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Some properties of Multiplication Operators acting on Banach spaces of measurable functions

Algunas propiedades del Operador Multiplicación actuando sobre espacios de Banach de funciones medibles

Julio C. Ramos Fernández[1,a]

Abstract. The purpose of this article is to survey the recent results about the properties of multiplication operators acting on Banach spaces of measurable functions. Most of them was presented by the author in a conference held in Bogotá 2016, on the occasion of celebrating the "IV UN Encuentro de Matemáticas" at the "Universidad Nacional de Colombia".

Keywords: Multiplication Operator, measurable functions, Köthe spaces.

Resumen. El propósito de este artículo es el de divulgar resultados recientes sobre las propiedades del operador multiplicación actuando en espacios de Banach de funciones medibles. La mayoría de estos resultados fueron presentados por el autor en una conferencia, la cual tuvo lugar en Bogotá 2016, en ocasión de la celebración del "IV UN Encuentro de Matemática" en la Universidad Nacional de Colombia.

Palabras claves: Operador Multiplicación, funciones medibles, Espacios de Köthe.

Mathematics Subject Classification: 47B38, 46E30.

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[1]Departamento de Matemáticas, Universidad de Oriente, Cumaná, Venezuela
[a]jcramos@udo.edu.ve

1. Introduction and Preliminaries

This note is an overview of the one talk given by the author at "Departamento de Matemáticas" of the "Universidad Nacional de Colombia" in Bogotá on the occasion of celebrating the "IV UN Encuentro de Matemtica - 2016". This talk was an invitation to Professors and Students of that Department to make some research in the topic of Operator Theory on Banach spaces of measurable functions.

Through this note, we consider real extended measurable functions whose domain of definition is a σ-finite measure space (Ω, Σ, μ). We recall that a σ-algebra Σ is a non-empty collection of subsets of the set Ω, which is closed under complement and it is closed under union of countably infinite many subsets of Ω. An element of Σ is called measurable set. A measure μ defined on the σ-algebra Σ is a real extended and nonnegative function such that $\mu(\emptyset) = 0$ and

$$\mu\left(\bigcup_{k=1}^{\infty}\right) = \sum_{k=1}^{\infty} \mu(E_k),$$

for all $\{E_k\}_{k\in\mathbb{N}}$ of pairwise disjoint sets of Σ. Furthermore, we are going to suppose that the measure μ is complete, that is, if $A \subset B$, $B \in \Sigma$ and $\mu(B) = 0$, then $A \in \Sigma$ and therefore $\mu(A) = 0$. The measure space is called σ-finite if Ω may be expressed as disjoint enumerable union of measurable subsets all of them having finite measure. The set of all measurable functions on (Ω, Σ, μ) is denote as $L_0(\mu) = L_0(\Omega, \Sigma, \mu)$. In this space, two functions f and g are equal if they are equal almost everywhere (a.e.), that is, if $\mu(\{t \in \Omega : f(t) \neq g(t)\}) = 0$. Hence $L_0(\mu)$ consists of equivalence classes defined by the relation $f \sim g$ if and only if $f = g$ a.e.

Since multiplication of measurable functions is also a measurable function, we can propose the following problems:

- **Problem 1.** We fix a function $u \in L_0(\mu)$ and now we take functions f belonging to a subspace X of $L_0(\mu)$. The problem, in this case, consists in to describe the set

  $$Y = \{u \cdot f : f \in X\}.$$

  Clearly, Y is a subspace of $L_0(\mu)$. But, what other properties does Y have?

- **Problem 2.** We consider fixed two subspaces X and Y of $L_0(\mu)$. The question is: What properties must the function $u \in L_0(\mu)$ have in order to $u \cdot f$ belongs to Y for all function $f \in X$?. The set

  $$Mult(X, Y) = \{u \in L_0(\mu) : u \cdot f \in Y \ \forall f \in X\}$$

  is called multipliers from X to Y. We can see that $Mult(X, Y)$ is a subspace of $L_0(\mu)$.

In both cases, given a function $u \in L_0(\mu)$, the relation M_u which maps each function $f \in X$ to the element $u \cdot f \in Y$, defines a linear operator which it is known as *multiplication operator* with symbol u. In the case that $X = Y = L_0(\mu)$ we have the following very useful properties:

Proposition 1.1. *The operator $M_u : L_0(\mu) \to L_0(\mu)$ is injective or $1-1$ if and only if $\mu(Z_u) = 0$, where $Z_u = \{t \in \Omega : u(t) = 0\}$.*

Proof. Let us suppose first that $\mu(Z_u) > 0$. Since Ω is σ-finite, we can assume that $0 < \mu(Z_u) < \infty$, then the measurable function $\mathbf{1}Z_u$, the characteristic function of the set Z_u, is not the null function and furthermore it satis

es $M_u \mathbf{1}Z_u = u \cdot \mathbf{1}Z_u = 0$. This means that $\ker(M_u) = \{f \in L_0(\mu) : M_u(f) = 0\} \neq \{0\}$ and M_u is not $1-1$ on $L_0(\mu)$.

Conversely, if $\mu(Z_u) = 0$ and $M_u f = 0$, then $u \cdot f = 0\mu$−a.e. This implies that

$$A_f := \{t \in \Omega : f(t) \neq 0\} \subset Z_u \cup \{t \in \Omega : u(t)f(t) \neq 0\}$$

and since the measure μ is complete, we can conclude that $\mu(A_f) = 0$; that is, $f = 0\mu$−a.e. and $\ker(M_u) = \{0\}$. □

Proposition 1.2. *If $M_u : L_0(\mu) \to L_0(\mu)$ is surjective or onto, then it is injective.*

Proof. If $M_u : L_0(\mu) \to L_0(\mu)$ is not $1-1$, then by Proposition 1, the set Z_u has positive measure. Since Ω is a σ-finite space, we can assume that $0 < \mu(Z_u) < 1$. Thus, the non-null function $\mathbf{1}Z_u \in L_0(\mu)$ does not belong to Ran $(M_u) = \{M_u(f) : f \in L_0(\mu)\}$, because in other case, we can find a measurable function $f \in L_0(\mu)$ such that $\mathbf{1}Z_u = u \cdot f$. Hence for any $t \in Z_u$ we have $1 = \mathbf{1}Z_u(t) = u(t) \cdot f(t) = 0$ which is impossible. This shows that $M_u : L_0(\mu) \to L_0(\mu)$ is not onto. □

As a direct consequence of the above proposition, we have:

Corollary 1.3. *The operator $M_u : L_0(\mu) \to L_0(\mu)$ is bijective or invertible if and only if* Ran $(M_u) = L_0(\mu)$.

