Pasting and reversing operations over some rings

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Este artículo examina dos operaciones, pegado e inversión, definidas como aplicaciones naturales sobre algunos anillos, tal como polinomios y operadores diferenciales lineales, el cual es un anillo diferenciable. Se obtienen algunas propiedades de estos operadores sobre estos anillos. Este último resultado nos permite establecer algunas propiedades de los números naturales que aparecen en matemática recreativa.

Palabras claves: anillos diferenciales, operadores, pegado, polinomios, inversión.

This paper examines two operations, pasting and reversing, defined as a natural mappings over some rings, such as polynomials and linear differential operators, which is a differential ring. We obtain some properties of these operations over these rings. This result allows us to establish some properties of natural numbers appearing in recreational mathematics.

Keywords: Differential rings, operators, pasting, polynomials, reversing.

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Introduction

Pasting and Reversing are common processes. The first attempt, due to the first author, intended to interpret these natural processes as mathematical operations dates back to 1992. One year later, he gave some lectures about the case of the natural numbers in several Colombian mathematical meetings, such as *Semana de Matemáticas y Física de la Universidad del Tolima*. These lectures were published ten years later; see [2]. Recently, in 2008, these operations of pasting and reversing were applied to obtain families of *simple permutations*; see [1, 3].

Following the same structure presented in [1, 2, 3], in this work we introduce *pasting and reversing operations for the case of the ring of polynomials*. Pasting and reversing operations provide new ways of expressing some properties of natural numbers presented in [18]. Finally, we start the study of some properties of differential rings. Following *differential Galois theory*, see [19], we start the analysis of these operations over linear differential operators. There is a wide bibliography where the definition of ring, as well as its properties, can be found. In particular, the ring theory is presented in [10].

We say that R is a differential ring if there exists a derivation ∂ such that, $\forall a, b \in R$, we have

$$\begin{aligned} \partial(a+b) &= \partial a + \partial b\,, \\ \partial(a \cdot b) &= \partial a \cdot b + a \cdot \partial b\,; \end{aligned}$$

for more details see [19]. In particular, we are interested in the ring of polynomials $\mathbb{C}[x]$ and in linear differential operators $\mathcal{L}$ defined as

$$\mathcal{L} := \sum_{k=0}^{n} a_k\, \partial^k\,,$$

where $a_k \in K$, and K is a differential field. The set of operators $\mathcal{L}$ is a differential ring.

1 Pasting and reversing over polynomials

In this section we introduce the operations of pasting and reversing over the ring of polynomials and over the set of natural numbers, where some aspects of recreational mathematics are shown.

1.1 Polynomial case

We only consider polynomials P such that $x \nmid P(x)$. For convenience, we write P as follows:

$$P(x) = \sum_{k=0}^{n} a_{n-k} x^{n-k}.$$

Clearly, $1+\deg(P) = Ç(P)$ is the number of coefficients of the polynomial P.

Definition 1. (Reversing, palindromic and antipalindromic polynomials.) *Consider $P \in \mathbb{C}[x]$ written as*

$$P(x) = \sum_{k=0}^{n} a_{n-k} x^{n-k}. \tag{1}$$

The reversing of P, denoted by $\tilde{P}$, is given by

$$\tilde{P}(x) = \sum_{k=0}^{n} b_{n-k}\, x^{n-k}\,, \tag{2}$$

where $b_{n-k} = a_k$ and $k = 0, 1, \cdots, n$. The polynomial P is called palindromic whether $\tilde{P} = P$, respectively, the polynomial P is called antipalindromic whether $\tilde{P} = -P$.

Thus, Definition 1 leads us to the following results.

Proposition 2. *Consider the polynomials P and $\tilde{P}$ as in equations* (1), (2) *respectively. The following statement holds.*

1. $\tilde{P}(x) = x^n P\left(\frac{1}{x}\right)$, *where* $n+1 = Ç(P)$.
2. $\tilde{P}\left(\frac{1}{\alpha}\right) = 0$ *if and only if* $P(\alpha) = 0$.
3. $\tilde{P}(x) = (-1)^n(\alpha_1 x - \beta_1)(\alpha_2 x - \beta_2)\cdots(\alpha_n x - \beta_n)$ *if and only if* $P(x) = (\beta_1 x - \alpha_1)\cdots(\beta_n x - \alpha_n)$.
4. $\tilde{\tilde{P}} = P$.
5. $Ç(P) = Ç(\tilde{P})$.
6. $\widetilde{P+Q} = \tilde{P} + \widetilde{Q}$, *for* $Ç(P) = Ç(Q)$.
7. $\widetilde{P \cdot Q} = \tilde{P} \cdot \widetilde{Q}$.

Proof.

1. By equation (1), we can see that

$$x^n\, P\left(\frac{1}{x}\right) = x^n \sum_{k=0}^{n} a_{n-k} \left(\frac{1}{x}\right)^{n-k},$$

hence

$$x^n\, P\left(\frac{1}{x}\right) = \sum_{k=0}^{n} a_{n-k}\, x^k = \sum_{k=0}^{n} a_k\, x^{n-k}\,.$$

In this way, by equation (2), we have $x^n P\left(\frac{1}{x}\right) = \tilde{P}(x)$.

