

## On the uniqueness of the limit amplitude for the induced vibrations in stratified flows<sup>1</sup>

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For three-dimensional exponentially stratified flows we make a review of our previous results and present a new corollary which consist of a detailed analysis for the classes of uniqueness of the solutions of the relevant equation for the limit amplitude of the induced vibrations caused by a periodic external force. Our main result is that for a frequency outside of the spectrum the limit amplitude is unique, while for a frequency belonging to the spectrum the limit amplitude loses the uniqueness.

Keywords: partial differential equations, Sobolev spaces, stratified fluid, limit amplitude.

Se hace una revisión de nuestros resultados anteriores acerca de flujos tridimensionales exponencialmente estratificados y se presenta un nuevo corolario que consiste en un análisis detallado para las clases de unicidad de las soluciones de la ecuación relevante para la amplitud límite de las vibraciones inducidas causadas por una fuerza externa periódica. Nuestro resultado principal es que para una frecuencia fuera del espectro la amplitud límite es única, mientras que para una frecuencia dentro del espectro se pierde la unicidad de la amplitud límite.

Palabras claves: ecuaciones diferenciales a derivadas parciales, espacios de Sobolev, fluidos estratificados, amplitud límite.

MSC: 35Q35, 35B05, 35PO5, 76O26.

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## 1 Introduction

The objective of this paper is to summarize up the results obtained in previous articles and to draw in detail some new corollaries. Particularly, here we study the relationship between the uniqueness of the limit amplitude of the induced vibrations, and the spectrum of normal vibrations in stratified flows, as well as to establish the classes of uniqueness of the equation for the limit amplitude in  $L_{\vec{p}}(\mathbb{R}^3)$ .

We consider a system of equations of the form

$$\begin{aligned} \frac{\partial v_1}{\partial t} + \frac{\partial p}{\partial x_1} &= F_1(x, t), \\ \frac{\partial v_2}{\partial t} + \frac{\partial p}{\partial x_2} &= F_2(x, t), \\ \frac{\partial v_3}{\partial t} + gv_4 + \frac{\partial p}{\partial x_3} &= F_3(x, t), \\ \frac{\partial v_4}{\partial t} - \frac{N^2}{g}v_3 &= F_4(x, t), \\ \frac{\partial v_1}{\partial x_1} + \frac{\partial v_2}{\partial x_2} + \frac{\partial v_3}{\partial x_3} &= 0, \end{aligned}$$

in the domain  $\{x \in \mathbb{R}^3, t > 0\}$ , where  $\vec{v}(x, t)$  is a velocity field with components  $v_1(x, t), v_2(x, t), v_3(x, t)$ ,  $p(x, t)$  is the scalar field of the dynamic pressure,  $v_4(x, t)$  is the dynamic density and  $g, N$  are positive constants. The equations (1) are deduced in [1], [13]–[15] under the assumption that the function of stationary distribution of density is performed by the function  $\rho e^{-Nx_3}$ .

The system (1) can be considered as describing linearized motions of three-dimensional fluid in a homogeneous gravity field, where there are the external mass forces acting. We note that, without loss of generality, we may assume  $g = N = 1$ . In that way, instead of system (1), we will consider the following system:

$$\begin{aligned}
 \frac{\partial v_1}{\partial t} + \frac{\partial p}{\partial x_1} &= F_1(x, t), \\
 \frac{\partial v_2}{\partial t} + \frac{\partial p}{\partial x_2} &= F_2(x, t), \\
 \frac{\partial v_3}{\partial t} + v_4 + \frac{\partial p}{\partial x_3} &= F_3(x, t), \\
 \frac{\partial v_4}{\partial t} - v_3 &= F_4(x, t), \\
 \frac{\partial v_1}{\partial x_1} + \frac{\partial v_2}{\partial x_2} + \frac{\partial v_3}{\partial x_3} &= 0,
 \end{aligned} \tag{1}$$

in the domain

$$\{x = (x_1, x_2, x_3), t \in \mathbb{R}^3, t > 0\},$$

together with the initial conditions

$$\begin{aligned}
 \tilde{v}(x, t) |_{t=0} &= \tilde{v}^0(x), \\
 \tilde{v} &= \{\vec{v}, v_4\}, \\
 \vec{v} &= (v_1, v_2, v_3).
 \end{aligned} \tag{2}$$

We may observe that, despite an extensive study of stratified flows from a physical point of view (see, for example, [16]–[18], [24]–[25]), there have been relatively few works considering the mathematical aspect of the problem.