The objective is to study other topological properties of multiplication operator. We consider the operator $M_u : X \to Y$, where $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ are Banach subspaces of $L_0(\mu)$. The goal is to characterize the symbols or functions $u \in L_0(\mu)$ for which the operator $M_u : X \to Y$ will be continuous, compact, Fredholm, has closed range or has range finite. Also, it is interesting to estimate the norm and the essential norm of the operator $M_u : X \to Y$.

Allow us recall that a linear transformation $T : X \to Y$ is said to be continuous or bounded if there exists $M > 0$ such that $\|T_x\|_Y \leq M\|x\|_X$ for all $x \in X$. The norm of the operator $T : X \to Y$ is defined as

$$\|T\| = \sup\{\|T_x\|_Y : \|x\|_X = 1\}. \tag{1}$$

The collection of all bounded operators from X into Y is denoted by $\mathcal{B}(X,Y)$, which is a Banach space with the norm of operators defined in (1). When $X=Y$, we write $\mathcal{B}(X)$ instead $\mathcal{B}(X,X)$. An operator $T:X\to Y$ has closed range if $T(X)$ is a closed subset of $(Y,\|\cdot\|_Y)$. The operator $T\in\mathcal{B}(X,Y)$ is an isomorphism or bounded below if there exists $M>0$ such that $\|T_x\|_Y\geq M\|x\|_X$ for all $x\in X$. It is known that $T\in\mathcal{B}(X,Y)$ is bounded below if and only if T is $1-1$ and has closed range. An operator $T\in\mathcal{B}(X,Y)$ has finite range if $\dim(T(X))<\infty$; while the transformation $T:X\to Y$ is compact if $T(B)$ is a compact subset of $(Y,\|\cdot\|_Y)$, where $B=\{x\in X:\|x\|_X\leq 1\}$. It is known that all operator having finite range is a compact operator. The class of all compact operators from X into Y is denoted by $\mathcal{K}(X,Y)$. The essential norm of an operator $T\in\mathcal{B}(X,Y)$, denoted as $\|T\|_e$, is the distance from T to $\mathcal{K}(X,Y)$, that is,

$$\|T\|_e=\inf\{\|K-T\|:K:X\longrightarrow Y \text{ is compact}\}.$$

The operator $T\in B(X,Y)$ is called Fredholm if $\dim(\ker(T))<\infty$ and $\dim(Y=\mathrm{Ran}(T))<\infty$. The reader interested in studying the properties of the operators de

ned in this section may consult the texts of Conway 1990 [12], Kato 1995 [18] and Müller 2007 [25].

2. Multiplication Operators on $L_p(\mu)$ Spaces

In this section we gather some studies about the properties of multiplication operators acting between $L_p(\mu)$ spaces. In order to put the notations accordingly with the various articles that we cite later, we fix a σ-finite measure space (Ω,Σ,μ), where the measure μ is suppose to be complete. The multiplication operator with symbol $u\in L_0(\mu)$ will be denoted by M_u. Let us recall the definition and some properties of the classical Lebesgue spaces $L_p(\mu)$, where $p>0$ is a parameter fixed. A function $f\in L_0(\mu)$ belongs to the space $L_p(\mu)$, with $0<p\leq\infty$, if $\|f\|_p<\infty$, where

$$\|f\|_p=\begin{cases}\left(\int_\Omega|f(t)|^p d\mu(t)\right)^{1/p}, & \text{if } 0<p<\infty,\\ \\ \inf\{\lambda>0:|f|\leq\lambda\mu-\text{a.e.}\}, & p=\infty.\end{cases}$$

The following facts are well known.

Proposition 2.1. *Suppose that $0<p\leq\infty$, then $L_p(\mu)$ spaces satisfy the following properties:*

- *For any $1\leq p\leq\infty$, $(L_p(\mu),\|\cdot\|_p)$ is a Banach space.*
- *Solid. Let $p>0$ be fixed. If $f,\ g\in L_0(\mu)$, $|f|\leq|g|\mu$-a.e. and $g\in L_p(\mu)$, then $f\in L_p(\mu)$ and $\|f\|_p\leq\|g\|_p$.*

- *If $E \in \Sigma$ and $\mu(E) < 1$ then $\mathbf{1}_E$, the characteristic function of the set E, belongs to the space $L_p(\mu)$.*

- *Fatou's property. If $\{f_n\}$ is a bounded sequence in $L_p(\mu)$ such that $0 \leq f_n \nearrow f \in L_0(\mu)$, then $f \in L_p(\mu)$ and $\|f_n\|_p \to \|f\|_p$.*

- *$L_p(\mu)$ is order continuous, that is, $\|f_n\|_p \to 0$ whenever $f_n \searrow 0$.*

The space $L_1(\mu)$ was introduced by Lebesgue in his Doctoral Thesis in 1902 [20]. The case $p = 2$ was studied in 1907, independently, by Riesz [32] and Fréchet [13]. The more general case $p > 1$ was introduced by Riesz in 1909 [33], when he established his duality theorem $(L_p(\mu))^* = L_q(\mu)$, where $1 < p < \infty$ and $\frac{1}{p} + \frac{1}{q} = 1$. There are a lot of good references where we can consult interesting properties of the classical $L_p(\mu)$ spaces, we recommend the excellent text of Lindenstrauss and Tzafriri [22] and the recent texts of Castillo [6] and Castillo and Rafeiro [10].

