Set–theoretical entropies of generalized shifts

Entropías conjuntistas de cambios generalizados

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Abstract. In the following text for arbitrary X with at least two elements, nonempty set Γ and self-map $\varphi : \Gamma \to \Gamma$ we prove the set-theoretical entropy of generalized shift $\sigma_\varphi : X^\Gamma \to X^\Gamma$ ($\sigma_\varphi((x_\alpha)_{\alpha\in\Gamma}) = (x_{\varphi(\alpha)})_{\alpha\in\Gamma}$ (for $(x_\alpha)_{\alpha\in\Gamma} \in X^\Gamma$)) is either zero or infinity, moreover it is zero if and only if φ is quasi-periodic.

We continue our study on contravariant set-theoretical entropy of generalized shift and motivate the text using counterexamples dealing with algebraic, topological, set-theoretical and contravariant set-theoretical positive entropies of generalized shifts.

Keywords: Bounded map, Contravariant set-theoretical entropy, Quasi-periodic, Set-theoretical entropy.

Resumen. En el siguiente texto, para X arbitraria con al menos dos elementos, Γ un conjunto no vacío y una función $\phi : \Gamma \to \Gamma$, demostramos que la entropía conjuntista del cambio generalizado $\sigma_\varphi : X^\Gamma \to X^\Gamma$ ($\sigma_\varphi((x_\alpha)_{\alpha\in\Gamma}) = (x_{\varphi(\alpha)})_{\alpha\in\Gamma}$ (para $(x_\alpha)_{\alpha\in\Gamma} \in X^\Gamma$)) es cero o infinito, además es cero si y solo si φ es casi periódica.

Continuamos nuestro estudio sobre la entropía conjuntista contravariante de cambios generalizados y motivamos este escrito usando contraejemplos que tratan con entropias positivas de cambios generalizados algebraicas, topológicas, conjuntistas y conjuntistas contravariantes.

Palabras claves: función acotada, entropía conjuntista contravariante, casi periodicidad, entropía conjuntista.

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1. Introduction

Amongst the most powerful tools in ergodic theory and dynamical systems we may mention one-sided shift $\{1,\dots,k\}^{\mathbb{N}} \underset{(x_n)_{n\geq 1}\mapsto (x_{n+1})_{n\geq 1}}{\to} \{1,\dots,k\}^{\mathbb{N}}$ and two-sided shift $\{1,\dots,k\}^{\mathbb{Z}} \underset{(x_n)_{n\in\mathbb{Z}}\mapsto (x_{n+1})_{n\in\mathbb{Z}}}{\to} \{1,\dots,k\}^{\mathbb{Z}}$ [8]. Now suppose X is an arbitrary set with at least two elements, Γ is a nonempty set, and $\varphi:\Gamma\to\Gamma$ is arbitrary, then $\sigma_\varphi : X^\Gamma \to X^\Gamma$ with $\sigma_\varphi((x_\alpha)_{\alpha\in\Gamma}) = (x_{\varphi(\alpha)})_{\alpha\in\Gamma}$ (for $(x_\alpha)_{\alpha\in\Gamma}\in X^\Gamma$) is a *generalized shift*. Generalized shifts have been introduced for the first time in [3]. It's evident that for self-map $\varphi:\Gamma\to\Gamma$ and generalized shift $\sigma_\varphi : X^\Gamma\to X^\Gamma$ if X has a group (resp. vector space, topological) structure, then $\sigma_\varphi : X^\Gamma\to X^\Gamma$ is a group homomorphism (resp. linear map, continuous (in which X^Γ considered under product topology)), so many dynamical [4] and non-dynamical [7] properties of generalized shifts have been studied in several texts. In this text our main aim is to study set-theoretical and contravariant set-theoretical entropy of generalized shifts. We complete our investigations with a comparative study regarding set-theoretical, contravariant set-theoretical, topological and algebraic entropies of generalized shifts.

For self-map $g : A\to A$ and $x,y\in A$ let $x\leq_g y$ if and only if there exists $n\geq 0$ with $g^n(x)=y$, then $(A,\leq_g)$ is a preordered (reflexive and transitive) set. Note that for set A by $|A|$ we mean the cardinality of A if it is finite and ∞ otherwise.

Although one may obtain the following lemma using [7], we establish it here directly.

Note 1.1. (Bounded self-map, Quasi-periodic self-map) For self-map $g : A\to A$ the following statements are equivalent (consider preordered set $(A,\leq_g)$):

1. there exists $N\geq 1$ such that for all totally preordered subset I of A (reflexive, transitive and for all $x,y\in A$ we have $x\leq_g y$ or $y\leq_g x$) we have $|I|\leq N$ (i.e., $g:A\to A$ is *bounded* [7]),
2. $\sup\{|\{g^n(x):n\geq 0\}| : x\in A\}<\infty$,
3. there exists $n>m\geq 1$ with $g^n=g^m$ (g is *quasi-periodic*).

Proof. "(1) $\Rightarrow$ (2)" Suppose there exists $N\geq 1$ such that for all totally preordered subset I of A we have $|I|\leq N$. Choose $x\in A$, then $\{g^n(x):n\geq 0\}$ is a totally preordered subset of A, thus $|\{g^n(x):n\geq 0\}|\leq N$, hence $\sup\{|\{g^n(y):n\geq 0\}| : y\in A\}\leq N<\infty$.

"(2) $\Rightarrow$ (3)" Suppose $\sup\{|\{g^n(x):n\geq 0\}| : x\in A\} = N<\infty$, then for all $x\in A$ we have $\{g^n(x):n\geq 0\}=\{x,g(x),\dots,g^{N-1}(x)\}$ and there exists $n_x\in\{0,\dots,N-1\}$ with $g^N(x)=g^{n_x}(x)$, thus $g^{N-n_x}(g^N(x))=g^N(x)$ and

$$\forall z\in g^N(A)\ \exists i\in\{1,\dots,N\}\ (g^i(z)=z)$$

so for all $z\in g^N(A)$ we have $g^{N!}(z)=z$, thus for all $x\in A$ we have $g^{N!+N}(x)=g^N(x)$.

"(3) $\Rightarrow$ (1)" Suppose there exist $n > m \geq 1$ with $g^n = g^m$ and I is a totally preordered subset of A, choose distinct $x_1, \ldots, x_k \in I$ and suppose $x_1 \leq_g x_2 \leq_g \cdots \leq_g x_k$. For all $i \in \{1, \ldots, k\}$ there exists $p_i \geq 0$ with $x_i = g^{p_i}(x_1)$, so $\{x_1, \ldots, x_k\} \subseteq \{g^i(x_1) : i \geq 0\} = \{g^i(x_1) : i \in \{0, \ldots, n\}\}$ and $k \leq n+1$. Hence $|I| \leq n+1$ which completes the proof. □

Convention. In the following text suppose X is an arbitrary set with at least two elements, Γ is a nonempty set, and $\varphi : \Gamma \to \Gamma$ is arbitrary.

1.1. Background on set-theoretical entropy

For self-map $g : A \to A$ and $a \in A$, the set $\{g^n(a) : n \geq 0\}$ is the *orbit* of a, we say $a \in A$ is a *wandering point* (or *non-quasi periodic point*) of g, if $\{g^n(a) : n \geq 0\}$ is infinite, or equivalently $\{g^n(a)\}_{n\geq 1}$ is a one-to-one sequence. We denote the collection of all wandering points of $g : A \to A$ with $W(g)$.

For $g : A \to A$ denote the *infinite orbit number of* g by $\mathfrak{o}(g)$ and define it with $\sup(\{0\} \cup \{k \geq 1 : \exists a_1, \ldots, a_k \in W(g)\ (\{g^n(a_1)\}_{n\geq 1}, \ldots, \{g^k(a)\}_{n\geq 1}$ are pairwise disjoint sequences$)\})$, i.e. $\mathfrak{o}(g) = \sup(\{0\} \cup \{k \geq 1 :$ there exists k pairwise disjoint infinite orbits$\})$. So $W(g) \neq \varnothing$ if and only if $\mathfrak{o}(g) \geq 1$. On the other hand for finite subset D of A the following limit exists [2]:

$$\mathrm{ent}_{\mathrm{set}}(g, D) = \lim_{n\to\infty} \frac{|D \cup g(D) \cup \cdots \cup g^{n-1}(D)|}{n}.$$

Now we call $\sup\{\mathrm{ent}_{\mathrm{set}}(g, D) : D$ is a finite subset of $A\}$ the *set-theoretical entropy of* g and denote it with $\mathrm{ent}_{\mathrm{set}}(g)$. Moreover $\mathrm{ent}_{\mathrm{set}}(g) = \mathfrak{o}(g)$ [2].

