Lecturas Matemáticas
Volumen **38** (2) (2017), páginas 65-73
ISSN 0120-1980

More about the Stieltjes function from which the discrete classical orthogonal polynomials are characterized

Más acerca de la función de Stieltjes a partir de la cual son caracterizados los polinomios ortogonales clásicos discretos

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ABSTRACT. In this work we consider a new way of constructing the Stieltjes function from which the discrete classical orthogonal polynomials (Charlier, Krawtchouk, Meixner, and Hahn polynomials) are characterized. In addition, the hypergeometric series representations for the Stieltjes function are obtained for one discrete classical case.

Key words: Stieltjes functions, discrete classical orthogonal polynomials, hypergeometric series representation.

RESUMEN. En este trabajo consideramos una nueva forma de construir la función de Stieltjes a partir de la cual son caracterizados los polinomios ortogonales clásicos discretos (Charlier, Krawtchouk, Meixner, y Hahn polynomials). Además, se obtienen las representaciones en series hipergeométricas de la función de Stieltjes para un caso clásico discreto.

Palabras clave: Función de Stieltjes, polinomios ortogonales clásicos discretos, representación en series hipergeométricas.

2010 AMS Mathematics Subject Classification. Primary 33C45, 33C47; Secondary 42C05, 33C20, 33C45, 33D45.

1. Introduction

It is well known that the discrete classical orthogonal polynomials are introduced as a class of orthogonal polynomials associated with discrete moment functionals. They satisfy a hypergeometric type difference equation; their finite differences constitute an orthogonal polynomial family; they can be expressed by a Rodrigues-type formula; their associated orthogonalizing weights satisfy a Pearson-type difference equation. In general, these properties, among others, characterize the classical orthogonal polynomials (see for instance [6, 10, 11]).

Apart from this, in [4] the authors proved the following characterization: a discrete measure is classical if and only if the Stieltjes function defined by

$$S(z) \equiv \sum_{k\geq 0} \frac{m_k}{[z]_{k+1}}, \quad z \notin \Omega, \tag{1}$$

in terms of falling factorials $[x]_n \equiv x(x-1)\cdots(x-n+1)$ for discrete classical orthogonal polynomials, is solution of a non-homogeneous version of the corresponding Pearson-type difference equation. Here, m_k denotes the generalized moment of order k defined as

$$m_k \equiv \sum_{0\leq x\leq|\Omega|-1} [x]_k \, \rho(x), \quad k = 0, 1, \ldots, \tag{2}$$

where $\rho(x)$ is the discrete classical weight function with positive jumps at $x \in \Omega \subset \mathbb{R}$ and $|\Omega|$ denotes the number of elements in Ω (see [1, pp. 116-117] for more details).

Curiously, in [3] the authors had already extended this result. They introduced the q-analogue of the Stieltjes function, given by

$$S_q(z) \equiv \sum_{k\geq 0} \frac{u_k^q}{q^k \, [z]_q^{(k+1)}}, \quad |q| < 1, \tag{3}$$

in terms of q-falling factorials, defined by

$$[s]_q^{(k)} \equiv \prod_{0\leq j\leq k-1} \frac{q^{s-j}-1}{q-1}, \quad \text{for} \quad k > 0, \quad \text{and} \quad [s]_q^{(0)} = 1.$$

Here u_k^q denotes the q-moment of order k given by

$$u_k^q \equiv \sum_{0\leq s\leq N} [s]_q^{(k)} \, \rho_q(s) \triangle x(s-1/2), \quad k = 0, 1, \ldots, \quad \triangle f(s) = f(s+1) - f(s),$$

being $\rho_q(s)$ the weight function corresponding to the q-orthogonal polynomials (q-*Charlier*, q-*Krawtchouk*, q-*Meixner* and q-*Hahn*). Moreover, in [3] was checked that (3) can be rewritten as

$$S_q(z) = \sum_{0\leq s\leq N-1} \frac{\rho_q(s) \triangle x(s-1/2)}{x(z) - x(s)}. \tag{4}$$

The purpose of this work is to get a result analogue to (4). Thus, as consequence we deduce the hypergeometric series representations for the Stieltjes function for one discrete classical case.