The study of multiplication operator in spaces of measurable functions is outlined in the texts of the mid-twentieth century. In particular, in the book of Halmos, A Hilbert space problem book, edited by D. Van Nostrand Co. in (1967), appears in the Chapter 7 some properties of this operator acting on the Hilbert space $L_2(\mu)$. In the Problem 64, see [14] of this text, we can see the relation between this operator and diagonals operators and that all essentially bounded symbol define a bounded multiplication operator on $L_2(\mu)$ into itself. The converse of this last assertion is proposed in Problem 65 of this book; while the invertibility and the spectrum of this operator is established in Problem 67. In fact, in this book appears the following problem:

Problem 67 in [14]. The multiplication operator M_u on $L_2(\mu)$ (for a σ-finite measure) induced by u is an invertible operator if and only if u is an invertible function. Consequence: the spectrum of a multiplication is the essential range of the multiplier.

Multiplication operators having closed range acting on $L_2(\mu)$ were characterized by Singh and Kumar in 1977 [34]. They showed the following result:

Theorem 2.2 (Theorem 2.1 in [34])**.** *Let $M_u \in \mathcal{B}(L_2(\mu))$. Then M_u has closed range if and only if u is bounded away from zero on $Z^u = \Omega \backslash \{x \in \Omega : u(x) = 0\}$.*

The set Z^u is also known as the support of u and it is denoted by $\text{supp}(u)$. The phrase u is bounded away from zero on Z^u means that there exists $\delta > 0$ such that $|u(x)| \geq \delta$ for all $x \in Z^u$. As a consequence of this last result, in this same article, Singh and Kumar also characterize composition operators having closed range on $L_2(\mu)$. Subsequently, these same authors in 1979, see [35] characterized all compact multiplication (and composition) operators acting on $L_2(\mu)$. The results obtained by Singh and Kumar can be summarized in the following way:

Theorem 2.3 ([35])**.** *Let $M_u \in \mathcal{B}(L_2(\mu))$, then:*

1. Lemma 1.1 in [35]. *M_u is compact if and only if Z_ϵ^u is finite dimensional for every $\epsilon > 0$, where Z_ϵ^u denotes the subspace of $L_2(\mu)$ consisting of all those functions which vanish outside $\Omega_\epsilon^u = \{x \in \Omega : |u(x)| > \epsilon\}$.*

2. Corollary 1.1 in [35]. *Let μ be a non-atomic measure, then M_u is compact if and only if M_u is the zero operator.*

3. Corollary 1.1 in [35]. *Let $\Omega = \Omega_1 \cup \Omega_2$ be the decomposition of Ω into non-atomic and atomic parts respectively. If M_u is compact then $u = 0$ almost everywhere on Ω_1*

An atom of a measure μ is an element $B \in \Sigma$ with $\mu(B) > 0$ such that for each $A \in \Sigma$, if $A \subset B$ then either $\mu(A) = 0$ or $\mu(A) = \mu(B)$. A measure with no atoms is called non-atomic (we refer to [4] for properties of non-atomic measures).

The compactness results in [35] were extended by Takagi in 1992, see [36]. In particular, Takagi showed that Lemma 1.1 in [35] is also true for $M_u \in \mathcal{B}(L_p(\mu))$ with $p \geq 1$. However, the most complete work about the properties of multiplication operator acting between $L_p(\mu)$ spaces appeared in 1999 and it is due to Takagi and Yokouchi, see [37]. The results obtained by the last authors can be summarized as follows:

Theorem 2.4 (Continuity Results in [37]). *Suppose that $u \in L_0(\mu)$, the*

1. *Case I: $p = q$ (Theorem 1.2 in [37]). Suppose $1 \leq p < \infty$, then u induces a multiplication operator M_u from $L_p(\mu)$ into itself if and only if $u \in L_\infty(\mu)$. In this case $\|M_u\| = \|u\|_\infty$.*

2. *Case II: $p > q$ (Theorem 1.3 in [37]). Suppose $1 \leq q < p < 1$, then u induces a multiplication operator M_u from $L_p(\mu)$ into $L_q(\mu)$ if and only if $u \in L_r(\mu)$, where $\frac{1}{p} + \frac{1}{r} = \frac{1}{q}$. In this case $\|M_u\| = \|u\|_r$.*

3. *Case III: $p < q$ (Theorem 1.4 in [37]). Suppose $1 \leq p < q < 1$, then u induces a multiplication operator M_u from $L_p(\mu)$ into $L_q(\mu)$ if and only if u satisfies the following two conditions:*

   (a) $u(x) = 0$ *for μ-almost all $x \in B$,*

   (b) $\sup_{n\in\mathbb{N}} \dfrac{|u(A_n)|^s}{\mu(A_n)} < \infty$, *where* $\frac{1}{q} + \frac{1}{s} = \frac{1}{p}$

   In this case $\|M_u\| = \sup_{n\in\mathbb{N}} \dfrac{|u(A_n)|}{\mu(A_n)^{\frac{1}{s}}}$.

In the Case III of the above theorem, the authors use the fact that all σ-finite measurable space (Ω, Σ, μ) can be uniquely decomposed as follows:

$$\Omega = \left(\bigcup_{n\in\mathbb{N}} A_n \right) \cup B,$$

where $\{A_n\}_{n\in\mathbb{N}}$ is a countable collection of disjoint atoms and B is a non-atomic set. Furthermore, they adopt the notation

$$f(A) = \frac{1}{\mu(A)} \int_A f d\mu$$

for all $f \in L_0(\mu)$ and each atom A.

