb{C}(y_1, \cdots, y_m)$.*

Definition 3.14. Let G act rationally in Z. A rational quotient is a dominant rational map, $\pi\colon Z \dashrightarrow Q$ such that $\pi^*\colon \mathbb{C}(Q) \to \mathbb{C}(Z)$ induces an isomorphism $\mathbb{C}(Q) \simeq \mathbb{C}(Z)^G$.

It is clear that rational quotients are models of $\mathbb{C}(Z)^G$, and therefore they are unique up to birrational equivalence. The existence of rational quotients follows from the Theorem 3.8. If G acts regularly on Z, and U is an invariant dense open subset such that the geometric quotient U/G exists, then any rational quotient of Z is birrationally equivalent of U/G.

3.4 Computation of rational invariants

In this section G will be a $\mathbb{C}$–algebraic group acting rationally on an affine irreducible $\mathbb{C}$–variety Z. We also assume that the action is good; therefore, there is an underlying regular action on some $\bar{Z}$ which contains Z as a dense open subset. By the Lemma 3.5 we may assume that the action of G in $\bar{Z}$ is regular and separated.

In order to clarify the notation of this section it is useful to distinguish the two components of the Cartesian product $Z \times Z$. Therefore we will write $Z \times \hat{Z}$, where Z and $\hat{Z}$ represent two copies of the affine variety Z.

Let us consider the (rational) graph map Ψ_α and its image, the graph $\Gamma_\alpha \subset Z \times \hat{Z}$. Since the action of G is separated, the graph Γ_α is a closed affine sub–$\mathbb{C}$–variety in $\mathbb{C}[Z \times \hat{Z}]$. The graph ideal is:

$$\mathfrak{I}(\Gamma_\alpha) = \{f \in \mathbb{C}[Z \times \hat{Z}] \,|\, f \circ \Psi_\alpha = 0\}\,.$$

Let us consider the canonical embedding $\mathbb{C}[Z \times \hat{Z}] \subset \mathbb{C}(Z)[\hat{Z}]$. The elements $f(z, \hat{z}) \in \mathbb{C}(Z)[\hat{Z}]$ are polynomials in the second variables $\hat{z}$ with rational coefficients in the first variables z. We will call the *generic graph ideal* to the extension,

$$\mathfrak{J} = \mathfrak{I}(\Gamma_\alpha) \cdot \mathbb{C}(Z)[\hat{Z}]\,.$$

In order to compute the rational invariants, we will prove that the generic graph ideal is generated by a polynomial with rational invariant coefficients. Let us remind the following general algebraic lemma from [17], page 155.

Lemma 3.17. *Suppose $\mathbb{L}$ is an extension of the field $\mathbb{F}$ and G is a group of its automorphisms. Suppose V is a vector space over $\mathbb{F}$ and W is a subspace of the vector space $\mathbb{L} \otimes_{\mathbb{F}} V$ which is invariant under the natural action of G. Then W is generated by invariant elements.*

The above lemma, applied to the particular case of $\mathbb{C} \subset \mathbb{C}(Z_1)$, where the algebraic group G is seen as a group of algebraic automorphisms of $\mathbb{C}(Z_1)$ gives us the following result.

Theorem 3.18. *Let us consider the ideal $\mathfrak{J} = \mathfrak{I}(\Gamma_\alpha) \cdot \mathbb{C}(Z)[\hat{Z}]$. There is a system of generators of $\mathfrak{J}$ in $\mathbb{C}(Z_1)^G[Z_2]$. If $\{F_1 \cdots F_p\}$ is a finite set of generators of $\mathfrak{J}$ in $\mathbb{C}(Z_1)^G[Z_2]$ and,*

$$F_i(z, \hat{z}) = \sum_{k=1}^{n_i} f_{i,k}(z) u_{i,k}(\hat{z})\,, \tag{12}$$

with

$$f_{i,k}(z) \in \mathbb{C}(Z)^G,\ u_{i,k}(\hat{z}) \in \mathbb{C}[\hat{Z}]\,,$$

then $\{f_{11}(z), \cdots, f_{k,n_k}(z)\}$ separates orbits in general position and therefore $\mathbb{C}(Z)^G = \mathbb{C}(f_{11}(z), \cdots, f_{k,n_k}(z))$.

Proof. Let $\mathbb{L} = \mathbb{C}(Z)$. We consider the action of G in Z, and therefore as a group of $\mathbb{C}$–algebra automorphisms of $\mathbb{C}(Z)$. The ideal $\mathfrak{J}$ is a G–invariant $\mathbb{C}(Z)$–subspace of $\mathbb{C}(Z) \otimes_{\mathbb{C}} \mathbb{C}[\hat{Z}] = \mathbb{C}(Z)[\hat{Z}]$. By the Lemma

3.17 it is generated as a vector space over $\mathbb{C}(Z)$ by G–invariant elements. Among those elements we can choose a finite set of generators, $\{F_1 \cdots F_p\}$ as in equation (12).

Let us see that the $f_{i,j}(z)$ separate orbits in general position. We replace Z by a suitable principal open subset Z^* such that:

i) $f_{i,k}(z) \in \mathbb{C}[Z^*]$.

ii) $F_i(z_1, z_2) \in \mathfrak{I}(\Gamma_\alpha)$ for all $i = 1, \cdots, p$.

iii) $\mathfrak{I}(\Gamma_\alpha) = (F_1(z_1, z_2), \cdots, F_p(z_1, z_2))$.

Under these conditions, the orbit of a point $p \in Z$ is defined by the equations for the variables z,

$$F_i(p,\, z) = \sum_k f_{i,k}(p)\, u_{i,k}(z)\,,$$

with $i = 1, \cdots, p$. Thus, it follows that two points p and q in Z^* have the same orbit if and only if $f_{i,k}(p) = f_{i,k}(q)$ for all i, k. $\square$

The theorem above says that $\mathbb{C}(Z_1)^G$ is the defining field of the ideal $\mathfrak{I}$. In order to compute a system of generators of an ideal with coefficients in its defining field, we can make use of *Gröbner basis*. Now, let us consider that Z is birrational to $\mathbb{C}^m$, therefore we can assume that $\mathbb{C}[Z]$ is the ring of polynomials in m variables and take any *monomial order* in $\mathbb{C}[\hat{Z}] = \mathbb{C}[\hat{z}_1, \cdots, \hat{z}_m]$.

Theorem 3.19 (Hubert-Kogan, 2007)**.** *Let us consider a good rational action of G in $Z = \mathbb{C}^n$. Let $\{F_1, \cdots, F_p\}$ be a reduced Gröbner basis of the generic graph ideal $\mathfrak{I}$:*

$$F_i = \sum_\alpha f_{i,\alpha}(z)\, \hat{z}^\alpha\,,$$

with $f_{i,\alpha}(z) \in \mathbb{C}(Z)$. Then, the coefficients $\{f_{i,\alpha}(z)\}$ form a set of rational invariants separating orbits in general position.

Example 3.3 (Scaling transformations)**.** Let us consider the group $\mathbb{C}^*$ acting on $\mathbb{C}^2$ by $\lambda \cdot (x, y) = (\lambda x, \lambda y)$. The graph Γ_α is the image of the graph map $\Psi_\alpha\colon (\lambda, \xi, \eta) \mapsto (x, y, \hat{x}, \hat{y}) = (\lambda\xi, \lambda\eta, \xi, \eta)$. We consider the variables λ, ξ, η as parameters and we eliminate them. We obtain $\mathfrak{I} = (\hat{x}y - \hat{y}x)$. A reduced Gröbner basis is $\{\hat{x} - \frac{x}{y}\hat{y}\}$ and therefore $\mathbb{C}(x, y)^G = \mathbb{C}\left(\frac{x}{y}\right)$.

Example 3.4 (Quasi homogeneous transformations)**.** Let p and q be coprime integer numbers. Let us consider the action of $\mathbb{C}^*$ on $\mathbb{C}^2$ by $\lambda \cdot (x, y) = (\lambda^p x, \lambda^q y)$. As above, we compute the parametric equations for Γ_α, $(x, y, \hat{x}, \hat{y}) = (\lambda^p \xi, \lambda^q \eta, \xi, \eta)$ and we eliminate the parameters obtaining $\mathfrak{J} = (\hat{x}^q y^p - \hat{y}^p x^q)$. Using lexicographic order, a reduced Gröbner basis is $\{\hat{x}^q - \frac{x^q}{y^p}\hat{y}^p\}$, and therefore $\mathbb{C}(x, y)^G = \mathbb{C}\left(\frac{x^q}{y^p}\right)$.

Example 3.5 (Rotations)**.** The group $SO(1, \mathbb{C})$ is identified with the unit circle. $SO(1, \mathbb{C}) = \{(a, b) \in \mathbb{C}^2 | a^2 + b^2 = 1\}$. It acts in $\mathbb{C}^2$ in the following way, $(a, b) \cdot (x, y) = (ax - by, bx + ay)$. The parametric equations of Γ_α are now $(x, y, \hat{x}, \hat{y}) = (a\xi - b\eta, b\xi + a\eta)$. The elimination of the parameters gives us $\mathfrak{J} = (\hat{x}^2 + \hat{y}^2 - x^2 - y^2)$. If follows that $\mathbb{C}(x, y)^G = \mathbb{C}(x^2 + y^2)$

3.5 Cross sections and replacement invariants

Let us consider G acting separatedly and regularly on Z; let s be the dimension of the orbits.

Definition 3.15. A cross section of degree d of the action of G in Z is a closed subvariety $\subset Z$ that intersects generic orbits in d simple points. That is, there exist a G–invariant dense open subset $U \subset Z$ such that any orbit contained in U intersects S in d simple points.

Let $S \subset Z$ a variety of codimension s. Let us consider the graph $\Gamma_\alpha \subset Z \times \hat{Z}$. We consider $\hat{S}$ a copy of the variety in S in $\hat{Z}$, and its ideal $\mathfrak{I}(\hat{S}) \subset \mathbb{C}[\hat{Z}]$. We call the *generic graph section ideal* of S to,

$$\mathfrak{J}_S = \mathfrak{I}(\hat{S}) \cdot \mathbb{C}(Z)[\hat{Z}] + \mathfrak{I}(\Gamma_\alpha) \cdot \mathbb{C}(Z)[\hat{Z}]\,.$$

The following result holds.

