Δ^c-rings and its basic properties

Δ^c-anillos y sus propiedades básicas

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Abstract. In this paper we introduce a c-derivation δ^c, Δ^c-rings and some basic properties. Finally, we prove that the radical of a Δ^c-ideal is also a Δ^c-ideal.

Keywords: c-derivation, Δ^c-rings, Δ^c-ideals.

Resumen. En este artículo introducimos una c-derivación δ^c, Δ^c-anillos y algunas propiedades básicas. Finalmente, como resultado general se demuestra que el radical de un Δ^c-ideal es también un Δ^c-ideal.

Palabras claves: c-derivación, Δ^c-anillos, Δ^c-ideal.

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1. Introduction

Differential rings and differential fields are rings and fields (respectively) with a derivation δ, which is a linear map that satisfies the Leibniz product rule. There is a branch of mathematics called "Differential algebra" that studies these algebraic objects and their applications to differential equations. Differential algebra was introduced and developed by Joseph Ritt and also developed by Ellis Robert Kolchin, a doctoral student of Ritt. Also differential Galois theory is a branch of abstract algebra that studies fields equipped with a derivation function and the solutions of differential equations over a differential base field [4].

In this paper, we follow the same steps as in [1], but with a c-derivation δ^c, which is a map $\delta^c : A \to A$ that satisfies

$$\begin{aligned} \delta^c(xy) &= \delta^c(x)c(y) + \delta^c(y)c(x), \\ \delta^c(x+y) &= \delta^c(x) + \delta^c(y), \end{aligned}$$

where c is a fixed ring homomorphism in a commutative ring A with an identity element containing $\mathbb{Q}$ and with characteristic 0. Note that δ^c depends on c.

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In addition, we introduce the notion of a Δ^c-ring which is a ring A with a c-derivation. A concept of a (α, β)-derivation for bimodules was introduced in [2].

We give a few examples of Δ^c-rings, where it is shown that δ^c and c not neccesarily commute.

On the other hand, we study also some sets defined in [3] with the c-derivation. However, we find that the power rule and the quotient rule of a c-derivation are different and do not satisfy the generalization of the Leibniz rule product. Thus we have to add more conditions, in order to define an ideal, a subring, a multiplicatively closed subsets, and others.

Once introduced the basic properties of commutative algebra, we prove that the radical of a Δ^c-ideal is also a Δ^c-ideal.

Throughout this paper the word ring shall mean a commutative ring with an identity element, containing $\mathbb{Q}$ and with characteristic 0. In particular we work on polynomial rings with one or two variables.

2. Δ^c-Rings, Δ^c-Subrings and Δ^c-Ring Homomorphisms

In this section, we start with our main definition, the c-derivation δ^c. Then we prove the power rule and quotient rule of this c-derivation, and we give the definition of a Δ^c-ring, Δ^c-Subrings and Δ^c-Ring Homomorphisms.

Definition 2.1. Let A be a ring, and $c : A \to A$ be a ring homomorphism. A *c-derivation* on A is a map $\delta^c : A \to A$ such that

$$\begin{aligned} \delta^c(xy) &= \delta^c(x)c(y) + \delta^c(y)c(x) \\ \delta^c(x+y) &= \delta^c(x) + \delta^c(y) \end{aligned}$$

for all $x, y \in A$.

Note that δ^c depends on the ring homomorphism c.

Lemma 2.2. *Let A be a ring and δ^c a c-derivation on A. Then the power rule is given by $\delta^c(x^n) = n\delta^c(x)c(x^{n-1})$ where $x \in A$ and $n \in \mathbb{N}$.*

Proof. The proof follows by induction over n. Indeed, for $n = 1$, $\delta^c(x) = \delta^c(x)c(1)$, so assume that $\delta^c(x^k) = k\delta^c(x)c(x^{k-1})$ for all $k < n$, thus

$$\begin{aligned} \delta^c(x^n) &= \delta^c(x^{n-1}x) \\ &= \delta^c(x^{n-1})c(x) + \delta^c(x)c(x^{n-1}) \\ &= (n-1)\delta^c(x)c(x^{n-2})c(x) + \delta^c(x)c(x^{n-1}) \\ &= n\delta^c(x)c(x^{n-1}). \end{aligned}$$