The fundamental solution for the operator of internal waves in stratified flows was first constructed in [19].

The solutions for a Cauchy problem for the homogeneous system (1), for the viscous case of intrusion for homogeneous system were constructed in [20], and the uniqueness of the homogeneous Cauchy problem for the viscous case was studied in [21]. The spectral properties of the differential operator of (1) were studied in [4]–[6]. And, the Cauchy problem for homogeneous system (1) with the initial conditions restrictions  $\frac{\partial v_1^0}{\partial x_2} - \frac{\partial v_2^0}{\partial x_1} = 0$  and  $v_4^0 = 0$ , was considered in [22]. The boundary value problem for the half-space in case of the homogeneous system (1) was considered in [23]. For the problem (1), (2) the existence and uniqueness theorems were proved in [2]. In particular, it was proved that the first component of the solution is represented by

$$\begin{aligned}
v_1(x, t) &= v_1^0(x)\Phi_1(t) - (v_1^0 * G_1)_{\mathbb{R}^3} - (v_1^0 * G_2)_{\mathbb{R}^3} - (v_2^0 * G_3)_{\mathbb{R}^3} \\
&\quad - (v_4^0 * G_4)_{\mathbb{R}^3} + \int_0^t \Phi_1(t - \beta)F_1(x, \beta)d\beta - (F_1 * G_1)_{\mathbb{R}_T^4} \\
&\quad - (F_1 * G_2)_{\mathbb{R}_T^4} - (F_2 * G_3)_{\mathbb{R}_T^4} - (F_4 * G_4)_{\mathbb{R}_T^4},
\end{aligned}$$

where

$$\mathbb{R}_T^4 = \{x \in \mathbb{R}^3, 0 \leq t \leq T\},$$

$$(v_i^0 * G_j)_{\mathbb{R}^3} = \int_{\mathbb{R}^3} v_i^0(y)G_j(x - y, t)dy,$$

$$(F_i * G_j)_{\mathbb{R}_T^4} = \int_0^t \int_{\mathbb{R}^3} F_i(y, \beta)G_j(x - y, t - \beta)dyd\beta,$$

where  $i = 1, 2, 4, j = 1, 2, 3, 4$ , and

$$G_1(x, t) = \Delta \left( \frac{1}{4\pi} \frac{1}{r} J_0 \left( \frac{\rho t}{r} \right) - \frac{1}{4\pi} \frac{1}{r} \int_0^t J_1(t - \tau) J_0 \left( \frac{\rho \tau}{r} \right) d\tau \right),$$

$$\begin{aligned}
G_2(x, t) &= \frac{1}{4\pi} \int_0^t \int_0^\alpha J_0(\alpha - \tau) \left[ \frac{(3x_2^2 - r^2)}{r^5} J_0 \left( \frac{\rho \tau}{r} \right) \right. \\
&\quad \left. + \frac{(5x_2^2 - r^2)\rho \tau}{r^6} J_0' \left( \frac{\rho \tau}{r} \right) + \frac{x_3^2 x_2^2 \tau^2}{r^7} J_0'' \left( \frac{\rho \tau}{r} \right) \right] d\tau d\alpha,
\end{aligned}$$

$$\begin{aligned}
G_3(x, t) &= \frac{1}{4\pi} \int_0^t \int_0^\alpha J_0(\alpha - \tau) \left[ \frac{3x_1 x_2}{r^5} J_0 \left( \frac{\rho \tau}{r} \right) \right. \\
&\quad \left. + \frac{5x_1 x_2 \rho \tau}{r^6} J_0' \left( \frac{\rho \tau}{r} \right) + \frac{x_1 x_2 x_3^2 \tau^2}{r^7} J_0'' \left( \frac{\rho \tau}{r} \right) \right] d\tau d\alpha,
\end{aligned}$$

$$\begin{aligned}
G_4(x, t) &= \frac{1}{4\pi} \int_0^t J_0(\alpha - \tau) \left[ \frac{3x_1 \rho}{r^5} J_0 \left( \frac{\rho \tau}{r} \right) \right. \\
&\quad \left. + \frac{x_1 \tau (3x_3^2 - 2x_1^2 - 2x_2^2)}{r^6} J_0' \left( \frac{\rho \tau}{r} \right) \right. \\
&\quad \left. - \frac{x_1 \rho \tau^2 (x_1^2 + x_2^2)}{r^7} J_0'' \left( \frac{\rho \tau}{r} \right) \right] d\tau,
\end{aligned}$$

$\rho^2 = x_3^2$ ,  $r^2 = \sum_{k=1}^3 x_k^2$ , and  $J_0$  is the Bessel function of order zero.