On the conditions that the symbol $u \in L_0(\mu)$ must have in order to its associated multiplication operator M_u will have closed range from $L_p(\mu)$ into $L_q(\mu)$, Takagi and Yokouchi showed that:

Theorem 2.5 (Closed Range Results in [37]). *Suppose that $u \in L_0(\mu)$, then*

1. *Case I: $p = q$ (Theorem 2.3 in [37]). Suppose that $1 \leq p < \infty$, and let M_u be a multiplication operator from $L_p(\mu)$ into itself. Then M_u has closed range if and only if there exists a constant $\delta > 0$ such that $|u(x)| \geq \delta$ for μ-almost all $x \in \operatorname{supp}(u)$.*

2. *Case II: $p > q$ (Theorem 2.4 in [37]). Suppose $1 \leq q < p < \infty$, and let M_u be a multiplication operator from $L_p(\mu)$ into $L_q(\mu)$. Then the following assertions are equivalent:*

   (i) M_u has closed range,

   (ii) M_u has finite range,

   (iii) $u(x) = 0$ for μ-almost all $x \in B$ and the set $\{n \in \mathbb{N} : u(A_n) \neq 0\}$ is finite.

3. *Case III: $p < q$ (Theorem 2.5 in [37]). Suppose $1 \leq p < q < \infty$, and let M_u be a multiplication operator from $L_p(\mu)$ into $L_q(X)$. Then the following assertions are equivalent:*

   (i) M_u has closed range,

   (ii) M_u has finite range,

   (iii) the set $\{n \in \mathbb{N} : u(A_n) \neq 0\}$ is finite.

It is very interesting the fact that for $p \neq q$, all multiplication operator from $L_p(\mu)$ into $L_q(\mu)$ having closed range is also a compact operator, since it is a well known result that all operator with finite range is a compact operator.

It might be interesting to address in the near future, the following problems about the properties of multiplication operator acting between $L_p(\mu)$ spaces.

Open Problem 1. Suppose that $u \in L_0(\mu)$ and let M_u be a multiplication operator from $L_p(\mu)$ into $L_q(\mu)$, where $p, q \in [1, \infty)$.

1. Characterize all the symbols $u \in L_0(\mu)$ such that $M_u : L_p(\mu) \to L_q(\mu)$ is a compact operator.

2. Estimate the essential norm of $M_u : L_p(\mu) \to L_q(\mu)$.

3. When $M_u : L_p(\mu) \to L_q(\mu)$ is a Fredholm operator?

Observe that compact multiplication operators from $L_p(\mu)$ into itself was studied by Takagi in [36]. In [17] Jabbarzadeh and Pourreza showed that if the measure μ is non-atomic on $L_2(\Omega)$, then M_u is a Fredholm operator from $L_2(\mu)$ into itself if and only if there exists a $\delta > 0$ such that $|u(x)| \geq \delta$ μ-almost everywhere on Ω.

We finish this section with the followin problem.

Open Problem 2. Suppose that $u \in L_0(\mu)$. To study the properties of M_u acting from $L_p(\Omega.\Sigma_1, \mu_1)$ into $L_q(\Omega, \Sigma_2, \mu_2)$, where $p, q \in [1, \infty)$.

In [17], Jabbarzadeh and Pourreza studied the continuity part of this problem for the case $1 \leq q < p < \infty$ using the conditional expectation operator.

3. Multiplication Operators between Orlicz spaces

In 1932 Orlicz [28] gave an important generalization of the space $L_p(\mu)$. His generalization is based by changing the function $\varphi p(t) = |t|^p$, with $t \in \mathbb{R}$, in the definition of the $L_p(\mu)$-norm, by a more general function φ with similar properties to φp. More precisely, suppose that (Ω, Σ, μ) is a σ-finite measure space, where μ is a complete measure. A function $f \in L_0(\mu)$ belongs to the Orlicz space $L_\varphi(\mu)$ if

$$N_\varphi\left(\frac{f}{\lambda}\right) := \int_\Omega \varphi\left(\frac{|f(t)|}{\lambda}\right) d\mu(t) < \infty,$$

for some $\lambda > 0$, where $\varphi : [0, \infty) \to [0, \infty)$ is a Young function, that is, φ is a convex and increasing function such that $\varphi(t) = 0$if and only if $t = 0$, and $\varphi(t) \to \infty$ as $t \to \infty$. It is known that $L_\varphi(\mu)$ is a Banach with the Luxemburg norm

$$\|f\|_\varphi = \inf\left\{\lambda > 0 : N_\varphi\left(\frac{f}{\lambda}\right) \leq 1\right\}.$$

The space $L_\varphi(\mu)$ is solid, norm continuous and satis

es the Fatou property. Furthermore, clearly $\mathbf{1}_E \in L_\varphi(\mu)$ for all $E \in \Sigma$ such that $\mu(E) < 1$. Some important reference to study Orlicz spaces are the texts of Rao and Ren [30, 31].

Properties of multiplication operator acting on Orlicz spaces were studied by Komal and Gupta in 2001, [19]. They charecterized all the symbols $u \in L_0(\mu)$ which induce invertible, compact and Fredholm multiplication operators M_u from $L_\varphi(\mu)$ into itself. They show that the results obtained by Singh and Kumar in [34, 35] and Takagi [36] are also valid for M_u from $L_\varphi(\mu)$ into itself. More precisely, they showed the following results:

Theorem 3.1 ([19])**.** *Suppose that $u \in L_0(\mu)$, the*

1. *Theorem 2.1 in [19]. $M_u \in B(L_\varphi(\mu))$ if and only if $u \in L_\infty(\mu)$. Moreover, $\|M_u\| = \|u\|_\infty$.*

2. *Theorem 2.2 in [19]. The set of all multilication operators on $L_\varphi(\mu)$ is a maximal Abelian subalgebra of $B(L_\varphi(\mu))$.*

3. *Theorem 2.3 in [19]. Suppose that $M_u \in B(L_\varphi(\mu))$. Then M_u is an invertible operator in $L_\varphi(\mu)$ if and only if u is invertible in $L_\infty(\mu)$.*

4. *Theorem 3.1 in [19]. Let $M_u \in B(L_\varphi(\mu))$. Then M_u is a compact operator if and only if Z_ϵ^u is finite dimensional for each $\epsilon > 0$.*

5. *Theorem 4.1 in [19]. Let $M_u \in B(L_\varphi(\mu))$. Then M_u has closed range if and only if there exists $\delta > 0$ such that $|u(x)| \geq \delta$ for μ-almost all $x \in \operatorname{supp}(u)$.*

6. *Theorem 4.2 in [19]. Suppose that μ is non-atomic measure and $M_u \in B(L_\varphi(\varphi))$. M_u is Fredholm on $L_\varphi(\mu)$ if and only is M_u is invertible on $L_\varphi(\mu)$.*

However, the following is an open problem:

Open Problem 3. Suppose that $u \in L_\infty(\mu)$. Calculate the essential norm of $M_u : L_\varphi(\mu) \to L_\varphi(\mu)$.

