Dynamics of a discrete predator-prey system with nonconstant death rate

Dinámica de un sistema depredador-presa discreto con tasa de mortalidad no constante

Cosme Duque[1,a], **Jahnett Uzcategui**[1,b]

Abstract. In this paper, we will consider a discrete non-autonomous predator-prey system with nonconstant death rate. We give sufficient conditions in order to get a dissipative and permanent system. By using the continuation theorem based on the system Gaines and Mawhin's coincidence degree, we study the existence of positive periodic solutions when the coeficients of system are periodic. A numerical example is given to illustrate the effectiveness of the main result.

Keywords: predator-prey with nonconstant death rate, difference equations, periodic solutions, coincidence degree.

Resumen. En este artículo consideramos un sistema depredador-presa discreto no autónomo con tasa de mortalidad no constante. Damos condiciones suficientes para que el sistema sea disipativo y permanente. Usando el Teorema de Continuación basado en el grado de coincidencia de Gaines y Mawhin, estudiamos la existencia de soluciones periódicas positivas cuando los coeficientes del sistema son periódicos. Un ejemplo numérico es dado para ilustrar la efectividad del resultado principal.

Palabras claves: depredator-presa con tasa de mortalidad no constante, ecuaciones en diferencias, soluciones periódicas, grado de coincidencia.

Mathematics Subject Classification: Primary 34K13, Secondary 92B05.

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1. Introduction

In recent decades, many research efforts have been put into investigation of population dynamics of predator-prey ecosystem, see, for example, [10, 11, 12, 14, 15, 16]. When investigating such biological phenomenon arising from predator-prey ecosystem, there are many factors which affect dynamical properties of biological and mathematical models, between this factors we have

[1]Departamento de Matemáticas, Universidad de Los Andes, Mérida, Venezuela
[a]duquec@ula.ve
[b]jahnettu@ula.ve

the functional response, and more recently the non-constant death rate in the predator, see [2, 3, 4, 6, 7, 13].

Concretely, Cavani and Farkas in [2] introduce the following predator-prey system

$$\begin{aligned} x' &= (c - dx)x - \frac{axy}{\beta + x}, \\ y' &= -M(y)y + \frac{bxy}{\beta + x}, \end{aligned} \tag{1}$$

where $x(t)$, $y(t)$ represent the population density of prey and predator at time t, respectively, $c > 0$ is the specific growth rate of prey in the absense of predation and without environment limitation; in the absense of predator the prey population grows logistically to carrying capacity c/d; the functional response of the predator is of Holling tipe II, see, for example, [10, 11, 12, 14, 15, 16], i.e., the rate at which an individual predator consumes prey assumes that predators do not interfere with one another's activities; thus competition among predator for food occurs only via the depletion of prey. The parameter $a > 0$ is the maximum number of prey population that can be eaten per predator population per time and $b > 0$ describes the efficiency of the predator in converting consumed prey into predator offspring, $\beta > 0$ is the satiation coefficient or conversion rate. The specific mortality of predator in absence of prey

$$M(y) = \frac{\gamma + \delta y}{1 + y} = \delta + \frac{\gamma - \delta}{1 + y}, \quad 0 < \gamma < \delta \tag{2}$$

depends on the quantity of predator; γ is the mortality at low density, and γ is the maximal mortality with the natural assumption $\gamma < \delta$. The advantage of the present model over the more often used models is that here the predator mortality is neither a constant nor an unbounded function, yet it is increasing with quantity. The predator-prey system with non-constant mortalty death rate (1) have been studied in the literature, see [2, 4, 6, 13].

When the environmental fluctuation is taken into account, the model must be nonautonomous, therefore we get the following version of (1):

$$\begin{aligned} x' &= (c(t) - d(t)x)x - \frac{a(t)xy}{\beta(t) + x}, \\ y' &= -M(y)y + \frac{b(t)xy}{\beta(t) + x}, \end{aligned} \tag{3}$$

with

$$M(y) = \frac{\gamma(t) + \delta(t)y}{1 + y} = \delta(t) + \frac{\gamma(t) - \delta(t)}{1 + y}, \quad 0 < \gamma(t) < \delta(t), \tag{4}$$

where all the variables and parameters have the same biological meanings as in (1), except that the parameters are time dependent now. However, it is well know that the discrete time models governed by difference equations are more appropriate that the continuous ones when the population have nonoverlapping generations see, for example, [1, 6, 8, 15]. In addition, discrete time models can also provide efficient computational models of continuous models for numerical simulations. In this work we will concentrate in to show the existence of positive periodic solutions of the discrete analogue of the predator-prey system (3).

The principal aim of this article is to propose a discrete analouge of system (3) and explore its dynamics. Concretely, we will show the permanence and the existence of positive periodic solutions of the discrete analogue of the predator-prey system (3).

Following the clues in [5], with the help of differential equations with piecewise constant arguments, one can reach its discrete analogous

$$\begin{aligned} x(k+1) &= x(k)\exp\left\{c(k)-d(k)x(k)-\frac{a(k)y(k)}{\beta(k)+x(k)}\right\}, \\ y(k+1) &= y(k)\exp\left\{-M(y(k))+\frac{b(k)x(k)}{\beta(k)+x(k)}\right\}, \end{aligned} \tag{5}$$

where

$$M(y(k))=\frac{\gamma(k)+\delta(k)y(k)}{1+y(k)}=\delta(k)+\frac{\gamma(k)-\delta(k)}{1+y(k)},$$

$0<\gamma(k)<\delta(k)$, $k\in\mathbb{N}$. In the following, we will focus our attention on system (5). Considering the biological significance, we consider (5) with positive initial values and assume that the parameters in system (5) are nonnegative.

2. Permanence

In this section we will show the permanence for system (5), which means that every solution belonging to positive initial conditions is bounded.

In the following discussion, we always assume that $a(k)$, $b(k)$, $c(k)$, $d(k)$, $\beta(k)$, $\gamma(k)$ and $\delta(k)$ are bounded nonnegative sequences.