2. Main results

In this section the main results of this contribution are stated and proved.

Definition 1. The ordinary hypergeometric series [2, 5, 8, 11] with variable z is defined by

$$ {}_rF_s\left(\begin{array}{c} a_1,\dots,a_r \\ \\ b_1,\dots,b_s \end{array}\middle|\, z\right) \equiv \sum_{k\geq 0} \frac{(a_1,\dots,a_r)_k}{(b_1,\dots,b_s)_k}\frac{z^k}{k!}, \tag{5} $$

where

$$ (a_1,\dots,a_r)_k \equiv \prod_{1\leq i\leq r} (a_i)_k $$

and $(\cdot)_n$ denotes the *Pochhammer symbol* [5], also called the shifted factorial, defined by

$$ (x)_n \equiv \prod_{0\leq j\leq n-1} (x+j)\,, \quad n\geq 1, \quad (x)_0 = 1. $$

Moreover, $\{a_i\}_{i=1}^r$ and $\{b_j\}_{j=1}^s$ are complex numbers subject to the condition that $b_j \neq -n$ with $n \in \mathbb{N}\setminus\{0\}$ for $j = 1, 2, \dots, s$.

Indeed, here we will use the *Chu-Vandermonde identity* [8], given by

$$ {}_2F_1\left(\begin{array}{c} -n, b \\ \\ c \end{array}\middle|\, 1\right) = \frac{(c-b)_n}{(c)_n}, \quad n = 0, 1, 2, \dots \tag{6} $$

The results of this paper are valid for the discrete classical orthogonal polynomials (Charlier, Krawtchouk, Meixner, and Hahn polynomials), which are defined by [1, pp. 116-117]

$$ C_n^{\mu}(x) \equiv (-\mu)^n\, {}_2F_1\left(\begin{array}{c} -n, -x \\ \\ - \end{array}\middle|\, -\mu^{-1}\right), \quad \mu > 0, $$

$$ M_n^{\gamma,\mu}(x) \equiv \frac{(\gamma)_n\,\mu^n}{(\mu-1)^n}\, {}_2F_1\left(\begin{array}{c} -n, -x \\ \\ \gamma \end{array}\middle|\, 1-\mu^{-1}\right), \quad \gamma > 0, \quad 0 < \mu < 1, $$

$$ K_n^{p,N}(x) \equiv \frac{(-p)^n\, N!}{(N-n)!}\, {}_2F_1\left(\begin{array}{c} -n, -x \\ \\ -N \end{array}\middle|\, p^{-1}\right), \quad 0 < p < 1, \quad n \leq N-1, $$

$$ \begin{aligned} h_n^{\alpha,\beta,N}(x) &\equiv \frac{(1-N)_n\,(\beta+1)_n}{(\alpha+\beta+n+1)_n}\, {}_3F_2\left(\begin{array}{c} -x, \alpha+\beta+n+1, -n \\ \\ 1-N, \beta+1 \end{array}\middle|\, 1\right), \\ \alpha, \beta &\geq -1, \quad n \leq N-1, \end{aligned} \tag{7} $$

respectively, which satisfy the orthogonality condition

$$\langle P_n, P_m \rangle = d_n^2 \delta_{mn}, \quad m, n = 0, \ldots, |\Omega|,$$

where

	d_n^2	$\rho(x)$	Ω
$C_n^{\mu}(x)$	$n!\mu^n$	$\dfrac{e^{-\mu}\mu^x}{\Gamma(x+1)}$	$[0,+\infty)$
$M_n^{\gamma,\mu}(x)$	$\dfrac{n!(\gamma)_n \mu^n}{(1-\mu)^{\gamma+2n}}$	$\dfrac{\mu^x \Gamma(\gamma+x)}{\Gamma(\gamma)\Gamma(x+1)}$	$[0,+\infty)$
$K_n^{p,N}(x)$	$\dfrac{n!N!p^n(1-p)^n}{(N-n)!}$	$\dfrac{N!p^x(1-p)^{N-x}}{\Gamma(N+1-x)\Gamma(x+1)}$	$[0,N]$
$h_n^{\alpha,\beta,N}(x)$	dh_n^2	$\dfrac{\Gamma(N+\alpha-x)\Gamma(\beta+x+1)}{\Gamma(N-x)\Gamma(x+1)}$	$[0,N-1]$