The properties of multiplication operator M_u from $L_\varphi(\mu)$ into $L_\psi(\mu)$, where φ, ψ are two different Young functions is still developing. There is a recent work due to Chawziuk et. al. 2016 [11], where the authors characterize the multipliers between two Orlicz spaces. They showed the following interesting result:

Theorem 3.2 (Theorem 3.1 in [11])**.** *The multiplication operator M_u maps $L_\varphi(\mu)$ into $L_\psi(\mu)$ if and only if*

$$\int_\Omega \chi K(t,1)d\mu(t) < \infty$$

for some $K > 1$, where $\chi K(t,u)$ is, for fixed $t \in \Omega$, the function complementary in the sense of Young to the function $\varphi \circ \frac{K}{u}\psi$ with respect to u.

In this same article [11], the authors gave conditions in order to the operator $M_u : L_\varphi(\mu) \to L\psi(\mu)$ will be equi-absolutely continuous. Still is a open problem:

Open Problem 4. To establish necessary and sufficient conditions in order to the operator $M_u : L_\varphi(\mu) \to L\psi(\mu)$ will be compact, Fredholm, has finite range or has closed range.

4. Multiplication Operators acting on other spaces of measurable functions

Techniques developed by Singh and Kumar in [34, 35] and Takagi in [36] have been modified by a big numerous of researchers which have extended (and continue extending) their results to other spaces of measurable functions which are generalizations of the classical $L_p(\mu)$ spaces. Listed below are some of these generalizations. First we recall the definition and some of the properties of the Lorentz spaces.

There is another interesting generalization of spaces $L_p(\mu)$ made by Lorentz in 1950 [23]. His generalization is based in terms of the distribution and the non-increasing rearrangement functions. More precisely, the Lorentz space $L_{(p,q)}(\mu)$, with $1 < p \leq \infty$ and $1 \leq q \leq 1$ fixed parameters, consist of all functions $f \in L_0(\mu)$ such that $\|f\|^*_{p,q} < \infty$, where

$$\|f\|^*_{p,q} = \begin{cases} \left(\dfrac{q}{p}\int_0^\infty \left(t^{\frac{1}{p}} f^*(t)\right)^q \dfrac{dt}{t}\right)^{1/q}, & 1 < p < \infty,\ 1 \leq q < \infty, \\ \sup\limits_{t>0}\left\{t^{\frac{1}{p}} f^*(t)\right\}, & 1 < p \leq \infty,\ q = \infty. \end{cases}$$

Here $f^*(t) = \inf\{s > 0 : \mu_f(s) \leq t\}$ is the non-increasing rearrangement of f and

$$\mu_f(s) = \mu(\{x \in \Omega : |f(x)| > s\})$$

is the distribution function of f. As before we are considering a σ-finite measure space (Ω, Σ, μ), where the measure μ is complete. The relation $\| \cdot \|^*_{p,q}$ is a seminorm for $L_(p, q)(\mu)$. The Lorentz space is a Banach space with the norm

$$\|f\|^*_{p,q} = \begin{cases} \left(\dfrac{q}{p}\int_0^\infty \left(t^{\frac{1}{p}} f^{**}(t)\right)^q \dfrac{dt}{t}\right)^{1/q}, & 1 < p < \infty,\ 1 \leq q < \infty, \\ \sup\limits_{t>0}\left\{t^{\frac{1}{p}} f^{**}(t)\right\}, & 1 < p \leq \infty,\ q = \infty. \end{cases}$$

where, for $t > 0$, the relation $f^{**}(t)$ is defined by

$$f^{**}(t) = \frac{1}{t}\int_0^t f^*(s)ds$$

and it is called the maximal function of f^*. This maximal function was introduced by Calderón [5] in 1964. It is known that the quantities $\|f\|^*_{p,q}$ and $\|f\|_{p,q}$ are equivalent for all $f \in L_{(p,q)}(\mu)$. An obligatory reference for the study of Lorentz spaces is the celebrated article of Hunt [16] published in 1966. We also recommend the recent text due to Castillo and Rafeiro [10].

The properties of multiplication operators M_u acting from a Lorentz space $L_{(p,q)}(\mu)$ into itself were established in 2006 by Arora, Datt and Verma [1].

Using the same techniques due to Singh and Kumar [34, 35] and Takagi [36], Arora, Datt and Verma show that the same results obtained by Komal and Gupta in [19], see Theorem 5 above, are also valid by changing $L_\varphi(\mu)$ in Theorem 5 by $L_{(p,q)}(\mu)$. Hence, we may enunciate the same Problems 2 and 3 by changing the Orlicz space $L_\varphi(\mu)$ by a Lorentz space $L_{(p,q)}(\mu)$. In 2009, Arora, Datt and Verma [2], show the existence of non-null compact multiplication operators acting on the Lorentz sequence spaces. They showed the following result:

Theorem 4.1 (Theorem 2.4 in [2])**.** *Let $M_u \in B(l_{(p,q)})$, $1 < p \leq \infty$, $1 \leq q \leq \infty$. A necessary and sufficient condition for M_u to be compact is that $|u(n)| \to 0$ as $n \to \infty$.*

Here, $l_{(p,q)}$ is the Lorentz sequence space which is obtained when $\Omega = \mathbb{N}$, $\Sigma = P(\mathbb{N})$ and μ is the counting measure.