1.2. Background on contravariant set-theoretical entropy

Suppose self-map $g : A \to A$ is onto and finite fibre (i.e., for all $a \in A$, $g^{-1}(a)$ is finite), then for finite subset D of A the following limit exists [5]:

$$\mathrm{ent}_{\mathrm{cset}}(g, D) = \lim_{n\to\infty} \frac{|D \cup g^{-1}(D) \cup \cdots \cup g^{-(n-1)}(D)|}{n}$$

Now let $\mathrm{ent}_{\mathrm{cset}}(g) := \sup\{\mathrm{ent}_{\mathrm{cset}}(g, D) : D$ is a finite subset of $A\}$. If $k : A \to A$ is an arbitrary finite fibre map, then for *surjective cover* of k, i.e. $\mathrm{sc}(k) := \bigcap\{k^n(A) : n \geq 1\}$, the map $k\restriction_{\mathrm{sc}(k)}: \mathrm{sc}(k) \to \mathrm{sc}(k)$ is an onto finite fibre map and $\mathrm{ent}_{\mathrm{cset}}(k) := \mathrm{ent}_{\mathrm{cset}}(k\restriction_{\mathrm{sc}(k)})$ is the *contravariant set-theoretical entropy* of k. Moreover we say $\{x_n\}_{n\geq 1}$ is a $k-$*anti-orbit sequence* (or simply anti-orbit sequence) if for all $n \geq 1$ we have $k(x_{n+1}) = x_n$ and define *infinite anti-orbit number of* k as $\mathfrak{a}(k) = \sup(\{0\} \cup \{j \geq 1 :$ there exists j pairwise disjoint infinite anti-orbits$\})$. Moreover $\mathrm{ent}_{\mathrm{cset}}(k) = \mathfrak{a}(k)$ [5].

2. Set-theoretical entropy of $\sigma_\varphi : X^\Gamma \to X^\Gamma$

In this section we prove that for generalized shift $\sigma_\varphi : X^\Gamma \to X^\Gamma$, $\mathrm{ent}_{\mathrm{set}}(\sigma_\varphi) \in \{0,\infty\}$ and $\mathrm{ent}_{\mathrm{set}}(\sigma_\varphi) = 0$ if and only if φ is quasi-periodic.

Lemma 2.1. *If $W(\varphi) \neq \varnothing$, then $W(\sigma_\varphi) \neq \varnothing$.*

Proof. Consider distinct points $p, q \in X$ and $\theta \in W(\varphi)$, thus $(\varphi^n(\theta))_{n\geq 0}$ is a one-to-one sequence. Let:

$$x_\alpha := \begin{cases} p & \alpha \in \{\varphi^{2^n}(\theta) : n \geq 1\}\,, \\ q & \text{otherwise}\,, \end{cases}$$

then $(x_\alpha)_{\alpha\in\Gamma} \in W(\sigma_\varphi)$, otherwise there exists $s > t \geq 1$ such that $\sigma_\varphi^s((x_\alpha)_{\alpha\in\Gamma}) = \sigma_\varphi^t((x_\alpha)_{\alpha\in\Gamma})$, thus $x_{\varphi^s(\alpha)} = x_{\varphi^t(\alpha)}$ for all $\alpha \in \Gamma$. In particular, $x_{\varphi^{s+i}(\theta)} = x_{\varphi^{t+i}(\theta)}$ for all $i \geq 0$. Choose $j \geq 1$ with $j + s \in \{2^n : n \geq 1\}$. We have the following cases:

Case 1: $j+t \notin \{2^n : n \geq 1\}$. In this case we have $p = x_{\varphi^{s+j}(\theta)} = x_{\varphi^{t+j}(\theta)} = q$, which is a contradiction.

Case 2: $j+t \in \{2^n : n \geq 1\}$. In this case using $j+t > j+s \in \{2^n : n \geq 1\}$ we have $j + t \geq 3$. There exist $k \geq 1$ and $l \geq 2$ with $j + s = 2^k$ and $j + t = 2^l$. Let $i = 2j + s$, then $i + s = 2(j + s) = 2^{k+1} \in \{2^n : n \geq 1\}$ and $i + t = 2^l + 2^k = 2^l(1 + 2^{k-l}) \notin \{2^n : n \geq 1\}$ (note that $k > l$ and $1 + 2^{k-l}$ is odd). So $p = x_{\varphi^{s+i}(\theta)} = x_{\varphi^{t+i}(\theta)} = q$, which is a contradiction.

Using the above two cases, we have $(x_\alpha)_{\alpha\in\Gamma} \in W(\sigma_\varphi)$. □

Lemma 2.2. *If $W(\varphi) \neq \varnothing$, then $\mathfrak{o}(\sigma_\varphi) = \infty$.*

Proof. Consider $\theta \in \Gamma$ with infinite $\{\varphi^n(\theta) : n \geq 0\}$ and choose distinct $p, q \in X$, thus $(\varphi^n(\theta))_{n\geq 0}$ is a one-to-one sequence. For $s \geq 1$ let:

$$x^s_\alpha := \begin{cases} p & \alpha \in \{\varphi^n(\theta) : \exists k \geq 0\, (ks + \frac{k(k+1)}{2} < i \leq ks + \frac{k(k+1)}{2} + s)\}\,, \\ q & \text{otherwise}\,, \end{cases}$$

so:

$$(x^s_{\varphi(\theta)}, x^s_{\varphi^2(\theta)}, x^s_{\varphi^3(\theta)}, \cdots)$$
$$= (\underbrace{p, \cdots, p}_{s \text{ times}}, q, \underbrace{p, \cdots, p}_{s \text{ times}}, q, q, \underbrace{p, \cdots, p}_{s \text{ times}}, q, q, q, \underbrace{p, \cdots, p}_{s \text{ times}}, q, q, q, q, \cdots).$$

Let $x^s := (x^s_\alpha)_{\alpha\in\Gamma}$. Now we have the following steps:

Step 1. For $s \geq 1$, the sequence $(\sigma^n_\varphi(x^s))_{n\geq 1}$ is one-to-one: Consider $j > i \geq 0$, then $i < js + \frac{j(j+1)}{2} + s$, so there exists $t \geq 1$ with $i+t = js + \frac{j(j+1)}{2} + s$ moreover

$$js + \frac{j(j+1)}{2} + s < \underbrace{js + \frac{j(j+1)}{2} + s + (j-i)}_{j+t} < (j+1)s + \frac{(j+1)(j+2)}{2}\,,$$

which show $x^s_{\varphi^{i+t}(\theta)} = p$ and $x^s_{\varphi^{j+t}(\theta)} = q$ and:

$$x^s_{\varphi^{i+t}(\theta)} \neq x^s_{\varphi^{j+t}(\theta)} \,. \tag{*}$$

Using (*) we have $\sigma^i_\varphi((x^s_\alpha)_{\alpha\in\Gamma}) \neq \sigma^j_\varphi((x^s_\alpha)_{\alpha\in\Gamma})$, thus $(\sigma^n_\varphi(x^s))_{n\geq 1}$ is a one-to-one sequence.

Step 2. $(\sigma^n_\varphi(x^1))_{n\geq1}$, $(\sigma^n_\varphi(x^2))_{n\geq1}$, $(\sigma^n_\varphi(x^3))_{n\geq1}, \ldots$ are pairwise disjoint sequences: consider $s \geq r \geq 1$ and $i, j \geq 0$ with $\sigma^i_\varphi(x^s) = \sigma^j_\varphi(x^r)$. Choose $m \geq 0$ with $i + m = is + \frac{i(i+1)}{2} + 1$, now we have:

$$\begin{aligned}
\sigma^i_\varphi(x^s) = \sigma^j_\varphi(x^r) &\Rightarrow (\forall \alpha \in \Gamma\ (x^s_{\varphi^i(\alpha)} = x^r_{\varphi^j(\alpha)}))\\
&\Rightarrow (\forall n \geq 0\ (x^s_{\varphi^{i+n}(\theta)} = x^r_{\varphi^{j+n}(\theta)}))\\
&\Rightarrow (\forall k \geq 0\ (x^s_{\varphi^{i+m+k}(\theta)} = x^r_{\varphi^{j+m+k}(\theta)}))\\
&\Rightarrow (\forall k \in \{0, \ldots, s-1\}\ (p = x^s_{\varphi^{i+m+k}(\theta)} = x^r_{\varphi^{j+m+k}(\theta)}))
\end{aligned}$$

using $x^r_{\varphi^{j+m}(\theta)} = x^r_{\varphi^{j+m+1}(\theta)} = \cdots = x^r_{\varphi^{j+m+s-1}(\theta)} = p$ and the way of definition of x^r we have $s \leq r$, thus $s = r$, and $\sigma^i_\varphi(x^s) = \sigma^j_\varphi(x^s)$ which leads to $i = j$ by Step 1.