□

Lemma 2.3. *Let A be a ring, and δ^c a c-derivation on A. Let $y \in A$ be an invertible element, such that $c(y)^2 \neq 0$. Then the quotient rule is given by*

$$\delta^c(xy^{-1}) = \delta^c\left(\frac{x}{y}\right) = \left(\frac{\delta^c(x)c(y) - \delta^c(y)c(x)}{c(y)^2}\right).$$

Proof. For an invertible element $x \in A$, we have

$$0 = \delta^c(xx^{-1}) = \delta^c(x)c(x^{-1}) + \delta^c(x^{-1})c(x)$$

so

$$\delta^c(x^{-1}) = \delta^c\left(\frac{1}{x}\right) = -\frac{\delta^c(x)}{c(x^2)}.$$

Thus,

$$\begin{aligned}
\delta^c(xy^{-1}) &= \delta^c(x)c(y^{-1}) + \delta^c(y^{-1})c(x) \\
&= \delta^c(x)c(y^{-1}) + \frac{\delta^c(y)}{c(y)^2}c(x) \\
&= \frac{\delta^c(x)c(y) - \delta^c(y)c(x)}{c(y)^2}.
\end{aligned}$$

□

We must prove that the quotient rule is well defined, so we invite the lector to section (6) for a proof.

We fix a ring homomorphism c, a c-derivation $\delta^c : A \to A$ and $\Delta^c = \{\delta^c\}$.

Definition 2.4. A differential ring is a triple (A, c, Δ^c), where A is a ring with a set of derivations Δ^c.

From now on we used Δ^c instead of the word differential.

Definition 2.5. Let A be a Δ^c-ring, and $S \subseteq A$, then S is a Δ^c-subring if it is a subring of A and preserves the c-derivation, i.e. $\delta^c(S) \subseteq S$, $c(S) \subseteq S$ for $\delta^c \in \Delta^c$.

Definition 2.6. An element $a \in A$ is called a constant if $\delta^c(a) = 0$. Let C_A denote the set of all constant elements.

Lemma 2.7. *If A is a Δ^c-ring and $c(C_A) \subseteq C_A$ then C_A is a Δ^c-subring.*

Proof. Let us see that C_A is a subring, indeed let $a, b \in C_A$ then

i) $\delta^c(a + b) = \delta^c(a) + \delta^c(b) = 0$ then $a + b \in C_A$.

ii) $\delta^c(ab) = \delta^c(a)c(b) + \delta^c(b)c(a) = 0$ then $ab \in C_A$.

iii) $\delta^c(1) = 0$ then $1 \in C_A$.

And $\delta^c(C_A) \subseteq C_A$, because if $a \in C_A$ then $\delta^c(a) = 0 \in C_A$, so C_A is a Δ^c-subring. $\square$

First, we show the existence of c-derivations.

Lemma 2.8. *Let R be a ring. Fix a ring homomorphism $c : R[x, y] \to R[x, y]$, then there exists a unique c-derivation on $R[x, y]$ whose constants set is R.*

Proof. The existence follows inmediately. Thus we prove uniqueness. Let $\partial^c, \delta_c : R[x, y] \to R[x, y]$ be c-derivations such that

$$\begin{aligned} \partial^c(x) &= \delta^c(x), \\ \partial^c(y) &= \delta^c(y). \end{aligned}$$

Then

$$\begin{aligned} \partial^c(x^i y^j) &= \partial^c(x^i)c(y^j) + \partial^c(y^j)c(x^i) \\ &= \partial^c(x^i)c(y^j) + jc(y^{j-1})\partial^c(y)c(x^i) \\ &= \delta^c(x^i)c(y^j) + jc(y^{j-1})\delta^c(y)c(x^i) \\ &= \delta^c(x^i y^j). \end{aligned}$$

So that

$$\partial^c\left(\sum_{i,j} a_{ij}x^i y^j\right) = \sum_{i,j} c(a_{ij}x^i y^j) = \sum_{i,j} c(a_{ij})\partial^c(x^i y^j) = \sum_{i,j} c(a_{ij})\delta^c(x^i y^j).$$

This implies that $\partial^c(x^n) = \delta^c(x^n)$ for all $n \in \mathbb{N}$. $\square$

Definition 2.9. Let A be a Δ^c-ring with δ^c_A the c-derivation of A, and B be a Δ^h-ring with δ^h_B the h-derivation of B. Then a Δ^c-ring homomorphism is a map $f : A \to B$ that is a ring homomorphism, and is compatible with δ^c_A and δ^h_B, i.e. $f(\delta^c_A(x)) = \delta^h_B(f(x))$ for all $x \in A$.