2. Due to $x \nmid P(x)$, $\alpha \neq 0$. Now, by item **1**, taking $x = \left(\frac{1}{\alpha}\right)$, we have

$$\tilde{P}\left(\frac{1}{\alpha}\right) = \left(\frac{1}{\alpha}\right)^n P(\alpha)\,.$$

By hypothesis $P(\alpha) = 0$, for instance $\tilde{P}\left(\frac{1}{\alpha}\right) = 0$. We proceed in a similar way for the converse.

3. From item **1**, we have

$$\tilde{P}(x) = x^n \left(\beta_1 \left(\frac{1}{x}\right) - \alpha_1\right) \cdots \left(\beta_n \left(\frac{1}{x}\right) - \alpha_n\right)\,.$$

Thus,

$$\tilde{P}(x) = (\beta_1 - \alpha_1\, x) \cdots (\beta_n - \alpha_n\, x)\,,$$

or, equivalently,

$$\tilde{P}(x) = (-1)^n\, (\alpha_1\, x - \beta_1) \cdots (\alpha_n\, x - \beta_n)\,.$$

We proceed in a similar way for the converse.

4. Assume $P(x)$ and $\tilde{P}(x)$ as in item **3**. Thus, we have

$$\tilde{\tilde{P}}(x) = (-x)^n \left(\alpha_1 \left(\frac{1}{x}\right) - \beta_1\right) \cdots \left(\alpha_n \left(\frac{1}{x}\right) - \beta_n\right),$$

therefore

$$\tilde{\tilde{P}}(x) = (\beta_1\, x - \alpha_1) \cdots (\beta_n\, x - \alpha_n)\,,$$

as needed.

5. From item **3** we observe that $\deg(P) = \deg(\tilde{P})$, thus $Ç(P) = Ç(\tilde{P})$.

6. Assume that

$$\begin{aligned} P(x) &= \sum_{k=0}^{n} a_{n-k}\, x^{n-k}\,, \\ Q(x) &= \sum_{k=0}^{n} b_{n-k}\, x^{n-k}\,. \end{aligned}$$

Setting $R = P + Q$, we have

$$R(x) = \sum_{k=0}^{n} c_{n-k} x^{n-k}\,,$$

with $c_j = a_j + b_j$ and $j = 0, \cdots, n$. Now, by equation (2) it follows that

$$\widetilde{R}(x) = \sum_{k=0}^{n} c_k\, x^{n-k} = \sum_{k=0}^{n} a_k\, x^{n-k} + \sum_{k=0}^{n} b_k\, x^{n-k}\,,$$

which means, again by equation (2), that $\widetilde{R} = \tilde{P} + \widetilde{Q}$. Thus, we conclude that $\widetilde{P+Q} = \tilde{P} + \widetilde{Q}$.

7. Assume that

$$\begin{aligned} P(x) &= (\beta_1\, x - \alpha_1) \cdots (\beta_n\, x - \alpha_n)\,, \\ Q(x) &= (\gamma_1\, x - \mu_1) \cdots (\gamma_m\, x - \mu_m)\,. \end{aligned}$$

Setting $R = P \cdot Q$, we have

$$R(x) = (\beta_1\, x - \alpha_1) \cdots (\beta_n\, x - \alpha_n)\, (\gamma_1\, x - \mu_1) \cdots (\gamma_m\, x - \mu_m)\,.$$

By item **3** of Proposition (2), we have

$$\begin{aligned}\widetilde{R}(x) &= (-1)^{n+m}\,(\alpha_1\,x-\beta_1)\cdots(\beta_n\,x-\alpha_n)\\ &\quad\times(\gamma_1\,x-\mu_1)\cdots(\gamma_m\,x-\mu_m)\\ &= [(-1)^{n}\,(\alpha_1\,x-\beta_1)\cdots(\beta_n\,x-\alpha_n)]\\ &\quad\times[(-1)^{m}\,(\gamma_1\,x-\mu_1)\cdots(\gamma_m\,x-\mu_m)]\\ &= \tilde{P}(x)\cdot\widetilde{Q}(x)\,.\end{aligned}$$

as desired. □

Remark 3. *From the beginning we assumed $x \nmid P(x)$, for instance, items* **2** *and* **5** *are false whether $x \mid P(x)$. We recall that items* **4** *and* **5** *can also be proven using only Definition 1, i.e., equations* (1) *and* (2)*.*

There are some specific known cases in which we can use the reversing operation over special families of polynomials such as the *Bessel polynomials*; see [5, 9].

Proposition 2 and Definition 1 lead us to the following results; see also [14].

Proposition 4. *Let P be a palindromic or antipalindromic polynomial with roots $\alpha_1,\cdots,\alpha_n$, with $n+1 = Ç(P)$. Then,*

$$\alpha_{k+j} = 1/\alpha_j, \quad Ç(P)\in\{2k+1,2k+2\}, \quad j=1,\cdots,k.$$

Furthermore, if $Ç(P) = 2k+2$ and P is palindromic (respectively antipalindromic), then $\alpha_{2k+1} = -1$ (respectively $\alpha_{2k+1} = 1$).