We use the following notation:

$$g^l=\min_{k\in\mathbb{N}} g(k) \quad \text{and} \quad g^u=\max_{k\in\mathbb{N}} g(k),$$

where $\{g(k)\}$ is a bounded sequence of real numbers defined for $k\in\mathbb{N}$.

Definition 2.1. System (5) is said to be permanent if there exist positive constants λ and Λ, with $0<\lambda<\Lambda$, such that

$$\min\left\{\liminf_{k\longrightarrow+\infty} x(k), \liminf_{k\longrightarrow+\infty} y(k)\right\}\geq\lambda,\ \max\left\{\limsup_{k\longrightarrow+\infty} x(k), \limsup_{k\longrightarrow+\infty} y(k)\right\}\leq\Lambda,$$

for all solutions of (5) with initial values positives.

Theorem 2.2. *Let* $(x(k), y(k))$ *be a solution of* (5) *with* $x(0) > 0$ *and* $y(0) > 0$. *If* $\delta^l > b^u$, *then*

$$\limsup_{k \longrightarrow +\infty} x(k) \leq x^*, \quad \limsup_{k \longrightarrow +\infty} y(k) \leq y^*, \tag{6}$$

where

$$x^* = \frac{1}{d^l} \exp\{c^u - 1\} \text{ and } y^* = \frac{1}{\delta^l - b^u} \exp\{c^u - 1\}. \tag{7}$$

Proof. To prove $\limsup_{k \to \infty} x(k) \leq x^*$, we first assume that there exists a k_0 such that $x(k_0 + 1) \geq x(k_0)$.

By using the first equation of system (5) we obtain

$$x(k+1) \leq x(k) \exp\{c(k) - d(k)x(k)\},$$

particulary for $k = k_0$,

$$x(k_0) \leq x(k_0 + 1) \leq x(k_0) \exp\{c(k_0) - d(k_0)x(k_0)\}.$$

It follows that $c(k_0) - d(k_0)x(k_0) \geq 0$ and therefore $x(k_0) \leq c(k_0)/d(k_0) \leq c^u/d^l$. Then

$$\begin{aligned} x(k_0 + 1) &\leq x(k_0) \exp\{c(k_0) + d(k_0)x(k_0)\} \\ &\leq \frac{c^u}{d^l} \frac{x(k_0)}{c(k_0)/d(k_0)} \exp\left\{ c^u \left[1 - \frac{x(x_0)}{c(k_0)/d(k_0)} \right] \right\} \\ &\leq \frac{1}{d^l} \exp\{c^u - 1\} = x^*, \end{aligned}$$

where we used $\max_{x \in \mathbb{R}} x \exp\{r(1-x)\} = \frac{exp\{r-1\}}{r}$ for $r > 0$.

We claim that $x(k) \leq x^*$ for $k \geq k_0$. In fact, if there exists an integer $n_0 \geq k_0$ such that $x(n_0) > x^*$, then $n_0 \geq k_0 + 2$ and letting $\tilde{n}_0$ be the least integer between k_0 and n_0 such that $x(\tilde{n}_0) = \max_{k_0 \leq k \leq n_0}\{x(k)\}$, then $\tilde{n}_0 \geq k_0 + 2$ and $x(\tilde{n}_0) \geq x(\tilde{n}_0 - 1)$. The above argument produces that $x(\tilde{n}_0) \leq x^*$, a contradiction. This proves the claim.

Now, we assume that $x(k) \geq x(k+1)$ for all $k \in \mathbb{N}$. In particular, $\lim_{k \to \infty} x(k)$ exists, denoted by $\overline{x}$. We claim that $\overline{x} \leq x^*$. In fact, assume that $\overline{x} > x^*$. Taking limit in the first equation in system (5) gives

$$\lim_{k \longrightarrow \infty} \left[c(k) - d(k) - \frac{a(k)y(k)}{\beta(k) + x(k)} \right] = 0,$$

which is a contradiction since

$$\begin{aligned} 0 &= \lim_{k \longrightarrow \infty} \left[c(k) - d(k) - \frac{a(k)y(k)}{\beta(k) + x(k)} \right] \leq \lim_{k \longrightarrow \infty} \left[c(k) - d(k)x(k) \right] \\ &\leq c^u \left(1 - \frac{\overline{x}}{x^*} \right) < 0. \end{aligned}$$

This proves the claim, so $\limsup_{k\to\infty} x(k) \leq x^*$.

Similary to the above analysis, next we prove $\limsup_{k\to\infty} y(k) \leq y^*$. Assume that there exists a k_0 such that $y(k_0+1) \geq y(k_0)$.

By using the second equation of system (5) we obtain

$$y(k+1) \leq y(k)\exp\left\{\frac{1}{1+y(k)}\big[b(k)-(\delta(k)-b(k))y(k)\big]\right\},$$

particulary for k_0,

$$y(k_0) \leq y(k_0+1) \leq y(k_0)\exp\left\{\frac{1}{1+y(k_0)}\big[b(k_0)-(\delta(k_0)-b(k_0))y(k_0)\big]\right\}.$$

It follows that $b(k_0)-(\delta(k_0)-b(k_0)y(k_0))>0$ and therefore

$$y(k_0) < \frac{b(k_0)}{\delta(k_0)-b(k_0)} < \frac{b^u}{\delta^l-b^u}.$$

Then

$$\begin{aligned}
y(k_0+1) &\leq y(k_0)\exp\left\{\frac{1}{1+y(k_0)}\big[b(k_0)-(\delta(k_0)-b(k_0))y(k_0)\big]\right\}\\
&\leq \left[\frac{b^u}{\delta^l-b^u}\right]\left[\frac{y(k_0)}{b^u/(\delta^l-b^u)}\right]\exp\left\{b^u\left[1-\frac{y(k_0)}{b^u/(\delta^l-b^u)}\right]\right\}\\
&\leq \frac{1}{\delta^l-b^u}\exp\{b^u-1\} = y^*.
\end{aligned}$$

Hence $y(k) \leq y^*$ for $k \geq k_0$.