Additionally, in [2], using the Calderón–Zygmund Theorem ([28]), the following estimate was established:

$$\|\tilde{v}\|_{W_{p,t,x}^{k+1,l}(\mathbb{R}_T^4)} \leq C \left( \|\tilde{v}^0\|_{W_p^l(\mathbb{R}^3)} + \|\tilde{F}\|_{W_{p,t,x}^{k,l}(\mathbb{R}_T^4)} \right),$$

where the constant  $C$  depends only on  $p$  and  $T$ ;  $W_{p,t,x}^{k,l}(\mathbb{R}_T^4)$  are Sobolev spaces ([3]) of functions having  $k$  derivatives with respect to  $t$  and  $l$  derivatives with respect to  $x$  which are  $p$ th power summable and  $\mathbb{R}_T^4 = \{x \in \mathbb{R}^3, 0 \leq t \leq T\}$ . Again, for non-homogeneous problem (1), without loss of generality, we can consider  $\tilde{v}^0(x) = 0$  in (2).

After reducing the system (1) to the equivalent scalar equation, we observe that all the components of the solution of (1) satisfy the equation

$$\frac{\partial^2}{\partial t^2} \Delta_3 u(x, t) + \Delta_2 u(x, t) = F(x, t), \tag{3}$$

where  $\Delta_3 = \sum_{i=1}^3 \frac{\partial^2}{\partial x_i^2}$ ,  $\Delta_2 = \sum_{i=1}^2 \frac{\partial^2}{\partial x_i^2}$ . For  $\tilde{v}^0(x)$  we will associate the equation (3) to the initial conditions

$$u(x, 0) = \frac{\partial u}{\partial t}(x, 0) = 0. \tag{4}$$

Let  $F(x, t)$  be a periodic external mass force of the following form:

$$F(x, t) = f(x) e^{-i\omega t},$$

with  $\omega \geq 0$ . In this way, the equation

$$\frac{\partial^2}{\partial t^2} \Delta_3 u(x, t) + \Delta_2 u(x, t) = f(x) e^{-i\omega t}, \tag{5}$$

together with the conditions (4), describes the induced vibrations of stratified fluid which start at the moment  $t = 0$  and which are caused by harmonic oscillations of point sources distributed with the density  $f(x)$ . In [2] the following result was proved.

**Theorem 1.** *If  $f \in C_0^\infty(\mathbb{R}^3)$ ,  $\omega \neq 1$ , then for the solution  $u(x, t)$  of the problem (4)–(5) for  $t \rightarrow \infty$  there exists (pointwise with respect to  $x \in \mathbb{R}^3$ ) the function  $U(x)$  such that  $U(x) = \lim_{t \rightarrow \infty} e^{i\omega t} u(x, t) \in C^\infty(\mathbb{R}^3)$ ,  $U = f * E_\omega$ , and  $U$  is a classical solution of the equation of the stabilized equations*

$$(1 - \omega^2)\Delta_2 U - \omega^2 \frac{\partial^2 U}{\partial x_3^2} = f(x), \quad (6)$$

$x \in \mathbb{R}^3$ , where  $E_\omega(x)$  is a singular solution of the equation which is obtained from (5) for  $u(x, t) = v(x)e^{-i\omega t}$ :

$$(1 - \omega^2) \left( \frac{\partial^2 v}{\partial x_1^2} + \frac{\partial^2 v}{\partial x_2^2} \right) - \omega^2 \frac{\partial^2 v}{\partial x_3^2} = \delta(x). \quad (7)$$

Following the widely accepted terminology ([29]), the function  $U(x)$  is called a “*limit amplitude of the solution of the problem (4)–(5)*”. In this way, for the problem (4)–(5) Theorem 1 states that there occurs an phenomenon of stabilized vibrations and that there exist a limit amplitude for the frequency  $\omega \neq 0, 1$ .