Proposition 3.20. *The variety $S \subset Z$ is a cross section if and only if $\mathfrak{J}_S$ is a zero dimensional radical ideal of $\mathbb{C}(Z)[\hat{Z}]$. The degree of S is the dimension of $\mathbb{C}(Z)[\hat{Z}]/\mathfrak{J}_S$ as $\mathbb{C}(Z)$–vector space.*

Since the generic graph ideal $\mathfrak{J}$ is defined over the field $\mathbb{C}(Z)^G$ and the section ideal $I(\hat{S})$ is defined over the field of constants $\mathbb{C}$ it follows that the generic graph section ideal is also defined over $\mathbb{C}(Z)^G$. In many cases to compute the Gröbner basis of a zero dimensional ideal is rather easier, and then cross sections can be useful to compute rational ideals.

Theorem 3.21 (Hubert-Kogan, 2007). *Let us consider a good rational action of G and S a cross section. in $Z = \mathbb{C}^n$. Let $\{F_1, \cdots, F_p\}$ be a reduced Gröbner basis of the generic graph section ideal $\mathfrak{J}_S$:*

$$F_i = \sum_\alpha f_{i,\alpha}(z)\hat{z}^\alpha\,,$$

with $f_{i,\alpha}(z) \in \mathbb{C}(Z)$. Then, the coefficients $\{f_{i,\alpha}(z)\}$ form a set of rational invariants separating orbits in general position.

Theorem 3.22 (Hubert-Kogan, 2007). *Assume Z is birrational to $\mathbb{C}^m$, and the generic orbit of G in Z is of dimension s. Then, to each point $(a_{ij})_{1\leq i\leq s;\, 0\leq j\leq m}$ outside an algebraic subset of $\mathbb{C}^{s(m+1)}$ we can associate a linear cross section:*

$$S:\, a_{i0} - \sum_{j=1}^m a_{ij}z_j\,,$$

with $i = 1, \cdots, s$, where $z_1, \cdots, z_m$ are the affine coordinates in Z.

Example 3.6 (Quasi homogeneous transformations). Let us revisit the example 3.4 of quasihomogeneous transformations, $\lambda\cdot(x, y) = (\lambda^p x, \lambda^q y)$, with p, q coprime integers. The line $S = \{y = 1\}$ is a cross section of degree q. We can add the equation of $\hat{S}$ to the equations of Γ_α to obtain the generic graph section ideal $\mathfrak{J}_S = (\hat{x}^q y^p - \hat{y}^p x^q, \hat{y} - 1)$ and therefore obtain a reduced Gröbner basis, $\left\{\hat{y} - 1, \hat{x}^q - \frac{x^q}{y^p}\hat{y}^p\right\}$, containing the rational invariants as coefficients.

Replacement invariants

As before let us consider a good rational action of G on an affine variety Z, which comes from a regular and separated action on $\bar{Z}$. Let us consider a cross section S.

Definition 3.16. An algebraic invariant $f(z)$ is an algebraic element over the field $\mathbb{C}(Z)^G$. The field of algebraic invariants is the algebraic closure $\overline{\mathbb{C}(Z)^G}$.

Remark 3.23. Let us assume that $\mathbb{C}$ is the field of complex numbers. Let S be a cross section of degree d. There is a closed subset X of orbits which does not intersect S in d simple points. Let us define $Z^* = Z \setminus X$ and $S^* = S \setminus X$. We have a cross section S^* of degree d of the action of

G in Z^* in the sense of Lie group actions. Therefore, there is a natural map of analytic manifolds $\rho\colon S^* \to Z^*/G$, which is a covering of degree d.

Theorem 3.24. *Let S be an irreducible cross section of degree d. Then the inclusion map $i\colon S \to Z$ induces an algebraic extension $\mathbb{C}(Z)^G \to \mathbb{C}(S)$ of degree d. The rational functions on S are algebraic invariants.*

Proof. Let us consider $\rho\colon Z \dashrightarrow Q$ a birrational model of Z/G. Then we have a rational map $\rho_0\colon S \dashrightarrow Q$, which is a covering of degree d and produces an algebraic extension $\mathbb{C}(Q) \hookrightarrow \mathbb{C}(S)$ of degree d. By the identification $\mathbb{C}(Q) \simeq \mathbb{C}(Z)^G$ we finish the proof. $\square$

The generic graph section ideal $\mathfrak{I}_S$ defines an algebraic subset in $S^* \subset \hat{Z}(\overline{\mathbb{C}(Z)})$, the algebraic variety $\hat{Z}$ with coordinates in the algebraic closure $\overline{\mathbb{C}(Z)}$.

Definition 3.17. A replacement invariant is a point of S^*.

By definition $\mathfrak{I}_S$ is a zero dimensional radical ideal. The variety S^* consists of d simple points, and therefore there exist exactly d different replacement invariants.

Proposition 3.25. *Let ξ be a replacement invariant, then $\xi \in Z(\mathbb{C}(S)) \subset Z(\overline{\mathbb{C}(Z)^G})$. In particular if Z is birrational to $\mathbb{C}^n$ then $\xi = (\xi_1, \cdots, \xi_n)$ with $\xi_i \in \mathbb{C}(S)$ algebraic invariants of degree d.*

Proof. It comes from a canonical isomorphism $\mathbb{C}(S) = \mathbb{C}(\hat{S}) \simeq \mathbb{C}(Z)[\hat{Z}]/\mathfrak{I}_S$. $\square$

The following result, which allows to rewrite rational invariants as functions of the replacement invariants, can be seen as an algebraic form of the Thomas replacement theorem [20].

Theorem 3.26 (Thomas replacement)**.** *Let Z be $\mathbb{C}^n$ and let $f(z_1, \cdots, z_n) \in \mathbb{C}(Z)^G$ be a rational invariant. Then, for any replacement invariant $(\xi_1, \cdots, \xi_n)$ we have,*

$$f(z_1, \cdots, z_n) = f(\xi_1, \cdots, \xi_n) \in \overline{\mathbb{C}(Z)^G}.$$

Example 3.7 (Quasi homogeneous transformations)**.** Let us revisit the examples 3.4 and 3.6 of quasihomogeneous transformations, $\lambda \cdot (x, y) = (\lambda^p x, \lambda^q y)$, with p, q coprime integers. The line $S = \{y = 1\}$ is a cross section of degree q. We compute $\mathfrak{I}_S = (\hat{x}^q y^p - \hat{y}^p x^q, \hat{y} - 1)$. The replacement invariants are the solutions $(\hat{x}, \hat{y})$ of the above equations. We obtain $\left(1, \sqrt[q]{\frac{x^q}{y^p}}\right)$, q different solutions for the q different determinations of the q–th root. By Thomas replacement theorem, for any rational invariant $f(x, y)$ we have,

$$f(x,y) = f\left(1, \sqrt[q]{\frac{x^q}{y^p}}\right).$$

4 Algebraic Lie–Tresse theorem

In this section we develop a general theory of differential invariants and give an algebraic version of Lie–Tresse theorem, the finiteness of the differential invariant algebra. Usually, differential invariants are developed using jet bundles. Here we develop the theory for Weil bundles; it has two advantages: many natural operations are easier to handle in the Weil bundles context, and we avoid Tresse derivatives. Since the jet bundles are geometric quotients of Weil bundles, any rational function defined on a jet bundle lifts naturally to a Weil bundle. Therefore there is no loss of generality in the use of Weil bundles instead of jet bundles. In the last part we will expose our results in the jet bundle language.

4.1 Weil algebras

Let $\mathbb{C}$ be an algebraically closed field of zero characteristic.

Definition 4.1. A *Weil* $\mathbb{C}$*–algebra* is a local (rational[4]) finite dimensional commutative $\mathbb{C}$–algebra.

From now on we will drop the prefix $\mathbb{C}$–, and we will assume that all the algebras and algebra morphism are linear over $\mathbb{C}$. Let A be a Weil algebra. We denote by $\mathfrak{m}_A$ the maximal ideal of A. There is a canonical decomposition $A = \mathbb{C} \oplus \mathfrak{m}_A$. We denote by ω_A its rational point, which is the projection $\omega_A \colon A \to \mathbb{C}$.

Example 4.1. Let $\mathbb{C}^{(1,1)} = \mathbb{C}[[\varepsilon]]/\varepsilon^2$ be the algebra of dual numbers, with multiplication $(a+b\varepsilon)(c+d\varepsilon) = ac+(ad+bc)\varepsilon$. It is a 2–dimensional

[4] This assumption is not necessary in our definition, since we assume that $\mathbb{C}$ is a algebraically closed field. However, it appears in the general definiton of Weil algebra over any field.

Weil algebra. Its maximal ideal is the vector space spanned by ε. The rational point is $\mathbb{C}^{(1,1)} \mapsto \mathbb{C}$, $a + b\varepsilon \mapsto a$.

Since A is finite dimensional its maximal ideal $\mathfrak{m}_A$ has a finite number of generators. The minimal cardinal of a system of generators of $\mathfrak{m}_A$ is called the *width* of A, and coincides with the dimension of the vector space $\mathfrak{m}_A/\mathfrak{m}_A^2$. The elements of $a \in \mathfrak{m}_A$ are nilpotent. In fact, there is a number l such that $\mathfrak{m}_A^{l+1} = (0)$ and $\mathfrak{m}_A^l \neq (0)$. This number l is called the *order* of A. Every element $a \in A$ which does not belong to $\mathfrak{m}_A$ is invertible.

Example 4.2. Let $\mathbb{C}^{(m,r)} = \mathbb{C}[[\varepsilon_1, \cdots, \varepsilon_r]]/(\varepsilon_1, \cdots, \varepsilon_r)^{r+1}$ be the algebra series in the variables $\varepsilon_1, \cdots, \varepsilon_m$ truncated to the degree r:

$$\begin{aligned} a(\varepsilon) &= \sum_{|\alpha| \leq r} a_\alpha \frac{\varepsilon_1^{\alpha_1} \cdots \varepsilon_m^{\alpha_m}}{\alpha_1! \cdots \alpha_m!}, \\ \alpha &= (\alpha_1, \cdots, \alpha_m) \in \mathbb{Z}_+^n, \\ |\alpha| &= \sum_{i=1}^n \alpha_i . \end{aligned}$$

The multiplication law is given by

$$a(\varepsilon) \cdot b(\varepsilon) = \sum_{|\alpha|+|\beta| \leq r} a_\alpha \, b_\beta \frac{\varepsilon_1^{\alpha_1} \cdots \varepsilon_m^{\alpha_m}}{\alpha_1! \cdots \alpha_m!} \frac{\varepsilon_1^{\beta_1} \cdots \varepsilon_m^{\beta_m}}{\beta_1! \cdots \beta_m!} .$$

We have that $\mathbb{C}^{(m,r)}$ is a Weil algebra of width m and order r, its maximal ideal $\mathfrak{m}_{(m,r)}$ consists of the truncated power series $a(\varepsilon)$ with zero independent term, *i.e.* such that $a(0) = 0$.