If f is bijective then f is an Δ^c-isomorphism.

Lemma 2.10. *Let A be a Δ^c-ring, B be a Δ^h-ring and C be a Δ^l-ring. If $f : A \to B$ is $\Delta^{c,h}$-ring homomorphism and $g : B \to C$ is $\Delta^{h,l}$-ring homomorphism then their composition $g \circ f : A \to C$ is a $\Delta^{c,l}$-ring homomorphism.*

Proof. We have

$$\begin{gathered} A \xrightarrow{f} B \xrightarrow{g} C \\ x \mapsto f(x) \mapsto g(f(x)) \end{gathered} \tag{1}$$

where $g\left((f\delta^c_A(x))\right) = g(\delta^h_B(f(x)))$ and $g(\delta^h_B(y)) = \delta^l_C(g(y))$. So

$$\delta^l_C(g(f(x))) = g(\delta^h_B f(x)) = g(f(\delta^c_A(x))).$$

$\square$

3. Examples of Δ^c-Rings

Now, we give some examples of Δ^c-rings.

Example 3.1. Consider the ring $A = \mathbb{Z}/p\mathbb{Z}[x]$ for some prime p. We can make A into a Δ^c-ring by considering the Frobenius endomorphism c defined by,

$$\begin{array}{rcl} c : \mathbb{Z}/p\mathbb{Z}[x] & \to & \mathbb{Z}/p\mathbb{Z}[x] \\ f & \mapsto & f^p, \end{array} \tag{2}$$

where $f = a_0 + a_1x + \cdots + a_nx^n$ and f^p means f to power p. And with $\delta^c : \mathbb{Z}/p\mathbb{Z}[x] \to \mathbb{Z}/p\mathbb{Z}[x]$ an additive map defined by

i) $\delta^c(x) = 1$.

ii) $\delta^c(x^n) = nc(x^{n-1})\delta^c(x) = nc(x^{n-1})$.

iii) $\delta^c(a) = 0$ for all $a \in \mathbb{Z}/p\mathbb{Z}$.

iv) $\delta^c(fg) = \delta^c(f)c(g) + \delta^c(g)c(f) = \delta^c(f)g^p + \delta^c(g)f^p$.

Note that $c|_{C_A} \neq \text{id}$, i.e. $\delta(x^p) = 0$. Also note that $\delta^c c \neq c\delta^c$. In fact,

$$\begin{aligned} \delta^c\left(c(f)\right) &= \delta^c(a_0^p + a_1^p x^p + \cdots + a_n^p x^{pn}) \\ &= pa_1^{p^2} x^{p-1} + \cdots + pna_n^{p^2} x^{pn-1} \\ &= 0. \end{aligned}$$

But on the other hand,

$$\begin{aligned} c(\delta^c(f)) &= c(a_1^p + 2a_2^p x^p + \cdots + na_n^p x^{p(n-1)}) \\ &= a_1^{p^2} + 2a_2^p x^{p^2} + \cdots + na_n^{p\,2} x^{(pn-p)^2}. \end{aligned}$$

Thus $\delta^c c \neq c\delta^c$.

Example 3.2. Let $A = \mathbb{C}[t]$, and define c by,

$$\begin{array}{rcl} c : \mathbb{C}[t] & \to & \mathbb{C}[t] \\ f & \mapsto & \overline{f}, \end{array} \tag{3}$$

then,

$$\begin{aligned} c(f+g) &= \overline{f+g} = \overline{f} + \overline{g} = c(f) + c(g) \\ c(fg) &= \overline{fg} = \overline{f}\,\overline{g} = c(f)c(g) \\ c(1) &= 1. \end{aligned}$$

Hence c is a homomorphism. Now consider $\delta^c : \mathbb{C}[t] \to \mathbb{C}[t]$ defined by,

i) $\delta^c(t) = 1$.

ii) $\delta^c(t^n) = nc(t^{n-1})\delta^c(t) = nc(t^{n-1})$.

iii) $\delta^c(z) = 0$ for all $z \in \mathbb{C}$.

