Placing the roots of cubics using linear fractional transformation

Colocación de las raíces cúbicas usando la transformación fraccional lineal

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Abstract. In this paper we make use of linear fractional transformation for placing the roots of cubic equation at desired locations.

Keywords: Linear fractional transformation, cubic equation, roots.

Resumen. En este artículo se hace uso de la transformación fraccional lineal (transformación de Möbius) para la colocación de las raíces de la ecuación cúbica en lugares deseados.

Palabras claves: transformación fraccional lineal, transformaciones de Möbius, ecuación de tercer grado, raíces.

Mathematics Subject Classification: 12E12.

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1. Introduction

The linear fractional transformation in its general form is expressed as, $y = (Ax+B)/(Cx+D)$, where the quantities A, B, C, and D are invariants linking the variables x and y. The quantities A, B, C and D are in general complex numbers such that $AD \neq BC$. This transformation is known by various names –like Möbius transformation, linear fractional transformation, bilinear transformation, or, conformal mapping– involves translation, rotation, magnification, and inversion [1]. The two variables (x and y) in the transformation are connected as $Cxy - Ax + Dy - B = 0$ they are such that one value of the either variable corresponds to one and only one value of the other variable [2].

In this paper we use the linear fractional transformation to transform a cubic equation in x to another cubic equation in y such that the roots of the second y can be placed at desired locations, as explained in the next section.

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2. The proposed method

Since a general cubic equation can be transformed to a reduced one by a linear transformation, we consider here the following reduced cubic equation in x without loss of any generality,

$$x^3 + ax + b = 0, \tag{1}$$

where a and b are coefficients in (1) and $ab \neq 0$. The linear fractional transformation of the form,

$$y = (x + c)/(dx + e), \tag{2}$$

is used here to transform (1). The quantities, c, d, and e are unknown numbers to be determined later. We rearrange the transformation (2) as,

$$x = -(ey - c)/(dy - 1) \tag{3}$$

for eliminating x from (1) as shown below. By replacing (3) into (1) we get

$$-\left[(ey - c)/(dy - 1)\right]^3 - a\left[(ey - c)/(dy - 1)\right] + b = 0. \tag{4}$$

Rearranging of (4) results in

$$(ey - c)^3 + a(dy - 1)^2(ey - c) - b(dy - 1)^3 = 0.$$

Expanding and arranging the above expression in descending powers of y leads to

$$fy^3 + gy^2 + hy + j = 0, \tag{5}$$

where f, g, h, and j are given by:

$$\begin{aligned} f &= e^3 + ad^2e - bd^3, \\ g &= -3ce^2 - a(2de + cd^2) + 3bd^2, \\ h &= 3c^2e + a(2cd + e) - 3bd, \\ j &= -c^3 - ac + b. \end{aligned} \tag{6}$$

Equation (5) is a transformed cubic equation in y. Rearranging it as,

$$fy^3 - fy^2 + (f+g)y^2 - (f+g)y + (f+g+h)y - (f+g+h) + f+g+h+j = 0, \tag{7}$$

and stipulating the condition

$$f + g + h + j = 0, \tag{8}$$

it transforms (7) as:

$$(y - 1)\left[y^2 + \frac{f+g}{f}y + \frac{f+g+h}{f}\right] = 0; \quad f \neq 0. \tag{9}$$

Observe that one root of (9) is already placed at $y = 1$, and the locations of the remaining two roots depend on the quadratic factor in (9). The coefficients of this quadratic factor are functions of c, d, and e, which are to be determined. Yet this purpose, let us look at the condition (8); by making use of the expressions in (6), this condition yields,

$$(e-c)^3 + a(d-1)^2(e-c) - b(d-1)^3 = 0. \tag{10}$$

Notice that (10) is a cubic equation in c, d, and e. In order to solve it, first we rearrange it as,

$$e^3 - c^3 + (e-c)[a(d-1)^2 - 3ce] - b(d-1)^3 = 0, \tag{11}$$

and force the condition,

$$e = a(d-1)^2/3c, \quad c \neq 0, \tag{12}$$

on equation (11) to convert it as,

$$(d-1)^6 - \frac{27bc^3}{a^3}(d-1)^3 - \frac{27c^6}{a^3} = 0, \tag{13}$$

which is a quadratic equation in $(d-1)^3$. So by solving it we determine $(d-1)^3$ in terms of c^3 as:

$$(d-1)^3 = \frac{27c^3}{a^3}\left(\frac{b}{2} \pm \sqrt{\frac{b^2}{4} + \frac{a^3}{27}}\right). \tag{14}$$

Taking cube root of (14) yields the principal cube root as,

$$d - 1 = kc, \tag{15}$$

and so $d = kc + 1$, where k is given by

$$k = \frac{3}{a}\left(\frac{b}{2} \pm \sqrt{\frac{b^2}{4} + \frac{a^3}{27}}\right)^{1/3}. \tag{16}$$

Notice that since $ab \neq 0$, $k \neq 0$; we have and $d \neq 1$. Eliminating $d-1$ from (12) using (15) results in

$$e = mc, \quad m = ak^2/3. \tag{17}$$

We have obtained expressions [(15) and (17)] for d and e, in terms of c; hence by use of proper value of c we can place the two roots of cubic equation (9) as desired, as illustrated in the following case studies.