If $y(k) \geq y(k+1)$ for $k \in \mathbb{N}$, then $\lim_{k\to\infty} y(k)$ exists, dentoted by $\overline{y}$. We claim that $\overline{y} \leq y^*$, in fact, if $\overline{y} > y^*$ then taking limit in the second equation in system (5) gives

$$\lim_{k\longrightarrow\infty}\left[-\left(\delta(k)+\frac{\gamma(k)-\delta(k)}{1+y(k)}\right)+\frac{b(k)x(k)}{\beta(k)+x(k)}\right]=0,$$

but

$$\begin{aligned}
0 &= \lim_{k\longrightarrow\infty}\left[-\left(\delta(k)+\frac{\gamma(k)-\delta(k)}{1+y(k)}\right)+\frac{b(k)x(k)}{\beta(k)+x(k)}\right]\\
&\leq \lim_{k\longrightarrow\infty}\frac{1}{1+y(k)}\big[b(k)-(\delta(k)-b(k))y(k)\big]\\
&\leq \frac{b^u}{1+\overline{y}}\left[1-\frac{\overline{y}}{y^*}\right] < 0,
\end{aligned}$$

which is a contradiction. This proves the claim. So $\limsup_{k\to\infty} y(k) \leq y^*$. $\square$

Theorem 2.3. *Assume that* $\delta^l > b^u$, $c^l\beta^l - a^u y^* > 0$ *and* $b^l x_* - \gamma^u(\beta^u + x_*) > 0$, *then*

$$\liminf_{k\longrightarrow\infty} x(k) \geq x_*, \quad \liminf_{k\longrightarrow\infty} y(k) \geq y_*, \tag{8}$$

where

$$\begin{aligned} x_* &= \frac{c^l}{d^u}\left[1 - \frac{a^u y^*}{c^l\beta^l}\right]\exp\left\{c^l - \frac{a^u y^*}{\beta^l} - d^u x^*\right\} \\ y_* &= \frac{1}{\delta^u}\left[\frac{b^l x_*}{\beta^u + x_*} - \gamma^u\right]\exp\left\{\frac{b^l x_*}{\beta^u + x_*} - \gamma^u - \delta^u y^*\right\}, \end{aligned} \tag{9}$$

and x^*, y^* *are the same as in theorem 2.2.*

Proof. Let $\varepsilon > 0$ such that $c^l\beta^l - a^u(y^* + \varepsilon) > 0$ and $b^l(x_* - \varepsilon) - \gamma^u(\beta^u + (x_* - \varepsilon)) > 0$, according to theorem 2.2, there exists k^* such that

$$x(k) < x^* + \varepsilon, \text{ and } y(k) < y^* + \varepsilon \quad \text{for} \quad k \geq k^*.$$

To prove $\liminf_{k\to\infty} \geq x_*$, we first assume that there exists a $k_0 \geq k^*$ such that $x(k_0 + 1) \leq x(k_0)$. By using the first equation of system (5) we obtain

$$x(k+1) \geq x(k)\exp\left\{c(k) - \frac{a(k)(y^* + \varepsilon)}{\beta(k)} - d(k)x(k)\right\},$$

particulary for $k = k_0$,

$$x(k_0) \geq x(k_0 + 1) \geq x(k_0)\exp\left\{c(k_0) - \frac{a(k_0)(y^* + \varepsilon)}{\beta(k_0)} - d(k_0)x(k_0)\right\}.$$

It follows that $c(k_0) - a(k_0)(y^* + \varepsilon)/\beta(k_0) - d(k_0)x(k_0) \leq 0$ and therefore

$$x(k_0) \geq \frac{1}{d(k_0)}\left[c(k_0) - \frac{a(k_0)(y^* + \varepsilon)}{\beta(k_0)}\right] \geq \frac{c^l}{d^u}\left[1 - \frac{a^u(y^* + \varepsilon)}{c^l\beta^l}\right] := \Delta_1.$$

Then

$$\begin{aligned} x(k_0 + 1) &\geq x(k_0)\exp\left\{c(k_0) - \frac{a(k_0)(y^* + \varepsilon)}{\beta(k_0)} - d(k_0)x(k_0)\right\} \\ &\geq \Delta_1\exp\left\{c^l - \frac{a^u(y^* + \varepsilon)}{\beta^l} - d^u(x^* + \varepsilon)\right\} := x_\varepsilon. \end{aligned}$$

We claim that $x(k) \geq x_\varepsilon$ for $k \geq k_0$. In fact, if there exists an integer $n_0 \geq k_0$ such that $x(n_0) < x_\varepsilon$, then $n_0 \geq k_0 + 2$ and letting $\tilde{n}_0$ be the least integer between k_0 and n_0 such that $x(\tilde{n}_0) = \min_{k_0\leq k\leq n_0}\{x(k)\}$, then $\tilde{n}_0 \geq k_0 + 2$ and $x(\tilde{n}_0) \leq x(\tilde{n}_0 - 1)$. The above argument produces that $x(\tilde{n}_0) \geq x_\varepsilon$, a contradiction. This proves the claim.

Now, we assume that $x(k+1) \geq x(k)$ for all $k \in \mathbb{N}$. In particular, $\lim_{k\to\infty} x(k)$ exists, denoted by $\underline{x}$. We claim that $\underline{x} \geq \Delta_1$. In fact, assume that $\underline{x} < \Delta_1$. Taking limit in the first equation in system (5) gives

$$\lim_{k\longrightarrow\infty}\left[c(k) - d(k) - \frac{a(k)y(k)}{\beta(k)+x(k)}\right] = 0,$$

which is a contradiction since

$$0 = \lim_{k\longrightarrow\infty}\left[c(k) - d(k) - \frac{a(k)y(k)}{\beta(k)+x(k)}\right] \geq \lim_{k\longrightarrow\infty}\left[c(k) - d(k)x(k) - \frac{a(k)y(k)}{\beta(k)}\right]$$

$$\geq c^l\left[1 - \frac{a^u(y^*+\varepsilon)}{c^l\beta^l}\right] - d^u\underline{x} = d^u(\Delta_1 - \underline{x}) > 0.$$

This proves the claim. Note that $x^* \geq c^u/d^l \geq c^l/d^u$, implies $\Delta_1 \geq x_\varepsilon$ and since $\lim_{\varepsilon\to 0} x_\varepsilon = x_*$ we have $\liminf_{k\to\infty} x(k) \geq x_*$.