The proof of Theorem 1 (see [2] for details) is based on the following idea. As it follows from the properties of the solution for the system (1)–(2), the solution of the scalar problem (4)–(5) exists, is unique and is represented by

$$u(x, t) = e^{-i\omega t} \int_{\mathbb{R}^3} f(x + y) M_\omega(y, t) dy,$$

where

$$M_\omega(y, t) = -\frac{1}{2\pi^2|y|} \int_0^t \left( \int_\lambda^1 \frac{\sin(\tau w)}{\sqrt{1 - w^2}\sqrt{w^2 - \lambda^2}} dw \right) e^{i\omega\tau} d\tau.$$

We conclude that for every compact  $K$  in  $\mathbb{R}^3$  the estimate holds:

$$|M_\omega(x, t) - E_\omega(x)| \leq \frac{C(\omega, K)}{\sqrt{t}},$$

where

$$E_\omega(x) = -\frac{1}{\pi|x|} \begin{cases} \frac{-1}{\sqrt{\omega^2-1}\sqrt{\omega^2-\lambda^2}}, & \text{for } \omega > 1, \\ \frac{i}{\sqrt{1-\omega^2}\sqrt{\omega^2-\lambda^2}}, & \text{for } \lambda < \omega < 1, \\ \frac{1}{\sqrt{\omega^2-1}\sqrt{\lambda^2-\omega^2}}, & \text{for } \omega < \lambda. \end{cases}$$

Then we prove that the function  $E_\omega(x)$  coincides with the singular (fundamental) solution ([9]) of the equation  $(1 - \omega^2)\Delta_2 v - \omega^2 \frac{\partial^2 v}{\partial x_3^2} = \delta(x)$  almost everywhere.

It is natural to call the function  $V_\omega(x, t) = \int_{\mathbb{R}^3} f(x + y)M_\omega(y, t)dy$  an “amplitude of the solution of the problem (4)–(5)”, since  $u(x, t) = V_\omega(x, t)e^{-i\omega t}$ . Finally, we prove that for  $t \rightarrow \infty$  the amplitude stabilizes to the function  $U(x)$  which is defined by

$$U(x) = \int_{\mathbb{R}^3} f(x + y)E_\omega(y)dy = \int_{\mathbb{R}^3} f(y)E_\omega(x - y)dy.$$

However, the important question of the uniqueness of the limit amplitude for stratified flows, has not been considered in previous works. In general, the solution  $U$  is not the unique solution of equation (6).

In this paper we investigate various aspects related to the uniqueness of the solutions of (6), depending on  $\omega$ . If we consider the problem of the normal vibrations of the type  $\tilde{v}(x, t) = u(x)e^{-i\lambda t}$  and  $p(x, t) = q(x)e^{-i\lambda t}$  for the homogeneous system

$$\begin{aligned} \frac{\partial v_1}{\partial t} + \frac{\partial p}{\partial x_1} &= 0, \\ \frac{\partial v_2}{\partial t} + \frac{\partial p}{\partial x_2} &= 0, \\ \frac{\partial v_3}{\partial t} + v_4 + \frac{\partial p}{\partial x_3} &= 0, \\ \frac{\partial v_4}{\partial t} - v_3 &= 0, \\ \frac{\partial v_1}{\partial x_1} + \frac{\partial v_2}{\partial x_2} + \frac{\partial v_3}{\partial x_3} &= 0. \end{aligned} \tag{8}$$

Then we obtain the following problem with the spectral parameter  $\lambda$

$$\begin{aligned}
-i\lambda u_1 + \frac{\partial q}{\partial x_1} &= 0, \\
-i\lambda u_2 + \frac{\partial q}{\partial x_2} &= 0, \\
-i\lambda u_3 + u_4 + \frac{\partial q}{\partial x_3} &= 0, \\
-i\lambda u_4 - u_3 &= 0, \\
\frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2} + \frac{\partial u_3}{\partial x_3} &= 0.
\end{aligned} \tag{9}$$

In [4]–[6] it was proved that the essential spectrum of the normal vibrations for the differential operator defined in (9) is the interval of the real axis  $[-1, 1]$ , the points  $\{\pm 1\}$  are eigenvalues of infinite multiplicity and outside the interval  $[-1, 1]$  there are no spectral elements.

Let us observe that equation (7) is elliptic for  $\omega \notin [-1, 1]$  and hyperbolic for  $\omega \in [-1, 1]$ . Therefore it seems natural to express a conjecture that the solution of the equation for the limit amplitude (6), for certain class of functions, will be unique for  $\omega \notin [-1, 1]$ , and not unique for  $\omega \in [-1, 1]$ . In this work we prove that this conjecture is true. We investigate first the case  $\omega \notin [-1, 1]$ . For the homogeneous equation

$$(\omega^2 - 1) \left( \frac{\partial^2 v}{\partial x_1^2} + \frac{\partial^2 v}{\partial x_2^2} \right) + \omega^2 \frac{\partial^2 v}{\partial x_3^2} = 0. \tag{10}$$

we prove the following result.