We also denote $\mathbb{C}^{(m,\infty)}$ the algebra $\mathbb{C}[[\varepsilon_1, \cdots, \varepsilon_m]]$ of formal series in m variables. The algebra $\mathbb{C}^{(m,\infty)}$ is not a Weil algebra (it has infinite dimension) but it is a rational local algebra. Its maximal ideal $\mathfrak{m} = (\varepsilon_1, \cdots, \varepsilon_r)$ consists of formal series having zero independent terms.

A Weil algebra morphism is an algebra morphism $\varphi \colon A \to B$. In general, any algebra morphism sends nilpotent elements nilpotent elements, Thus, any Weil algebra morphism φ is local, *i.e.* it sends the maximal ideal of A to the maximal ideal of B, $\varphi(\mathfrak{m}_A) \subseteq \mathfrak{m}_B$.

Proposition 4.1. *If A has order $\leq r$ and width $\leq m$ then there is a surjective Weil algebra morphism $\pi \colon \mathbb{C}^{(m,r)} \to A$. In particular all Weil algebras of width $\leq m$ are quotients of $\mathbb{C}^{(m,\infty)}$.*

Proof. The width of A is less or equal than m, so that we can find a system of generators of $\mathfrak{m}_A$ with m elements. Let $a_1, \cdots, a_m$ form a system of generators of of $\mathfrak{m}_A$. There is a unique algebra morphism,

$$\begin{aligned} \varphi \quad : \quad & \mathbb{C}^{(m,\infty)} \to A\,, \\ & \varepsilon_i \mapsto a_i\,, \end{aligned}$$

that sends ε_i to a_i for $1 \leq i \leq m$. This morphism is surjective. The ideal $(\varepsilon_1, \cdots, \varepsilon_m)^l$ is in the kernel of φ since $\varphi((\varepsilon_1, \cdots, \varepsilon_m)^l) = \varphi((\varepsilon_1, \cdots, \varepsilon_m))^l \subseteq \mathfrak{m}_A^l = 0$. Thus, φ induces a surjective morphism $\bar{\varphi}\colon \mathbb{C}^{(m,r)} \to A$. □

Groups of automorphisms

Theorem 4.2. *Let A and B be Weil algebras. Let $a_1, \cdots, a_m$ be a minimal system of generators of $\mathfrak{m}_A$ and $b_1, \cdots, b_n$ be a basis of B as vector space. Then there is a natural structure of affine variety in the set* $\mathrm{Hom}_{\mathbb{C}-alg}(A, B)$ *such that coefficients* $\mathbb{C}_{ij}\colon \mathrm{Hom}_{\mathbb{C}-alg}(A, B) \to \mathbb{C}$ *defined by,*

$$\varphi(a_i) = \sum_{j=1}^{n} c_{ij}(\varphi) b_j, \quad \forall \varphi \in \mathrm{Hom}_{\mathbb{C}-alg}(A, B)\,,$$

are regular functions.

Proof. Any algebra automorphism is a linear map, then we have $\mathrm{Hom}_{\mathbb{C}-alg}(A, B) = \mathrm{Hom}_{\mathbb{C}}(A, B) = A^* \otimes_{\mathbb{C}} B$. The coefficients c_{ij} are regular functions in $A^* \otimes B$ by definition. We just need to prove that $\mathrm{Hom}_{\mathbb{C}-alg}(A, B)$ is a Zariski closed subset of $A^* \otimes B$. Let us complete the generating system $a_1, \cdots, a_m$ to a linear basis $a_1, \cdots, a_m, a_{m+1} \cdots, a_M$ of A. We have that the $M \times n$ coefficients,

$$\phi_{a_i} = \sum_{j=1}^{n} c_{ij}(\phi) b_j, \quad \forall \phi \in \mathrm{Hom}_{\mathbb{C}}(A, B)\,,$$

is a linear system of coordinates in $A^* \otimes_{\mathbb{C}} B$. Thus, in particular, the $m \times n$ first coefficients c_{ij}, $1 \leq i \leq m$ are regular functions.

We have that the linear map ϕ is an algebra homomorphism if and only if $\phi(a_i a_j) = \phi(a_i)\phi(a_j)$ for all $1 \leq i \leq m$. The algebra structure in A and B is determined by some structure constants,

$$\begin{aligned} a_i\, a_j &= \sum_{k=1}^{M} \lambda_{ij}^k\, a_k \,, \\ b_i\, b_j &= \sum_{k=1}^{n} \mu_{ij}^k\, b_k \,. \end{aligned}$$

Substituting, we obtain,

$$\sum_{k=1}^{M} \sum_{s=1}^{n} \lambda_{ij}^k\, c_{ks}\, b_s = \sum_{l=1}^{n} \sum_{r=1}^{n} \sum_{s=1}^{n} c_{jr}\, c_{il}\, \mu_{rl}^s\, b_s \,,$$

and therefore,

$$\sum_{s=1}^{n} \left(\sum_{k=1}^{M} \lambda_{ij}^k\, c_{ks} - \sum_{l=1}^{n} \sum_{r=1}^{n} c_{jr}\, c_{il}\, \mu_{rl}^s \right) b_s = 0 \,.$$

We obtain the system of algebraic equations

$$\sum_{k=1}^{M} \lambda_{ij}^k\, c_{ks} - \sum_{l=1}^{n} \sum_{r=1}^{n} c_{jr}\, c_{il}\, \mu_{rl}^s = 0, \,,$$

with $1 \le i \le m$, $1 \le j \le m$, $1 \le s \le n$. Thus, $\mathrm{Hom}_{\mathbb{C}-alg}(A, B)$ is an algebraic subset of $A^* \otimes_{\mathbb{C}} B$. □

Remark 4.3. Taking into account that Weil algebras decompose $A = \mathbb{C} \oplus \mathfrak{m}_A$ and that Weil algebra morphisms are local, we have a lower dimensional embedding $\mathrm{Hom}_{\mathbb{C}-alg}(A, B) \subset \mathfrak{m}_A^* \otimes_{\mathbb{C}} \mathfrak{m}_B$.

Let us consider now the complex analytic case. Let $(\mathbb{C}^n, 0)$ denote the pointed affine space, and $f, g\colon (\mathbb{C}^n, 0) \to (\mathbb{C}^m, 0)$ a germ of an analytic map at zero. Let us consider $x_1, \cdots, x_n$ and $y_1, \cdots, y_m$ the canonical coordinate systems on the source and target respectively. The map f is determined by its Taylor series,

$$y_i = f_i(x_1, \cdots, x_m) = \sum_{|\alpha| \ge 1} \frac{\partial^{|\alpha|} f_i}{\partial x_1^{\alpha_1} \cdots \partial x_n^{\alpha_n}} \frac{x_1^{\alpha_1} \cdots x_n^{\alpha_n}}{\alpha_1! \cdots \alpha_n!} \,.$$

We say that f and any other analytic map $g\colon (\mathbb{C}^n, 0) \to (\mathbb{C}^m, 0)$ *coincide up to order* r if the cofficients of their Taylor development up to order r

coincide. Let $a = a(y_1, \cdots, y_m)$ be a formal series $a(y) \in \mathbb{C}[[y_1, \cdots, y_m]]$. By replacing y in the above expression we define

$$\begin{aligned} & f^*(a)(x_1, \cdots, n) \\ = \; & a(f_1(x_1, \cdots, x_n), \cdots, f_m(x_1, \cdots, x_m)) \in \mathbb{C}[[x_1, \cdots, x_n]] . \end{aligned}$$

The map

$$f^*\colon \mathbb{C}[[x_1, \cdots, x_n]] \to \mathbb{C}[[y_1, \cdots, y_m]] ,$$

is a local map, $f^*(\,(x_1, \cdots, x_n)\,) = (y_1, \cdots, y_n)$. We can truncate it up to order r, obtaining a Weil algebra morphism $f^*_{(r)}\colon \mathbb{C}^{(m,r)} \to \mathbb{C}^{(n,r)}$. It is clear that f and g coincide up to order r if and only if $f^*_{(r)} = g^*_{(r)}$ —it also follows from here that the notion of *coincidence up to order* r is independent of the choice of coordinates—. On the other hand, for each Weil algebra morphism $\varphi\colon \mathbb{C}^{(m,r)} \to \mathbb{C}^{(n,r)}$ there is a unique polynomial map of degree $\leq r$, $f\colon (\mathbb{C}^n, 0) \to (\mathbb{C}^m, 0)$ such that $f^* = \varphi$. Therefore, we have the identification,

$$\begin{aligned} \mathrm{Hom}_{\mathbb{C}-alg}(\mathbb{C}^{(m,r)}, \mathbb{C}^{(r,m)}) \;\; = \;\; & \{ \text{classes of analytic maps} \\ & \;\; f\colon (\mathbb{C}^n, 0) \to (\mathbb{C}^m, 0) \\ & \;\; \text{up to order } r \text{ coincidence} \} . \end{aligned}$$

If we apply the Theorem 4.2 to the case $A = B$, and we consider only algebra automorphism we obtain the following result.

Corollary 4.4. *Let A be a Weil algebra, the group* $\mathrm{Aut}(A)$ *of Weil algebra automorphisms has a natural structure of algebraic group defined over* $\mathbb{C}$. *Let n be the dimension of $\mathfrak{m}_A$, then there is an inyective morphism* $\mathrm{Aut}(A) \subset \mathrm{GL}(n, \mathbb{C})$.

The group $\mathrm{Aut}(\mathbb{C}^{(m,r)})$ is called the r–th prolongation of the general linear group of rank n and is denoted by $\mathrm{GL}^{(r)}(m, \mathbb{C})$. In particular $\mathrm{Aut}(\mathbb{C}^{(m,1)}) = \mathrm{GL}(m, \mathbb{C})$. The elements of $\mathrm{Aut}(\mathbb{C}^{(m,r)})$ are Taylor series of degree r of analytic transformations, invertible near 0, of $\mathbb{C}^m$. Note that the order of the composition is reversed, $(f \circ g)^*_r = g^*_{(r)} \circ f^*_{(r)}$.