and

$$dh_n^2 = \frac{n!\Gamma(\alpha+n+1)\Gamma(\beta+n+1)\Gamma(\alpha+\beta+N+n+1)}{(\alpha+\beta+2n+1)(N-n-1)!\Gamma(\alpha+\beta+n+1)\Gamma(\alpha+\beta+n+1)_n^2}.$$

Theorem 1. *Let $\rho(x)$ be the jump function with positive jumps at $x \in \Omega \subset \mathbb{R}$ for a discrete classical measure. Then, the Stieltjes function $S(x)$ defined by (1) can be rewritten as*

$$S(z) = \sum_{0 \le x \le |\Omega|-1} \frac{\rho(x)}{z-x}. \tag{8}$$

Proof. In fact, taking (2) and

$$[z]_{k+1} = z[z-1]_k \tag{9}$$

into account, we deduce

$$S(z) = z^{-1} \sum_{0 \le x \le |\Omega|-1} \rho(x) \sum_{k \ge 0} \frac{[x]_k}{[z-1]_k}.$$

Since $[x]_k = (-1)^k(-x)_k$, we have

$$\begin{aligned} S(z) &= z^{-1} \sum_{0 \le x \le |\Omega|-1} \rho(x) \sum_{k \ge 0} \frac{(-x)_k}{(-z+1)_k} \\ &= z^{-1} \sum_{0 \le x \le |\Omega|-1} \rho(x)\, {}_2F_1\left(\begin{array}{c} -x, 1 \\ \\ -z+1 \end{array} \middle| 1 \right). \end{aligned}$$

Using the Chu-Vandermonde identity (6), we obtain

$$S(z) = z^{-1} \sum_{0 \le x \le |\Omega|-1} \rho(x) \frac{(-z)_x}{(-z+1)_x}. \tag{10}$$

Then, by simplifying, we obtain the desired result. This completes the proof. □

Observe that the Stieltjes function (8) is connected with the function of the second kind [11, eq. 79] on uniform lattice, for the specific case $n=0$. For this case we have

$$Q_0\left(z\right)=\frac{B_0}{\rho(z)}\sum_{0\leq x\leq|\Omega|-1}\frac{\rho(s)}{s-z}.$$

Therefore $S\left(z\right)=-Q_0\left(z\right)\rho(z)/B_0$, where B_0 is defined in [11]. Notice also that the Stieltjes function (8) is a rational function with poles in $[0,|\Omega|-1]$, which reveals its singularities.

In this paper we will work, in particular, with the *discrete Hahn polynomials* [11, 12, 13] of order n, $n=0,1,\ldots,N-1$, defined by

$$\begin{aligned}\tilde{h}_n^{\mu,\nu,N}\left(x\right)&\equiv\left(N+\nu-1\right)_n\left(N-1\right)_n\\&\times\ {}_3F_2\left(\begin{array}{c|c}-n,-x,2N+\mu+\nu-n-1 & \\ & 1\\ N+\nu-1,N-1 & \end{array}\right),\quad\mu,\nu>-1.\end{aligned}\tag{11}$$

These polynomials satisfy the following orthogonal condition

$$\sum_{0\leq k\leq N}\tilde{h}_m^{\mu,\nu,N}\left(k\right)\tilde{h}_n^{\mu,\nu,N}\left(k\right)\rho_n\left(k\right)=d_n^2\delta_{m,n},\quad 0\leq m,n\leq N-1,$$

where $\delta_{m,n}$ denotes the Dirac function. Otherwise,

$$\rho_n\left(x\right)=\frac{1}{\Gamma\left(x+1\right)\Gamma\left(x+\mu+1\right)\Gamma\left(N+\nu-x\right)\Gamma\left(N-n-x\right)},\tag{12}$$

and

$$\begin{aligned}d_n^2=&\frac{\Gamma\left(2N+\mu+\nu-n\right)}{\left(2N+\mu+\nu-2n-1\right)\Gamma\left(N+\mu+\nu-n\right)}\\&\times\frac{1}{\Gamma\left(N+\mu-n\right)\Gamma\left(N+\nu-n\right)\Gamma\left(n+1\right)\Gamma\left(N-n\right)}.\end{aligned}$$

Notice that there is a simple connection between the polynomials $\tilde{h}_n^{\mu,\nu,N}\left(x\right)$ and the Hahn polynomials (7) used in [4]; see [11, pp. 119-120] for more details.