Proof. By Definition 1 and Proposition 2, $P(\alpha_j) = 0$ implies that

$$\tilde{P}\left(\frac{1}{\alpha_j}\right) = \pm P\left(\frac{1}{\alpha_j}\right) = 0\,,$$

for $j = 1,\cdots,Ç(P)-1$. Thus, for $Ç(P) = 2k+1$ we can arrange $\alpha_{k+j} = 1/\alpha_j$, $j = 1,\cdots,k$. In the same way, for $Ç(P) = 2k+2$, we have $\alpha_{k+j} = 1/\alpha_j$, $j = 1,\cdots,k$ and $\alpha_{2k+1} = 1/\alpha_{2k+1}$, hence $\alpha_{2k+1} = \pm 1$. If $P = \tilde{P}$, then the signs of its coefficients must be preserved, so that α_{2k+1} must be -1. Finally, if $\tilde{P} = -P$, then the signs of the coefficients must be interchanged, so that α_{2k+1} must be 1. □

Proposition 5. *The following statements hold.*

1. *The sum of two palindromic polynomials, with the same degree, is also a palindromic polynomial.*

2. *The product of two palindromic polynomials is also a palindromic polynomial.*

3. *The sum of two antipalindromic polynomials, with the same degree, is also an antipalindromic polynomial.*

4. *The product of two antipalindromic polynomials is a palindromic polynomial.*

5. *The product of a palindromic polynomial with an antipalindromic polynomial is an antipalindromic polynomial.*

Proof.

1. Let P and Q be palindromic polynomials. By item **6** of Proposition 2 we have $\widetilde{P+Q} = \tilde{P} + \widetilde{Q} = P + Q$. Consequently, $P + Q$ is a palindromic polynomial.

2. Let P and Q be palindromic polynomials. By item **7** of Proposition 2 we have $\widetilde{P \cdot Q} = \tilde{P} \cdot \widetilde{Q} = P \cdot Q$. Consequently, $P \cdot Q$ is a palindromic polynomial.

3. Let P and Q be antipalindromic polynomials. By item **6** of Proposition 2 we have $\widetilde{P+Q} = \tilde{P} + \widetilde{Q} = -P - Q = -(P + Q)$. Consequently, $P + Q$ is an antipalindromic polynomial.

4. Let P and Q be antipalindromic polynomials. By item **7** of Proposition 2 we have $\widetilde{P \cdot Q} = \tilde{P} \cdot \widetilde{Q} = (-P) \cdot (-Q) = PQ$. Consequently, $P \cdot Q$ is a palindromic polynomial.

5. Let P be a palindromic polynomial and let be Q an antipalindromic polynomial. By item **7** of Proposition 2 we have $\widetilde{P \cdot Q} = \tilde{P} \cdot \widetilde{Q} = P \cdot (-Q) = -P \cdot Q$. Consequently, $P \cdot Q$ is an antipalindromic polynomial. □

The following definition corresponds to a natural example of *orthogonal polynomials*; see [6, 15, 16].

Definition 6. (Chebyshev polynomials of the first kind.) *The Chebyshev polynomials of the first kind, denoted by* T_n, *are defined by the trigonometric identity*

$$T_n(w) = \cos(n \arccos w) = \cosh(n \operatorname{arccosh} w)\,,$$

with $n \in \mathbb{Z}_+$, *which is equivalent to the identities:*

$$\begin{aligned} T_n(\cos(\alpha)) &= \cos(n\alpha)\,, \\ T_n(\cosh(\alpha)) &= \cosh(n\alpha)\,. \end{aligned}$$

Lemma 7. *If* $w = \frac{1}{2}(z + \frac{1}{z})$, *then* $\frac{1}{2}(z^n + \frac{1}{z^n}) = T_n(w)$

Proof. By Definition 6, we write

$$\begin{aligned} T_1(w) &= \cos(\alpha) = w\,, \\ &\vdots \\ T_n(w) &= \cos(n\alpha)\,, \end{aligned}$$

which lead us to

$$T_n(w) = \frac{e^{in\alpha} + e^{-in\alpha}}{2} = \frac{\left(e^{i\alpha}\right)^n + \left(e^{i\alpha}\right)^{-n}}{2} = \frac{1}{2}\left(z^n + z^{-n}\right).$$

In particular, $w = \frac{1}{2}(z + \frac{1}{z})$. □

The following result has been suggested by V. Sokolov.

Proposition 8. *Let* P_{2n} *be a palindromic polynomial with coefficients* a_i, $0 \le i \le 2n$. *Then*

$$\frac{P_{2n}(z)}{2z^n} = \sum_{k=0}^{n} a_{n-k}\, T_k(w)\,, \tag{3}$$

where $w = \frac{1}{2}\left(z + \frac{1}{z}\right)$.