Using the above two steps $(\sigma^n_\varphi(x^1))_{n\geq1}$, $(\sigma^n_\varphi(x^2))_{n\geq1}$, $(\sigma^n_\varphi(x^3))_{n\geq1}, \ldots$ are pairwise disjoint infinite sequences which leads to $\mathfrak{o}(\sigma_\varphi) = \infty$. □

Lemma 2.3. *Let $W(\varphi) = \varnothing$, and φ is not quasi-periodic, then $\mathfrak{o}(\sigma_\varphi) = \infty$.*

Proof. Since $W(\varphi) = \varnothing$, for all $\alpha \in \Gamma$, $\{\varphi^n(\alpha) : n \geq 0\}$ is finite. Since φ is not quasi-periodic we have $\sup\{|\{\varphi^n(\alpha) : n \geq 0\}| : \alpha \in \Gamma\} = \infty$. Thus there exist $\theta_1, \theta_2, \ldots \in \Gamma$ such that for all $i \geq 1$ the set $\{\theta_i, \varphi(\theta_i), \ldots, \varphi^i(\theta_i)\}$ has $i + 1$ elements, moreover for all $j \neq i$ we have $\{\theta_i, \varphi(\theta_i), \ldots, \varphi^i(\theta_i)\} \cap \{\theta_j, \varphi(\theta_j), \ldots, \varphi^j(\theta_j)\} = \varnothing$. For $n \geq 1$ suppose u_n is the nth prime number and choose distinct $p, q \in X$, now for $m \geq 1$ let:

$$x^m_\alpha = \begin{cases} p & \alpha \in \{\varphi^n(\theta_{u^t_m}) : t \geq 1, 1 \leq n < u^t_m\}\,, \\ q & \text{otherwise}\,, \end{cases}$$

so:

$$
\begin{aligned}
(p,q) &= (x^1_{\varphi(\theta_2)}, x^1_{\varphi^2(\theta_2)})\\
(p,p,p,q) &= (x^1_{\varphi(\theta_4)}, x^1_{\varphi^2(\theta_4)}, x^1_{\varphi^3(\theta_4)}, x^1_{\varphi^4(\theta_4)})\\
(p,p,p,p,p,p,p,q) &= (x^1_{\varphi(\theta_8)}, x^1_{\varphi^2(\theta_8)}, x^1_{\varphi^3(\theta_8)}, x^1_{\varphi^4(\theta_8)}, x^1_{\varphi^5(\theta_8)}, x^1_{\varphi^6(\theta_8)}, x^1_{\varphi^7(\theta_8)},\\
&\quad x^1_{\varphi^8(\theta_8)})\\
&\vdots\\
(p,p,q) &= (x^2_{\varphi(\theta_3)}, x^2_{\varphi^2(\theta_3)}, x^2_{\varphi^3(\theta_3)})\\
(p,p,p,p,p,p,p,p,q) &= (x^2_{\varphi(\theta_9)}, x^2_{\varphi^2(\theta_9)}, x^2_{\varphi^3(\theta_9)}, x^2_{\varphi^4(\theta_9)}, x^2_{\varphi^5(\theta_9)}, x^2_{\varphi^6(\theta_9)}, x^2_{\varphi^7(\theta_9)},\\
&\quad x^2_{\varphi^8(\theta_9)}, x^2_{\varphi^9(\theta_9)})\\
&\vdots\\
(p,p,p,p,q) &= (x^3_{\varphi(\theta_5)}, x^3_{\varphi^2(\theta_5)}, x^3_{\varphi^3(\theta_5)}, x^3_{\varphi^4(\theta_5)}, x^3_{\varphi^5(\theta_5)})\\
&\vdots\\
(\underbrace{p,\cdots,p}_{u_m-1 \text{ times}},q) &= (x^m_{\varphi(\theta_{u_m})}, x^m_{\varphi^2(\theta_{u_m})}, \cdots, x^m_{\varphi^{u_m}(\theta_{u_m})})\\
(\underbrace{p,\cdots,p}_{u_m^2-1 \text{ times}},q) &= (x^m_{\varphi(\theta_{u_m^2})}, x^m_{\varphi^2(\theta_{u_m^2})}, \cdots, x^m_{\varphi^{u_m^2}(\theta_{u_m^2})})\\
(\underbrace{p,\cdots,p}_{u_m^3-1 \text{ times}},q) &= (x^m_{\varphi(\theta_{u_m^3})}, x^m_{\varphi^2(\theta_{u_m^3})}, \cdots, x^m_{\varphi^{u_m^3}(\theta_{u_m^3})})\\
&\vdots
\end{aligned}
$$

For $m \geq 1$, let $x^m := (x^m_\alpha)_{\alpha\in\Gamma}$. Now we have the following steps:

Step 1. For $m \geq 1$, the sequence $(\sigma^n_\varphi(x^m))_{n\geq 1}$ is one-to-one: Consider $j \geq i \geq 1$ with $\sigma^i_\varphi(x^m) = \sigma^j_\varphi(x^m)$, choose $t, l \geq 1$ such that $j + l = u^t_m$, so:

$$
\begin{aligned}
\sigma^i_\varphi(x^m) = \sigma^j_\varphi(x^m) &\Rightarrow (\forall \alpha \in \Gamma\ (x^m_{\varphi^i(\alpha)} = x^m_{\varphi^j(\alpha)}))\\
&\Rightarrow (\forall k \geq 0\ \forall s \geq 1\ (x^m_{\varphi^{i+k}(\theta_{u^s_m})} = x^m_{\varphi^{j+k}(\theta_{u^s_m})}))\\
&\Rightarrow (x^m_{\varphi^{i+l}(\theta_{u^t_m})} = x^m_{\varphi^{j+l}(\theta_{u^t_m})} = x^m_{\varphi^{u^t_m}(\theta_{u^t_m})} = q)
\end{aligned}
$$

using $x^m_{\varphi^{i+l}(\theta_{u^t_m})} = q$ and $1 \leq i+l \leq j+l = u^t_m$ considering the way of definition of x^m we have $i + l = u^t_m = j + l$, thus $i = j$ and the sequence $(\sigma^n_\varphi(x^m))_{n\geq 1}$ is one-to-one.

Step 2. $(\sigma^n_\varphi(x^1))_{n\geq 1}$, $(\sigma^n_\varphi(x^2))_{n\geq 1}$, $(\sigma^n_\varphi(x^3))_{n\geq 1}, \ldots$ are pairwise disjoint sequences: Consider $r, m \geq 1$ and $i \geq j \geq 1$ with $\sigma^i_\varphi(x^m) = \sigma^j_\varphi(x^r)$. Choose $l, t \geq 1$ with $i + l = u^t_m - 1$, so:

$$
\begin{aligned}
\sigma_\varphi^i(x^m)=\sigma_\varphi^j(x^r) &\Rightarrow (\forall\alpha\in\Gamma\ (x^m_{\varphi^i(\alpha)}=x^r_{\varphi^j(\alpha)}))\\
&\Rightarrow (\forall k\geq 0\ \forall s\geq 1\ (x^m_{\varphi^{i+k}(\theta_{u^s_m})}=x^m_{\varphi^{j+k}(\theta_{u^s_r})}))\\
&\Rightarrow p=x^m_{\varphi^{i+l}(\theta_{u^t_m})}=x^r_{\varphi^{j+l}(\theta_{u^t_m})}\\
&\Rightarrow \varphi^{j+l}(\theta_{u^t_m})\in\{\varphi^v(u^w_r):w\geq 1,1\leq v<u^w_r\}\\
&\Rightarrow (\exists w\geq 1\ (\varphi^{j+l}(\theta_{u^t_m})\in\{\varphi^v(u^w_r):1\leq v<u^w_r\}))\\
&\Rightarrow (\exists w\geq 1\ \{\varphi^v(\theta_{u^t_m}):1\leq v<u^t_m\}\cap\{\varphi^v(u^w_r):1\leq v<u^w_r\})\\
&\qquad\qquad\qquad (\text{since } 1\leq j+l\leq i+l<u^t_m)\\
&\Rightarrow (\exists w\geq 1\ u^t_m=u^w_r)\ (\text{use the way of choosing } \theta_v\text{s})\\
&\Rightarrow u_m=u_r\ (\text{since } u_m \text{ and } u_r \text{ are prime numbers})\\
&\Rightarrow m=r
\end{aligned}
$$

thus $m=r$ and $\sigma_\varphi^i(x^m)=\sigma_\varphi^j(x^m)$ which leads to $i=j$ by Step 1.