Similarly to the above analysis, next we prove $\liminf_{k\to\infty} y(k) \geq y_*$.

Since $\liminf_{k\to\infty} x(k) \geq x_*$, there exists $k_* \geq k^*$ such that $x_* - \varepsilon < x(k)$ for $k \geq k_*$. If there exists a $k_0 \geq k_*$ such that $y(k_0+1) \leq y(k_0)$, then by using the second equation of system (5) we obtain

$$y(k+1) \geq y(k)\exp\left\{\frac{b(k)(x_*-\varepsilon)}{\beta(k)+(x_*-\varepsilon)} - \gamma(k) - \delta(k)y(k)\right\},$$

particulary for k_0,

$$y(k_0) \geq y(k_0+1) \geq y(k_0)\exp\left\{\frac{b(k_0)(x_*-\varepsilon)}{\beta(k_0)+(x_*-\varepsilon)} - \gamma(k_0) - \delta(k_0)y(k_0)\right\}.$$

It follows that

$$\frac{b(k_0)(x_*-\varepsilon)}{\beta(k_0)+(x_*-\varepsilon)} - \gamma(k_0) - \delta(k_0)y(k_0) \leq 0$$

and therefore

$$y(k_0) \geq \frac{1}{\delta(k_0)}\left[\frac{b(k_0)(x_*-\varepsilon)}{\beta(k_0)+(x_*-\varepsilon)} - \gamma(k_0)\right] \geq \frac{1}{\delta^u}\left[\frac{b^l(x_*-\varepsilon)}{\beta^u+(x_*-\varepsilon)} - \gamma^u\right] := \Delta_2.$$

Then

$$\begin{aligned} y(k_0+1) &\geq y(k_0)\exp\left\{\frac{b(k_0)(x_*-\varepsilon)}{\beta(k_0)+(x_*-\varepsilon)} - \gamma(k_0) - \delta(k_0)y(k_0)\right\} \\ &\geq \Delta_2\exp\left\{\frac{b^l(x_*-\varepsilon)}{\beta^u+(x_*-\varepsilon)} - \gamma^u - \delta^u(y^*+\varepsilon)\right\} =: y_\varepsilon. \end{aligned}$$

Hence $y(k) \geq y_\varepsilon$ for $k \geq k_0$.

If $y(k+1) \geq y(k)$ for $k \in \mathbb{N}$, then $\lim_{k\to\infty} y(k)$ exists, denoted by $\underline{y}$. We claim that $\underline{y} \geq \Delta_2$. In fact, if $\underline{y} < \Delta_2$ then by taking limit in the second equation in system (5) gives

$$\begin{aligned}
0 &= \lim_{k\to\infty} \left\{ -\left[\delta(k) + \frac{\gamma(k)-\delta(k)}{1+y(k)}\right] + \frac{b(k)x(k)}{\beta(k)+x(k)} \right\} \\
&\geq \lim_{k\to\infty} \left\{ -\left[\gamma(k) + \delta(k)y(k)\right] + \frac{b(k)x(k)}{\beta(k)+x(k)} \right\} \geq -\gamma^u + \frac{b^l(x_* - \varepsilon)}{\beta^u + (x_* - \varepsilon)} - \delta^u \underline{y} \\
&\geq \delta^u(\Delta_2 - \underline{y}) > 0,
\end{aligned}$$

which is a contradiction. Note that $y^* \geq b^u/(\delta^l - b^u)$, therefore $\Delta_2 \geq y_\varepsilon$ and since $\lim_{k\to\infty} y_\varepsilon = y_*$ we have $\liminf_{k\to\infty} y(k) \geq y_*$. This concludes the proof. $\square$

From theorems 2.2 and 2.3 we have the permanence of system (5).

3. Existence of positive periodic solutions

In this section we will confine ourselves to the case when the parameters in system (5) are periodic functions of the time variables having a common integer period.

Let $\mathbb{Z}$, $\mathbb{Z}^+$, $\mathbb{R}^+$ y $\mathbb{R}^2$ denote the set of all integers, nonnegative integer, nonnegative real numbers, and two-dimensional Euclidean vector space, respectively.

For convenience in the following discussion, we will use the notation below:

$$I_\omega = \{0,1,2,...,\omega-1\}, \;\; \overline{g} = \frac{1}{\omega}\sum_{k=0}^{\omega-1} g(k), \;\; g^u = \max_{k\in I_\omega} g(k), \;\; g^l = \min_{k\in I_\omega} g(k),$$

where $\{g(k)\}$ is a ω-periodic sequence of real numbers defined for $k \in \mathbb{N}$.

In system (5), we always assume that $a, b, c, d, \beta, \gamma, \delta : \mathbb{N} \longrightarrow \mathbb{R}^+$, are ω-periodic, where ω, a fixed positive integer, denotes the prescribed common period of the parameter in (5).

The exponential form of the equations in (5) assures that the forward trajectory $(x(k), y(k))$ of the system with respect to any initial condition $x(0) > 0$, $y(0) > 0$, remains in the positive quadrant of the plane for all times. In the remainder of this paper, for biological reasons, we only consider solutions $(x(k), y(k))$ with $x(0) > 0$, $y(0) > 0$.