Thus $\mathbb{C}[t]$ is a Δ^c-ring and if $a_0 \in \mathbb{C}$, $c(a_0) = \overline{a_0} \neq a_0$, thus $c|_{\mathbb{C}} \neq id$.

Note that, $\delta^c c = c\delta^c$.

Example 3.3. Consider the ring $\mathbb{C}[x, y]$, and the following ring homomorphism,

$$\begin{array}{ccc} c : \mathbb{C}[x, y] & \to & \mathbb{C}[x, y] \\ x & \mapsto & y, \\ y & \mapsto & x, \end{array} \tag{4}$$

and $c|_{\mathbb{C}} = id_{\mathbb{C}}$. Let δ^c be the following c-derivation

i) $\delta^c(x) = 1$.

ii) $\delta^c(x^n) = nc(x^{n-1})\delta^c(x) = nc(x^{n-1})$.

iii) $\delta^c(y) = 0$.

iv) $\delta^c(z) = 0$ for all $z \in \mathbb{C}$.

Then $\mathbb{C}[x, y]$ is a Δ^c-ring and $\delta^c c \neq c\delta^c$.

It follows by definition that $\mathbb{C}[x, y]$ is a Δ^c-ring. Let us see that $\delta^c c \neq c\delta^c$. So let $f \in \mathbb{C}[x, y]$, $f = \sum_{i,j} d_{ij} x^i y^j$ where $d_{ij} \in \mathbb{C}$. Then

$$\begin{aligned} \delta^c c\left(\sum_{i,j} d_{ij} x^i y^j\right) &= \delta^c\left(\sum_{i,j} d_{ij} c(x^i) c(y^j)\right) \\ &= \sum_{i,j} d_{ij} \delta^c\left((y^i)(x^j)\right) \\ &= \sum_{i,j} d_{ij}\left(\delta^c(y^i) c(x^j) + \delta^c(x^j) c(y^i)\right) \\ &= \sum_{i,j} d_{ij}\left(j y^{j-1} x^i\right) \end{aligned}$$

and,

$$
\begin{aligned}
c\delta^c\left(\sum_{i,j} d_{ij}x^iy^j\right) &= c\left(\sum_{i,j} d_{ij}\left(\delta^c(x^iy^j)\right)\right)\\
&= \sum_{i,j} d_{ij}c\left(\delta^c(x^i)c(y^j)+\delta^c(y^j)c(x^i)\right)\\
&= \left(\sum_{i,j} d_{ij}c\left(\delta^c(x^iy^j)\right)\right)\\
&= \sum_{i,j} d_{ij}c\left(iy^{i-1}x^j\right)\\
&= \sum_{i,j} d_{ij}ix^{i-1}y^j
\end{aligned}
$$

Thus $\delta^c c \neq c\delta^c$.

4. Δ^c-Ideals and Quotient Δ^c-Rings

In this section we define Δ^c-Ideals and Quotient Δ^c-Rings.

Definition 4.1. A Δ^c-ideal $\mathfrak{a}$ of a Δ^c-ring A is a subset $\mathfrak{a} \subseteq A$ such that $\mathfrak{a}$ is an ideal, $c(\mathfrak{a}) \subseteq \mathfrak{a}$, and for $\delta^c \in \Delta^c$, $\delta^c(\mathfrak{a}) \subseteq \mathfrak{a}$.

Example 4.2. Let A be a Δ^c-ring and B be a Δ^h-ring. Let $f : A \to B$ be a $\Delta^{c,h}$-ring homomorphism. If $c(ker(f)) \subseteq ker(f)$ then $\ker(f)$ is a Δ^c-ideal.