Lemma 1. *Let x, μ, ν and N be the parameters of the discrete Hahn polynomials $\tilde{h}_n^{\mu,\nu,N}\left(x\right)$, such that, μ, $\nu>-1$, $n\leq N$ and $x\in[0,N-1]$. Then, the generalized moment of order k corresponding to these polynomials has the form*

$$m_k=m_0\frac{\left(-1\right)^k\left(1-N-\nu\right)_k\left(1-N\right)_k}{\left(2-2N-\mu-\nu\right)_k},\tag{13}$$

where

$$m_0=\frac{\left(N+\mu+\nu\right)_{N-1}}{\Gamma\left(N+\mu\right)\Gamma\left(N+\nu\right)\left(N-1\right)!}.$$

Proof. In fact, using (2) and (12) we get

$$\begin{aligned} m_k &= \sum_{0\le x\le N-1} \frac{[x]_k}{\Gamma(x+1)\Gamma(N-x)\Gamma(x+\mu+1)\Gamma(N+\nu-x)} \\ &= \sum_{k\le x\le N-1} \frac{1}{(x-k)!(N-x-1)!\Gamma(x+\mu+1)\Gamma(N+\nu-x)}. \end{aligned}$$

Clearly,

$$\begin{aligned} m_k &= \sum_{0\le n\le N-k-1} \frac{1}{n!(N-k-1-n)!\Gamma(\mu+k+n+1)\Gamma(N+\nu-k-n)} \\ &= \frac{1}{(N-k-1)!} \sum_{0\le n\le N-k-1} \frac{\binom{N-k-1}{n}}{\Gamma(\mu+k+n+1)\Gamma(N+\nu-k-n)}. \end{aligned}$$

On the other hand, from

$$\frac{1}{\Gamma(\mu+k+n+1)} = \frac{(-1)^{N-k-1-n}(1-N-\mu)_{N-k-1-n}}{\Gamma(N+\mu)}$$

and

$$\frac{1}{\Gamma(N+\nu-k-n)} = \frac{(-1)^{n}(1-N-\nu+k)_{n}}{\Gamma(N+\nu-k)}$$

we deduce

$$\begin{aligned} m_k = {} & \frac{(-1)^{N-k-1}[\Gamma(N+\mu)\Gamma(N+\nu-k)]^{-1}}{(N-k-1)!} \\ & \times \sum_{0\le n\le N-k-1} \binom{N-k-1}{n} (1-N-\nu+k)_n (1-N-\mu)_{N-k-1-n}. \end{aligned}$$

Then, applying [7]

$$(x+y)_n = \sum_{0\le k\le n} \binom{n}{k} (x)_k (y)_{n-k}, \quad n\in\mathbb{N},$$

we obtain

$$m_k = \frac{(-1)^{N-k-1}(-N-\mu-\nu-N+k+1+1)_{N-k-1}}{\Gamma(N+\mu)\Gamma(N+\nu-k)(N-k-1)!}.$$