Let X, Y be normed vector spaces, $L : DomL \subset X \longrightarrow Y$ be a linear mapping, $N : X \longrightarrow Y$ be a continuos mapping. The mapping L will be called Fredholm mapping of index zero if dim Ker$L$ = codim Im$L < +\infty$ and ImL is closed in Y. If L is a Fredholm mapping of index zero and there exist contiunuos projections $P : X \longrightarrow X$ and $Q : Y \longrightarrow Y$ such that ImP = Ker$L$, ImL = Ker$Q$ = Im$(I-Q)$, it follows that $L\mid_{\mathrm{Dom}L\cap \mathrm{Ker}P}: (I-P)X \longrightarrow \mathrm{Im}L$ is invertible. We denote the inverse of the map by K_P. If Ω is an open bounded subset of X, the mapping N will be called L-compact on $\overline{\Omega}$ if $QN(\overline{\Omega})$ is bounded and $K_P(I-Q)N : \overline{\Omega} \longrightarrow X$ is compact. Since $\mathrm{Im}Q$ is isomorphic to $\mathrm{Ker}L$, there exists an isomorphism $J : \mathrm{Im}Q \longrightarrow \mathrm{Ker}L$.

Lemma 3.1. *(Continuation Theorem [9]). Let L be a Fredholm mapping of index zero and N be L-compact on $\overline{\Omega}$. Suppose:*

i) For each $\lambda \in (0,1)$, every solution of $Lx = \lambda Nx$ is such that $x \in \partial\Omega$;

ii) $QNx \neq 0$ for each $x \in \partial\Omega \cap \mathrm{Ker}L$ and the Brouwer degree $\deg\{JQN, \Omega \cap \mathrm{Ker}L, 0\} \neq 0$.

Then the operator equation $Lx = Nx$ has at least one solution lying in $\mathrm{Dom}L \cap \overline{\Omega}$.

Lemma 3.2. *([5]) Let $g : Z \longrightarrow \mathbb{R}$ be ω-periodic, i.e., $g(k+\omega) = g(k)$. Then for any fixed k_1, $k_2 \in I_\omega$, and any $k \in \mathbb{N}$, one has*

$$g(k) \leq g(k_1) + \sum_{s=0}^{\omega-1} |g(s+1) - g(s)|, \qquad g(k) \geq g(k_2) - \sum_{s=0}^{\omega-1} |g(s+1) - g(s)|.$$

Define

$$l_2 = \{u = u(k) : u(k) \in \mathbb{R}^2, k \in \mathbb{N}\}.$$

For $a = (a_1, a_2)^T \in \mathbb{R}^2$, define $|a| = \max\{a_1, a_2\}$. Let $l^\omega \subset l_2$ denote the subspace of all ω-periodic sequences equipped with the usual supremum norm $||\cdot||$, i.e.,

$$||u|| = \max_{k \in I_\omega} |u_1(k)| + \max_{k \in I_\omega} |u_2(k)|, \text{ for any } u = \{u(k) : k \in \mathbb{N}\} \in l^\omega.$$

It is not difficult to show that l^ω is a finite dimensional Banach space. Let

$$l_0^\omega = \left\{ u = \{u(k)\} \in l^\omega : \sum_{k=0}^{\omega-1} u(k) = 0, k \in \mathbb{N} \right\},$$

$$l_c^\omega = \{u = \{u(k)\} \in l^\omega : u(k) = h \in \mathbb{R}^2, k \in \mathbb{N}\};$$

then it follows that l_0^ω and l_c^ω are both closed linear subspaces of l^ω and $l^\omega = l_0^\omega \oplus l_c^\omega$, $\dim l_c^\omega = 2$.

Now, we are ready to present and prove the main result of this paper.

Theorem 3.3. *If $\overline{\delta} > \overline{b}$, $\overline{c} - \overline{\left(\frac{a}{\beta}\right)} \left(\frac{\overline{b}}{\overline{\delta} - \overline{b}}\right) \exp(2\overline{\delta}\omega) > 0$ and $\overline{\gamma} < \dfrac{\overline{b}\xi \exp(-2\overline{c}\omega)}{\beta^u + \xi \exp(-2\overline{c}\omega)}$, where $\xi = \frac{1}{d}\left[\overline{c} - \overline{\left(\frac{a}{\beta}\right)} \left(\frac{\overline{b}}{\overline{\delta} - \overline{b}}\right) \exp(2\overline{\delta}\omega)\right]$, then system (5) has at least one positive ω- periodic solution.*

Proof. First let $x(k) = \exp\{u(k)\}$, $y(k) = \exp\{v(k)\}$, so that (5) becomes

$$\begin{aligned} u(k+1) - u(k) &= c(k) - d(k)\exp\{u(k)\} - \frac{a(k)\exp\{v(k)\}}{\beta(k) + \exp\{u(k)\}} \\ v(k+1) - v(k) &= -\left(\delta(k) + \frac{\gamma(k) - \delta(k)}{1 + \exp\{v(k)\}}\right) + \frac{b(k)\exp\{u(k)\}}{\beta(k) + \exp\{u(k)\}} \end{aligned} \tag{10}$$

In this manner, we can exploit some information about the continuation theorem and prove our result in a more direct way. Now, let us define $X = Y = l^{\omega}$, $(Ly)(k) = y(k+1) - y(k)$, and

$$(Ny)(k) = \begin{bmatrix} c(k) - d(k)\exp\{u(k)\} - \dfrac{a(k)\exp\{v(k)\}}{\beta(k) + \exp\{u(k)\}} \\ \\ -\left(\delta(k) + \dfrac{\gamma(k) - \delta(k)}{1 + \exp\{v(k)\}}\right) + \dfrac{b(k)\exp\{u(k)\}}{\beta(k) + \exp\{u(k)\}} \end{bmatrix}$$

for any $y \in X$ and $k \in \mathbb{N}$. It is trivially easy to see that L is a bounded linear operator and

$$\mathrm{Ker}L = l_c^{\omega}, \mathrm{Im}L = l_0^{\omega},$$

as well as

$$\mathrm{dimKer}L = 2 = \mathrm{codimIm}L.$$

Since $\mathrm{Im}L$ is closed in Y, it follows that L is a Fredholm mapping of index zero.

Define

$$Pu = \frac{1}{\omega}\sum_{s=0}^{\omega-1} y(s),\ y \in X, \qquad Qz = \frac{1}{\omega}\sum_{s=0}^{\omega-1} z(s),\ z \in Y.$$

It is not difficult to show that P and Q are continuos projectors such that $\mathrm{Im}P = \mathrm{Ker}L$ and $\mathrm{Im}L = \mathrm{Ker}Q = \mathrm{Im}(I - Q)$.