**Theorem 2.** *Let  $\omega^2 > 1$ ,  $1 \leq p < \infty$ . If  $u \in L_p(\mathbb{R}^3)$  satisfies (10), then  $u(x) = 0$  almost everywhere.*

*Proof.* According to Schwartz theorem (see [7]), for  $u \in L_p(\mathbb{R}^3)$  every linear functional

$$\int_{\mathbb{R}^3} u(x) \varphi(x) dx, \quad \varphi \in S,$$

is an element of  $S'$ .

After applying the Fourier transform to (10), we obtain

$$(\omega^2 - 1)(\xi_1^2 + \xi_2^2)\widehat{u} + \omega^2 \xi_3^2 \widehat{u} = 0, \tag{11}$$



where  $\widehat{u}$  is the Fourier image of  $u$ . From (11) it follows that the support of  $\widehat{u}(\xi)$  consists of the only point  $\{0\}$ . Then, from the theory of generalized functions ([8], [9]) we have

$$\widehat{u}(\xi) = \sum_{|\alpha| \leq N} C_\alpha D^\alpha \delta(\xi),$$

which implies

$$u(x) = \sum_{|\alpha| \leq N} C'_\alpha x^\alpha.$$

In other words,  $u$  is a polynomial of  $x$ . Since  $u \in L_p(\mathbb{R}^3)$ , we finally have  $u(x) = 0$  almost everywhere in  $\mathbb{R}^3$  and the theorem is proved.  $\square$

**Theorem 3.** *If  $f(x) \in W_1^1(\mathbb{R}^3)$  and  $\omega^2 > 1$ , then the limit amplitude  $U(x)$  belongs to the class of uniqueness  $L_q(\mathbb{R}^3)$  for  $3 < q < \infty$ .*

*Proof.* After applying a contraction transform for  $x_3$ , the equation (6), without loss of generality, may be considered as

$$\frac{\partial^2 U}{\partial x_1^2} + \frac{\partial^2 U}{\partial x_2^2} + \frac{\partial^2 U}{\partial x_3^2} = f(x), \tag{12}$$

with  $x \in \mathbb{R}^3$ . Since the singular solution of the Laplace equation in  $\mathbb{R}^3$  is  $-\frac{1}{4\pi|x|}$ , then, from theorem 2 we obtain that the limit amplitude  $U(x)$  is performed by

$$U(x) = -\frac{1}{4\pi} \int_{\mathbb{R}^3} \frac{f(y)}{|x-y|} dy.$$

From Hardy–Littlewood inequalities (see [10]) and Sobolev inequalities of potential type ([3]) we have that for  $1 < p < 3/2$ ,  $3 < q < \infty$ , the property holds:

$$\|U\|_{L_q(\mathbb{R}^3)} \leq C \|f\|_{L_p(\mathbb{R}^3)},$$

where  $p$  and  $q$  are related by  $\frac{1}{q} = \frac{1}{p} - \frac{2}{3}$ . Finally, from Sobolev inclusion theorems ([3]) we have

$$W_1^1(\mathbb{R}^3) \subset L_p(\mathbb{R}^3),$$

with  $1 < p < 3/2$ , which concludes the proof.  $\square$

For  $\omega^2 < 1$  the homogeneous equation (6)

$$\frac{\partial^2 u}{\partial x_1^2} + \frac{\partial^2 u}{\partial x_2^2} - \left( \frac{\omega^2}{1 - \omega^2} \right) \frac{\partial^2 u}{\partial x_3^2} = 0,$$

analogously to the case  $\omega^2 > 1$ , can be transformed by a contraction of the axis  $x_3$ , to the wave equation in  $\mathbb{R}^3$ .

$$\frac{\partial^2 u}{\partial x_1^2} + \frac{\partial^2 u}{\partial x_2^2} - \frac{\partial^2 u}{\partial x_3^2} = 0. \quad (13)$$

with  $x \in \mathbb{R}^3$ . The singular solution (see [9]) of (13) is the function  $\frac{-1}{4\pi\sqrt{x_1^2+x_2^2-x_3^2}}$ . Let us consider the following norm in the non-isotropic space  $L_{\vec{p}}(\mathbb{R}^3)$ ,  $\vec{p} = (p, p, p_3)$

$$\|f\|_{\vec{p}(\mathbb{R}^3)} = \left( \int_{\mathbb{R}^2} \left( \int_{\mathbb{R}^1} |f(x)|^{p_3} dx_3 \right)^{p/p_3} dx' \right)^{1/p},$$

where  $x' = (x_1, x_2)$ .