Thus, using the identity

$$(-x)_k = (-1)^k (x-k+1)_k,$$

we obtain

$$m_k = \frac{(N+\mu+\nu)_{N-k-1}}{\Gamma(N+\mu)\Gamma(N+\nu-k)(N-k-1)!},$$

so that

$$m_0 = \frac{(N+\mu+\nu)_{N-1}}{\Gamma(N+\mu)\,\Gamma(N+\nu)\,(N-1)!}.$$

Finally, taking

$$(N+\mu+\nu)_{N-k-1} = \frac{(-1)^k\,(N+\mu+\nu)_{N-1}}{(2-2N-\mu-\nu)_k},$$

$$\frac{1}{\Gamma(N+\nu-k)} = \frac{(-1)^k\,(1-N-\nu)_k}{\Gamma(N+\nu)},$$

and

$$\frac{1}{(N-k-1)!} = \frac{(-1)^k\,(1-N)_k}{(N-1)!}$$

into account, we deduce (13). This completes the proof. □

Theorem 2. *Let $\rho(x)$ be the jump function with positive jumps at $x \in [0, N-1]$ for the discrete Hahn polynomials $\tilde{h}_n^{\mu,\nu,N}(x)$, with μ, $\nu > -1$ and $n \le N$. Then, the Stieltjes function $S^{\tilde{h}}(x)$ defined by (1) can be rewritten as follows:*

$$\begin{aligned} S^{\tilde{h}}(z) &= \frac{m_0}{z}\,{}_3F_2\left(\left.\begin{array}{c} 1-N-\nu, 1-N, 1 \\ \\ 2-2N-\mu-\nu, 1-z \end{array}\right| 1\right) \\ &= \frac{1}{\Gamma(N)\,\Gamma(\mu+1)\,\Gamma(N+\nu)\,z}\,{}_3F_2\left(\left.\begin{array}{c} 1-N-\nu, 1-N, -z \\ \\ \mu+1, 1-z \end{array}\right| 1\right). \end{aligned}$$

Proof. According to

$$\frac{1}{\Gamma(N-k)} = \frac{(-1)^k\,(1-N)_k}{\Gamma(N)},$$

$$\frac{1}{\Gamma(k+\mu+1)} = \frac{1}{\Gamma(\mu+1)\,(\mu+1)_k},$$

and

$$\frac{1}{\Gamma(N+\nu-k)} = \frac{(-1)^k\,(1-N-\nu)_k}{\Gamma(N+\nu)},$$

we have that $\rho(k) = \rho_0(k)$ can be rewritten as

$$\rho(k) = \frac{1}{\Gamma(N)\,\Gamma(\mu+1)\,\Gamma(N+\nu)}\,\frac{(1-N-\nu)_k\,(1-N)_k}{(\mu+1)_k\,(1)_k}.$$

As a consequence, from (10) we have

$$S^{\tilde{h}}(z) = \frac{1}{\Gamma(N)\,\Gamma(\mu+1)\,\Gamma(N+\nu)\,z} \times \sum_{k\ge 0} \frac{(1-N-\nu)_k\,(1-N)_k\,(-z)_k}{(\mu+1)_k\,(1-z)_k}\,\frac{1^k}{k!}. \qquad (14)$$

In addition, using (9) and (1) we deduce

$$S^{\tilde{h}}(z) = z^{-1} \sum_{k \geq 0} (-1)^k \frac{m_k}{(1-z)_k}.$$

Then, taking Lemma 1 into account, we obtain

$$S^{\tilde{h}}(z) = \frac{m_0}{z} \sum_{k \geq 0} \frac{(1-N-\nu)_k (1-N)_k (1)_k}{(2-2N-\mu-\nu)_k (1-z)_k} \frac{1^k}{k!}. \tag{15}$$

Now it is only necessary to prove that (14) and (15) coincide, for which it is enough to take into account the following transformation formula [9, pp. 93]

$${}_3F_2\left(\left.\begin{array}{c} a, b, e+n-1 \\ \\ d, e \end{array}\right| 1\right) = \frac{\Gamma(d)\Gamma(d-a-b)}{\Gamma(d-a)\Gamma(d-b)}\, {}_3F_2\left(\left.\begin{array}{c} a, b, 1-n \\ \\ a+b-d+1, e \end{array}\right| 1\right),$$

for $a = 1-N-\nu$, $b = 1-N$, $e = 1-z$, $n = 0$ and $d = \mu+1$. This completes the proof. $\square$

Acknowledgements

The authors are grateful to the ClaveMat project, supported by the European Union: www.clavemat.com.

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Recibido en marzo de 2017. Aceptado para publicación en junio de 2017.

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