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A novel Tschirnhaus method to get only the true solutions of quartic equations

Un nuevo método de Tschirnhaus para obtener solamente las verdaderas soluciones de ecuaciones cuarto grado

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Abstract. The traditional Tschirnhaus method for solving quartic equations produces false solutions along with true solutions, without offering any clue to identify the true solutions. Thus the true solutions have to be picked by trial and error only. In this paper, we present a new Tschirnhaus method for solving quartic equations, which produces only the true solutions.

***Key words*:** Tschirnhaus transformation, quartic equation, resolvent cubic equation, true solutions, false solutions.

Resumen. El método tradicional de Tschirnhaus para resolver ecuaciones de cuarto grado produce soluciones falsas junto con las verdaderas y no da pista alguna para reconocer las verdaderas soluciones. Por tanto se debe recurrir a ensayo y error para poder identificarlas. En este artículo presentamos un nuevo método de Tschirnhaus para resolver ecuaciones de cuarto grado que produce solamente las verdaderas soluciones.

***Palabras clave*:** Transformación de Tschirnhaus, ecuación de cuarto grado, ecuación cúbica resolvente, soluciones verdaderas, soluciones falsas.

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The Tschirnhaus transformation, proposed by the German mathematician Ehrenfried Walther von Tschirnhaus (1683), is a polynomial expression linking two variables. For example, a quadratic Tschirnhaus transformation, $x^2 + Ax + B = y$ links the variables x and y through constants A and B, which are complex numbers in general. When a quadratic Tschirnhaus transformation is employed to transform a N-th degree polynomial equation in x, such a $x^N + a_{N-1}x^{N-1} + a_{N-2}x^{N-2} + \cdots + a_0 = 0$, to another polynomial equation in y of same degree, $y^N + b_{N-1}y^{N-1} + b_{N-2}y^{N-2} + \cdots + b_0 = 0$, the two coefficients of the equation in y (normally b_{N-1} and b_{N-2}) can be made zero by a proper choice of constants A and B, resulting in an equation without these terms. Thus, using a quadratic Tschirnhaus transformation, a cubic equation can be transformed to a binomial cubic equation, $y^3 + b_0 = 0$, leading to its solution; a quartic equation can be transformed to an even-powered quartic equation, $y^4 + b_2y^2 + b_0 = 0$, so it can be solved. In the same manner, a cubic Tschirnhaus transformation is expected to remove three terms in a polynomial equation, a quartic Tschirnhaus transformation–four terms, and so on. In this way, Tschirnhaus believed that, using his transformation, equations of any degree can be transformed to binomial equations, and hence they can be solved [1].

However, the later works of Ruffini (1799), Abel (1824), and Galois (1832) have proved that the polynomial equations of degree five and above cannot be solved by radicals [2]. Unfortunately, the traditional Tschirnhaus method for solving quartic equations produces true as well as false solutions, without offering a way to identify the true solutions [1, 3]. Very recently, a Tschirnhaus method (though a bit involved) is described in [4], which identifies at least one true solution of cubic equations; but for quartic equations as such, so far there is no method in the literature which can identify true solutions.

This paper presents a new method using a quadratic Tschirnhaus transformation for solving quartic equations, which is capable of producing only the true solutions. The method follows the traditional path for the first half of the journey, but differs in the later half as we will see in the following paragraphs.

Let us consider, without loss of any generality, a depressed quartic equation,

$$x^4 + ax^2 + bx + c = 0, \tag{1}$$

where a, b, and c are coefficients. We define a special quadratic Tschirnhaus transformation,

$$x^2 = 2dx + f^2 - d^2 + y, \tag{2}$$

where d and f are two unknown numbers in (2) and y is a new variable. Notice that (2) is a quadratic equation in x, hence solving for x yields the two expressions

$$x = d \pm \sqrt{f^2 + y} \tag{3}$$

Using of (3) in (1) to eliminate x yields two expressions in y,

$$y^2 + gy + h = \mp\sqrt{f^2 + y}(4dy + j), \tag{4}$$

where g, h, and j are given by:

$$g = 6d^2 + 2f^2 + a, \tag{5}$$

$$h = d^4 + 6d^2f^2 + f^4 + ad^2 + af^2 + bd + c, \tag{6}$$

$$j = 4d^3 + 4df^2 + 2ad + b. \tag{7}$$

Observe that squaring the two expressions in (4) results in one expression in y,

$$\begin{aligned} &y^4 + (2g - 16d^2)y^3 + [g^2 + 2h - 8d(j + 2df^2)]y^2 \\ &+[2gh - j(j + 8df^2)]y + h^2 - f^2j^2 = 0. \end{aligned} \tag{8}$$

In the traditional method [1], the coefficients of y and y^3 in (8) are set to zero, rendering (8) to be a quadratic equation in y^2, so that it can be solved. However this method generates true as well as false solutions, without giving any clue to identify the true solutions. The reason for the generation of the both true and false solutions is that the transformed quartic equation (8) produces an octic equation in x, when back-substitution using (2) is done. The octic equation has two factors one is the given quartic equation (1), whose solutions are termed as true solutions, and the second one is another quartic equation, whose solutions are termed as false solutions of (1).