4.2 Partial differential fields

Let R be a commutative ring and M be a R–module. We recall that a *derivation* of R with values on M is an additive[5] map $\partial\colon R \to M$, satisfying Leibniz formula, *i.e.* for all $a, b \in M$ we have $\partial(ab) = b\partial(a) + a\partial(b)$. The set of derivations from R in M is denoted by $\mathrm{Der}(R, M)$[6].

The commutator $[\partial, \partial']$[7] of two derivations $\partial, \partial' \in \mathrm{Der}(K, K)$ is also a derivation[8]. We say that two derivations $\partial, \partial' \in \mathrm{Der}(K, K)$ *commute* if $[\partial, \partial'] = 0$.

Definition 4.2. A partial differential field with m commuting derivations, (from now on a Δ*–field*) is a pair $\mathcal{K} = (K, \Delta_K)$ where K is a field and $\Delta_K = (\partial_1, \cdots, \partial_m) \in \mathrm{Der}(K, K)^m$ is a m–tuple of pairwise commuting derivations of K. The subfield $K^\Delta = \{f \in K | \partial_i f = 0 \text{ for } i = 1, \cdots, m\}$ is called the *field of constants* of $\mathcal{K}$.

Example 4.3. Let us consider $\mathbb{C}(z_1, \cdots, z_n)$ the field of rational functions in m variables. Then $(\mathbb{C}(z_1, \cdots, z_n), (\partial_{z_1}, \cdots, \partial_{z_m}))$ is a differential field with field of constants $\mathbb{C}$.

Definition 4.3. A Δ–field extension $\iota\colon \mathcal{K}(K, \Delta_K) \subseteq \mathcal{K}' = (K', \Delta_{K'})$ with $\Delta = (\delta_1, \cdots, \delta_m)$ and $(\Delta_{K'}) = (\delta'_1 \cdots, \delta'_m)$ is a field extension $\iota_*\colon K \subseteq K'$ such that $\iota_* \circ \delta_i = \delta'_i \circ \iota_*$ for all $i = 1, \cdots, m$.

Definition 4.4. Let $\mathcal{K} \subset \mathcal{K}'$ be a Δ–field extension. Let $f_1, \cdots, f_n$ be in $\mathcal{K}'$[9]. We denote by $\mathcal{K}\langle f_1, \cdots, f_m\rangle$ the smallest partial differential subfield of $\mathcal{K}'$ containing $f_1, \cdots, f_m$. We say that $\mathcal{K}'$ is Δ–finitely generated over $\mathcal{K}$ if $\mathcal{K}' = \mathcal{K}\langle f_1, \cdots, f_n\rangle$ for certain elements $f_1, \cdots, f_n \in \mathcal{K}'$.

The following theorem can be found —in a stronger form— in[9], page 112, Proposition 14.

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[5] $\partial(a + b) = \partial(a) + \partial(b)$.

[6] If R is a $\mathbb{C}$–algebra, then we write $\mathrm{Der}_{\mathbb{C}}(R, M)$ for the space of derivations vanishing on $\mathbb{C}$. If A is an R–algebra, then its R–module structure is given by the algebraic morphism $\phi\colon R \to A$ that fix the R–algebra structure in A. We denote by $\mathrm{Der}_\phi(R, A)$ the space of derivations from R to A, where the R–algebra (and henceforth R–module) structure of A is given by the morphism ϕ. We do it in order to distinguish different R–algebra structures in A. For instance, we have $T_pZ = \mathrm{Der}_p(\mathbb{C}[Z], \mathbb{C})$, where p is seen as the valuation morphism $p\colon \mathbb{C}[Z] \to \mathbb{C}$, $f \mapsto f(p)$.

[7] $[\partial, \partial'] = \partial \circ \partial' - \partial' \circ \partial$.

[8] The space $\mathrm{Der}(K, K)$ as a natural structure of Lie algebra.

[9] Let $(\mathcal{K}' = (K', \Delta_{K'})$. By abuse of notation, we say that f belongs to $\mathcal{K}'$ or $f \in \mathcal{K}'$ for elements f of K'.

Theorem 4.5. *Let $\mathcal{K} \subseteq \mathcal{L} \subseteq \mathcal{K}'$ be differential field extensions of characteristic zero. Let us assume that $\mathcal{K}'$ is Δ–finitely generated over $\mathcal{K}$. Then $\mathcal{L}$ is Δ–finitely generated over $\mathcal{K}$.*

4.3 Weil bundles

Let Z be an affine $\mathbb{C}$–variety, and A a $\mathbb{C}$–algebra. We denote by $Z(A)$ the set $\mathrm{Hom}_{\mathbb{C}-alg}(\mathbb{C}[Z], A)$. An element $p^A \in Z(A)$ is called an A–point of Z. We will see than under certain assumptions on A the set $Z(A)$ admits a natural structure of variety.

Example 4.4. There is a canonical identification of $Z(\mathbb{C})$ with Z. For each point $p \in Z$ there is a morphism $\omega_p \colon \mathbb{C}[Z] \to Z$ which to each regular function f assigns its value $f(p)$.

Definition 4.5. Let $f \in \mathbb{C}[Z]$ be a regular map. We call the A–prolongation of f to the map $f^A \colon Z(A) \to A$ defined by $f^A(p^A) = p^A(f)$. Let $\omega \in A^*$ be any linear form $\omega \colon A \to \mathbb{C}$. Then, $\omega \circ f^A \colon Z(A) \to \mathbb{C}$ is called a $\mathbb{C}$–component of f. The set of $\mathbb{C}$–components of regular functions is a vector space isomorphic to $A^* \otimes_{\mathbb{C}} \mathbb{C}[Z]$, and therefore we write $\omega \otimes f$ for $\omega \circ f^A$.

Theorem 4.6. *Let A be a finite dimensional $\mathbb{C}$–algebra. Then, there is a natural structure of affine $\mathbb{C}$–variety in $Z(A)$ such that all the $\mathbb{C}$–components of A–prolongations of functions $f \in \mathbb{C}[Z]$ are regular functions in $Z(A)$.*

Proof. Let $a_1, \cdots, a_s$ be a basis of A, and then let us consider $\{\omega_1, \cdots, \omega_s\}$ its dual basis. We have then a coordinate map $A \simeq \mathbb{C}^s$, $a \mapsto (\omega_1(a), \cdots, \omega_s(a))$.

First, we can see that if $Z = \mathbb{C}^n$. For any choice of elements $b_1, \cdots, b_n \in A$ there is a homomorphism $\mathcal{C}[z_1, \cdots, z_n] \to A$ such that z_i mapsto b_i. Then $Z(A) = \mathrm{Hom}_{\mathbb{C}-alg}(\mathbb{C}[z_1, \cdots, z_n], A) = A^n$. The $\mathbb{C}$–components of the coordinate functions correspond to the new coordinate functions if we identify A with $\mathbb{C}^s$. Thus, $Z(A) \simeq \mathbb{C}^{ns}$.

Let us now consider $Z \subset \mathbb{C}^n$ with equations

$$Z : \ P_1(z_1, \cdots, z_n) = 0, \cdots, P_k(z_1, \cdots, z_m) = 0 .$$

Let $\bar{z}_i \in \mathbb{C}[Z]$ denote the coordinate functions z_i in $\mathbb{C}^n$ restricted to Z. A n–tuple $(b_1, ldots, b_n)$ of elements of A defines a homomorphism such that $z_i \mapsto b_i$ if and only if

$$P_1(b_1, \cdots, b_n) = 0, \cdots, P_k(b_1, \cdots, b_m) = 0\,.$$

Taking the $\mathbb{C}$–components of those equations we obtain the $s \times k$ polynomial equations of $Z(A)$ in $\mathbb{C}^{sn}$. □

Remark 4.7. In general we have $Z(A) = \mathrm{Spec}_{\mathbb{C}}(\mathbb{C}[\mathbb{C}[Z] \otimes_{\mathbb{C}} A^*])$ where $\mathbb{C}[\mathbb{C}[Z] \otimes_{\mathbb{C}} A^*]$ denotes the ring spanned by all the $\mathbb{C}$–components of all the A–prolongations of regular functions.

Example 4.5. Let us consider $Z(\mathbb{C} \times \mathbb{C})$. Each regular function has two components, so we are duplicating the number of coordinates. We have $Z(\mathbb{C} \times \mathbb{C}) = Z \times Z$. In general $Z(\mathbb{C}^k) = Z^k$.

The assignation $(Z, A) \rightsquigarrow Z(A)$ is in fact *a functor* in both variables, the variety Z and the $\mathbb{C}$–algebra A. Let $\varphi\colon A \to B$ be an algebra morphism, then $\varphi^A\colon Z(A) \to Z(B)$ is defined by $\varphi^A(p^A) = \varphi \circ p^A$. If $F\colon Z \to Y$ is a regular map, then we have a morphism $F^*\colon \mathbb{C}[Y] \to \mathbb{C}[Z]$ defined by $F^*(f) = f \circ F$. Thus, we can define $F^A\colon Z(A) \to Y(A)$ as $F^A(p^A) = p^A \circ F^*$.

Exercise 4.8. Prove the following sentences:

a. If Z is smooth, then so is $Z(A)$ for any finite dimensional algebra A.

b. Let A and B finite dimensional, then $Z(A \times B) = Z(A) \times Z(B)$.

c. Assume that A is finite dimensional, and $F\colon Z \to Y$ a regular map, then the prolongation $F^A\colon Z(A) \to Y(A)$ is also a regular map.

d. Assume that $\varphi\colon A \to B$ is a morphism of finite dimensional algebras, then for any variety Z the map $\varphi^A\colon Z(A) \to Z(B)$ is a regular map.

Let A be a Weil algebra or order r, then the rational point $\omega_A\colon A \to \mathbb{C}$ produces a map $\pi Z(A) \to Z$. This map is a bundle, the so–called *Weil bundle* of A–points in Z, also called *nearby points of type* A. Given a A–point p^A and its projection $p = \pi_A(p^A)$, we say that p^A is *nearby* the point p.

Let us consider $p \in Z$, and $\mathfrak{m}_p \subset \mathbb{C}[Z]$ the ideal of all regular functions vanishing at p. Then, the fiber of the bundle is, $\pi_A^{-1}(p) =$

$\mathrm{Hom}_{\mathbb{C}-alg}(\mathbb{C}[Z]/\mathfrak{m}_p^{r+1}, A)$. If p is a smooth point and the dimension of Z at p is n, then we also have $\mathbb{C}[Z]/\mathfrak{m}_p^{r+1} \simeq \mathbb{C}^{(m,r)}$.