By the above two steps $(\sigma_\varphi^n(x^1))_{n\geq1}$, $(\sigma_\varphi^n(x^2))_{n\geq1}$, $(\sigma_\varphi^n(x^3))_{n\geq1},\dots$ are pairwise disjoint infinite sequences which leads to $\mathfrak{o}(\sigma_\varphi)=\infty$. □

Theorem 2.4. *The following statements are equivalent:*

1. $W(\sigma_\varphi)=\varnothing$ *(i.e.,* $\mathfrak{o}(\sigma_\varphi)=\mathrm{ent}_{\mathrm{set}}(\sigma_\varphi)=0$*),*
2. φ *is quasi-periodic,*
3. $\mathrm{ent}_{\mathrm{set}}(\sigma_\varphi)<\infty$ *(i.e.,* $\mathfrak{o}(\sigma_\varphi)<\infty$*).*

Proof. "(1) ⇒ (2)": Suppose $W(\sigma_\varphi)=\varnothing$, thus by Lemma 2.1 we have $W(\varphi)=\varnothing$. Since $W(\sigma_\varphi)=\varnothing$, we have $\mathfrak{o}(\sigma_\varphi)=0$. Using $\mathfrak{o}(\sigma_\varphi)=0$, $W(\varphi)=\varnothing$ and Lemma 2.3, φ is quasi-periodic.

"(2) ⇒ (1)": If there exist $n>m\geq1$ with $\varphi^n=\varphi^m$, then for all $(x_\alpha)_{\alpha\in\Gamma}\in X^\Gamma$ we have $\sigma_\varphi^n((x_\alpha)_{\alpha\in\Gamma})=(x_{\varphi^n(\alpha)})_{\alpha\in\Gamma}=(x_{\varphi^m(\alpha)})_{\alpha\in\Gamma}=\sigma_\varphi^m((x_\alpha)_{\alpha\in\Gamma})$ which shows $(x_\alpha)_{\alpha\in\Gamma}\notin W(\sigma_\varphi)$.

"(3) ⇒ (2)": By Lemmas 2.2 and 2.3, if φ is not quasi-periodic, then $\mathfrak{o}(\sigma_\varphi)=\infty$. □

Corollary 2.5. *By Theorem 2.4 we have:*

$$\mathrm{ent}_{\mathrm{set}}(\sigma_\varphi)=\begin{cases}0 & \varphi \text{ is quasi}-\text{periodic}\,,\\ \infty & \text{otherwise}\,.\end{cases}$$

3. Contravariant set-theoretical entropy of $\sigma_\varphi : X^\Gamma \to X^\Gamma$

For $\alpha,\beta\in\Gamma$ let $\alpha\Re\beta$ if and only if there exists $n\geq1$ with $\varphi^n(\alpha)=\varphi^n(\beta)$. Then $\Re$ is an equivalence relation on Γ, moreover it's evident that $\alpha\Re\beta$ if and only if $\varphi(\alpha)\Re\varphi(\beta)$. In this section we prove that for all $x \in X^\Gamma$, $\sigma_\varphi^{-1}(x)$ is finite, if and only if either $\Gamma = \varphi(\Gamma)$ or "X and $\Gamma \setminus \varphi(\Gamma)$ are finite". Moreover if for all $x \in X^\Gamma$, $\sigma_\varphi^{-1}(x)$ is finite, then $\text{ent}_{\text{cset}}(\sigma_\varphi) \in \{0, \infty\}$ with $\text{ent}_{\text{cset}}(\sigma_\varphi) = 0$ if and only if there exists $n \geq 1$ such that $\varphi^n(\alpha)\Re\alpha$ for all $\alpha \in \Gamma$.

Remark 3.1. The generalized shift $\sigma_\varphi : X^\Gamma \to X^\Gamma$ is on-to-one (resp. onto) if and only if $\varphi : \Gamma \to \Gamma$ is onto (resp. one-to-one) [3, 2].

Remark 3.1. The generalized shift $\sigma_\varphi : X^\Gamma \to X^\Gamma$ is on-to-one (resp. onto) if and only if $\varphi : \Gamma \to \Gamma$ is onto (resp. one-to-one) [3, 2].

Note 3.2. Consider $\widetilde{\varphi} : \underset{[\alpha]_\Re \mapsto [\varphi(\alpha)]_\Re}{\frac{\Gamma}{\Re} \to \frac{\Gamma}{\Re}}$ and note that (for $\alpha \in \Gamma$ let $[\alpha]_\Re = \{y \in \Gamma : x\Re y\}$ and $\frac{\Gamma}{\Re} = \{[\lambda]_\Re : \lambda \in \Gamma\}$):

$$\mathfrak{f} : \underset{(x_\alpha)_{\alpha\in\Gamma} \mapsto (x_\alpha)_{[\alpha]_\Re \in \frac{\Gamma}{\Re}}}{\text{sc}(\sigma_\varphi) \to X^{\frac{\Gamma}{\Re}}}$$

is well-defined, since for $(x_\alpha)_{\alpha\in\Gamma} \in \text{sc}(\sigma_\varphi)$ and $\theta, \beta \in \Gamma$ with $\theta\Re\beta$, there exists $n \geq 1$ and $(y_\alpha)_{\alpha\in\Gamma} \in X^\Gamma$ with $\varphi^n(\theta) = \varphi^n(\beta)$ and $(y_{\varphi^n(\alpha)})_{\alpha\in\Gamma} = \sigma_\varphi^n((y_\alpha)_{\alpha\in\Gamma}) = (x_\alpha)_{\alpha\in\Gamma}$, thus $x_\theta = y_{\varphi^n(\theta)} = y_{\varphi^n(\beta)} = x_\beta$. Now we have:

1. $\mathfrak{f} : \text{sc}(\sigma_\varphi) \to X^{\frac{\Gamma}{\Re}}$ is one-to-one.
2. The following diagram commutes:

$$\begin{array}{ccc} \text{sc}(\sigma_\varphi) & \xrightarrow{\sigma_\varphi \restriction_{\text{sc}(\sigma_\varphi)}} & \text{sc}(\sigma_\varphi) \\ \mathfrak{f} \downarrow & & \downarrow \mathfrak{f} \\ X^{\frac{\Gamma}{\Re}} & \xrightarrow{\sigma_{\widetilde{\varphi}}} & X^{\frac{\Gamma}{\Re}} \end{array}$$

3. Using Remark 3.1, since $\widetilde{\varphi} : \frac{\Gamma}{\Re} \to \frac{\Gamma}{\Re}$ is one-to-one, $\sigma_{\widetilde{\varphi}} : X^{\frac{\Gamma}{\Re}} \to X^{\frac{\Gamma}{\Re}}$ is onto.

Lemma 3.3. *The generalized shift $\sigma_\varphi : X^\Gamma \to X^\Gamma$ is finite fibre, if and only if at least one of the following conditions hold:*

- *X and $\Gamma \setminus \varphi(\Gamma)$ are finite,*
- *$\Gamma = \varphi(\Gamma)$.*

Proof. First suppose for all $x \in X^\Gamma$, $\sigma_\varphi^{-1}(x)$ is finite and $\Gamma \neq \varphi(\Gamma)$. Choose $p \in X$, for all $q = (q_\alpha)_{\alpha\in\Gamma\setminus\varphi(\Gamma)} \in X^{\Gamma\setminus\varphi(\Gamma)}$ let:

$$x_\alpha^q := \begin{cases} q_\alpha & \alpha \in \Gamma \setminus \varphi(\Gamma)\,, \\ p & \text{otherwise}\,, \end{cases}$$

then $\sigma_\varphi((x_\alpha^q)_{\alpha\in\Gamma}) = (p)_{\alpha\in\Gamma}$. So $\underset{q \mapsto (x_\alpha^q)_{\alpha\in\Gamma}}{X^{\Gamma\setminus\varphi(\Gamma)} \to \sigma_\varphi^{-1}((p)_{\alpha\in\Gamma})}$ is one-to-one, using finiteness of $\sigma_\varphi^{-1}((p)_{\alpha\in\Gamma})$, $X^{\Gamma\setminus\varphi(\Gamma)}$ is finite too. Both sets $\Gamma \setminus \varphi(\Gamma), X$ are finite since $X^{\Gamma\setminus\varphi(\Gamma)}$ is finite, X has at least two elements and $\Gamma \setminus \varphi(\Gamma) \neq \varnothing$.