Therefore, we need to adopt an approach different from the one used by the traditional method, to get only the true solutions. So we prescribe that one solution of (8) be zero (i.e., $y = 0$). This requires the constant term, $h^2 - f^2j^2$, in (8) has to vanish, implying

$$(h + fj)(h - fj) = 0. \tag{9}$$

From (9) it is clear that either $h + fj = 0$, or $h - fj = 0$. Let us choose $h + fj = 0$, so further use of (6) and (7) in $h + fj = 0$ yields

$$(d + f)^4 + a(d + f)^2 + b(d + f) + c = 0. \tag{10}$$

Use of $y = 0$ in the transformation (3) yields two x values, $x = d \pm f$, as the probable solutions of quartic equation (1). But a comparison of (1) and (10) reveals that only $x = d + f$ can be the solution of (1), eliminating the other possibility. Thus the true solution of (1) is confined to $x = d + f$. So now we need to obtain d and f, which satify (10). For this purpose, let us expand (10),

$$d^4 + 4d^3f + 6d^2f^2 + 4df^3 + f^4 + ad^2 + 2adf + af^2 + bd + bf + c = 0.$$

We rearrange the above equation such that the left-hand-side is a perfect square:

$$(d^2 + 2df)^2 = -(2f^2 + a)\left(d^2 + \frac{4f^3 + 2af + b}{2f^2 + a}d + \frac{f^4 + af^2 + bf + c}{2f^2 + a}\right). \tag{11}$$

The quadratic term in d in the right-hand-side of (11) can also be made a perfect square,

$$\left[d + \frac{4f^3 + 2af + b}{2(2f^2 + a)}\right]^2,$$

if the condition

$$\frac{(4f^3 + 2af + b)^2}{4(2f^2 + a)} - (f^4 + af^2 + bf + c) = 0 \tag{12}$$

is satisfied. Further simplification of (12) leads to:

$$f^6 + (a/2)f^4 - cf^2 + [(b^2 - 4ac)/8] = 0. \tag{13}$$

Notice that (13) is a cubic equation in f^2, known as resolvent cubic equation. We employ the recently published [4] Tschirnhaus method for solving cubics to obtain only the true solutions of (13), which yields three values of f^2 or six values of f. Now (11) becomes a perfect square,

$$(d^2 + 2df)^2 = -k\{d + [(2fk + b)/2k]\}^2, \tag{14}$$

where $k = 2f^2 + a$. Taking the square root of (14) yields two quadratic equations:

$$d^2 + (2f - \sqrt{-k}\,)d - \sqrt{-k}\,[(2fk + b)/2k] = 0, \tag{15}$$

$$d^2 + (2f + \sqrt{-k}\,)d + \sqrt{-k}\,[(2fk + b)/2k] = 0. \tag{16}$$

Solving the quadratic equations (15) and (16), we determine four values of d for each f. The four true solutions of the quartic equation (1) are then obtained using $x = d + f$. Now, let us solve one numerical example. Consider the depressed quartic equation

$$x^4 + 3x^2 - 6x + 10 = 0.$$

We get the resolvent cubic equation (13) in f^2, which are $f^6 + 1.5f^4 - 10f^2 - 10.5 = 0$. Solving it yields three values of f^2, namely 3, -1, and -3.5, and six values of f, which are

$$\pm 1.732050807568, \quad \pm i, \quad \pm 1.870828693387.$$

Choosing $f = i$, we obtain the two quadratic equations [(15) and (16)]

$$d^2 + id + 1 + 3i = 0, \quad d^2 + 3id - 1 - 3i = 0.$$

Solving the above quadratic equations, we get four values of d, which turn out to be $1 - 2i$, $-1 + i$, 1, and $-1 - 3i$. Using these values of d with $f = i$ in $x = d + f$, all the four true solutions of given quartic equation, $x^4 + 3x^2 - 6x + 10 = 0$, are determined, $1 - i$, $-1 + 2i$, $1 + i$, and $-1 - 2i$. Use of other values of f also yields true solutions; readers are invited to verify this.

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