This last result was generalized in 2016 by Castillo, Ramos-Fernández and Salas-Brown in [29]. They calculate the essential norm of multiplication operator on Lorentz sequence space; more precisely, they showed the following:

Theorem 4.2 (Main Theorem in [29])**.** *Let $M_u \in B(l_{(p,q)})$, $1 < p \leq \infty$, $1 \leq q \leq \infty$. Then*

$$\|M_u\|_e = \lim \sup_{n\to\infty} |u(n)|.$$

Clearly Theorem 8 implies Theorem 7 since $M_u : l_{(p,q)} \to l_{(p,q)}$ is compact if and only if $\|M_u\|_e = 0$.

About the properties of multiplication operators acting on Lorentz spaces, we can formulate the following problem:

Open Problem 5. Study the properties of multiplication operators acting between two different Lorentz spaces.

In 1996, Nakai [27] characterized the pointwise multipliers between two different Lorentz spaces under certain conditions on the parameters which define the spaces.

A big numerous of researchers have adapted the techniques developed by Singh and Kumar [34, 35] and Takagi [36] and they have concluded that Theorem 5 is also valid for other spaces which are generalizations of the classical $L_p(\mu)$ space. About this topic, the author of this note and his collaborators have obtained the following results:

1. In 2015, Castillo, Vallejo-Narvaez and Ramos-Fernández [9] showed that Theorem 5 is also valid for Weak $L_p(\mu)$ spaces, where $f \in L_0(\mu)$ belongs to Weak $L_p(\mu)$ if

$$S_p(f) := \sup_{\lambda>0} \mu(\{x \in \omega : |f(x)|Z\lambda\}) < \infty$$

Weak $L_p(\mu)$ is a Banach space with the norm $\|f\|_{(p,1)} = S_p^{1/p}(f)$.

2. In 2015, Castillo, Chaparro and Ramos-Fernández in [7], established that Theorem 5 holds for Weighted Lorentz-Orlicz spaces $L_{\varphi,w}(\mu)$, where $f \in L_0(\mu)$ belongs to $L_{\varphi,w}(\mu)$ if

$$N_{\varphi,w}\left(\frac{f^*}{\lambda}\right) := \int_0^\infty \varphi\left(\frac{f` * (t)}{\lambda}\right) w(t)dt < \infty$$

for some $\lambda > 0$. Here, $\varphi : [0,\infty) \to [0,\infty)$ is a Young function and the weight function $w : [0,\infty) \to [0,\infty)$ is an integrable function. The space $L_{\varphi,w}(\mu)$ is Banach with the norm

$$\|f\|_{\varphi,w} = \inf\left\{\lambda > 0 : N_{\varphi,w}\left(\frac{f^*}{\lambda}\right) \leq 1\right\},$$

where it is understood that $\inf(\emptyset) = +\infty$.

3. In 2015, Castillo, Ramos-Fernández and Rafeiro in [8], showed that Theorem 5 is also true for variable Lebesgue spaces $L^{p(\cdot)}(\mu)$, where $f \in L_0(\mu)$ belongs to $L_{\varphi,w}(\mu)$ if

$$N_{p(\cdot)}(\mu)(f) := \int_\Omega |f(t)|^{p(\cdot)}\ dt < \infty.$$

The space $L^{p(\cdot)}(\mu)$ is Banach space with the Luxemburg norm

$$\|f\|_{p(\cdot)} = \inf\left\{\lambda > 0 : N_{p(\cdot)}\left(\frac{f^*}{\lambda}\right) \leq 1\right\}.$$

It is important to remark that in the articles cited before also are established new properties of the spaces cited therein and also they give new proof of known properties of those spaces.

5. Multiplication operators acting on Köthe-type spaces

The fact that the same results about the properties of multiplication operators are valid for too many spaces generalizing the classical $L_p(\mu)$ spaces make us think that they are part of a more general result. Indeed, the spaces mentioned in the above section are examples of certain Banach lattice which are known as Köthe spaces or Banach function spaces which we define below.

Let (Ω, Σ, μ) be a complete σ-finite measure space and let $L_0(\mu) = L_0(\Omega, \Sigma, \mu)$ be the space of all equivalence classes of μ-measurable real-valued functions endowed with the topology of convergence in measure relative to each set of finite measure. A real Banach space $(X; \|\cdot\|_X)$ consisting of equivalence classes modulo equality almost everywhere of locally integrable real-valued functions on Ω is called a *Köthe function space* if the following conditions hold:

(i) If $|f(t)| \geq |g(t)|$ a.e. on Ω, with f measurable and $g \in X$, then $f \in X$ and $\|f\|_X \leq \|g\|_X$ (often this property is refer as the space is solid).

(ii) For every $A \in \Sigma$ with $\mu(A) < \infty$, the characteristic function $\mathbf{1}_A$ of A belongs to X.

A function f is said *locally integrable* if belongs to $L_0(\mu)$ and $\int_A |f| d\mu < \infty$ for each $A \in \Sigma$ such that $\mu(A) < \infty$. Clearly, every Köthe function space is a Banach lattice in the obvious order ($g \geq 0$ if $g(t) \geq 0$ a.e.). By (i), every Köthe function space is a σ-order complete space, see [21]. The spaces mentioned in the above Section are examples of Köthe spaces. For more properties of Köthe spaces, we refer the reader to [21] and [22].