Proof. Let $x \in \ker(f)$, so

$$f(\delta^c_A(x)) = \delta^h_B(f(x)) = \delta^h_B(0) = 0$$

thus $\delta^c_A(x) \in \ker(f)$. And using the fact that $c(ker(f)) \subseteq ker(f)$, then $\ker f$ is a Δ^c-ideal. $\square$

Example 4.3. Let A be a Δ^c-ring and B be a Δ^h-ring. Let $f : A \to B$ be a $\Delta^{c,h}$-ring homomorphism. Then if $h(Imf) \subseteq Imf$ then $Im(f) = f(A)$ is a Δ^h-ideal. Furthermore $f(A)$ is a Δ^h-subring.

Proof. Let $y \in f(A)$, so $f(x) = y$ for some $x \in A$ and $\delta^h_B(f(x)) = f(\delta^c_A(x)) \in f(A)$. Then using the fact that $f(A)$ is a subring of B, $h(Imf) \subseteq Imf$, and that $\delta^h_B(f(x)) \subseteq f(A)$ we have that $f(A)$ is a Δ^h-subring. $\square$

Proposition 4.4. *Let A be a Δ^c-ring and $\mathfrak{a}$ be a Δ^c-ideal of A. Then $A/\mathfrak{a}$ has unique structure of Δ^c-ring.*

Proof. [1] It follows as in (Proposition 1.7.7, page 13), but here we give a detailed proof for δ^c. Recall that $A/\mathfrak{a}$ has a ring structure, called the quotient ring of A module $\mathfrak{a}$, with the addition as $(x+\mathfrak{a})+(y+\mathfrak{a})=(x+y)+\mathfrak{a}$ and the multiplication as $(x+\mathfrak{a})(y+\mathfrak{a})=(xy)+\mathfrak{a}$, where $x,y\in A$. Now using the fact that A is a Δ^c-ring, define δ^c in $A/\mathfrak{a}$ by,

$$\begin{aligned} \delta^c : A/\mathfrak{a} &\to A/\mathfrak{a} \\ (x+\mathfrak{a}) &\mapsto \delta^c(x)+\mathfrak{a}. \end{aligned} \tag{5}$$

So if $x+\mathfrak{a}=y+\mathfrak{a}$ then $x-y\in\mathfrak{a}$, thus $\delta^c(x)-\delta^c(y)=a$ for $a\in\mathfrak{a}$, then $\delta^c(x)+\mathfrak{a}=\delta^c(y)+\mathfrak{a}$. Hence δ^c is well defined.

On the other hand define the homomorphism c in $A/\mathfrak{a}$ by

$$\begin{aligned} c : A/\mathfrak{a} &\to A/\mathfrak{a} \\ x+\mathfrak{a} &\mapsto c(x)+\mathfrak{a}, \end{aligned} \tag{6}$$

which is an homomorphism, because

i) $c(x+y)+\mathfrak{a}=c(x)+c(y)+\mathfrak{a}$.

ii) $c(xy)+\mathfrak{a}=c(x)c(y)+\mathfrak{a}$.

iii) $c(1)+\mathfrak{a}=1+\mathfrak{a}$.

Hence it defines a c-derivation on A/a by

$$\begin{aligned} \delta^c\left((x+\mathfrak{a})(y+\mathfrak{a})\right) &= \delta^c\left((xy+\mathfrak{a})\right)=\delta^c(x)c(y)+\delta^c(y)c(x)+\mathfrak{a} \\ \delta^c\left((x+y)+\mathfrak{a}\right) &= \delta^c(x+y)+\mathfrak{a}=\delta^c(x)+\delta^c(y)+\mathfrak{a}. \end{aligned}$$

□

Consider the surjective ring homomorphism $\phi : A\to A/a$ which maps each $x\in A$ to $x+\mathfrak{a}$. We claim that ϕ is a Δ^c-ring homomorphism. In fact, let δ^c be a c-derivation in A, then $\delta^c(\phi(x))=\delta^c(x+\mathfrak{a})=\delta^c(x)+\mathfrak{a}$ and $\phi(\delta^c(x))=\delta^c(x)+\mathfrak{a}$, thus $\delta^c\phi=\phi\delta^c$.