Furthermore, the generalized inverse (to L) $K_P : \mathrm{Im}L \longrightarrow \mathrm{Ker}P \cap \mathrm{Dom}L$ exists and is given by

$$K_P(z) = \sum_{s=0}^{\omega-1} z(s) - \frac{1}{\omega}\sum_{s=0}^{\omega-1} z(s).$$

Obviously, QN and $K_P(I-Q)N$ are continuos. Since X is a finite-dimensional Banach space, using the Arzela-Ascoli theorem, it is not difficult to show that $\overline{K_P(I-Q)N(\overline{\Omega})}$ is compact for any open bounded set $\Omega \subset X$. Moreover, $QN(\overline{\Omega})$ is bounded. Thus, N is L-compact on $\overline{\Omega}$ with any open bounded set $\Omega \subset X$.

For the application of the continuation theorem, we must search for an appropriate open, bounded set Ω. Corresponding to the operator equation

$Ly = \lambda NY$, $\lambda \in (0,1)$, we have

$$\begin{aligned} u(k+1)-u(k) &= \lambda\left[c(k)-d(k)\exp\{u(k)\}-\frac{a(k)\exp\{v(k)\}}{\beta(k)+\exp\{u(k)\}}\right], \\ v(k+1)-v(k) &= \lambda\left[-\left(\delta(k)+\frac{\gamma(k)-\delta(k)}{1+\exp\{v(k)\}}\right)+\frac{b(k)\exp\{u(k)\}}{\beta(k)+\exp\{u(k)\}}\right]. \end{aligned} \tag{11}$$

Suppose that $(u(k),v(k))^T \in X$ is an arbitrary solution of system (11) for a certain $\lambda \in (0,1)$.

Summing on both sides of (11) from 0 to $\omega-1$ with respect to k, we reach

$$\begin{aligned} 0 &= \sum_{k=0}^{\omega-1}[u(k+1)-u(k)] \\ &= \lambda\sum_{k=0}^{\omega-1}\left[c(k)-d(k)\exp\{u(k)\}-\frac{a(k)\exp\{v(k)\}}{\beta(k)+\exp\{u(k)\}}\right], \\ 0 &= \sum_{k=0}^{\omega-1}[v(k+1)-v(k)] \\ &= \lambda\sum_{k=0}^{\omega-1}\left[-\left(\delta(k)+\frac{\gamma(k)-\delta(k)}{1+\exp\{v(k)\}}\right)+\frac{b(k)\exp\{u(k)\}}{\beta(k)+\exp\{u(k)\}}\right], \end{aligned} \tag{12}$$

that is,

$$\begin{aligned} \bar{c}\omega &= \sum_{k=0}^{\omega-1}\left[d(k)\exp\{u(k)\}+\frac{a(k)\exp\{v(k)\}}{\beta(k)+\exp\{u(k)\}}\right], \\ \bar{\delta}\omega &= \sum_{k=0}^{\omega-1}\left[\frac{\delta(k)-\gamma(k)}{1+\exp\{v(k)\}}+\frac{b(k)\exp\{u(k)\}}{\beta(k)+\exp\{u(k)\}}\right]. \end{aligned} \tag{13}$$

From, (11) and (13) we obtain

$$\begin{aligned} \sum_{k=0}^{\omega-1}|u(k+1)-u(k)| &\le \lambda\sum_{k=0}^{\omega-1}\left[|c(k)|-d(k)\exp\{u(k)\}-\frac{a(k)\exp\{v(k)\}}{\beta(k)+\exp\{u(k)\}}\right] \\ &= 2\lambda\bar{c}\omega < 2\bar{c}\omega \\ \sum_{k=0}^{\omega-1}|v(k+1)-v(k)| &\le \lambda\sum_{k=0}^{\omega-1}\left[\delta(k)+\frac{\gamma(k)-\delta(k)}{1+\exp\{v(k)\}}+\frac{b(k)\exp\{u(k)\}}{\beta(k)+\exp\{u(k)\}}\right] \\ &= 2\lambda\bar{\delta}\omega < 2\bar{\delta}\omega. \end{aligned} \tag{14}$$

Now, since $(u(k), v(k))^T \in X$, there exist $\xi_i, \eta_i \in I_w$ such that

$$\begin{aligned} u(\xi_1) &= \min_{k\in I_\omega} u(k), \quad u(\eta_1) = \max_{k\in I_\omega} u(k) \\ v(\xi_2) &= \min_{k\in I_\omega} v(k), \quad v(\eta_2) = \max_{k\in I_\omega} v(k) \end{aligned} \tag{15}$$

It follows from (13) and (15) that

$$\overline{c}\omega \geq \sum_{k=0}^{\omega-1} d(k)\exp\{u(k)\} \geq \sum_{k=0}^{\omega-1} d(k)\exp\{u(\xi_1)\} = \overline{d}\omega\exp\{u(\xi_1)\},$$

which reduces to

$$u(\xi_1) \leq \ln\left[\frac{\overline{c}}{\overline{d}}\right] := L_1,$$

and hence, from Lemma 3.2 and (14) we obtain

$$u(k) \leq u(\xi_1) + \sum_{s=0}^{\omega-1} |u(s+1) - u(s)| \leq L_1 + 2\overline{c}\omega := H_1. \tag{16}$$

On the other hand, from (13) and (15) we also have

$$\overline{\delta}\omega \leq \sum_{k=0}^{\omega-1}\left[\frac{\delta(k)}{1+\exp\{v(\xi_2)\}} + b(k)\right] = \frac{\overline{\delta}\omega}{1+\exp\{v(\xi_2)\}} + \overline{b}\omega,$$

which reduces to

$$v(\xi_2) \leq \ln\left[\frac{\overline{b}}{\overline{\delta} - \overline{b}}\right] := L_2.$$