The proof of the following two theorems was given in [27], although originally the model considered in [27] is different, since it is related to rotational (not stratified) fluid. However, we would like to present a detailed proof of them, not only in order to give more divulging character to this article, but also because of the interest and novelty of the followed conclusions.

**Theorem 4.** *Let  $G(x) = \int_{\mathbb{R}^3} \frac{f(y)}{\sqrt{|x'-y'^2-(x_3-y_3)^2}} dy$ ,  $f(x) \in W_1^1(\mathbb{R}^3)$ . Then there exists a constant  $C > 0$  such that*

$$\|G\|_{L_{\vec{p}}(\mathbb{R}^3)} \leq C \|f\|_{W_1^1(\mathbb{R}^3)},$$

with  $\vec{p} = (p, p, 2)$ ,  $4 < p < \infty$ .

*Proof.* For the function

$$H(x, y') = \int_{\mathbb{R}^1} \frac{f(y)}{\sqrt{|x' - y'|^2 - (x_3 - y_3)^2}} dy_3$$

from the Young’s convolution inequality (see [11]), we have

$$\left( \int_{\mathbb{R}^1} |H(x, y')|^2 dx_3 \right)^{1/2} \leq \|f(y')\|_{L_q(\mathbb{R}_{y_3}^1)} \left( \int_{\mathbb{R}^1} \frac{dy_3}{||x' - y'|^2 - y_3^2|^{s/2}} \right)^{1/s},$$

where  $1 \leq q, s < \infty, \frac{1}{2} = \frac{1}{q} + \frac{1}{s} - 1$ .

We observe that for  $1 < s < 2$  the last integral can be calculated as

$$\left( \int_{\mathbb{R}^1} \frac{dy_3}{||x' - y'|^2 - y_3^2|^{s/2}} \right)^{1/s} = \frac{C}{|x' - y'|^{1/s'}},$$

where  $C$  is a constant and  $s' = \frac{s}{s-1}$ . In this way, we have

$$\left( \int_{\mathbb{R}^1} |H(x, y')|^2 dx_3 \right)^{1/2} \leq C \frac{h(y')}{|x' - y'|^{1/s'}},$$

$$h(y') = \left( \int_{\mathbb{R}^1} |f(y', y_3)|^q dy_3 \right)^{1/q}. \tag{14}$$

Using (14) and Minkowsky inequality, we obtain:

$$\begin{aligned}
& \left( \int_{\mathbb{R}^1} |G(x)|^2 dx_3 \right)^{\frac{1}{2}} \\
&= \left( \int_{\mathbb{R}^1} \left| \int_{\mathbb{R}^2} \left( \int_{\mathbb{R}^1} \frac{f(y) dy_3}{\sqrt{|x' - y'|^2 - (x_3 - y_3)^2}} \right) dy' \right|^2 dx_3 \right)^{\frac{1}{2}} \\
&= \left( \int_{\mathbb{R}^1} \left| \int_{\mathbb{R}^2} H(x, y') dy' \right|^2 dx_3 \right)^{\frac{1}{2}} \\
&\leq \int_{\mathbb{R}^2} \left( \int_{\mathbb{R}^1} |H(x, y')|^2 dx_3 \right) dy' \\
&\leq C \int_{\mathbb{R}^2} \frac{h(y')}{|x' - y'|^{1/s'}} dy'.
\end{aligned}$$

Let us denote  $I = \int_{\mathbb{R}^2} \frac{h(y')}{|x' - y'|^{1/s'}} dy'$ ,  $s' = \frac{s}{s-1}$ . From Sobolev properties for potential integrals ([3]) we obtain that the estimate holds:

$$\|I\|_{L_p(\mathbb{R}^2)} \leq C \|h\|_{L_r(\mathbb{R}^2)},$$

and  $1 < s < 2$ ,  $1 < r < p < \infty$ ,  $4 < p < \infty$ ,  $\frac{1}{p} = \frac{1}{r} + \left(\frac{1+s}{2s}\right)$ .

Finally, we have

$$\|G\|_{L_{\vec{p}}(\mathbb{R}^3)} \leq C \|h\|_{L_r(\mathbb{R}^2)} \leq C \|f\|_{L_{\vec{r}}(\mathbb{R}^3)}, \quad (15)$$

where

$$\begin{aligned}
\vec{p} &= (p, p, 2) \\
\vec{r} &= (r, r, q),
\end{aligned} \quad (16)$$

and  $4 < p < \infty$ ,  $\frac{3}{2} = \frac{1}{q} + \frac{1}{s}$ ,  $1 + \frac{1}{s} + \frac{2}{p} = \frac{2}{r}$ ,  $1 < s < 2$ .