Notation. The Weil bundle of $\mathbb{C}^{(m,r)}$–points of Z will be denoted $\pi_{(m,r)}\colon T^{(r,m)}Z \to Z$. Their elements will be called nearby points of type (m,r) or (m,r)–points.

Example 4.6. Let us consider $\pi_{(1,1)}\colon T^{(1,1)}Z \mapsto Z$. The Weil algebra $\mathbb{C}^{(1,1)}$ is a $\mathbb{C}$–vector space with basis $\{1, \varepsilon\}$, let $\{\omega_0, \omega_1\}$ be its dual basis. We consider $p^{(1,1)} \in Z(\mathbb{C}^{(1,1)})$, a $(1,1)$–point nearby to $p \in Z$. We have:

- **a.** Let f be a regular function. Then, $(f \otimes \omega_0)(v) = f(p))$. Therefore we have $f \otimes \omega_0 = \pi^*_{(1,1)}(f)$. Considering the algebraic embedding $\pi^*_A\colon \mathbb{C}[Z] \subset \mathbb{C}[Z(A)]$ we have $f_0 = f$.
- **b.** The map $D_{p^{(1,1)}}\mathbb{C}[Z] \to \mathbb{C}$ which sends $f \mapsto (f \otimes \omega_1)(p^{(1,1)})$ is a derivation. We have a one–to–one map $\pi^{-1}(p) \to T_pZ$ which sends the $(1,1)$–point $p^{(1,1)}$ to the tangent vector at p, $D_{p^{(1,1)}}$, seen as a derivation $D_{p^{(1,1)}}f = (f \otimes \omega_1)(p^{(1,1)})$.

We denote $f \otimes \omega_1$ as $\dot{f}$, and $f \otimes \omega_0$ as f. Then, the decomposition in $\mathbb{C}$–components is written: $f^{(1,1)} = f + \dot{f}\varepsilon$. Let $z_1, \cdots, z_n$ be a system of coordinates in Z. The functions $z_1, \cdots, z_n, \dot{z}_1, \cdots, \dot{z}_n$ form a system of coordinates in $T^{(1,1)}Z$. The bundle $T^{(1,1)}Z$ is identified with the tangent bundle TM. The $(1,1)$–point with coordinates $(z_1, \cdots, z_n, \dot{z}_1, \cdots, \dot{z}_n)$ corresponds to the vector $\sum_{i=1}^n \dot{z}_i \left(\frac{\partial}{\partial z_i}\right)_p$ where $p = (z_1, \cdots, z_n)$.

Example 4.7. Let us consider the Weil bundle $\pi_{(m,r)}\colon T^{(m,r)}Z \mapsto Z$. A basis of $\mathbb{C}^{(m,r)}$ as $\mathbb{C}$–vector space is $\left\{\frac{\varepsilon_1^{\alpha_1}\cdots\varepsilon_m^{\alpha_m}}{\alpha_1!\cdots\alpha_m!}\right\}_{0\leq|\alpha|\leq r}$. For a regular function $f \in \mathbb{C}[Z]$ we denote by $f_{:\alpha}$ the $\mathbb{C}$–component given by the coefficient of $\frac{\varepsilon_1^{\alpha_1}\cdots\varepsilon_m^{\alpha_m}}{\alpha_1!\cdots\alpha_m!}$ in the prolongation $f^{(m,r)}\colon T^{(m,r)Z} \to \mathbb{C}^{(m,r)}$. We have the following decomposition in $\mathbb{C}$–components:

$$
\begin{aligned}
f^{(m,r)} &= \sum_{|\alpha|\leq l} f_{:\alpha}\,\frac{\varepsilon_1^{\alpha_1}\cdots\varepsilon_m^{\alpha_m}}{\alpha_1!\cdots\alpha_m!}\,, \\
f_{:\alpha} &: \quad T^{(m,r)}Z \to \mathbb{C}\,.
\end{aligned}
$$

In the complex analytic case, let U be a small neighbohood of 0 in $\mathbb{C}^m$ and $F\colon U \to Z$ be a holomorphic map which sends 0 to $p \in Z$. Let us consider $\varepsilon_1, \cdots, \varepsilon_m$ as Cartesian coordinates in U. We consider the map $\mathbb{C}[Z] \to \mathbb{C}[[\varepsilon_1, \cdots, \varepsilon_m]]$ that assigns to each $f \in \mathbb{Z}$ the Taylor series of $f \circ F$ at 0. By truncation we obtain maps $t^{(r)}F \colon \mathbb{C}[Z] \to \mathbb{C}^{(m,r)}$. The map $t^{(r)}F$ is a (m,r)–point near to p called *the r–th Taylor series* of F. Let us consider $z_1, \cdots, z_n$ affine coordinates in Z, and $F \colon (\varepsilon_1, \cdots, \varepsilon_m) \to (z_1(\varepsilon_1, \cdots, \varepsilon_m), \cdots, z_n(\varepsilon_1, \cdots, \varepsilon_m))$. Then,

$$f_{:\alpha}(t^{(r)}F) = \left(\frac{\partial^{|\alpha|}}{\varepsilon_1^{\alpha_1} \cdots \varepsilon_m^{\alpha_m}}\right)_{\varepsilon=0} f(z_1(\varepsilon_1, \cdots, \varepsilon_m), \cdots, z_n(\varepsilon_1, \cdots, \varepsilon_m)) \,.$$

Remark 4.9. Let Z be smooth of dimension n. As we have seen above, the fiber $\pi^{-1}_{(m,r)}(p) \subset T^{(m,r)}Z$ corresponds to the Taylor series of degree r of all the germs of holomorphic maps $(\mathbb{C}^m, 0) \to (Z, p)$. A *frame of order* r is the r–th order Taylor series of a germ of holomorphic locally invertible map $(\mathbb{C}^n, 0) \to (Z, p)$. The *bundle of frames* of order r, $R^{(r)}Z \to Z$ is a Zariski open dense subset of $T^{(n,r)}Z \to Z$. The non–degeneracy condition is as follows, let $z_1, \cdots, z_n$ be a system of coordinates in Z, then:

$$R^{(r)}Z = \{p^{(n,r)} \in T^{(n,r)}(Z) \mid \det(z_{i:\epsilon_j}(p^{(n,r)})) \neq 0\} \,,$$

where ϵ_i represents the multi–index $(\delta_{1i}, \cdots, \delta_{ni})$ having all its components zero except the i–th with values 1.

4.4 Total derivatives

By the definition of Weil algebras $\mathbb{C}^{(m,r)}$ we have that $\mathbb{C}^{(m,r)}/\mathfrak{m}^k_{(m,r)} = \mathbb{C}^{(m,k-1)}$ for $k = 1, \cdots, r$. Therefore, we have that the Weil algebras $\mathbb{C}^{(m,r)}$ form a *projective system*,

$$\cdots \to \mathbb{C}^{(m,r+1)} \to \mathbb{C}^{(m,r)} \to \cdots \to \mathbb{C}^{(m,1)} \to \mathbb{C}^{(m,0)} = \mathbb{C} \,.$$

In the projective limit we have,

$$\mathbb{C}^{(m,\infty)} = \varprojlim_{r} \mathbb{C}^{(r,m)} \,.$$

We consider an irreducible affine variety Z and take the nearby point Weil bundles, getting a projective system the bundles of nearby points form a projective system:

$$\cdots \to T^{(m,r+1)}Z \to T^{(m,r)}Z \to \cdots \to T^{(m,1)}M \to M\,.$$

The projective limit

$$T^{(m,\infty)}Z = \varprojlim_{r} T^{(m,r)}Z\,,$$

is the set of (m,∞)–points in Z, $T^{(m,\infty)}Z$. The above projective limit gives to $T^{(m,\infty)}Z$ the structure of a *pro–algebraic variety.*[10] Considering this structure, the ring of regular functions in $T^{(m,r)}Z$ is,

$$\mathbb{C}[T^{(m,\infty)}Z] = \bigcup_{r=0}^{\infty} \mathbb{C}[T^{(m,r)}Z]\,.$$

In particular, let us consider $Z = \mathbb{C}^n$ and a system of coordinates $z_1, \cdots, z_m$. Then we have

$$\mathbb{C}[T^{(m,\infty)}Z] = \mathbb{C}[z_1, \cdots, z_n, z_{1:\epsilon_1}, \cdots, z_{n:\epsilon_m}, \cdots] = \mathbb{C}[z_{i:\alpha}]\,.$$

On the other hand, we have that the ring $\mathbb{C}^{(m,\infty)} = \mathbb{C}[[\varepsilon_1, \cdots, \varepsilon_m]]$ is endowed with the derivations $\frac{\partial}{\partial \varepsilon_i}$ for $i = 1, \cdots, m$. Let f be a regular function in Z, let us write its prolongation to $\mathbb{C}^{(m,\infty)}$ decomposed in its infinite $\mathbb{C}$–components:

$$f^{(m,\infty)} = \sum_{\alpha\in\mathbb{Z}^n_+} f_{:\alpha} \frac{\varepsilon_1^{\alpha_1}\cdots\varepsilon_m^{\alpha_m}}{\alpha_1!\cdots\alpha_m!}\,.$$

We define the *total derivative* of $f_{:\alpha}$ with respect to ε_i, denoted as $\frac{\mathfrak{d} f_{:\alpha}}{\mathfrak{d}\varepsilon_i}$ as the component in $\frac{\varepsilon_1^{\alpha_1}\cdots\varepsilon_m^{\alpha_m}}{\alpha_1!\cdots\alpha_m!}$ of $\frac{\partial f^{(m,\infty)}}{\partial\varepsilon_i}$. Therefore, we have:

$$\frac{\partial f^{(m,\infty)}}{\partial\varepsilon_i} = \sum_{\alpha\in\mathbb{Z}^m_+} \frac{\mathfrak{d} f_{:\alpha}}{\mathfrak{d}\varepsilon_i} \frac{\varepsilon_1^{\alpha_1}\cdots\varepsilon_m^{\alpha_m}}{\alpha_1!\cdots\alpha_m!}\,.$$

We obtain $\frac{\mathfrak{d} f_{:\alpha}}{\mathfrak{d}\varepsilon_i} = f_{:\alpha+\epsilon_i}$. By extension to all regular functions in $\mathbb{C}[T^{(m,\infty)}Z]$ we define *total derivative operators.*

[10] A pro–algebraic variety is a projective limit of algebraic varieties.

$$\frac{\partial}{\partial \varepsilon_i} : \mathbb{C}[T^{(m,\infty}Z] \to \mathbb{C}[T^{(m,\infty)}Z]\,.$$

Note that the total derivatives of functions in $\mathbb{C}[T^{(m,r)}Z]$ are in $\mathbb{C}[T^{(m,r+1)}Z]$. For a system $z_1, \cdots, z_n$ of affine coordinates in Z it is possible to write these total derivative operators as formal sums,

$$\frac{\partial}{\partial \varepsilon_i} = \sum_{j,\alpha} z_{j:\alpha+\epsilon_i} \frac{\partial}{\partial z_{j:\alpha}}\,.$$

Let us consider Z irreducible. Then $C[T^{(m,\infty)}Z]$ is a domain and the total derivative operators extend to the quotient field $C(T^{(m,\infty)}Z)$. We obtain the structure of Δ–field $\left(\mathbb{C}(T^{(m,\infty)}Z), \left(\frac{\partial}{\partial\varepsilon_1}, \cdots, \frac{\partial}{\partial\varepsilon_m}\right)\right)$. Note that if $z_1, \cdots, z_m$ is a system of affine coordinates, then $\mathbb{C}(T^{(m,\infty)}Z)$ is Δ–generated over $\mathbb{C}$ by $z_1, \cdots, z_n$.