Proof. By hypothesis, $P_{2n}(z) = a_{2n}z^{2n} + a_{2n-1}z^{2n-1} + \cdots + a_1 z + a_0$. Now, due to $P_{2n} = \tilde{P}_{2n}$, we have $a_i = a_{2n-i}$, $i = 0, \cdots, 2n$. Thus, dividing $P_{2n}(z)$ by $2z^n$, we obtain

$$\frac{P_{2n}(z)}{2z^n} = \frac{a_{2n}z^n + a_{2n-1}z^{n-1} + \cdots + a_{2n-1}z^{1-n} + a_{2n}z^{-n}}{2}\,.$$

Rearranging the common coefficients we have

$$\frac{P_{2n}(z)}{2z^n} = \frac{a_{2n}}{2}\left(z^n + \frac{1}{z^n}\right) + \frac{a_{2n-1}}{2}\left(z^{n-1} + \frac{1}{z^{n-1}}\right) + \cdots + \frac{1}{2}\,a_n\,.$$

With the change of variable $w = \frac{1}{2}(z + \frac{1}{z})$ and Lemma 7, the right-hand side is $a_{2n}T_n(w) + a_{2n-1}T_{n-1}(w) + \cdots + a_nT_0(w)$ when $a_i = a_{2n-i}$, $i = 0, .., 2n$. Thus we arrive at the expression given in equation (3). □

The concept of palindromic and antipalindromic polynomials is rather ancient. There are a lot of references about these polynomials using the concept of *Reciprocal Polynomials*; see for example [4, 7, 8, 11, 13, 17] and references therein. We recall, using the previous references, that P is the reciprocal of Q if $Q = \widetilde{\overline{P}} = P^*$,

$$\begin{aligned} P(z) &= a_n\,z^n + a_{n-1}\,z^{n-1} + \cdots + a_1\,z + a_0\,, \\ Q(z) &= \widetilde{\overline{P}}(z) = a_0\,\overline{z^n} + a_1\,\overline{z^{n-1}} + \cdots + a_{n-1}\,\overline{z} + a_n\,, \end{aligned}$$

where $z = a + bi$, $\overline{z} = a - bi$.

On the other hand, P is self–reciprocal if $P = P^* = \widetilde{\overline{P}}$. Furthermore, if $b = 0$, then $P = \tilde{P}$, which means that P is a palindromic polynomial. The same argument can be applied to antipalindromic polynomials.

Now, we introduce the definition of the operation of pasting over polynomials.

Definition 9. *Pasting of the polynomials P and Q, denoted by $P \diamond Q$, is given by*

$$P \diamond Q := x^{\text{Ç}(Q)}P + Q.$$

The following properties are consequences of Definition 9.

Proposition 10. *Let P, Q, R be polynomials. The following statements hold:*

1. $\tilde{P} \diamond \tilde{Q} = \widetilde{Q \diamond P}$

2. $(P \diamond Q) \diamond R = P \diamond (Q \diamond R)$

Proof.

1. Let P, Q be polynomials, where

$$\begin{aligned} P(x) &= \sum_{\ell=0}^{s} a_{s-\ell}\, x^{s-\ell}\,, \\ Q(x) &= \sum_{j=0}^{k} b_{k-j}\, x^{k-j}\,. \end{aligned}$$

Then, by Definition 9 and assuming $R = Q \diamond P$, we have

$$\begin{aligned} R(x) &= x^{s+1} \sum_{j=0}^{k} b_{k-j}\, x^{k-j} + \sum_{\ell=0}^{s} a_{s-\ell}\, x^{s-\ell} \\ &= \sum_{i=0}^{k+s+1} c_{k+s+1-i}\, x^{k+s+1-i}\,, \end{aligned}$$

where the coefficients c_m are given by

$$c_m = \begin{cases} a_m, & 0 \le m \le s, \\ b_m, & s+1 \le m \le k+s+1. \end{cases}$$

By Definition 1, we obtain

$$\begin{aligned} \tilde{P}(x) &= \sum_{\ell=0}^{s} a_{\ell}\, x^{s-\ell}\,, \\ \tilde{Q}(x) &= \sum_{j=0}^{k} b_{j}\, x^{k-j}\,, \\ \tilde{R}(x) &= \sum_{i=0}^{k+s+1} c_{i}\, x^{k+s+1-i}\,. \end{aligned}$$

By Definition 9, we have

$$\tilde{R}(x) = \sum_{i=0}^{k+s+1} c_i\, x^{k+s+1-i} = x^{k+1} \sum_{\ell=0}^{s} a_\ell\, x^{s-\ell} + \sum_{j=0}^{k} b_j\, x^{k-j}\,.$$

Therefore, $\widetilde{R} = \tilde{P} \diamond \widetilde{Q}$.

2. Let P, Q, R be polynomials such that

$$\begin{aligned} P(x) &= \sum_{i=0}^{k} a_{k-i}\, x^{k-i}\,, \\ Q(x) &= \sum_{i=0}^{j} b_{j-i}\, x^{j-i}\,, \\ R(x) &= \sum_{i=0}^{\ell} d_{l-i}\, x^{\ell-i}\,, \end{aligned}$$

where, $\text{Ç}(P) = k+1, \text{Ç}(Q) = j+1$ and $\text{Ç}(R) = \ell+1$. We write $(P \diamond Q) \diamond R$ according to Definition 1 as follows

$$x^{\ell+1}\left(x^{j+1}\sum_{i=0}^{k} a_{k-i}\, x^{k-i} + \sum_{i=0}^{j} b_{j-i}\, x^{j-i}\right) + \sum_{i=0}^{\ell} d_{\ell-i}\, x^{\ell-i}\,.$$

This can be rewritten as

$$x^{j+\ell+2}\sum_{i=0}^{k} a_{k-i}\, x^{k-i} + \left(x^{\ell+1}\sum_{i=0}^{j} b_{j-i}\, x^{j-i} + \sum_{i=0}^{\ell} d_{\ell-i}\, x^{\ell-i}\right)\,,$$

or, equivalently,

$$x^{j+\ell+2}\sum_{i=0}^{k} a_{k-i}\, x^{k-i} + \left(\sum_{i=0}^{j} b_{j-i}\, x^{j-i} \diamond \sum_{i=0}^{\ell} d_{\ell-i}\, x^{\ell-i}\right)\,.$$

Consequently,

$$(P \diamond Q) \diamond R = P \diamond (Q \diamond R)\,.$$

□

As a consequence of Proposition 4, which is adapted for pasting of polynomials, we present the following result.