Proposition 4.5. *Let A be Δ^c-ring, B be a Δ^h-ring and $f : A\to B$ be a Δ^c-ring homomorphism. If $c(kerf)\subseteq(kerf)$ then $A/\ker(f)\cong Im(f)$, under Δ^c-isomorphism.*

Proof. Consider the following function

$$\begin{aligned} g : A/\ker(f) &\to Im(f) \\ x+\ker(f) &\mapsto f(x). \end{aligned} \tag{7}$$

Now we check that g is well defined, i.e., if $x+\ker(f)=y+\ker(f)$, then $g(x+\ker(f))=f(x)=g(y+\ker(f))=f(y)$, but it is equivalent to the statement that $y=x+z$ where $z\in\ker(f)$, therefore $f(y)=f(x+z)=f(x)$.

i) It follows by definition that g is a homomorphism.

ii) g is surjective.

iii) g is injective; if $g(x+\ker(f)) = f(x) = g(y+\ker(f)) = f(y)$ then $0 = f(x-y)$, so $x-y \in \ker(f)$, i.e., $x+\ker(f) = y+\ker(f)$.

iv) That g is a Δ^c-homomorphism follows by:

$$g\left(\delta_A^c\left(x+\ker(f)\right)\right) = g(\delta_A^c(x){+}\ker(f)) = f(\delta_A^c(x)) = \delta_B^h(f(x)) = \delta_B^h\left(g(x+\ker(f))\right).$$

On the other hand, using the fact that $\delta_A^c(kerf) \subseteq (kerf)$ we have

$$\begin{aligned} g(\delta_A^c(x+kerf)) &= g(\delta_A^c(x)+\delta_A^c(kerf)) \\ &= g(\delta_A^c(x)+kerf) \\ &= f(\delta_A^c(x)) \\ &= \delta_B^h(f(x)) \\ &= \delta_B^h(g(x+\ker(f))). \end{aligned}$$

Thus $A/\ker(f) \cong Im(f)$. □

5. Δ^c-Prime Ideal And Δ^c-Maximal Ideal

In this section we define Δ^c-prime ideals and Δ^c-maximal ideals.

Definition 5.1. A prime Δ^c-ideal $\mathfrak{p}$ of a Δ^c-ring A is a subset $\mathfrak{p} \subseteq A$ such that $\mathfrak{p}$ is a prime ideal of A and a Δ^c-ideal.

Definition 5.2. Let A be a Δ^c-ring. A Δ^c-ideal $\mathfrak{m}$ in A is a maximal ideal if, $\mathfrak{m}$ is a Δ^c-ideal, $\mathfrak{m} \neq A$ and if $\mathfrak{a}$ is a Δ^c-ideal such that $\mathfrak{m} \subset \mathfrak{a} \subset A$ then $\mathfrak{m} = \mathfrak{a}$ or $\mathfrak{a} = A$.

Proposition 5.3. *Let A be a Δ^c-ring and $\mathfrak{m}$ be a maximal Δ^c-ideal. Then $\mathfrak{m}$ is a Δ^c-prime.*

Proof. It follows by definition. □

Let A be a Δ^c-ring then $\mathfrak{m}$ is a maximal Δ^c-ideal if and only if $A/\mathfrak{m}$ is a Δ^c-field.

Theorem 5.4. *For any Δ^c-ring A there exist at least one Δ^c-maximal ideal $\mathfrak{m}$.*

Proof. Let $\mathbf{M}$ be the set of all Δ^c-ideals of A without the Δ^c-ideal A. Notice that $\mathbf{M} \neq \emptyset$, because $0 \in \mathbf{M}$. And also let $\{\mathfrak{a}_j : j \in J\}$ be an increasing chain ideals in $\mathbf{M}$ where J is a set of indices. Let $\mathfrak{b} = \left(\bigcup_{j\in J} \mathfrak{a}_j\right)$, then

$$\delta^c\left(\bigcup_{j\in J} \mathfrak{a}_j\right) \subseteq \left(\bigcup_{j\in J} \mathfrak{a}_j\right),$$

because, if $x \in \left(\bigcup_{j\in J} \mathfrak{a}_j\right)$, then $x \in \mathfrak{a}_j$ for some $j \in J$, where $\mathfrak{a}_j$ is a Δ^c-ideal, so $\delta^c(x) \in \bigcup_{j\in J} \mathfrak{a}_j$. So $\mathfrak{b} \in \mathbf{M}$ and is an upper bound of $\{\mathfrak{a}_j : j \in J\}$, and by applying Zorn's Lemma on $\mathbf{M}$ we find that it has a maximal element. □

As a remark. Let A be a Δ^c-ring, and $\{\mathfrak{a}_j : j \in J\}$ a family of Δ^c-ideals. Then $\{\sum_{j\in J} a_j : a_j \in \mathfrak{a}_j\}$ is a Δ^c-ideal and the intersection of any family $\{\mathfrak{a}_j : j \in J\}$ of Δ^c-ideals is a Δ^c-ideal.