A Köthe space X is said to have the *Fatou property* if whenever $\{f_n\}$ is a norm-bounded sequence in X such that $0 \leq f_n \nearrow f \in L_0(\mu)$, then $f \in X$ and $\|f_n\|_X \to \|f\|_X$. Similarly, a Köthe space is said to be *order continuous* if $\|f_n\|_X \to 0$ whenever $f_n \searrow 0$. The assumption that every $g \in X$ is locally integrable implies that for every finite measurable set $A \in \Sigma$, the positive functional $g \mapsto \int_\Omega g(t)\mathbf{1}A(t) d\mu$ is well defined and thus bounded; i.e., it is an element of X^*, the (topological) dual space of X. In fact see [24], every $g \in X$ such that $fg \in L_1(\mu)$ for all $f \in X$ defines an element T_g of X^* by

$$T_g(f) = \int_\Omega fg d\mu (f \in X),$$

which is called an integral. The set of all measurable functions $\widehat{g}$ such that $f\widehat{g} \in L_1(\mu)$ for all $f \in X$ is denoted by X' and it is called the *Köthe dual* of X. The space X' is a Köthe space endowed with the norm

$$\|\widehat{g}\|_{X'} := \sup_{\|f\|_X \leq 1} \int_\Omega |f\widehat{g}| d\mu$$

It is known see [21] that if X is order continuous then X^* is isometrically equal to X'. Also, it is known that

$$\|f\|_X = \sup \left\{ \left|\widehat{h}(f)\right| : \widehat{h} \in X' \ \left\|\widehat{h}\right\|_{X'} = 1 \right\} \tag{2}$$

for every $f \in X$ (i.e. X' is a norming subspace of X^*) if and only if whenever $\{f_n\}$, f are nonnegative elements of such that $f_n \nearrow f$ a.e., we have $\|f_n\|_X \to \|f\|_X$. Observe that from the relation (2) we have

$$\left|\widehat{h}(f)\right| \leq \left\|\widehat{h}\right\|_{X'} \|f\|_X$$

for all $\widehat{h} \in X'$ and all $f \in X$. For more properties of Köthe spaces, we recommend the excellent books [22] and [21].

In 2006, Hudzik, Kumar and Kumar [15] published an important article about the properties of multiplication operators acting on Köthe spaces. They assume that $X = Xb$, that is, the simple functions are dense in X and that X has absolutely continuous norm. Hence its Banach dual space X^* and its associate space X' coincide. In this case, the Köthe space X is also known as a Banach function space. As consequence of the results obtained by the authors in [15], we can write the following:

Theorem 5.1 ([15])**.** *Let X be an order continuous Köthe space satisfying the Fatou property, then*

1. *Continuity (Theorem 1.1 in [15]). The multiplication operator M_u is a bounded operator on a Banach function space X if and only if $u \in L_\infty(\mu)$. Moreover, $\|M_u\| = \|u\|_\infty$.*

2. *Invertibility and spectrum (Proposition 2.1 in [15]). The operator M_u has a bounded inverse if and only if $0 \notin u_{ess}(\Omega) = \sigma(M_u)$, where*
$$u_{ess}(\Omega) = \{\lambda \in \mathbb{C} : \mu\left(\{s \in \Omega : |u(s) - \lambda| < \epsilon\}\right) \neq 0 \ \forall_\epsilon > 0\}$$
is the essential range of u and $\sigma(M_u) = \{\lambda \in \mathbb{C} : M_u - \lambda I : X \to X \text{ is not bijective}\}$ is the spectrum of M_u.

3. *Theorem 2.2 in [15]. The set of all bounded multiplication operators on a Banach function space X of $\mathbb{C}$-valued functions forms a maximal abelian subalgebra of $\mathcal{B}(X)$, the space of all bounded linear operators on X.*

4. *Closed Range (Theorem 2.3 in [15]). Let X be a Banach function space of $\mathbb{C}$-valued measurable functions on Σ and $M_u \in \mathcal{B}(X)$ for some $u \in L_\infty(\mu)$. Then M_u has closed range if and only if there exists some $\delta > 0$ such that $|u(x)| \geq \delta$, for μ-almost all $x \in \operatorname{supp}(u)$.*

5. *Compactness Theorem 2.4 in [15]). Let X be a Banach function space of $\mathbb{C}$-valued measurable functions on Ω and $M_u \in \mathcal{B}(X)$. Then M_u is a compact operator if and only if $X(N, \epsilon)$ is finite-dimensional, for each $\epsilon > 0$, where $N = N(u, \epsilon) = \{x \in \Omega : |u(x)| \geq \epsilon\}$ and $X(N, \epsilon) = \{f \in X : f(x) = 0 \ \forall x \in \Omega \setminus N\}$.*

6. *Fredholm (Theorem 2.5 in [15]). Suppose (Ω, Σ, μ) is a non-atomic measure space and $M_u \in \mathcal{B}(X)$, where X is a Banach function space having absolutely continuous norm. Then, M_u is Fredholm if and only if M_u is invertible if and only if there exists $\delta > 0$ such that $|u(x)| \geq \delta$ for μ-a.e. $x \in \Omega$.*

Although this article is quite general, it continues appearing results about this topic, for example, recently in 2016, Mursaleena, Aghajanib and Rajc [26] show that Theorem 5 is also valid for Cesàro function spaces $Ces_p(\Omega)$. Certainly, almost all the results shown in the article [26] are consequence of the results that appear in [15], see Theorem 9 because $Ces_p(\Omega)$ is a Banach function space. However, still there are some open problems about this subject, for instance:

Open Problem 6. Suppose that $u \in L_\infty(\mu)$. Calculate the essential norm of $M_u : X \to X$. Characterize when $M_u : X \to X$ is Fredholm for the more general case. Establish the properties of $M_u : X \to Y$ in the case of X, Y Köthe-type spaces on the same measurable space.

Finally, we note that there exist a lot of works about the properties of multiplication, composition and weighted composition operators acting on Köthe-type spaces as we can see in the reference of the articles cited in this note. Also there are great number of works about the properties of these operators acting on non-Köthe spaces, for instance, recently, in 2016 Astudillo-Villalba and Ramos-Fernández in [3] made a very complete study of the properties of multiplication operator acting on the classical space of functions of bounded variation.

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References

[1] S. C. Arora, G. Datt, and S. Verma, *Multiplication operators on Lorentz spaces*, Indian J. Math. **48** (2006), no. 3, 317–329.

[2] ______, *Operators on Lorentz sequences spaces*, Math. Bohem. **134** (2009), no. 1, 87–98.