Conversely, if $\Gamma=\varphi(\Gamma)$, then by Remark 3.1, $\sigma_\varphi : X^\Gamma \to X^\Gamma$ is one-to-one, so for all $x\in X^\Gamma$ the set $\sigma_\varphi^{-1}(x)$ has at most one element and is finite. Now suppose X and $\Gamma\setminus\varphi(\Gamma)$ are finite. For all $(x_\alpha)_{\alpha\in\Gamma},(y_\alpha)_{\alpha\in\Gamma}\in X^\Gamma$ we have:

$$\begin{aligned}(y_\alpha)_{\alpha\in\Gamma}\in\sigma_\varphi^{-1}(\sigma_\varphi((x_\alpha)_{\alpha\in\Gamma})) &\Rightarrow \sigma_\varphi((y_\alpha)_{\alpha\in\Gamma})=\sigma_\varphi((x_\alpha)_{\alpha\in\Gamma})\\ &\Rightarrow (y_{\varphi(\alpha)})_{\alpha\in\Gamma}=(x_{\varphi(\alpha)})_{\alpha\in\Gamma}\\ &\Rightarrow \forall\beta\in\varphi(\Gamma)\; y_\beta=x_\beta\\ &\Rightarrow (y_\alpha)_{\alpha\in\Gamma}\in\{(z_\alpha)_{\alpha\in\Gamma}\in X^\Gamma:\forall\alpha\in\varphi(\Gamma)\; z_\alpha=x_\alpha\}\end{aligned}$$

Hence

$$|\sigma_\varphi^{-1}(\sigma_\varphi((x_\alpha)_{\alpha\in\Gamma}))|\leq|\{(z_\alpha)_{\alpha\in\Gamma}\in X^\Gamma:\forall\alpha\in\varphi(\Gamma)z_\alpha=x_\alpha\}|=|X^{\Gamma\setminus\varphi(\Gamma)}|<\infty.$$

Thus for all $w\in X^\Gamma$, $\sigma_\varphi^{-1}(w)$ is finite. □

Lemma 3.4. *If $\sigma_\varphi : X^\Gamma\to X^\Gamma$ is finite fibre, then $\sigma_{\widetilde{\varphi}} : X^{\frac{\Gamma}{\Re}}\to X^{\frac{\Gamma}{\Re}}$ is finite fibre.*

Proof. Suppose for all $w\in X^\Gamma$, $\sigma_\varphi^{-1}(w)$ is finite, then by Lemma 3.3 we have the following cases:

Case 1. $\Gamma=\varphi(\Gamma)$: In this case we have $\widetilde{\varphi}(\frac{\Gamma}{\Re})=\{[\varphi(\alpha)]_\Re:\alpha\in\Gamma\}=\{[\alpha]_\Re:\alpha\in\varphi(\Gamma)\}=\{[\alpha]_\Re:\alpha\in\Gamma\}=\frac{\Gamma}{\Re}$.

Case 2. X and $\Gamma\setminus\varphi(\Gamma)$ are finite: For all $A\in\frac{\Gamma}{\Re}\setminus\widetilde{\varphi}(\frac{\Gamma}{\Re})$ we have $A\subseteq\Gamma\setminus\varphi(\Gamma)$ which leads to $|\frac{\Gamma}{\Re}\setminus\widetilde{\varphi}(\frac{\Gamma}{\Re})|\leq|\bigcup(\frac{\Gamma}{\Re}\setminus\widetilde{\varphi}(\frac{\Gamma}{\Re})|\leq|\Gamma\setminus\varphi(\Gamma)|$ and $\frac{\Gamma}{\Re}\setminus\widetilde{\varphi}(\frac{\Gamma}{\Re})$ is finite in this case.

Using the above two cases and Lemma 3.3, $\sigma_{\widetilde{\varphi}}$ is finite fibre. □

Lemma 3.5. *We have $\mathfrak{a}(\sigma_\varphi)=\mathfrak{a}(\sigma_{\widetilde{\varphi}})$. In particular by Lemma 3.4, if $\sigma_\varphi : X^\Gamma\to X^\Gamma$ is finite fibre, then* $\mathrm{ent}_{\mathrm{cset}}(\sigma_\varphi)=\mathrm{ent}_{\mathrm{cset}}(\sigma_{\widetilde{\varphi}})$.

Proof. For $s\geq1$ suppose $(y_n^1)_{n\geq1},\ldots,(y_n^s)_{n\geq1}$ are pairwise disjoint infinite $\sigma_\varphi-$anti-orbit sequences, then for all $i\in\{1,\ldots,s\}$ and $n\geq1$ we have $y_n^i\in\mathrm{sc}(\sigma_\varphi)$. Using Note 3.2, $\mathfrak{f}$ is one-to-one, thus $(\mathfrak{f}(y_n^1))_{n\geq1},\ldots,(\mathfrak{f}(y_n^s))_{n\geq1}$ are infinite pairwise disjoint sequences in $X^{\frac{\Gamma}{\Re}}$, moreover for all $i\in\{1,\ldots,s\}$ and $n\geq1$ we have $\sigma_{\widetilde{\varphi}}(\mathfrak{f}(y_{n+1}^i))=\mathfrak{f}(\sigma_\varphi(y_{n+1}^i))=\mathfrak{f}(y_n^i)$. Thus $(\mathfrak{f}(y_n^1))_{n\geq1},\ldots,(\mathfrak{f}(y_n^s))_{n\geq1}$ are infinite pairwise disjoint $\sigma_{\widetilde{\varphi}}-$anti-orbit sequences. Therefore $\mathfrak{a}(\sigma_{\widetilde{\varphi}})\geq\mathfrak{a}(\sigma_\varphi)$.

Now for $x=(x_\beta)_{\beta\in\frac{\Gamma}{\Re}}\in X^{\frac{\Gamma}{\Re}}$ let $w^x=(x_{[\alpha]_\Re})_{\alpha\in\Gamma}\in X^\Gamma$. So for all $x,y\in X^{\frac{\Gamma}{\Re}}$ if $w^x=w^y$, then $x=y$. Moreover for $x=(x_\beta)_{\beta\in\frac{\Gamma}{\Re}}\in X^{\frac{\Gamma}{\Re}}$ we have:

$$\begin{aligned}w^{\sigma_{\widetilde{\varphi}}(x)}=w^{(x_{\widetilde{\varphi}(\beta)})_{\beta\in\frac{\Gamma}{\Re}}}&=(x_{\widetilde{\varphi}([\alpha]_\Re)})_{\alpha\in\Gamma}\\ &=(x_{[\varphi(\alpha)]_\Re})_{\alpha\in\Gamma}=\sigma_\varphi((x_{[\alpha]_\Re})_{\alpha\in\Gamma})=\sigma_\varphi(w^x).\end{aligned}$$

For $t\geq1$ suppose $(y_n^1)_{n\geq1},\ldots,(y_n^t)_{n\geq1}$ are pairwise disjoint infinite $\sigma_{\widetilde{\varphi}}-$anti-orbit sequences, then $(w^{y_n^1})_{n\geq1},\ldots,(w^{y_n^t})_{n\geq1}$ are pairwise disjoint infinite $\sigma_\varphi-$anti-orbit sequences. Thus $\mathfrak{a}(\sigma_\varphi)\geq\mathfrak{a}(\sigma_{\widetilde{\varphi}})$. □

Lemma 3.6. *Suppose $\psi : \Gamma \to \Gamma$ is one-to-one and has at least one non-periodic point, then $\mathfrak{a}(\sigma_\psi) = \infty$, thus if $\sigma_\psi : X^\Gamma \to X^\Gamma$ is finite fibre too, then* $\mathrm{ent}_{\mathrm{cset}}(\sigma_\psi) = \infty$.