Now, for a more narrow variation of  $s$ :  $\frac{4}{3} < s < 2$ , we obtain from (16)

$$1 < q < \frac{4}{3},$$

$$1 < r < \frac{4}{3}.$$

For  $1 < r < \frac{4}{3}$ ,  $1 < q < \frac{4}{3}$ , the inclusion theorem ([3]) is valid:

$$W_1^1(\mathbb{R}^3) \subset L_{\vec{r}}(\mathbb{R}^3),$$

with  $\vec{r} = (r, r, q)$ . Thus from (15) we have:  $\|G\|_{L_{\vec{r}}(\mathbb{R}^3)} \leq C\|f\|_{W_1^1(\mathbb{R}^3)}$ ,  $\vec{p} = (p, p, 2)$ ,  $4 < p < \infty$ , which concludes the proof.  $\square$

**Theorem 5.** *Let  $u(x)$  be a solution of (13) which belongs to the class  $L_{\vec{p}}(\mathbb{R}^3)$ , with  $\vec{p} = (p, p, 2)$ ,  $4 < p < \infty$ . Then, there exists a non-trivial solution of (13) in that class of functions.*

*Proof.* By direct substitution it can be easily verified that for arbitrary  $A(\xi)$ , the function  $v(x', \xi) = A(\xi)J_0(\xi|x'|)$  for almost every  $\xi \in \mathbb{R}^1$ , satisfies the Helmholtz equation

$$\frac{\partial^2 v}{\partial x_1^2} + \frac{\partial^2 v}{\partial x_2^2} + \xi^2 v = 0, \tag{17}$$

where  $|x'| = \sqrt{x_1^2 + x_2^2}$ ;  $J_0(\xi r)$  is the Bessel function of order zero satisfying the Bessel equation

$$r^2 v'' + r v' + r^2 \xi^2 v = 0.$$

Let us choose  $A(\xi) \in C_0^\infty(\mathbb{R}^1)$  such that  $A(\xi) \equiv 0$  for  $|\xi| \leq 1$ , and  $A(\xi) \neq 0$  in a non-zero measure set in  $\mathbb{R}^1$ . Thus, the function

$$u(x) = \frac{1}{2\pi} \int_{\mathbb{R}^1} v(x', \xi) e^{-ix_3 \xi} d\xi, \tag{18}$$

represents a non-trivial solution of (13).

From the Hausdorff–Young inequality for Fourier transforms (see, for example, [12]), the estimate holds:

$$\|u\|_{L_2(\mathbb{R}_{x_3}^1)} \leq C \|v\|_{L_2(\mathbb{R}_\xi^1)}.$$

In this way, we obtain

$$\left\| \|u\|_{L_2(\mathbb{R}_{x_3}^1)} \right\|_{L_p(\mathbb{R}^2)} \leq C \left\| \|v\|_{L_2(\mathbb{R}_\xi^1)} \right\|_{L_p(\mathbb{R}^2)} \quad (19)$$

Since  $\int_1^\infty r |J_0(\xi r)|^p dr < \infty$  for  $p > 4$ , then the norm in the right side of (19) is finite. Finally, we have  $\|u\|_{L_{\vec{p}}(\mathbb{R}^3)} < \infty$ ,  $\vec{p} = (p, p, 2)$  and the theorem is proved.  $\square$

We can sum up the results of Theorems 1–5 as follows. Let  $f \in W_1^1(\mathbb{R}^3)$ . Then, the limit amplitude of vibrations in stratified flows which are induced by the external mass force of the type  $f(x)e^{-i\omega t}$ , is a unique solution of equation (6) for the case  $\omega^2 > 1$ .

If, however,  $\omega^2 < 1$  then the limit amplitude belongs to the class of non-uniqueness of equation (6). For  $\omega = \pm 1$ , from the explicit form of the solution ([2]) we can see that the solution is unboundedly growing, i.e., the resonance takes place.

On the other hand, we have seen that the essential spectrum of normal vibrations in stratified fluid is the interval  $[-1, 1]$ . Thus, we observe a remarkable relationship between the essential spectrum and the uniqueness of the limit amplitude of the vibrations induced by an external periodical force of frequency  $\omega$ .

When  $\omega$  does not belong to the spectrum, the solution of the equation for the limit amplitude is unique. When  $\omega$  belongs to the spectrum, the limit amplitude loses the uniqueness. At the boundary points of the spectrum, the effect of resonance takes place.