Proposition 11. *Let P be a polynomial. The linear polynomial $x+1$ is divisor of the polynomial $P \diamond \tilde{P}$.*

Proof. Owing to $Ç(P \diamond \tilde{P})$ is even and $P \diamond \tilde{P}$ is palindromic, then, by Proposition 4, -1 is a root of $P \diamond \tilde{P}$, so that $x+1 \mid P \diamond \tilde{P}$. □

1.2 The case of natural numbers

The properties presented above for the polynomial case are useful to demonstrate properties of natural numbers choosing $x = 10$. For natural numbers, Ç is called *digital cipher*; see [2].

We recall that, due to previous results, the reversing of $n \in \mathbb{N}$ is given by

$$\widetilde{n} = \sum_{j=0}^{r} a_j \, 10^{r-j} \, ,$$

where $n = \sum_{j=0}^{r} a_{r-j} 10^{r-j}$.

In a natural way, we introduce the concept of *palindromic numbers*: n is palindromic if and only if $n = \widetilde{n}$. In the same way, the pasting of n, $m \in \mathbb{N}$ is given by $10^{Ç(m)} n + m$.

For the case of natural numbers Propositions 2, 10 and 11, can be summarized in the following result.

Proposition 12. *Let n, m, $p \in \mathbb{N}$. The following statements hold:*

1. $\widetilde{\widetilde{n}} = n$

2. $\tilde{n} \diamond \tilde{m} = \widetilde{m \diamond n}$

3. $(m \diamond n) \diamond p = m \diamond (n \diamond p)$

4. *If n is palindromic and $Ç(n)$ is even, then* 11 *is a divisor of n.*

5. $11 | n \diamond \tilde{n}$.

In general, the properties presented in the polynomial case for the operations of the ring $(+, \cdot)$ are not true for natural numbers, although by imposing some restrictions we can get similar results.

Applying pasting and reversing operations to natural numbers we can rewrite some mathematical games such as the one presented in [18]. It will be convenient to introduce the following notation:

$$\Diamond_{k=0}^{n} a_k := a_0 \diamond a_1 \diamond \cdots \diamond a_n \, . \tag{4}$$

The following mathematical games can be found in [18], but here we use an approach based on the previous results.

1. $(\diamondsuit_{k=0}^{n} 9-k)\cdot 9+9-(n+2)=\diamondsuit_{k=0}^{n+1} 8$, where $n\leq 9$. This can be expanded as:

$$
\begin{aligned}
9\times 9+7 &= 88\\
98\times 9+6 &= 888\\
987\times 9+5 &= 8888\\
9876\times 9+4 &= 88888\\
98765\times 9+3 &= 888888\\
987654\times 9+2 &= 8888888\\
9876543\times 9+1 &= 88888888\\
98765432\times 9+0 &= 888888888\\
987654321\times 9-1 &= 8888888888
\end{aligned}
$$

2. We know that $1^2=1$. Now, for $0<n<9$, we have

$$(\diamondsuit_{k=0}^{n} 1)^2=\diamondsuit_{k=0}^{n}(k+1)\diamond\widetilde{\diamondsuit_{k=0}^{n-1}(k+1)}\,.$$

This can be expanded as:

$$
\begin{aligned}
1\times 1 &= 1\\
11\times 11 &= 121\\
111\times 111 &= 12321\\
1111\times 1111 &= 1234321\\
11111\times 11111 &= 123454321\\
111111\times 111111 &= 12345654321\\
1111111\times 1111111 &= 1234567654321\\
11111111\times 11111111 &= 123456787654321\\
111111111\times 111111111 &= 12345678987654321
\end{aligned}
$$

The pasting operation is also useful to obtain powers of natural numbers: $(n\diamond m)^k$, where $0\leq m\leq 9$ and $k\in\mathbb{N}$. The following proposition, which summarizes the propositions 2.1, 2.2, 2.3, 2.4, 2.5, 2.6, 2.7, 2.8 and 2.9 in [2], gives us some ideas to obtain similar results for $k>2$.

Proposition 13. *Let* $0 \leq m, p, q, r, s \leq 9$. *Then,*

$$(n \diamond m)^2 = n\left((n+p) \diamond q) + r\right) \diamond s\,, \tag{5}$$

where $p \diamond q = 2m$ *and* $r \diamond s = m^2$.

Proof. We can see that $(n \diamond m)^2 = (10n + m)^2 = 100n^2 + 20nm + m^2$, which leads us to $10n(10(n+p)+q)+10r+s$ thus proving the result. □

As an immediate consequence of Proposition 13, which summarizes the corollaries 2.1, 2.2, 2.3, 2.4, 2.5, 2.6, 2.7, 2.8 and 2.9 in [2], we present the following corollary.