4.5 Prolongation of groups and group actions

Theorem 4.10. *Let G be an affine $\mathbb{C}$–algebraic group. For any $\mathbb{C}$–algebra A the set $G(A)$ has a natural group structure. Let us assume that A is a finite dimensional algebra A, then $G(A)$ is an algebraic group.*

Proof. We apply the fact that the tensor product is a direct sum in the category of $\mathbb{C}$–algebras. Then $(G\times G)(A) = \mathrm{Hom}_{\mathbb{C}-alg}(\mathbb{C}[G]\otimes\mathbb{C}[G], A) = \mathrm{Hom}_{\mathbb{C}-alg}(\mathbb{C}[G], A) \times \mathrm{Hom}_{\mathbb{C}-alg}(\mathbb{C}[G], A) = G(A) \times G(A)$. We have that the composition law in G induces a composition law in $G(A)$ which in the finite dimensional case is given by regular functions. □

Example 4.8. Let us consider the multiplicative group $\mathbb{C}^*$. Then, $T^{(1,1)}\mathbb{C}^*$ is a two dimensional group with composition law defined by $(x + \dot{x}\varepsilon)(y + \dot{y}\varepsilon) = xy + (\dot{x}y + x\dot{y})\varepsilon$. If we take as coordinates $a = x$ and $b = x\dot{x}$ then we obtain an isomorphism $T^{(1,1)}\mathbb{C}^* \simeq \mathbb{C}^* \times \mathbb{C}$, since $(a + ba^{-1}\varepsilon)(a' + b'a'^{-1}\varepsilon) = (aa', (b + b')a^{-1}a'^{-1}\varepsilon)$.

Remark 4.11. The natural embedding $\mathbb{C} \subset A$ induces a map $G \subset G(A)$. When A is finite dimensional, G is an algebraic subgroup of $G(A)$. When A is a Weil algebra, the Weil bundle is a natural algebraic group morphism $\pi_A \colon G(A) \to G$.

Remark 4.12. In the case $G = \mathrm{GL}(n, \mathbb{C})$ then the corresponding group $\mathrm{GL}(n, A)$ is the group of invertible $n \times n$ matrices with coefficients in

A. In this case the nondegeneracy condition is not $\det(u_{ij}) \neq 0$ but $\omega_A(\det(u_{ij})) \neq 0$. That is, the determinant is an invertible element of A. The same reasoning applies to any algebraic linear group.

Theorem 4.13. *Let α be a regular action of G in Z. For each $\mathbb{C}$–algebra A the action α induces an action α^A of $G(A)$ en $Z(A)$. In particular $Z(A)$ is a G–variety.*

Let Z be an irreducible G–variety. We have that G acts in $Z(A)$ and therefore it acts on $\mathbb{C}[Z(A)]$ by the right side: $(f \cdot \sigma)(p) = f(\sigma \cdot p))$.

Definition 4.6. Let Z be an irreducible G–variety. A regular differential invariant of type A is an element $f \in \mathbb{C}[Z(A)]^G$.

$$C[Z(A)]^G = \{f \in C[Z(A)] \,|\, \forall \sigma \in G, \quad f \cdot \sigma = f\}.$$

In particular, a regular differential invariant of type (m,r) —of rank m and order r— is an element of $\mathbb{C}[T^{(m,r)}Z]^G$.

The action of G in $\mathbb{C}[Z(A)]$ lifts to an action in the field $\mathbb{C}(Z(A))$. Thus, G is identified with a subgroup of automorphisms of $\mathbb{C}(Z(A))$.

Definition 4.7. A rational differential invariant of type A is an element of $\mathbb{C}(Z(A))^G$. The elements of the algebraic closure $\overline{\mathbb{C}(Z(A))^G}$ are referred as *algebraic differential invariantes* of type A. In particular, a rational differential invariant of type (m,r) —of rank m and order r— is an element of $\mathbb{C}(T^{(m,r)}Z)^G$.

The field of differential invarians of any order is then defined,

$$\mathbb{C}(T^{(m,\infty)}Z)^G = \bigcup_{r=1}^{\infty} \mathbb{C}(T^{(m,r)}Z)^G.$$

Proposition 4.14. *Let f be a rational differential invariant of type (m,r). Then, for all $i = 1,\cdots,m$ the total derivative $\frac{\partial f}{\partial \varepsilon_i}$ is a differential invariant of type $(m,r+1)$. Therefore $\left(\mathbb{C}(T^{(m,\infty)}Z)^G, \left(\frac{\partial}{\partial\varepsilon_1},\cdots,\frac{\partial}{\partial\varepsilon_m}\right)\right)$ is a Δ–field.*

Proof. Note that for all $\sigma \in G$, $\frac{\partial f}{\partial \varepsilon_i} \cdot \sigma = \frac{\partial (f\cdot\sigma)}{\partial \varepsilon_i}$. Thus, we have that $f \cdot \sigma = f$ implies $\frac{\partial f}{\partial \varepsilon_i} \cdot \sigma = \frac{\partial f}{\partial \varepsilon_i}$. □

We can now state a first version of the Lie–Tresse theorem on the finiteness of differential invariants.

Theorem 4.15 (Algebraic Lie–Tresse, first version)**.** *Let Z be an irreducible G–variety. For each m there exist $r > 0$ and diffential invariants $I_1, \cdots, I_l \in \mathbb{C}(T^{(m,r)}Z)^G$, such that any rational differential invariant $J \in \mathbb{C}(T^{(m,\infty)}Z)^G$ of type (m,k) for any integer k, is a rational function in the total derivatives, $\frac{\mathfrak{d}^{|\alpha|}I_i}{\varepsilon^\alpha}$ with $0 \leq |\alpha| < \infty$. Therefore,*

$$C(T^{(m,\infty)}Z)^G = \mathbb{C}\left(\frac{\mathfrak{d}^{|\alpha|}I_1}{\varepsilon^\alpha}, \cdots, \frac{\mathfrak{d}^{|\alpha|}I_s}{\varepsilon^\alpha}\right)_{0\leq|\alpha|<\infty}.$$

Proof. By Theorem 4.5 the field of differential invariants of type (m,∞) is Δ–finitely generated over $\mathbb{C}$. Let $I_1, \cdots, I_s$ such that $\mathbb{C}\langle I_1, \cdots, I_{s'}\rangle = \mathbb{C}(T^{(m,\infty)}Z)^G$. □

Remark 4.16. The proof of the Theorem 4.15 does not make use of the algebraic structure of G or the regularity of the action. We have that, in a general setting, the field or rational invariants for a group or *pseudogroup* action in Z is always Δ–finitely generated.

Remark 4.17. Theorem 4.15 is in some sense weaker that the classical Lie–Tresse theorem. It needs to be improved in two ways.

- **a.** Characterize the maximum order r of the generating system of differential invariants.
- **b.** Characterize the order of derivatives of the generating set we need to apply to obtain all differential invariants of type $(m, r+k)$.

We will do both things in the statement of Theorem 4.28

4.6 Prolongation of ideals and subvarieties

Tangent structures

Let p^A be in $Z(A)$. There is a canonical identification of the space $T_{p^A}Z(A)$ with the space of derivations $\mathrm{Der}_{p^A}(\mathbb{C}[Z], A)$.

$$\begin{aligned} & \mathrm{Der}_{p^A}(\mathbb{C}[Z],\, A) \\ = \;& \{D:\, C[Z] \to A \,|\, D(f\,g) = p^A(f)\,D(g) + p^A(g)\,D(f)\}\,. \end{aligned}$$

Let r be the order of A. Let us denote by A_r the quotient algebra $A/\mathfrak{m}_A^r$. There is a projection, $\pi_r\colon Z(A) \to Z(A_r)$. For $p^A \mapsto p^{A_r}$, we denote by p the projection of p^A in Z. The tangent map induces an exact sequence,

$$0 \to \mathrm{Der}_{p^A}(\mathbb{C}[Z],\, \mathfrak{m}_A^r) \to \mathrm{Der}_{p^A}(\mathbb{C}[Z],\, A) \to \mathrm{Der}_{p^A}(\mathbb{C}[Z],\, A_r) \to 0\,.$$

Then $\mathfrak{m}_A^l$ verifies that for all $a \in A$ and $b \in \mathfrak{m}_A^l$ we have $ab = \omega_A(a)b$. Therefore the A–algebra structure in $\mathfrak{m}_A^r$ reduces to a $\mathbb{C}$–vector space structure and we have, $\mathrm{Der}_{p^A}(\mathbb{C}[Z], \mathfrak{m}_A^r) = \mathrm{Der}_p(\mathbb{C}[Z], \mathfrak{m}_A^r) = T_pZ \otimes_{\mathbb{C}} \mathfrak{m}_A^r$. The above sequence gives

$$0 \to T_pZ \otimes \mathfrak{m}_A^r \to T_{p^A}Z \to T_{p^{A_r}}Z \to 0\,.$$

Let p^A and q^A be in the same fiber of π_r. Then, the map $q^A - p^A\colon \mathbb{C}[Z] \to A$ turns out to be a derivation with values in $\mathfrak{m}_A^l$. Let $\bar\pi$ be the projection $Z(A_r) \to Z$ and $Q(A) \to Z(A_r)$ the pullback vector bundle $Q(A) = \bar\pi^*(TZ \otimes \mathfrak{m}_A^r)$ where $\bar\pi\colon Z(A_r) \to Z$ is the canonical projection. The following theorem holds (the interested reader can consult [1] for further results):

Theorem 4.18. *The bundle $Z(A) \to Z(A_r)$ is an affine bundle modeled over the vector bundle $Q(A) \to Z(A_r)$.*

For the particular case $Z(A) = T^{(m,r)}Z$ we denote $Q^{(m,r)}$ and we have

$$Q^{(m,r)}_{p^{(m,r)}} = T_pZ \otimes S^r[\varepsilon_1, \cdots, \varepsilon_m]\,,$$

where $S^r[\varepsilon_1, \cdots, \varepsilon_m]$ denotes the space of homogeneous polynomials of degree r.