6. Δ^c-Rings of Fractions

On Section 2, Lemma 2.3 we mentioned the quotient rule, but we didn't proved that it was well defined, so in this section we would do it.

Definition 6.1. Let A be a Δ^c-ring. A Δ^c- multiplicatively closed subset S of A is a subset $S \subseteq A$ such that

i) $1 \in S$,

ii) for all $a, b \in S$, $ab \in S$,

iii) $c(S) \subseteq S$.

Proposition 6.2. *Let A be a Δ^c-ring and $S \subseteq A$, be a Δ^c-multiplicatively closed subset. If $c(y)^2 \neq 0$ for any $y \in S$ then*

$$\begin{array}{rcl} \delta^c : S^{-1}A & \longrightarrow & S^{-1}A \\ \frac{x}{y} & \longrightarrow & \delta^c\left(\frac{x}{y}\right) = \frac{\delta^c(x)c(y) - \delta^c(y)c(x)}{c(y)^2}. \end{array} \tag{8}$$

δ^c is well defined.

Proof. Let $\frac{x}{y} = \frac{z}{w}$, then $(xw - yz)u = 0$ for some $u \in S$, so

$$\begin{aligned} 0 =& \delta^c((xw - yz)u) \\ =& \delta^c(u)c(xw - yz) + \delta^c(xw - yz)c(u). \end{aligned}$$

Multiplying by $c(u)$

$$0 = (c(u))^2\left(\delta^c(x)c(w) + \delta^c(w)c(x) - \delta^c(y)c(z) - \delta^c(z)c(y)\right)$$

Now, multiplying last equation by $c(w)$ and adding a zero

$$\begin{aligned}0 =&(c(u))^2\left(\delta^c(x)(c(w))^2+\delta^c(w)c(x)c(w)-\delta^c(w)c(y)c(z)+\delta^c(w)c(y)c(z)\right)\\&-(c(u))^2\left(\delta^c(y)c(w)c(z)+\delta^c(z)c(y)c(w)\right).\end{aligned}$$

Now, multiplying by $c(y)$ and adding another zero

$$\begin{aligned}0 =&(c(u))^2\left(\delta^c(x)(c(w))^2c(y)+\delta^c(w)\left(c(xw-yz)\right)c(y)+\delta^c(w)(c(y))^2c(z)\right)\\&-(c(u))^2\left(\delta^c(y)c(wzy)+\delta^c(y)c(xw^2)-\delta^c(y)c(xw^2)+\delta^c(z)(c(y))^2c(w)\right)\end{aligned}$$

thus

$$\begin{aligned}0 =&(c(u))^2\left(\delta^c(x)(c(w))^2c(y)+\delta^c(w)(c(y))^2c(z)\right)\\&-(c(u))^2\left(\delta^c(y)\left(c(wzy-xw^2)\right)+\delta^c(y)c(xw^2)+\delta^c(z)(c(y))^2c(w)\right)\end{aligned}$$

then we have that

$$\begin{aligned}0 =&(c(u))^2(\delta^c(x)(c(w))^2c(y)+\delta^c(w)(c(y))^2c(z)-\delta^c(z)(c(y))^2c(w)\\&-\delta^c(y)c(x)(c(w))^2)\\=&(c(u))^2\left((\delta^c(x)c(y)-\delta^c(y)c(x))(c(w))^2+(\delta^c(w)c(z)-\delta^c(z)c(w))(c(y))^2\right).\end{aligned}$$

Then it follows that

$$\delta^c\left(\frac{x}{y}\right)=\delta^c\left(\frac{z}{w}\right).$$

□

7. Δ^c-Radical Ideal

Finally in this section we have a main result. First of all, we define the radical of a Δ^c-ideal, and then prove that the radical of a Δ^c-ideal is a Δ^c-ideal. And as a corollary we have that the set of nilpotent elements of a Δ^c-ring is a Δ^c-ideal.