Proof. Suppose $\varphi : \Gamma \to \Gamma$ is one-to-one, and $\theta \in \Gamma$ is a non-periodic point of φ. Choose distinct $p, q \in X$ and for $m, n \geq 1$ let:

$$(x_n^m(0), x_n^m(1), x_n^m(2), \cdots) := (\underbrace{p, \cdots, p}_{m \text{ times}}, \underbrace{q, \cdots, q}_{n \text{ times}}, p, p, p, \cdots),$$

now let:

$$z_\alpha^{(n,m)} := \begin{cases} x_n^m(k) & k \geq 0, \alpha = \varphi^k(\theta), \\ p & \text{otherwise}. \end{cases}$$

Then for $z^{(n,m)} := (z_\alpha^{(n,m)})_{\alpha \in \Gamma}$, considering the sequences

$$(z^{(1,m)})_{m \geq 1}, (z^{(2,m)})_{m \geq 1}, (z^{(3,m)})_{m \geq 1}, \dots$$

we have:

- For $k, n, i, j \geq 1$ if $z^{(n,i)} = z^{(k,j)}$, then:

$$\begin{aligned} z^{(n,i)} = z^{(k,j)} &\Rightarrow (\forall \alpha \in \Gamma \; z_\alpha^{(n,i)} = z_\alpha^{(k,j)}) \\ &\Rightarrow (z_\theta^{(n,i)}, z_{\varphi(\theta)}^{(n,i)}, z_{\varphi^2(\theta)}^{(n,i)}, \cdots)(z_\theta^{(k,j)}, z_{\varphi(\theta)}^{(k,j)}, z_{\varphi^2(\theta)}^{(k,j)}, \cdots) \\ &\Rightarrow (x_n^i(0), x_n^i(1), \cdots) = (x_k^j(0), x_k^j(1), \cdots) \\ &\Rightarrow (\underbrace{p, \cdots, p}_{i \text{ times}}, \underbrace{q, \cdots, q}_{n \text{ times}}, p, \cdots) = (\underbrace{p, \cdots, p}_{j \text{ times}}, \underbrace{q, \cdots, q}_{k \text{ times}}, p, \cdots) \\ &\Rightarrow (i = j \wedge n = k) \end{aligned}$$

  Thus $(z^{(1,m)})_{m \geq 1}, (z^{(2,m)})_{m \geq 1}, (z^{(3,m)})_{m \geq 1}, \dots$ are pairwise disjoint infinite sequences.

- For all $n, m \geq 1$ and $\alpha \in \Gamma$ we have:

$$\begin{aligned} z_{\varphi(\alpha)}^{(n,m+1)} = q &\Leftrightarrow \varphi(\alpha) \in \{\varphi^i(\theta) : m+1 \leq i < m+1+n\} \\ &\Leftrightarrow \alpha \in \{\varphi^i(\theta) : m \leq i < m+n\} \\ &\Leftrightarrow z_\alpha^{(n,m)} = q \end{aligned}$$

  thus $\sigma_\varphi(z^{(n,m+1)}) = z^{(n,m)}$ and $(z^{(n,k)})_{k \geq 1}$ is an anti-orbit

Hence $(z^{(1,m)})_{m \geq 1}, (z^{(2,m)})_{m \geq 1}, (z^{(3,m)})_{m \geq 1}, \dots$ are pairwise disjoint infinite $\sigma_\varphi-$anti-orbit sequences and $\mathfrak{a}(\sigma_\varphi) = \infty$. □

Note 3.7. Suppose all points of Γ are periodic points of $\varphi : \Gamma \to \Gamma$, then $\sigma_\varphi : X^\Gamma \to X^\Gamma$ is bijective (note that $\varphi : \Gamma \to \Gamma$ is bijective and apply Remark 3.1) and using Corollary 2.5 we have:

$$\mathrm{ent}_{\mathrm{cset}}(\sigma_\varphi) = \mathrm{ent}_{\mathrm{set}}(\sigma_\varphi^{-1}) = \mathrm{ent}_{\mathrm{set}}(\sigma_{\varphi^{-1}}) = \begin{cases} 0 & \exists n \geq 1 \; \varphi^n = \mathrm{id}_\Gamma, \\ \infty & \text{otherwise}. \end{cases}$$

where for arbitrary A we have $\mathrm{id}_A : \underset{x\mapsto x}{A \to A}$.

Corollary 3.8. *Suppose $\varphi : \Gamma \to \Gamma$ is one-to-one and $\sigma_\varphi : X^\Gamma \to X^\Gamma$ is finite fibre, then*

$$\mathrm{ent}_{\mathrm{cset}}(\sigma_\varphi) = \mathrm{ent}_{\mathrm{set}}(\sigma_\varphi) = \begin{cases} 0 & \exists n \geq 1\ \varphi^n = \mathrm{id}_\Gamma\ , \\ \infty & \text{otherwise}\ . \end{cases}$$

Proof. Use Corollary 2.5, Lemma 3.6 and Note 3.7. □

Corollary 3.9. *If $\sigma_\varphi : X^\Gamma \to X^\Gamma$ is finite fibre, then:*

$$\mathrm{ent}_{\mathrm{cset}}(\sigma_\varphi) = \mathrm{ent}_{\mathrm{cset}}(\sigma_{\widetilde{\varphi}}) = \begin{cases} 0 & \exists n \geq 1\ (\widetilde{\varphi})^n = \mathrm{id}_{\frac{\Gamma}{\Re}}\ , \\ \infty & \text{otherwise}\ . \end{cases}$$

Proof. First we recall that $\widetilde{\varphi} : \frac{\Gamma}{\Re} \to \frac{\Gamma}{\Re}$ is one-to-one by Note 3.2. Use Corollary 3.8 and Lemma 3.5 to complete the proof. □

4. Other entropies: counterexamples

The main aim of this section is to compare positive topological, algebraic, set-theoretical and contravariant set-theoretical entropies in generalized shifts.

Remark 4.1. If G is an abelian group, $\theta : G \to G$ is a group homomorphism and $H$is a finite subset of G, then $\mathrm{ent}_{\mathrm{alg}}(\theta, H) = \lim_{n\to\infty} \frac{\log(|H \cup \theta(H) \cup \cdots \cup \theta^{n-1}(H)|)}{n}$ exists [5, 6] and we call $\mathrm{ent}_{\mathrm{alg}}(\theta) := \sup\{\mathrm{ent}_{\mathrm{alg}}(\theta, H) : H$ is a finite subgroup of $G\}$ the *algebraic entropy* of θ. Moreover if $\varphi : \Gamma \to \Gamma$ is finite fibre and X is a finite nontrivial group with identity e, then $\mathrm{ent}_{\mathrm{alg}}(\sigma_\varphi \restriction_{\bigoplus_\Gamma X}) = \mathrm{ent}_{\mathrm{cset}}(\varphi) \log|X|$ (as it has been mentioned in [1, Theorem 4.14] $\mathrm{ent}_{\mathrm{alg}}(\sigma_\varphi \restriction_{\bigoplus_\Gamma X})$ is equal to the product of string number of φ and $\log|X|$ this result has been evaluated in [5, Theorem 7.3.3] in the above form), where $\bigoplus_\Gamma X = \{(x_\alpha)_{\alpha\in\Gamma} \in X^\Gamma : \exists \alpha_1, \ldots, \alpha_n \in \Gamma\ \forall \alpha \in \Gamma \setminus \{\alpha_1, \ldots, \alpha_n\}\ (x_\alpha = e)\}$. Also by [7], $\mathrm{ent}_{\mathrm{alg}}(\sigma_\varphi) \in \{0, \infty\}$ with $\mathrm{ent}_{\mathrm{alg}}(\sigma_\varphi) = 0$ if and only if there exists $n > m \geq 1$ with $\varphi^n = \varphi^m$ (thus $\mathrm{ent}_{\mathrm{alg}}(\sigma_\varphi) = \mathrm{ent}_{\mathrm{set}}(\sigma_\varphi)$ by Corollary 2.5).