Corollary 14. *Let* $x = \alpha \diamond s$, *where* $s \in \{0, 1, 4, 5, 6, 9\}$ *and* $\alpha \in \mathbb{N}$. *If* $x - r = n((n+p)\diamond q)$, *where* $p \diamond q = 2m$, $r \in \{0, 1, 2, 3, 4, 6, 8\}$, $p \in \{0, 1\}$ *and* $q \in \{0, 2, 4, 6, 8\}$, *then* $\sqrt{x} = n \diamond m$.

Remark 15. *Using this approach we can recover the classical result about the square of odd natural numbers having five as a divisor:* $(n \diamond 5)^2 = n(n+1)\diamond 25$. *In the same way, for* $\alpha, n \in \mathbb{N}$, $\sqrt{\alpha \diamond 25} = n \diamond 5$ *if and only if* $\alpha = n(n+1)$. *Finally, the interested reader can obtain similar results for* $(n \diamond (m \diamond 5))^2$, *with* $0 \leq m \leq 9$, *as well as for their square roots.*

2 Pasting and reversing over differential operators

We consider linear differential operators

$$\mathcal{L} = a_n\,\partial^n + a_{n-1}\,\partial^{n-1} + \cdots + a_1\,\partial + a_0\,,$$

with $a_0 \neq 0$, $a_i \in K$, where $i = 0, 1, \cdots, n$ and K is a *differential field*; see [19]. Solutions of linear differential equations are related with the factorization of linear differential operators; see [12].

From now on the term *differential operator* will always mean *linear differential operator*. For convenience, we write $\mathcal{L}$ as follows:

$$\mathcal{L} = \sum_{k=0}^{n} a_{n-k}\,\partial^{n-k}\,.$$

As in previous cases, we denote by $\c{C}(\mathcal{L})$ the number of coefficients of the differential operator $\mathcal{L}$. For instance, if the order of $\mathcal{L}$ is n, then $\c{C}(\mathcal{L}) = n + 1$.

Definition 16. (Reversing, palindromics and antipalindromics of differential operators.) *Consider the differential operator*

$$\mathcal{L} = \sum_{k=0}^{n} a_{n-k}\, \partial^{n-k}\,, \tag{6}$$

with $a_0 \neq 0$ and $a_i \in K$. The reversing of $\mathcal{L}$, denoted by $\widetilde{\mathcal{L}}$, is given by

$$\widetilde{\mathcal{L}} = \sum_{k=0}^{n} b_{n-k}\, \partial^{n-k}\,, \tag{7}$$

with $b_{n-k} = a_k$ and $k = 0, 1, \cdots, n$. The differential operator $\mathcal{L}$ is called palindromic whether $\widetilde{\mathcal{L}} = \mathcal{L}$, respectively, the differential operator $\mathcal{L}$ is called antipalindromic whether $\widetilde{\mathcal{L}} = -\mathcal{L}$.

Definition 16 leads us to the following result.

Proposition 17. *Let $\mathcal{L}$ and $\widetilde{\mathcal{L}}$ be differential operators as in equations* (6) *and* (7)*, respectively. The following statements hold.*

1. $\widetilde{\widetilde{\mathcal{L}}} = \mathcal{L}$.
2. $\c{C}(\mathcal{L}) = \c{C}(\widetilde{\mathcal{L}})$.
3. $\widetilde{\mathcal{L} + \mathcal{R}} = \widetilde{\mathcal{L}} + \widetilde{\mathcal{R}}$, *for* $\c{C}(\mathcal{L}) = \c{C}(\mathcal{R})$.

Proof. Items **1** and **2** are consequences of the Definition 16. Item **3** is demonstrated in a way similar to the polynomial case. □

Remark 18. *In general, there is no relationship between* $\ker \mathcal{L}$ *and* $\ker \tilde{\mathcal{L}}$. *Using* differential Galois theory *[19], it can be shown that* $e^{-\frac{x^2}{2}} \in \ker(\partial^2 + 1 - x^2)$*, while there are not* Liouvillian functions *in* $\ker((1 - x^2)\partial^2 + 1)$.

As a particular case, we have the following result.

Proposition 19. *Assume $\c{C}(\mathcal{L}) = 2$, $y \in \ker \mathcal{L}$ and $u \in \ker \widetilde{\mathcal{L}}$. Then $(\partial \ln y)(\partial \ln u) = 1$.*

Proof. Solving the linear differential equations $\mathcal{L}y = 0$ and $\widetilde{\mathcal{L}}u = 0$, we obtain

$$e^{-\int \frac{a_0}{a_1}} \in \ker \mathcal{L}\,,$$

with $e^{-\int \frac{a_1}{a_0}} \in \ker \widetilde{\mathcal{L}}$. Thus, $(\partial \ln y)(\partial \ln u) = 1$. □

Proposition 17 and Definition 16 lead us to the following results.

Proposition 20. *Let $\mathcal{L}$ and $\mathcal{R}$ be palindromic and antipalindromic differential operators respectively, with $\c{C}(\mathcal{L}) = \c{C}(\mathcal{R}) = 2k$. Then, there exist differential operators $\mathcal{S}$ and $\mathcal{T}$ such that*

$$\mathcal{L} = \mathcal{S}(\partial + 1)\,,$$

with $\mathcal{R} = \mathcal{T}(\partial - 1)$.