## References

- [1] A. Brekhovskih and V. Goncharov, *Introduction to the Mechanics of Continuous Media* (Nauka, Moscow, 1982); in Russian.
- [2] A. Giniatoulline and E. Mayorga, *On the uniqueness, existence, stabilization and the limit amplitude for non-homogeneous system modelling internal non-stationary waves in stratified flows*, Pacific J. Appl. Math. **3**(4), (2011); accepted for publication.

- [3] S. Sobolev, *Some Applications of Functional Analysis in Mathematical Physics*, Translations of Mathematical Monographs **90** (AMS, 1991).
- [4] A. Giniatouline, *Essential spectrum of the operators generated by PDE systems of stratified fluids and  $L_p$ -estimates for the solutions*, Int. T. Comp. Sci. Appl. **2**, 38 (2005).
- [5] A. Giniatouline and C. Rincón, *On the spectrum of normal vibrations for stratified flows*, Comput. Fluid Dyn. J. **13**(2), 273 (2004).
- [6] A. Giniatouline, *Spectral properties of the operators generated by PDE systems of stratified ideal and compressible fluids*, Hokkaido Univ. Technical Report, Series in Maths. **111**, 56 (2006).
- [7] L. Schwartz, *Théorie des Distributions* (Paris, 1951).
- [8] R. Kanwal, *Generalized Functions: Theory and Technique* (Academic Press, New York, 1983).
- [9] V. S. Vladimirov, *Equations of Mathematical Physics* (Marcel Dekker, New York, 1971).
- [10] E. Stein, *Singular Integrals and Differentiability Properties of Functions* (Princeton University Press, Princeton, 1970).
- [11] G. Hardy, J. Littlewood and G. Polya, *Inequalities* (Cambridge University Press, Cambridge, 1952).
- [12] W. Beckner, *Inequalities in Fourier analysis*, Ann. Math. **102**(1), 159 (1975).
- [13] B. Cushman–Roisin and J. Beckers, *Introduction to Geophysical Fluid Dynamics* (Academic Press, New York, 2011).
- [14] D. Tritton, *Physical Fluid Dynamics* (Oxford University Press, Oxford, 1990).
- [15] P. Kundu, *Fluid Mechanics* (Academic Press, New York, 1990).
- [16] J. Flo, M. Ungari and J. Bush, *Spin-up from rest in a stratified fluid: boundary flows*, J. Fluid Mech. **472**, 51 (2002).
- [17] N. Makarenko, J. Maltseva and A. Kazakov, *Conjugate flows and amplitude bounds for internal solitary waves*, Nonlinear Proc. Geophys. **16**, 169 (2009).

- [18] B. Maurer, D. Bolster and P. Linden, *Intrusive gravity currents between two stably stratified fluids*, J. Fluid Mech. **647**, 53 (2010).
- [19] S. Sekersh-Zenkovich, *Construction of the fundamental solution for the operator of inner waves*, Dokl. Akad. Nauk **246**, 286 (1979).
- [20] V. Maslennikova and A. Giniatoulline, *On the intrusion problem in a viscous stratified fluid for three space variables*, Math. Notes **51**, 374 (1992).
- [21] A. Giniatoulline, *On the uniqueness of solutions in the class of increasing functions for a system describing the dynamics of a viscous weakly stratified fluid in three dimensional space*, Rev. Col. Mat. **31**, 71 (1997).
- [22] A. Giniatoulline and O. Zapata, *On some qualitative properties of stratified flows*, Contemp. Math. **434**, 193 (AMS, 2007).
- [23] A. Giniatoulline and E. Mayorga, *On some properties of the solutions of the problem modelling stratified ocean and atmosphere flows in the half-space*, Rev. Col. Mat. **43**, 43 (2009).
- [24] T. Dauxois and W. Young, *Near-critical reflection of internal waves*, J. Fluid Mech. **90**, 271 (1999).
- [25] V. Birman and E. Meiburg, *On gravity currents in stratified ambients*, Phys. Fluids **19**, 602 (2007).
- [26] B. Sutherland and W. Peltier, *The stability of stratified jets*, Geophys. Astrophys. Fluid Dyn. **66**, 101 (1992).
- [27] V. Maslennikova and A. Giniatoulline, *Spectral properties of operators for systems of hydrodynamics*, Siberian Math. J. **29**, 812 (1988).
- [28] A. Calderón and A. Zygmund, *On singular integrals*, Am. J. Math. **78**, 289 (1956).
- [29] A. Tikhonov and A. Samarski, *Equations of Mathematical Physics* (Dover, New York, 1990).