Using, again, Lemma 3.2 and (14) we obtain

$$v(k) \leq v(\xi_2) + \sum_{s=0}^{\omega-1} |v(s+1) - v(s)| \leq L_2 + 2\overline{\delta}\omega := H_2. \tag{17}$$

Now, from (13) and (15) it follows that

$$\begin{aligned} \overline{c}\omega &\leq \sum_{k=0}^{\omega-1}\left[d(k)\exp\{u(k)\} + \frac{a(k)}{\beta(k)}\exp\{v(k)\}\right] \\ &\leq \sum_{k=0}^{\omega-1}\left[d(k)\exp\{u(\eta_1)\} + \frac{a(k)}{\beta(k)}\left(\frac{\overline{b}}{\overline{\delta} - \overline{b}}\right)\exp(2\overline{\delta}\omega)\right] \\ &= \overline{d}\omega\exp\{u(\eta_1)\} + \overline{\left[\frac{a}{\beta}\right]}\left(\frac{\overline{b}}{\overline{\delta} - \overline{b}}\right)\exp(2\overline{\delta}\omega), \end{aligned}$$

so we know that

$$u(\eta_1) \geq \ln\left[\frac{1}{\overline{d}}\left(\overline{c} - \overline{\left[\frac{a}{\beta}\right]}\left(\frac{\overline{b}}{\overline{\delta} - \overline{b}}\right)\exp(2\overline{\delta}\omega)\right)\right] := l_1,$$

therefore, Lemma 3.2 and (14) imply

$$u(k) \geq u(\eta_1) - \sum_{s=0}^{\omega-1} |u(s+1) - u(s)| \geq l_1 - 2\overline{c}\omega := H_3. \tag{18}$$

If we set $\xi = \exp(l_1) = \frac{1}{\overline{d}}\left(\overline{c} - \overline{\left[\frac{a}{\beta}\right]}\left(\frac{\overline{b}}{\overline{\delta} - \overline{b}}\right)\exp(2\overline{\delta}\omega)\right) > 0$, then, from (18) we obtain $\exp\{u(k)\} \geq \xi \exp(-2\overline{c}\omega)$.

We can derive from (13) and (15) that

$$\begin{aligned} \overline{\delta}\omega &\geq \sum_{k=0}^{\omega-1}\left[\frac{\delta(k) - \gamma(k)}{1 + \exp\{v(k)\}} + \frac{b(k)\xi\exp(-2\overline{c}\omega)}{\beta(k) + \xi\exp(-2\overline{c}\omega)}\right] \\ &\geq \left[\frac{\overline{\delta} - \overline{\gamma}}{1 + \exp\{v(\eta_2)\}} + \frac{\overline{b}\xi\exp(-2\overline{c}\omega)}{\beta^u + \xi\exp(-2\overline{c}\omega)}\right]\omega, \end{aligned}$$

consequently,

$$v(\eta_2) \geq \ln\left[\frac{\overline{\delta} - \overline{\gamma}}{\overline{\delta} - \frac{\overline{b}\xi\exp(-2\overline{c}\omega)}{\beta^u + \xi\exp(-2\overline{c}\omega)}} - 1\right] := l_2.$$

From this, (14) and Lemma 3.2, we easily obtain

$$v(k) \geq v(\eta_2) - \sum_{s=0}^{\omega-1} |v(s+1) - v(s)| \geq l_2 - 2\overline{\delta}\omega := H_4. \tag{19}$$

Now, from (16), (17), (18) and (19) it follows that

$$\max_{k\in I_\omega} |u(k)| \leq \max\{|H_1|, |H_3|\} := B_1$$

and

$$\max_{k\in I_\omega} |v(k)| \leq \max\{|H_2|, |H_4|\} := B_2.$$

Obviously, B_1 and B_2 are independent of λ. Take $B = B_1 + B_2 + B_3$, where $B_3 > 0$ is taken sufficiently large such that $B_3 > |l_1| + |L_1| + |l_2| + |L_2|$.

Consider the algebraic equations

$$\begin{cases} \overline{c} - \overline{d}\exp\{u\} - \dfrac{1}{\omega}\displaystyle\sum_{k=0}^{\omega-1} \dfrac{\mu a(k)\exp\{v\}}{\beta(k)+\exp\{u\}} & = & 0 \\ -\overline{\delta} - \dfrac{1}{\omega}\displaystyle\sum_{k=0}^{\omega-1} \dfrac{\gamma(k)-\delta(k)}{1+\exp\{v\}} + \dfrac{1}{\omega}\displaystyle\sum_{k=0}^{\omega-1} \dfrac{b(k)\exp\{u\}}{\beta(k)+\exp\{u\}} & = & 0, \end{cases} \tag{20}$$

where $(u, v) \in \mathbb{R}^2$ and $\mu \in [0, 1]$ is a parameter.

Note that (20) is equivalent to

$$\begin{cases} \overline{c} - \overline{d}\exp\{u\} - \dfrac{1}{\omega}\displaystyle\sum_{k=0}^{\omega-1} \dfrac{\mu a(k)\exp\{v\}}{\beta(k)+\exp\{u\}} & = & 0 \\ -\overline{\delta} + \dfrac{\overline{\delta}-\overline{\gamma}}{1+\exp\{v\}} + \dfrac{1}{\omega}\displaystyle\sum_{k=0}^{\omega-1} \dfrac{b(k)\exp\{u\}}{\beta(k)+\exp\{u\}} & = & 0. \end{cases} \tag{21}$$

One can show that any solution (u^*, v^*) of (21), with $\mu \in [0, 1]$, satisfies

$$l_1 < u^* < L_1 \quad , \quad l_2 < v^* < L_2. \tag{22}$$

Let

$$\Omega = \{(u, v)^T \in X : \|(u, v)\| < B\},$$

then Ω is an open, bounded set in X and verifies requirement (a) of Lemma 3.1.