Proof. We see that $e^{-x} \in \ker(\partial + 1)$ and $e^{x} \in \ker(\partial - 1)$. Now, due to $\widetilde{\mathcal{L}} = \mathcal{L}$, $a_{2k-1-i} = a_i$ and $\partial^{2k-1-i}e^{-x} = -\partial^i e^{-x}$. In this way, $e^{-x} \in \ker \mathcal{L}$, which means that there exist $\mathcal{S}$ such that $\mathcal{L} = \mathcal{S}(\partial + 1)$. On the other hand, owing to $\widetilde{\mathcal{R}} = -\mathcal{R}$, $a_{2k-1-i} = -a_i$ and $\partial^{2k-1-i}e^{x} = \partial^i e^{x}$. In this way, $e^{x} \in \ker \mathcal{R}$, which means that there exist $\mathcal{T}$ such that $\mathcal{R} = \mathcal{T}(\partial - 1)$. □

We recall that differential operators $\mathcal{T}$ and $\mathcal{S}$ are *left divisors* of $\mathcal{L}$ and $\mathcal{R}$ respectively. In the same way, $\partial + 1$ and $\partial - 1$ are *right divisors* of $\mathcal{L}$ and $\mathcal{R}$ respectively. For further details see [12].

Proposition 21. *The following statements hold.*

1. *The sum of two palindromic differential operators, with the same order, is also a palindromic differential operator.*

2. *The sum of two antipalindromic differential operators, with the same order, is also an antipalindromic differential operator.*

Proof. We proceed exactly as in Proposition 5 for the polynomial case, using Definition 16 and item **3** of Proposition 17. □

Now we introduce the definition of pasting operation over differential operators.

Definition 22. *Pasting of the differential operators $\mathcal{L}$ and $\mathcal{R}$, denoted by $\mathcal{L} \diamond \mathcal{R}$, is given by: $\mathcal{L} \diamond \mathcal{R} := \mathcal{L}\partial^{\c{C}(\mathcal{R})} + \mathcal{R}$.*

The following properties, adapted from Proposition 10, are consequences of Definition 22.

Proposition 23. *Let $\mathcal{L}$, $\mathcal{R}$ and $\mathcal{S}$ be differential operators. The following statements hold:*

1. $\tilde{\mathcal{L}} \diamond \tilde{\mathcal{R}} = \widetilde{\mathcal{R} \diamond \mathcal{L}}$

2. $(\mathcal{L} \diamond \mathcal{R}) \diamond \mathcal{S} = \mathcal{L} \diamond (\mathcal{R} \diamond \mathcal{S})$

Proof.

1. Let $\mathcal{L}, \mathcal{R}$ be differential operators,

$$\begin{aligned} \mathcal{L} &= \sum_{\ell=0}^{s} a_{s-\ell}\,\partial^{s-\ell}\,, \\ \mathcal{R} &= \sum_{j=0}^{k} b_{k-j}\,\partial^{k-j}\,. \end{aligned}$$

Then, by Definition 22 and assuming $\mathcal{S} = \mathcal{R} \diamond \mathcal{L}$ we have

$$\begin{aligned} \mathcal{S} &= \left(\sum_{j=0}^{k} b_{k-j}\,\partial^{k-j} \right) \partial^{s+1} + \sum_{\ell=0}^{s} a_{s-\ell}\,\partial^{s-\ell} \\ &= \sum_{i=0}^{k+s+1} c_{k+s+1-i}\,\partial^{k+s+1-i}\,, \end{aligned}$$

where the coefficients c_m are given by

$$c_m = \begin{cases} a_m, & 0 \le m \le s, \\ b_m, & s+1 \le m \le k+s+1. \end{cases}$$

By Definition 16, we obtain

$$\begin{aligned} \widetilde{\mathcal{L}} &= \sum_{\ell=0}^{s} a_{\ell}\,\partial^{s-\ell}\,, \\ \widetilde{\mathcal{R}} &= \sum_{j=0}^{k} b_{j}\,\partial^{k-j}\,, \\ \widetilde{\mathcal{S}} &= \sum_{i=0}^{k+s+1} c_{i}\,\partial^{k+s+1-i}\,. \end{aligned}$$

Therefore, by Definition 22 we have

$$\widetilde{\mathcal{S}} = \sum_{i=0}^{k+s+1} c_i \, \partial^{k+s+1-i} = \left(\sum_{\ell=0}^{s} a_\ell \, \partial^{s-\ell} \right) \partial^{k+1} + \sum_{j=0}^{k} b_j \, \partial^{k-j} \, .$$

We conclude that $\widetilde{\mathcal{S}} = \widetilde{\mathcal{L}} \diamond \widetilde{\mathcal{R}}$.