Remark 4.2. Suppose Y is a compact topological space and $\mathcal{U}, \mathcal{V}$ are open covers of Y, let $\mathcal{U} \vee \mathcal{V} := \{U \cap V : U \in \mathcal{U}, V \in \mathcal{V}\}$ and $N(\mathcal{U}) := \min\{|\mathcal{W}| : \mathcal{W}$ is a finite subcover of $\mathcal{U}\}$. Now suppose $T : Y \to Y$ is continuous, then $\mathrm{ent}_{\mathrm{top}}(T, \mathcal{U}) := \lim_{n\to\infty} \frac{\log N(\mathcal{U} \vee T^{-1}(\mathcal{U}) \vee \cdots \vee T^{-(n-1)}(\mathcal{U}))}{n}$ exists [8] and we call $\mathrm{ent}_{\mathrm{top}}(T) := \sup\{\mathrm{ent}_{\mathrm{top}}(T, \mathcal{W}) : \mathcal{W}$ is a finite open cover of $Y\}$ the *topological entropy* of T. If X is a finite discrete topological space with at least two elements and X^Γ considered with product (pointwise convergence) topology, then $\mathrm{ent}_{\mathrm{top}}(\sigma_\varphi) = \mathrm{ent}_{\mathrm{set}}(\varphi) \log|X|$ [2].

In the rest let:

- $\mathcal{C}$ is the collection of all generalized shifts $\sigma_\psi : Y^\Gamma \to Y^\Gamma$ such that Y is a nontrivial finite discrete topological group (so $2 \leq |Y| < \infty$), Γ is a nonempty set and both maps $\psi : \Gamma \to \Gamma$, $\sigma_\psi : Y^\Gamma \to Y^\Gamma$ are finite fibre,
- $\mathcal{C}_{\text{top}}$ is the collection of all elements of $\mathcal{C}$ like $\sigma_\psi : Y^\Gamma \to Y^\Gamma$ such that $\text{ent}_{\text{top}}(\sigma_\psi) > 0$,
- $\mathcal{C}_{\text{dalg}}$ is the collection of all elements of $\mathcal{C}$ like $\sigma_\psi : Y^\Gamma \to Y^\Gamma$ such that $\text{ent}_{\text{alg}}(\sigma_\psi \restriction_{\underset{\Gamma}{\bigoplus} Y}) > 0$,
- $\mathcal{C}_{\text{cset}}$ is the collection of all elements of $\mathcal{C}$ like $\sigma_\psi : Y^\Gamma \to Y^\Gamma$ such that $\text{ent}_{\text{cset}}(\sigma_\psi) > 0$,
- $\mathcal{C}_{\text{set}}$ is the collection of all elements of $\mathcal{C}$ like $\sigma_\psi : Y^\Gamma \to Y^\Gamma$ such that $\text{ent}_{\text{set}}(\sigma_\psi) > 0$ (i.e., $\text{ent}_{\text{alg}}(\sigma_\psi) > 0$ by Remark 4.1).

Lemma 4.3. *We have $\mathcal{C}_{\text{top}} \subseteq \mathcal{C}_{\text{cset}} \subseteq \mathcal{C}_{\text{set}}$ and $\mathcal{C}_{\text{dalg}} \subseteq \mathcal{C}_{\text{set}}$. As a matter of fact for an element of $\mathcal{C}$ like $\sigma_\varphi : Y^\Gamma \to Y^\Gamma$ we have:*

$$\text{ent}_{\text{top}}(\sigma_\varphi) \leq \text{ent}_{\text{cset}}(\sigma_\varphi) \leq \text{ent}_{\text{set}}(\sigma_\varphi) \textit{ and } \text{ent}_{\text{alg}}(\sigma_\varphi \restriction_{\underset{\Gamma}{\bigoplus} Y}) \leq \text{ent}_{\text{set}}(\sigma_\varphi).$$

Proof.

- "$\text{ent}_{\text{top}}(\sigma_\varphi) \leq \text{ent}_{\text{cset}}(\sigma_\varphi)$" Suppose $\text{ent}_{\text{top}}(\sigma_\varphi) > 0$, then $\mathfrak{o}(\varphi) \log |Y| = \text{ent}_{\text{set}}(\varphi) \log |Y| = \text{ent}_{\text{top}}(\sigma_\varphi) > 0$, thus $\mathfrak{o}(\varphi) > 0$ and $W(\varphi) \neq \varnothing$. Choose $\alpha \in W(\varphi)$, then $\{\varphi^n(\alpha)\}_{n \geq 0}$ is a one-to-one sequence thus for all $n > m \geq 0$, $[\varphi^n(\alpha)]_{\Re} \neq [\varphi^m(\alpha)]_{\Re}$, so for all $n \geq 1$ we have $\widetilde{\varphi}^n([\alpha]_{\Re}) \neq [\alpha]_{\Re}$, hence by Corollary 3.9, $\text{ent}_{\text{cset}}(\sigma_\varphi) > 0$ and $\text{ent}_{\text{cset}}(\sigma_\varphi) = \infty (\geq \text{ent}_{\text{top}}(\sigma_\varphi))$.
- "$\text{ent}_{\text{cset}}(\sigma_\varphi) \leq \text{ent}_{\text{set}}(\sigma_\varphi)$" Suppose $\text{ent}_{\text{set}}(\sigma_\varphi) \neq \infty$, then $\text{ent}_{\text{set}}(\sigma_\varphi) = 0$ and there exists $n > m \geq 1$ with $\varphi^n = \varphi^m$, thus $\widetilde{\varphi}^n = \widetilde{\varphi}^m$, and using the fact that $\widetilde{\varphi}$ is one-to-one we have $\widetilde{\varphi}^{n-m} = \text{id}_{\frac{\Gamma}{\Re}}$, thus $\text{ent}_{\text{cset}}(\sigma_\varphi) = 0$ by Corollary 3.9.
- "$\text{ent}_{\text{alg}}(\sigma_\varphi \restriction_{\underset{\Gamma}{\bigoplus} Y}) \leq \text{ent}_{\text{set}}(\sigma_\varphi)$" Suppose $\text{ent}_{\text{alg}}(\sigma_\varphi \restriction_{\underset{\Gamma}{\bigoplus} Y}) > 0$, then $\mathfrak{a}(\varphi) \log |Y| = \text{ent}_{\text{cset}}(\varphi) \log |Y| = \text{ent}_{\text{alg}}(\sigma_\varphi \restriction_{\underset{\Gamma}{\bigoplus} Y}) > 0$, thus $\mathfrak{a}(\varphi) > 0$ and there exists a one-to-one anti-orbit sequence $\{\alpha_n\}_{n \geq 1}$ in Γ.For all $n > m \geq 1$ we have $\varphi^n(\alpha_{n+m}) = \alpha_m \neq \alpha_n = \varphi^m(\alpha_{n+m})$ and $\varphi^n \neq \varphi^m$, thus $\text{ent}_{\text{set}}(\sigma_\varphi) = \infty (\geq \text{ent}_{\text{alg}}(\sigma_\varphi \restriction_{\underset{\Gamma}{\bigoplus} Y}))$ by Corollary 2.5.

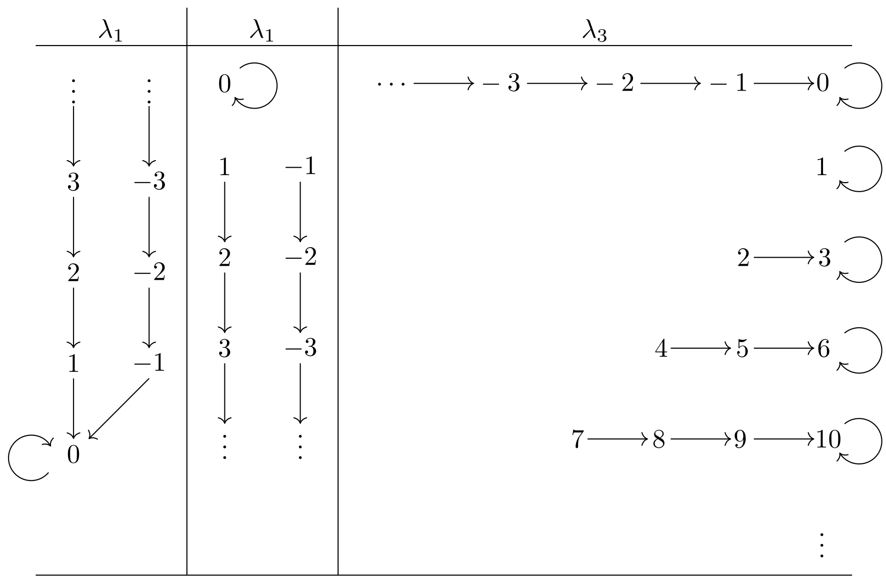
□

Table 4.4. We have the following table, in which the mark "$\surd$" means $p \leq q$ for the corresponding case for all $\sigma_\psi : Y^\Gamma \to Y^\Gamma$ in $\mathcal{C}$, also the mark "$\times$" indicates that there exists $\sigma_\psi : Y^\Gamma \to Y^\Gamma$ in $\mathcal{C}$ with $p > q$ in the corresponding case.