2.1. Case-1

The roots (y_1, y_2, and y_3) of (9) are such that $y_1 = 1$, $y_2 y_3 = 1$.

To achieve this, we note from (9) that the condition, $f + g + h = f$, has to be satisfied, and so (9) is expressed as,

$$(y-1)\left[y^2 + \frac{f+g}{f}y + 1\right] = 0. \tag{18}$$

Notice that $f + g + h = f$ implies $g + h = 0$; and use of expressions in (6) for g and h yields,

$$3ce(c-e) + 2ad(c-e) + a(e - cd^2) + 3bd(d-1) = 0. \tag{19}$$

Use of (15) and (17) to eliminate d and e from (19) results in a quadratic equation in c,

$$Pc^2 + Qc + R = 0, \tag{20}$$

where P, Q, and R are given by

$$P = 3m(m-1) + ak^2, \quad Q = k(2am - 3bk), \quad R = a(m-1) - 3bk. \tag{21}$$

By solving (20) we determine c, and then d and e are determined from (15) and (17). With these values of c, d, and e, (18) is satisfied, implying that the product, $y_2 y_3 = 1$.

Let us solve one numerical example. Consider the cubic equation,

$$x^3 - 6x - 9 = 0.$$

First we determine k using (16) as: $k = 0.5$; and then determine $m = -0.5$ by using expression in (17). From the expressions in (21), P, Q, and R are determined as: 0.75, 9.75, and 22.5. Using these values the quadratic, equation (20) is solved obtaining two values of c as: -3 and -10. Since for $c = -3$, $f = 0$ we discard this value of c, instead proceed with $c = -10$. From (15) and (17), d and e are determined as: -4 and 5. Next, using expressions in (6), f and g are found out as -931 and -882. Now the quadratic factor in (18) is equated to zero, obtaining the quadratic equation in y as:

$$y^2 + (37/19)y + 1 = 0,$$

solving which, y_2 and y_3 are determined as:

$$-\frac{37}{38} \pm \frac{5\sqrt{3}}{38}\, i.$$

Notice that $y_2 y_3 = 1$, as desired. By using the transformation (3), the corresponding roots of x are obtained as:

$$3, \; -\frac{3}{2} \mp \frac{\sqrt{3}}{2}\, i.$$

2.2. Case-2

The roots (y_1, y_2, and y_3) of (9) are such that: $y_1 = 1$, $y_2 + y_3 = 2$.

This can be achieved by stipulating the condition $f + g = -2f$ in (9), making (9) as,

$$(y - 1)\left[y^2 - 2y - (j/f)\right] = 0, \tag{22}$$

since $f + g + h = -j$. We note that $f + g = -2f$ means $3f + g = 0$; so using expressions in (6) for f and g, this condition yields,

$$3e^2(e - c) + 2ad(d - 1)e + ad^2(e - c) - 3bd^2(d - 1) = 0. \tag{23}$$

Using (15) and (17) we eliminate d and e from (23) resulting in a quadratic equation in c,

$$P'c^2 + Q'c + R' = 0, \tag{24}$$

where P', Q', and R' are given by:

$$P' = 3m^2(m - 1) + k^2(3am - a - 3bk),$$
$$Q' = 2k(2am - a - 3bk), \quad R' = a(m - 1) - 3bk. \tag{25}$$

Solving (24), c is determined and subsequently d and e are determined from (15) and (17). These values of c, d, and e render the sum of the two roots (y_2, and y_3) of cubic equation (22) to be equal to 2.

Let us solve one numerical example using the same cubic equation as in the Case-1, for placing the roots of y such that $y_2 + y_3 = 2$.

The values of k and m will be same as before. We determine P', Q', and R' from (25) as 6, 25.5, 22.5. Solving the quadratic equation (24), two values of c are obtained as: -1.25 and -3. Choosing $c = -1.25$, we determine d and e from (15) and (17) as: 0.375 and 0.625. Next, f and g are obtained from (6) as: 0.19140625 and -14.546875.

The quadratic factor in (22) is equated to zero, resulting in the quadratic equation, $y^2 - 2y + 76 = 0$, which when solved yields y_2 and y_3 as: $1 + 5\sqrt{3}i$ and $1 - 5\sqrt{3}i$. Note that $y_2 + y_3 = 2$.

3. Conclusions

In this paper we have shown that the roots of cubic equation can be placed at desired locations using linear fractional transformation. Two cases of placement of roots are illustrated along with numerical examples.

References

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