Definition 7.1. Let A be a Δ^c-ring and $\mathfrak{a}$ a Δ^c-ideal. The radical of $\mathfrak{a}$ is the set $r(\mathfrak{a})=\{x\in A : x^n\in\mathfrak{a}\text{for some } n\in\mathbb{N}\}$.

Theorem 7.2. *Let $\mathfrak{a}$ be any Δ^c-ideal of the Δ^c-ring A, then if c is an isomorphism with $c(\mathfrak{a})=\mathfrak{a}$ then the radical of $\mathfrak{a}$ is a Δ^c-ideal.*

Proof. Let $x\in r(\mathfrak{a})$, then $x^m\in\mathfrak{a}$ for some $m>0$. So we must prove that $\delta^c(x)^n\in\mathfrak{a}$ for some $n>0$. We claim that $c^{-1}\left(\delta^c(x)^{2k}\right)x^{n-k}\in\mathfrak{a}$ for $k=0,\dots,n$.

The proof follows by induction over k. So, if $k=0$, then $c^{-1}(1)\,x^n\in\mathfrak{a}$, and the assertion is true.

Now suppose that for all $k < n$, $c^{-1}\left(\delta^c(x)^{2k}\right)x^{n-k} \in \mathfrak{a}$, thus

$$(n-k)c(x^{n-k-1})\delta^c(x)^{2k+1} + 2k\delta^c(x)^{2k-1}\delta^c\left(c^{-1}(\delta^c(x)\right)c(x^{n-k}) \in \mathfrak{a}.$$

Next, we multiply last equation by $\delta^c(x)$,

$$(n-k)c(x^{n-k-1})\delta^c(x)^{2k+2} + 2k\delta^c(x)^{2k}\delta^c\left(c^{-1}(\delta^c(x)\right)c(x^{n-k}) \in \mathfrak{a}.$$

Notice that $\delta^c(x)^{2k}c(x^{n-k}) \in \mathfrak{a}$. So,

$$k\delta^c(x)^{2k}\delta^c\left(c^{-1}(\delta^c(x)\right)c(x^{n-k}) \in \mathfrak{a}.$$

Hence

$$(n-k)c(x^{n-k-1})\delta^c(x)^{2k+2} \in \mathfrak{a}$$

and using the fact that the assertion is true for all $k < n$, in particular for $k = n-1$ we have that,

$$\begin{aligned}\delta^c(x)^{2(n-1)+2} &\in \mathfrak{a}\\ \delta^c(x)^{2n} &\in \mathfrak{a}\\ c^{-1}(\delta^c(x)^{2n}) &\in \mathfrak{a}.\end{aligned}$$

□

Corollary 7.3. *Let A be a Δ^c-ring and $\eta(A)$ be the set of all nilpotent elements of A. Then $\eta(A)$ is a Δ^c-ideal.*

Proof. Notice that $c(\mathfrak{o}) = \mathfrak{o}$, thus $r(\mathfrak{o})$ is a Δ^c-ideal. □

Example 7.4. Let A be a ring and let $A[x]$ be the ring polynomials in an ideterminate x with coefficients in A. Define δ^c by

i) $\delta^c(x) = 1$.

ii) $\delta^c(x^n) = nc(x^{n-1})\delta^c(x) = nc(x^{n-1})$.

iii) $\delta^c(a) = 0$ for all $a \in A$.

and c be any homomorphism. Thus $A[x]$ is a Δ^c-ring. Then $\eta(A[x])$ is a Δ^c-ideal.

Now take $f \in \eta(A[x])$ then $f = a_0 + a_1x + \cdots + a_nx^n$ is nilpotent if and only if $a_0, \ldots, a_n$ are nilpotent. So

$$\begin{aligned}\delta^c(f) &= \delta^c(a_0 + a_1x + \cdots + a_nx^n)\\ &= c(a_1) + \cdots + nc(a_n)x^{n-1} \in \eta(A).\end{aligned}$$

Because, if a_i is nilpotent then $c(a_i)$ is also a nilpotent element, so $\delta^c(f)$ is a nilpotent element.

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