2. Let $\mathcal{L}$, $\mathcal{R}$, $\mathcal{S}$ be differential operators such that

$$\begin{aligned} \mathcal{L} &= \sum_{i=0}^{k} a_{k-i} \, \partial^{k-i} \, , \\ \mathcal{R} &= \sum_{i=0}^{j} b_{j-i} \, \partial^{j-i} \, , \\ \mathcal{S} &= \sum_{i=0}^{\ell} d_{\ell-i} \, \partial^{\ell-i} \, . \end{aligned}$$

where, $\c{C}(\mathcal{L}) = k+1, \c{C}(\mathcal{R}) = j+1$ and $\c{C}(\mathcal{S}) = \ell+1$. We write $(\mathcal{L} \diamond \mathcal{R}) \diamond \mathcal{S}$ by means of Definition 16 as follows

$$\left(\left(\sum_{i=0}^{k} a_{k-i} \, \partial^{k-i} \right) \partial^{j+1} + \sum_{i=0}^{j} b_{j-i} \, \partial^{j-i} \right) \partial^{\ell+1} + \sum_{i=0}^{\ell} d_{\ell-i} \, \partial^{\ell-i} \, .$$

This can be rewritten as:

$$\left(\sum_{i=0}^{k} a_{k-i} \, \partial^{k-i} \right) \partial^{j+\ell+2} + \left(\left(\sum_{i=0}^{j} b_{j-i} \, \partial^{j-i} \right) \partial^{\ell+1} + \sum_{i=0}^{\ell} d_{\ell-i} \, \partial^{\ell-i} \right) ,$$

or, equivalently, as

$$\left(\sum_{i=0}^{k} a_{k-i} \, \partial^{k-i} \right) \partial^{j+\ell+2} + \left(\sum_{i=0}^{j} b_{j-i} \, \partial^{j-i} \diamond \sum_{i=0}^{\ell} d_{\ell-i} \, \partial^{\ell-i} \right) .$$

Therefore,

$$(\mathcal{L} \diamond \mathcal{R}) \diamond \mathcal{S} = \mathcal{L} \diamond (\mathcal{R} \diamond \mathcal{S})\,.$$

□

As a consequence of Proposition 20, which is adapted for pasting of polynomials, we present the following result.

Proposition 24. *Let $\mathcal{L}$ be a differential operator. The operator $\partial + 1$ is a right divisor of the differential operator $\mathcal{L} \diamond \widetilde{\mathcal{L}}$.*

Proof. Owing to the fact that $\text{Ç}(\mathcal{L} \diamond \widetilde{\mathcal{L}})$ is even and $\mathcal{L} \diamond \widetilde{\mathcal{L}}$ is palindromic, by Proposition 20, we have $e^{-x} \in \ker P \diamond \tilde{P}$, so that $\mathcal{T}(\partial + 1) = P \diamond \tilde{P}$, for some differential operator $\mathcal{T}$. □

The following results are particular cases of differential operators, where the differential field is $K = \mathbb{C}$.

Proposition 25. *Consider $\mathcal{L}$ and $\widetilde{\mathcal{L}}$ as in equations* (6) *and* (7)*, respectively, with $K = \mathbb{C}$. The following statements hold.*

1. $\widetilde{\mathcal{L}} = \partial^n \sum_{k=0}^{n} a_{n-k} \partial^{k-n}$*, where* $\mathcal{L} = \sum_{k=0}^{n} a_{n-k} \partial^{n-k}$*.*
2. $x^k e^{-\lambda_i x} \in \ker \widetilde{\mathcal{L}}$ *if and only if* $x^k e^{\lambda_i x} \in \ker \mathcal{L}$*.*
3. $\widetilde{\mathcal{L}} = (-1)^n (\alpha_1 \partial - \beta_1)(\alpha_2 \partial - \beta_2) \cdots (\alpha_n \partial - \beta_n)$ *if and only if* $\mathcal{L} = (\beta_1 \partial - \alpha_1)(\beta_2 \partial - \alpha_2) \cdots (\beta_n \partial - \alpha_n)$*,* $\alpha_i, \beta_i \in \mathbb{C}$*.*
4. $\widetilde{\mathcal{L} \cdot \mathcal{R}} = \widetilde{\mathcal{L}} \cdot \widetilde{\mathcal{R}}$*.*
5. *If $\mathcal{L}$ is a palindromic (or antipalindromic) differential operator such that* $\{x^{i_1} e^{\lambda_1 x}, \cdots, x^{i_r} e^{\lambda_r x}\} \subset \ker \mathcal{L}$*, then* $e^{\lambda_{k+j} x} = e^{-\lambda_j x}$*, with* $r \in \{2k, 2k+1\}$ *and* $j = 1, \cdots, k$*.*
6. *The product of two palindromic differential operators is also a palindromic differential operator.*
7. *The product of two antipalindromic differential operators is a palindromic differential operator.*
8. *The product of a palindromic differential operator with an antipalindromic differential operator is an antipalindromic differential operator.*

Proof. It follows from the fact that the characteristic polynomial satisfies the same properties (see Propositions 2, 4, 5) and due to $\partial a = a \partial$ for all $a \in \mathbb{C}$. □

Final remarks

This paper is a starting point to develop several research projects such as the applications of pasting and reversing over:

1. vector spaces and matrices;

2. polynomials in several variables;

3. general differential operators;

4. *Ore extensions* or other kind of non–commutative polynomial rings, as for example Weyl algebras and Heisenberg algebras;

5. general difference and q–difference operators;

6. general simple permutations and combinatorial dynamics; and

7. in physics, particularly in *supersymmetric quantum mechanics*.

There are works in which this approach can be applied; see for example [4, 8, 14] for the polynomial case. In particular, palindromic and antipalindromic polynomials have been applied to analyze time-reversible systems and conserved quantities, see [14], with a similar approach. We hope that the material presented here can be useful for further studies.

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