p \ q	$\mathrm{ent}_{\mathrm{top}}(\sigma_\varphi)$	$\mathrm{ent}_{\mathrm{alg}}(\sigma_\varphi \restriction_{\bigoplus_\Gamma Y})$	$\mathrm{ent}_{\mathrm{cset}}(\sigma_\varphi)$	$\mathrm{ent}_{\mathrm{set}}(\sigma_\varphi)$
$\mathrm{ent}_{\mathrm{top}}(\sigma_\varphi)$	$\surd$	$\times$	$\surd$	$\surd$
$\mathrm{ent}_{\mathrm{alg}}(\sigma_\varphi \restriction_{\bigoplus_\Gamma Y})$	$\times$	$\surd$	$\times$	$\surd$
$\mathrm{ent}_{\mathrm{cset}}(\sigma_\varphi)$	$\times$	$\times$	$\surd$	$\surd$
$\mathrm{ent}_{\mathrm{set}}(\sigma_\varphi)$	$\times$	$\times$	$\times$	$\surd$

Proof. For all "$\surd$" marks use Lemma 4.3. In order to establish "$\times$" marks use the following counterexamples.

Define $\lambda_1, \lambda_2, \lambda_3 : \mathbb{Z} \to \mathbb{Z}$ with the following diagrams:



So:

$$\lambda_1(n) = \begin{cases} n-1 & n \geq 1\,, \\ 0 & n = 0\,, \\ n+1 & n \leq -1\,, \end{cases} \quad \lambda_2(n) = \begin{cases} n+1 & n \geq 1\,, \\ 0 & n = 0\,, \\ n-1 & n \leq -1\,, \end{cases} \quad \lambda_3(n) = \begin{cases} n+1 & n \leq -1\,, \\ 0 & n = 0, 1\,, \\ 3 & n = 2\,, \\ 3 & n = 3\,, \\ 5 & n = 4\,, \\ 6 & n = 5 \\ 6 & n = 6\,, \\ \vdots & \end{cases}$$

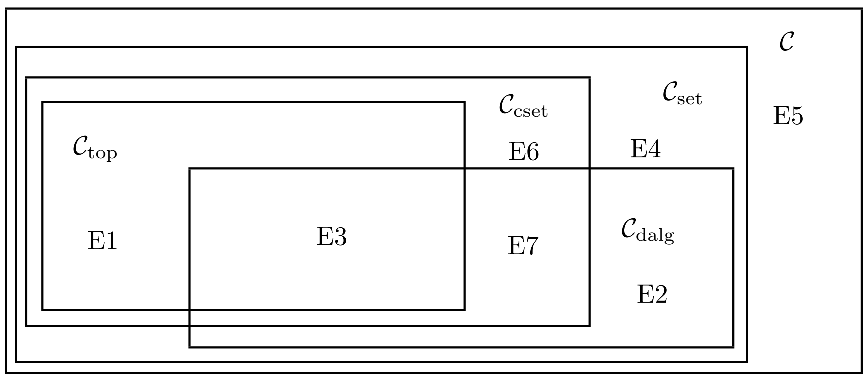
Then for discrete finite abelian group G with $|G| \geq 2$ and $\sigma_{\lambda_i} : G^{\mathbb{Z}} \to G^{\mathbb{Z}}$ we have:

- $\mathfrak{o}(\lambda_1) = \mathfrak{o}(\lambda_3) = 0$, $\mathfrak{o}(\lambda_2) = 2$, $\mathfrak{a}(\lambda_1) = 2$, $\mathfrak{a}(\lambda_2) = 0$, $\mathfrak{a}(\lambda_3) = 1$,

- $\text{ent}_{\text{top}}(\sigma_{\lambda_1}) = \text{ent}_{\text{top}}(\sigma_{\lambda_3}) = 0$, $\text{ent}_{\text{top}}(\sigma_{\lambda_2}) = 2\log|G|$,
- $\text{ent}_{\text{alg}}(\sigma_{\lambda_1} \restriction_{\bigoplus\limits_{\Gamma} G}) = 2\log|G|$, $\text{ent}_{\text{alg}}(\sigma_{\lambda_2} \restriction_{\bigoplus\limits_{\Gamma} G}) = 0$, $\text{ent}_{\text{alg}}(\sigma_{\lambda_3} \restriction_{\bigoplus\limits_{\Gamma} G}) = \log|G|$,
- $\text{ent}_{\text{cset}}(\sigma_{\lambda_1}) = \text{ent}_{\text{cset}}(\sigma_{\lambda_3}) = 0$, $\text{ent}_{\text{cset}}(\sigma_{\lambda_2}) = \infty$,
- $\text{ent}_{\text{set}}(\sigma_{\lambda_1}) = \text{ent}_{\text{set}}(\sigma_{\lambda_2}) = \text{ent}_{\text{set}}(\sigma_{\lambda_3}) = \infty$,

which complete the proof. □

Diagram 4.5. We have the following diagram:



where by "Ei" we mean counterexample $\sigma_{\mu_i} : G^{\Lambda_i} \to G^{\Lambda_i}$ for finite discrete abelian group G with $|G| \geq 2$.

- for $\Lambda_1 := \mathbb{Z}$ and $\mu_1 := \lambda_2$ as in Table 4.4, we have $\text{ent}_{\text{top}}(\sigma_{\mu_1}) = 2\log|G| > 0$ and $\text{ent}_{\text{alg}}(\sigma_{\mu_1} \restriction_{\bigoplus\limits_{\Lambda_1} G}) = 0$,
- for $\Lambda_2 := \mathbb{Z}$ and $\mu_2 := \lambda_1$ as in Table 4.4, we have $\text{ent}_{\text{alg}}(\sigma_{\mu_2} \restriction_{\bigoplus\limits_{\Lambda_2} G}) = 2\log|G| > 0$ and $\text{ent}_{\text{cset}}(\sigma_{\mu_2}) = 0$,
- for $\Lambda_3 := \mathbb{Z} \times \{0,1\}$ and

$$\mu_3(n,i) = \begin{cases} \mu_1(n) & i = 0\,, \\ \mu_2(n) & i = 1\,, \end{cases}$$

  we have $\text{ent}_{\text{top}}(\sigma_{\mu_3}) = \text{ent}_{\text{alg}}(\sigma_{\mu_3} \restriction_{\bigoplus\limits_{\Lambda_3} G}) = 2\log|G| > 0$,
- for $\Lambda_4 := \mathbb{N}$ and $\mu_4 = \lambda_3 \restriction_{\mathbb{N}}$ we have $\text{ent}_{\text{alg}}(\sigma_{\mu_4} \restriction_{\bigoplus\limits_{\Lambda_4} G}) = \text{ent}_{\text{cset}}(\sigma_{\mu_4}) = 0$ and $\text{ent}_{\text{set}}(\sigma_{\mu_4}) = \infty$,
- for $\Lambda_5 := \mathbb{Z}$ and $\mu_5(n) = -n$ $(n \in \mathbb{Z})$ we have $\text{ent}_{\text{set}}(\sigma_{\mu_5}) = 0$,

- for $\Lambda_6 := \mathbb{N}$ and $\mu_6 = (1,2)(3,4,5)(6,7,8,9)(10,11,12,13,14)\cdots$ we have $\mathrm{ent}_{\mathrm{alg}}(\sigma_{\mu_6} \restriction_{\bigoplus_{\Lambda_6} G}) = \mathrm{ent}_{\mathrm{top}}(\sigma_{\mu_6}) = 0$ and $\mathrm{ent}_{\mathrm{cset}}(\sigma_{\mu_6}) = \infty$,

- for $\Lambda_7 = (\mathbb{N} \times \{0\}) \cup (\mathbb{Z} \times \{1\})$ and

$$\mu_7(n,i) = \begin{cases} \mu_6(n) & i = 0\,, \\ \mu_2(n) & i = 1\,, \end{cases}$$

we have $\mathrm{ent}_{\mathrm{alg}}(\sigma_{\mu_7} \restriction_{\bigoplus_{\Lambda_7} G}) = 2\log|G| > 0$, $\mathrm{ent}_{\mathrm{cset}}(\sigma_{\mu_7}) = \infty$, $\mathrm{ent}_{\mathrm{set}}(\sigma_{\mu_7}) = 0$.

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