Weil theorem and Taylor embedding

Theorem 4.19 ([22])**.** *Let A and B be Weil algebras. Then, there are canonical isomorphisms $Z(A)(B) \simeq Z(B)(A) \simeq Z(A \otimes_{\mathbb{C}} B)$.*

Proof. Just note that $\mathbb{C}[Z(A \otimes_{\mathbb{C}} B] = \mathbb{C}[\mathbb{C}[Z] \otimes_{\mathbb{C}} A^* \otimes_{\mathbb{C}} B^*]$. □

For $r = s + t$ there is a canonical map,

$$\begin{aligned} \mathbb{C}^{(m,r)} &= \mathbb{C}[[\varepsilon]]/(\varepsilon)^{r+s+1} \\ &\to \mathbb{C}^{(m,r)} \otimes_{\mathbb{C}} \mathbb{C}^{(m,r)} = C[[\lambda,\mu]]/((\lambda)^{r+1} + (\mu)^{t+1})\,, \end{aligned}$$

that sends $f(\varepsilon) \mapsto f(\lambda+\mu)$. These morphism induce regular bundle maps,

$$i_{(m,t,s)} : T^{(m,r)}Z \to T^{(m,s)}\,T^{(m,t)}Z\,,$$

called the Taylor embeddings.

Example 4.9. The classical embedding $T^{(2)}Z \mapsto T(TZ)$. The second tangent bundle is a subbundle of the tangent bundle to the tangent bunndle.

Prolongation of ideals and subvarieties

Definition 4.8. Let $\mathfrak{a}$ be an ideal of $\mathbb{C}[Z]$, we call $\mathfrak{a}(A)$ the prolongation of $\mathfrak{a}$ to $\mathbb{C}[Z(A)]$ to the ideal spanned by all the $\mathbb{C}$–components of the prolongations of elements of $\mathfrak{a}$ to $Z(A)$.

Remark 4.20. In particular, if Y is a irreducible closed sub–variety of Z then the prolongation to A of the ideal ot Z, is the ideal of $Y(A)$ as a sub–variety of $Z(A)$.

Let us consider an ideal $\mathfrak{a} \subset \mathbb{C}[T^{(m,r)}Z]$. The Taylor embedding $t_{m,1,s}\colon T^{(m,r+1)}Z \subset T^{(m,1)}T^{(m,r)}Z$ induces a restriction morphism $t^*_{m,1,s}\colon \mathbb{C}[T^{(m,1)}T^{(m,r)}Z] \to \mathbb{C}[T^{(m,r+1)}Z]$.

Definition 4.9. Let $\mathfrak{a}$ be an ideal of $\mathbb{C}[T^{(m,r)}Z]$. The first prolongation $\mathfrak{a}(m,r+1) \subset \mathbb{C}[T^{(m,r+1)}Z]$ is the ideal $t^*_{m,1,s}(\mathfrak{a})\cdot\mathbb{C}[T^{(m,r+1)}Z]$. For $s \geq 1$ the ideal $\mathfrak{a}(m,r+s) \subset \mathbb{C}[T^{(m,r+s)}Z]$ is defined by succesive prolongations.

Definition 4.10. Let $\mathfrak{a} \subset \mathbb{C}[T^{(m,r)}Z]$ be an ideal. Let $\mathfrak{b}$ be the intersection $\mathfrak{a} \cap C[T^{(m,r-1)}Z]$. We define the *first iteration* of $\mathfrak{a}^{(1)}$ as the ideal $+\mathfrak{b}(m,r-1+1)$. By repetition of this processs we define the k–th iteration $\mathfrak{a}^{(k)} = (\mathfrak{a}^{(k-1)})^{(1)}$. We say that $\mathfrak{a}$ is r*–th order complete* if $\mathfrak{a}^{(1)} = \mathfrak{a}$.

We have a natural ascending chain $\mathfrak{a} \subseteq \mathfrak{a}^{(1)} \subset \cdots \subseteq \mathfrak{a}^{(k)} \subseteq \cdots$. By the Noetherian property of $\mathbb{C}[T^{(m,r)}Z]$ we have that this sequence ends —in fact, we need almost r iterations—. The ideal $\mathfrak{a}^{(r)}$ is the smallest r–th order complete ideal containing $\mathfrak{a}$.

Definition 4.11. Let $Y \subset T^{(m,r)}Z$ be an irreducible subvariety. The first prolongation $Y(m,r+1)$ is the intersection of $T^{(m,1)}Y \subset T^{(m,1)}T^{(m,r)}Z$ with $T^{(m,r+1)}Z \subset T^{(m,1)}T^{(m,r)}Z$.

Let $\mathfrak{a}$ be the ideal $\mathfrak{I}(Y)$ of Y. Then, the radical $\sqrt{\mathfrak{a}(m,r+1)}$ coincides with the ideal of $Y(m,r+1)$. A direct computation shows the following result:

Proposition 4.21. *Let $\mathfrak{a} \subset \mathbb{C}[T^{(m,r)}Z]$ be an ideal spanned by functions $f_1,\cdots,f_s$. The prolongation $\mathfrak{a}(m,r+1)$ is spanned by $f_1,\cdots,f_n,\frac{\partial f_1}{\partial\varepsilon_1},\cdots,\frac{\partial f_s}{\partial\varepsilon_m}$.*

Remark 4.22. In the language of differential algebra, the prolongation $\mathfrak{a}(m,\infty)$ is the Δ–ideal of the Δ–ring $\left(\mathbb{C}[T^{(m,\infty)}Z],\left(\frac{\partial}{\partial\varepsilon_1},\cdots,\frac{\partial}{\partial\varepsilon_m}\right)\right)$ spanned by $\mathfrak{a}$.

Let $\mathfrak{a}=(F_1,\cdots,F_n)$ be an ideal of $\mathbb{C}[T^{(m,r)}Z]$. We can decompose $F_i = P_i(z_{j;\alpha}) + R_i(z_{j;\beta})$ in such way that $P_i(z_{j;\alpha})$ depends only on the derivatives of higher order. Let $p^{(r,m)}$ be in $V(\mathfrak{a})$ and $p^{(r+1,m)}$ in the fiber of $p^{(r,m)}$. We have that $p^{(r+1,m)}$ is $V(\mathfrak{a}(m,r+1))$ if and only if it satisfy $\frac{\partial F_i}{\partial\varepsilon_k}(p^{(r+1,m)})=0$ for all i and k. Those equations turn out to be linear in the derivatives of higher order:

$$\sum_{k=1}^{m}\frac{\partial P_i}{\partial z_{j;\alpha}}(p^{m,r})\,z_{j;\alpha+\epsilon_k} = -\frac{\partial R_i(p^{(r,m)})}{\partial\varepsilon_k}\,.$$

The solution of the homogeneous equation gives us a subspace of $\mathcal{S}(\mathfrak{a},1)_{p^{(m,r)}} \subset Q^{(m,r)}_{p^{(m,r)}}$ called the *first symbol* of the ideal $\mathfrak{a}$ at $p^{(m,r)}$.

Let $Y \subset T^{(m,r)}Z$ be an irreducible variety and $p^{(m,r)} \in Y$. The *first symbol* of Y at $p^{(m,r)}$ is the space $\mathcal{S}(Y,1)_{p^{(m,r)}}$ defined by the symbol of the ideal $\mathfrak{I}(Y)$. If Y is not irreducible, then the symbol is computed separately for their irreducible components.

If Y is irreducible then there is a Zariski open subset $U \subset Y$ such that $\mathcal{S}(Y,1)\to U$ is a vector bundle. The rank d of this bundle is called the *dimension of the generic symbol*. For any $p^{(m,r)}\in Y$ outside U we have $\dim\mathcal{S}(Y,1)_{p^{(m,r)}} > d$.

Lemma 4.23. *Let $Y \subset T^{(m,r)}Z$ such that the ideal $\mathfrak{I}(Y)$ is r–th order complete. Let us consider $\pi_{r+1,r}\colon T^{(m,r+1)}Z \to T^{(m,r)}Z$ the canonical projection. There is a Zariski open subset $U\subset Y$ such that*

$$U(m,\,r+1) = \{p^{(m,r+1)} \in Y(m,r+1) \,|\, \pi_{r+1,r}(p^{(m,r+1)}) \in Y\}\,,$$

and one of the following holds:

a. *$U(m, r+1)$ is empty.*

b. *The map $\pi_{r+1,r} \colon U(m, r+1) \to U$ is an affine bundle of rank d.*

Proof. The equations of the prolongation are a system of linear equations in the derivatives of order $r+1$. If the ideal $\mathfrak{I}(Y)$ is r–th order complete then no new linear equations in derivatives of order $\leq r$ appear. In a Zariski open subset, this linear system is either incompatible (case **a**) or compatible with a r–dimensional affine space of solutions (case **b**). $\square$

We say that Y has *vanishing first symbol* if $\mathcal{S}(Y,1) = 0$, or equivalently the dimension of the generic symbol of Y is 0.

4.7 Effective algebraic Lie–Tresse theorem

Let Z be an irreducible G–variety. Without loss of generality, we assume that the actions of G in any of the spaces $T^{(m,r)}Z$ are separated, so that the orbits are closed. In this section we will state a second version for the Lie–Tresse theorem that includes a geometric characterization of the maximum order of a generating system of differential invariants.

Lemma 4.24. *Let $Y \subset T^{(m,r)}Z$ be an orbit for the action of G, then the ideal $\mathfrak{I}(Y)$ is r–th order complete.*

By Rosenlicht theorem, there is an open Zariski G–invariant subset $U \subseteq T^{(m,r)}Z$ endowed with regular differential invariants $I_1, \cdots, I_l$ — they are in fact rational invariants whose denominators vanish outside U— that separate the orbits.