When $(u, v) \in \partial\Omega \cap \mathrm{Ker}L$, (u, v) is a constant vector in $\mathbb{R}^2$ with $\|(u, v)\| = |u| + |v| = B$. Then

$$QN\begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} \overline{c} - \overline{d}\exp\{u\} - \dfrac{1}{\omega}\displaystyle\sum_{k=0}^{\omega-1} \dfrac{\mu a(k)\exp\{v\}}{\beta(k)+\exp\{u\}} \\ -\overline{\delta} + \dfrac{\overline{\delta}-\overline{\gamma}}{1+\exp\{v\}} + \dfrac{1}{\omega}\displaystyle\sum_{k=0}^{\omega-1} \dfrac{b(k)\exp\{u\}}{\beta(k)+\exp\{u\}} \end{bmatrix} \neq \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

that is, $QNx \neq 0$, $\forall\ x = (u, v)^T \in \partial\Omega \cap \mathrm{Ker}L$. So, the first part of (b) of Lemma 3.1 is valid.

Consider the homotopy for computing the Brouwer degree

$$A_\mu((u, v)^T) = \mu QN((u, v)^T) + (1-\mu)G((u, v)^T), \mu \in [0, 1],$$

where

$$G((u, v)^T) = \begin{bmatrix} \overline{c} - \overline{d}\exp\{u\} \\ \overline{\delta} - \dfrac{(\overline{\delta}-\overline{\gamma})}{1+\exp\{v\}} - \dfrac{1}{\omega}\displaystyle\sum_{k=0}^{\omega-1} \dfrac{b(k)\exp\{u\}}{\beta(k)+\exp\{u\}} \end{bmatrix}.$$

From (22) it follows that $0 \notin A_\mu(\partial\Omega \cap \mathrm{Ker}L)$, $\mu \in [0,1]$, and not it is difficult to show that the algebraic equation $G((u,v)^T) = 0$ has an unique solution in $\mathbb{R}^2$.

By the invariance property of homotopy, we have that

$$\begin{aligned} \deg(JQN, \Omega \cap \mathrm{Ker}L, 0) &= \deg(QN, \Omega \cap \mathrm{Ker}L, 0) \\ &= \deg(G, \Omega \cap \mathrm{Ker}L, 0) \\ &= \sum_{x \in G^{-1}(0)} \mathrm{sig} J_G(x) \neq 0, \end{aligned}$$

where $\deg(\cdot,\cdot,\cdot)$ is the Brouwer degree, $J = I_d$ since $\mathrm{Im}Q = \mathrm{Ker}L$ and the Jacobian of G is

$$J_G(x) = \det \begin{bmatrix} -\overline{d}\exp\{u\} & 0 \\ h(u) & \dfrac{(\overline{\delta} - \overline{\gamma})}{(1+\exp\{v\})^2}\exp\{u\} \end{bmatrix} < 0,$$

where $h(u)$ is the derivative of the second row of G respect to u.

By now, we have proved that Ω verifies all requirements of Lemma 3.1, then it follows that $Lx = Nx$ has at least one solution in $\mathrm{Dom}L \cap \overline{\Omega}$, that is to say, the system (10) has at least one ω periodic solution in $\mathrm{Dom}L \cap \overline{\Omega}$, say $(u^*(k), v^*(k))^T$.

Let $x^*(k) = \exp\{u^*(k)\}$ and $y^*(k) = \exp\{v^*(k)\}$, then $(x^*(k), y^*(k))^T$ is an ω periodic solution of system (5) with strictly positive components. This completes the proof. □

4. Numerical example

The following numerical example illustrates our results. Let us pick the coefficients

$$\begin{array}{ll} a(k) = 0.01(0.5 + 0.2\sin(\pi k/2)), & b(k) = 0.1(1 + 0.2\cos(\pi k/2)), \\ c(k) = 0.1(1 + 0.8\cos(\pi k/2)), & d(k) = 0.025(2 + \cos(\pi k/2)), \\ \beta(k) = 2(0.2 + 0.1\sin(\pi k/2)), & \gamma(k) = 0.0067(3.2 + 0.8\sin(\pi k/2)), \\ \delta(k) = 0.125(2 + \sin(\pi k/2)), & \end{array}$$

which are 4-periodic. It is easy to show that the conditions in Theorem 3.3 are verified, therefore the system (5) admits at least one ω-periodic solution. Our numerical simulation supports our theoretical findings and we can appreciate in Figure 1 that the solution tends to the 4-periodic solutions $(x^*(k), y^*(k))$.

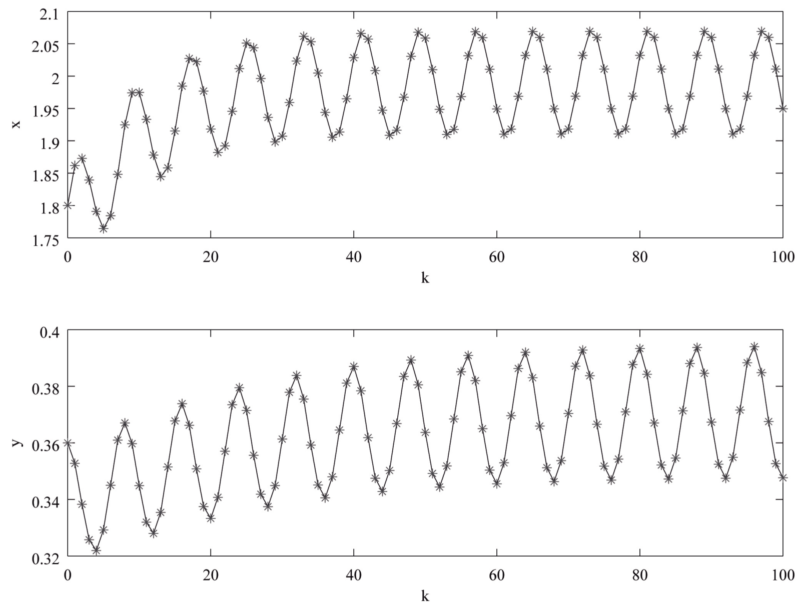

Figure 1: Solution of (5) with initial conditions $x(0) = 1.8$ and $y(0) = 0.36$.

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