[3] F. R. Astudillo-Villalba and J. C. Ramos-Fernández, *Multiplication operators on the space of functions of bounded variation*, To appear in Demonstratio Mathematica.

[4] C. Bennett and R. Sharply, *Interpolation of Operators*, Academic Press INC, Boston, 1991.

[5] A. P. Calderón., *Intermediate spaces and interpolation, the complex method*, Studia Math. **24** (1964), 133–190.

[6] R. E. Castillo, *Espacios L_p*, U. Nacional de Colombia, Bogotá, 2014.

[7] R. E. Castillo, H. C. Chaparro, and J. C. Ramos Fernández, *Orlicz-Lorentz spaces and their multiplication operators*, Hacet. J. Math. Stat. **44** (2015), no. 5.

[8] R. E. Castillo, J. C. Ramos Fernández, and H. Rafeiro, *Multiplication operators in variable Lebesgue spaces*, Rev. Colombiana Mat. **49** (2015), no. 2, 293–305.

[9] R. E. Castillo, F. A. Vallejo Narvaez, and J. C. Ramos Fernández, *Multiplication and composition operators on weak L_p spaces*, Bull. Malays. Math. Sci. Soc. **38** (2015), no. 3, 927–973.

[10] R. E. Castillo and H. Rafeiro, *An Introductory Course in Lebesgue Spaces*, CMS Books in Matehamtics, Springer International Publishing, Switzerland, 2016.

[11] T. Chawziuk, Y. Cui, Y. Estaremi, H. Hudzik, and R. Kaczmarek, *Composition and multiplication operators between Orlicz function spaces*, J. Inequal. Appl. **52** (2016), 18.

[12] J. B. Conway, *A course in functional analysis. Second edition. Graduate Texts in Mathematics*, Springer-Verlag, New York, 1990, 96.

[13] M. Fréchet, *Sur les ensembles de fonctions et les opérations linaires*, C. R. Acad. Sci. Paris **144** (1907), 14141416.

[14] P. Halmos, *Hilbert space problem book*, D. Van Nostrand Co., Inc., Princeton, N.J.-Toronto, Ont.-London, 1967.

[15] H. Hudzik, R. Kumar, and R. Kumar, *Matrix multiplication operators on Banach function spaces*, Proc. Indian Acad. Sci. Math. Sci. **116** (206), 71–81.

[16] R. Hunt, *On $L(p,q)$ spaces*, Enseignement Math. **12** (1996), no. 2, 249–276.

[17] M. R. Jabbarzadeh and E. Pourreza, *note on weighted composition operators on L_p-spaces*, Bull. Iranian Math. Soc. **29** (2003), no. 1, 47–54.

[18] T. Kato, *Perturbation theory for linear operators*, Reprint of the 1980 edition. Classics in Mathematics. Springer-Verlag, Berlin, 1995.

[19] B. S. Komal and S. Gupta, *Multiplication operators between Orlicz spaces*, Integral Equations Operator Theory **41** (2001), no. 3, 324–330.

[20] H. Lebesgue, *Integrale, Longueur, Aire*, Ph.D. thesis, University of Paris, 1902.

[21] P. Lin, *Köthe-Bochner function spaces*, Birkhäuser Boston, Inc., Boston, MA, 2004.

[22] J. Lindenstrauss and L. Tzafriri, *Classical Banach spaces. II. Function spaces*, 1979.

[23] G. G. Lorentz, *Some new functional spaces*, Ann. of Math. **51** (1950), no. 2, 37–55.

[24] ______, *On the theory of spaces* Λ, Pacific J. Math. **1** (1951), 411–429.

[25] V. Müller, *Spectral theory of linear operators and spectral systems in Banach algebras. Second edition. Operator Theory: Advances and Applications*, Birkhäuser Verlag, Basel, 2007, 139.

[26] M. Mursaleena, A. Aghajanib, and K. Rajc, *Multiplication Operators on Cesàro Function Spaces*, Filomat **30** (2016), no. 5, 1175–1184.

[27] E. Nakai, *Pointwise multipliers on the Lorentz spaces*, Mem. Osaka Kyoiku Univ. III Natur. Sci. Appl. Sci. **45** (1996), no. 1, 1–7.

[28] W. Orlicz, *Über eine gewisse Klasse von Räumen vom Typus B*, Bull. Int. Acad. Polon. Sci. A. (1932), 207–222.

[29] J. C. Ramos-Fernández R. E. Castillo and M. Salas-Brown, *The essential norm of Multiplication Operator on Lorentz sequence spaces*, Real Analysis Exchange **41** (2016), no. 1, 245–252.

[30] M. M. Rao and Z. D. Ren, *Theory of Orlicz spaces. Monographs and Textbooks in Pure and Applied MathematicsThe Quantum Gaudin System*, Marcel Dekker, Inc., New York, 1991, 146.

[31] ______, *Applications of Orlicz spaces. Monographs and Textbooks in Pure and Applied Mathematics*, Marcel Dekker, Inc., New York, 2002, 250.

[32] F. Riesz, *Sur une espèce de géométrie analytique des systèmes de fonctions sommables*, C. R. Acad. Sci. Paris **144** (1907), 1409–1411.

[33] ______, *Sur les opérations fonctionnelles linéaires*, C. R. Acad. Sci. Paris **149** (1909), 947–977.

[34] R. K. Singh and A. Kumar, *Multiplication operators and composition operators with closed ranges*, Bull. Austral. Math. Soc. **16** (1977), no. 2, 247–252.

[35] ______, *Compact composition operators*, J. Austral. Math. Soc. Ser. A **28** (1979), no. 3, 309–314.

[36] H. Takagi, *Compact weighted composition operators on* L^p, Proc. Amer. Math. Soc. **116** (1992), no. 2, 505–511.

[37] H. Takagi and K. Yokouchi, *Multiplication and composition operators between two* L^p*-spaces. Function spaces (Edwardsville, IL, 1998)*, Contemp. Math., Amer. Math. Soc., Providence, RI **232** (1999), 321–338.