Lemma 4.25. *Let $Y \subset U \subset T^{(m,r)}Z$ be a principal orbit with vanishing symbol. Let $I_1, \cdots, I_s$. Let us consider $Y' = \pi^{-1}_{r+1,r}Y$ be the preimage of the Y by $\pi_{r+1,r} \colon T^{(m,r+1)}Z \to T^{(m,r)}Z$. Let $I_1, \cdots, I_s$ be regular differential invariants of type (m,r) separating the orbits in U. The restriction of the functions $\frac{\partial I_i}{\partial \varepsilon_j}$ to the prolongation Y' separate the orbits of the action of G in Y'.*

Proof. Let $p^{(m,r)} \in Y$ with symbol equal to (0). Taking into account that Y is a G–invariant variety, we obtain that each $\sigma \in G$ takes the symbol $\mathcal{S}(Y,1)_{p^{(m,r)}}$ to $\mathcal{S}(Y,1)_{\sigma \cdot p^{(m,r)}}$ and therefore the symbol $\mathcal{S}(Y,1)_{q^{(m,r)}}$ is (0) for any $q^{(m,r)} \in Y$. Let $\mathfrak{a} = (I_1 - \lambda_1, \cdots, I_s - \lambda_s)$ the equations of the orbit given by differential invariants of type (m,r). The ideal $\mathfrak{a}$ is r–th order complete by lemma 4.24. Let us assume that $I_1, \cdots, I_{s'}$ are of type (m,r) and not of type $(m,r-1)$, and that $I_{s'+1},\cdots,I_s$ are of type $(m,r-1)$. Let us consider $p^{(m,r)} \in Y$ and μ_{ij} $s'\times m$ arbitrary constants. Then, the system of linear equations in $\pi^{-1}_{r+1,r}(p^{(r,m)})$ defined by

$$\frac{\partial I_i}{\partial \varepsilon_j} = \mu_{ij}\,,$$

with $i=1,\cdots,s'$, $j=1,\cdots,m$, is either incompatible for any $p^{(m,r)}\in Y$, or has a unique solution $p^{(m,r+1)}=\phi(\mu_{ij},p^{(m,r)})$ for any $p^{(m,r)}$. In the compatible case, $Y(\mu_{ij})=\{\phi(\mu_{ij},p^{(m,r)})\,|\,p^{(m,r)}\in Y\}$ is an orbit of G in $T^{(m,r+1)}Z$. Reciprocally, let $X\subset T^{(m,r+1)}Z$ be an orbit that projects onto Y. Since $\frac{\partial I_i}{\partial\varepsilon_j}$ are differential invariants of type $(m,r+1)$ they take constant values μ_{ij} along X. Finally $X=Y(\mu_{ij})$. □

Lemma 4.26. *Let Z be a G–variety. For any $m\geq 0$ there exist $r>0$ such that the following holds.*

- **a.** *Generic orbit in $T^{(m,r)}Z$ are principal, i.e. there is a G–invariant Zariski open subset $U\subset T^{(m,r)}Z$ such that for all $p^{(m,r)}$ in U the stabilizer $G^{p^{(r,m)}}$ is the identity.*
- **b.** *For generic orbits $G\cdot p^{(m,r)}$ the symbol $\mathcal{S}(G\cdot p^{(m,r)},1)$ vanish, i.e. there is a G–invariant Zariski open subset $U\subset T^{(m,r)}$ such that for all $p^{(m,r)}$ in U the first symbol $\mathcal{S}(G\cdot p^{(m,r)},1)$ vanishes.*

Lemma 4.27. *Let us fix $m\geq 0$, and let r_0 satisfy* **a** *and* **b** *in the Lemma 4.26. Then, for all $r\geq r_0$, conditions* **a** *and* **b** *are satisfied.*

Theorem 4.28 (Lie–Tresse, second version)**.** *Let Z be a G–variety. Let $r\geq 0$ such that the generic orbit in $T^{(m,r)}Z$ is principal with vanishing symbol. Then, there exist a complete set of rational differential invariants $I_1,\cdots,I_s$ in $\mathbb{C}(T^{(m,r)}Z)^G$ such that:*

- **a.** *For any $k\geq 0$, any differential invariant of type $(m,r+k)$ is a rational function in the functions $\frac{\partial^{|\alpha|}I_i}{\varepsilon^\alpha}$ with $0\leq|\alpha|\leq k$:*

$$\mathbb{C}(T^{(m,r+k)}Z)^G = \mathbb{C}\left(\frac{\partial^{|\alpha|}I_1}{\partial\varepsilon^\alpha},\cdots,\frac{\partial^{|\alpha|}I_s}{\partial\varepsilon^\alpha}\right)_{0\leq|\alpha|<k}.$$

- **b.** *The above differential invariants $\frac{\partial^{|\alpha|}I_i}{\partial\varepsilon^\alpha}$ separate generic orbits in $T^{(m,r+k)}Z$.*

Proof. Let r be such that the generic orbit in $T^{(m,r)}Z$ is principal and has vanishing symbol. From Roselicht theorem we know that there is a system of rational differential invariants $I_1, \cdots, I_s$ separating generic orbits in Z. From Lemma 4.25 it follows than the rational functions I_i and $\frac{\partial I_j}{\partial \varepsilon_j}$ separate generic orbits in $T^{(m,r+1)}Z$, and therefore they generate the field $\mathbb{C}(T^{(m,r+1)}Z)^G$. By Lemma 4.27 we can apply the same argument iteratively obtaining that the succesive total derivatives or order $\leq k$ separate generic orbits in $T^{(m,r+k)}Z$. By the Theorem 3.8 we have,

$$\mathbb{C}(T^{(m,r+k)}Z)^G = \mathbb{C}\left(\frac{\partial^{|\alpha|}I_1}{\partial\varepsilon^\alpha}, \cdots, \frac{\partial^{|\alpha|}I_s}{\partial\varepsilon^\alpha}\right)_{0\leq|\alpha|<k}.$$

□

Example 4.10. Let $Z = \mathbb{C}^2$ be the plane, and let us consider the group $\mathrm{SE}(2,\mathbb{C})$ of orientation preserving Euclidean movements,

$$\begin{pmatrix} z_1 \\ z_2 \end{pmatrix} \mapsto \begin{pmatrix} c \\ d \end{pmatrix} + \begin{pmatrix} a & -b \\ b & a \end{pmatrix}\begin{pmatrix} z_1 \\ z_2 \end{pmatrix},$$

with $a^2 + b^2 = 1$. Let us compute the differential invariants of type $(1, r)$. First, $\mathrm{SE}(2,\mathbb{C})$ acts transitively in $\mathbb{C}^2$ so that there is no differential invariants of type $(1, 0)$. Let us compute the prolongation of the action to $T^{(1,1)}\mathbb{C}^2 = T\mathbb{C}^2$ —which is isomorphic to $\mathbb{C}^4$ with coordinates $z_1, z_2, \dot{z}_1, \dot{z}_2$— to compute the differential invariants of type $(1, 1)$. We have,

$$\begin{pmatrix} \dot{z}_1 \\ \dot{z}_2 \end{pmatrix} \mapsto \begin{pmatrix} a & -b \\ b & a \end{pmatrix}\begin{pmatrix} \dot{z}_1 \\ \dot{z}_2 \end{pmatrix}.$$

We obtain a differential invariant of type $(1, 1)$, namely,

$$I_1 = \dot{z}_1^2 + \dot{z}_2^2\,.$$

The Zariski open subset $U_1 = \{I_1 \neq 0\}$ all orbits are principal, and separated by te values of I_1. Let us compute the symbol equations,

$$\begin{aligned} I_2 \quad &= \quad \frac{\partial I_1}{\partial\varepsilon} = 2\,\dot{z}_1\,\ddot{z}_1 + 2\,\dot{z}_2\,\ddot{z}_2 \\ &\rightsquigarrow \quad 2\,\dot{z}_1\,\ddot{z}_1 + 2\,\dot{z}_2\,\ddot{z}_2 = 0\,. \end{aligned}$$

This homogenous equation in the variables $\ddot{z}_1, \ddot{z}_2$ has a one dimensional space of solutions —in $(1,1)$-points in U_1—, and therefore the symbol does not vanish. We need to compute the following prolongation.

$$\begin{pmatrix} \ddot z_1 \\ \ddot z_2 \end{pmatrix} \mapsto \begin{pmatrix} a & -b \\ b & a \end{pmatrix} \begin{pmatrix} \ddot z_1 \\ \ddot z_2 \end{pmatrix}.$$

We compute a new invariant,

$$I_3 = \dot z_1\, \ddot z_2 - \dot z_2\, \ddot z_1\,.$$

Again we have that I_1, I_2, I_3 separate generic orbits in $T^{(1,2)}\mathbb{C}^2 \simeq \mathbb{C}^6$. The symbol equations give,

$$\begin{aligned} I_4 &= \frac{\partial^2 I_1}{\partial \varepsilon^2} = 2\,\dot z_1\, z_1^{(3)} + 2\,\dot z_2\, z_2^{(3)} + 2\,\ddot z_1^2 + 2\,\ddot z_2^2 \\ &\rightsquigarrow \quad 2\,\dot z_1\, z_1^{(3)} + 2\,\dot z_2\, z_2^{(3)} = 0\,, \end{aligned}$$

$$\begin{aligned} I_5 &= \frac{\partial^2 I_1}{\partial \varepsilon^2} = \dot z_1\, z_2^{(3)} - \dot z_2\, z_1^{(3)} \\ &\rightsquigarrow \quad \dot z_1\, z_2^{(3)} - \dot z_2\, z_1^{(3)} = 0\,. \end{aligned}$$

These homogeneous equations in $z_1^{(3)}, z_2^{(3)}$ have only trivial solution in U_1. Therefore, the symbol vanishes. We can conclude that I_1, I_2, I_3 and their derivatives up to order k allow us to express any rational differental invariant of type $(1, k{+}2)$. In particular the curvature κ of a planar curve is an algebraic differential invariant given by,

$$\kappa = \sqrt{\frac{I_3^2}{I_1^3}}\,.$$

Finally, taking into account that I_2 is defined as the derivative of I_1, we have that as differential fields,

$$\mathbb{C}(z_1,\, z_2,\, \dot z_1,\, \cdots)^{\mathrm{SE}(2,\mathbb{C})} = \mathbb{C}\langle \dot z_1^2 + \dot z_2^2,\, \dot z_1\, \ddot z_2 - \dot z_2\, \ddot z_1\rangle\,.$$

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