On α-Topological Vector Spaces and α-irresolute gauges

Sobre α-espacios vectoriales topológicos y α-calibres irresolutos

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Abstract. The aim of this paper is to provide a characterization of $\alpha-$irresolute maps that were introduced in [1]. If p is the gauge of an absolutely convex and absorbent subset U of an $\alpha-$topological vector space, then p is shown to be $\alpha-$irresolute if and only if U is an $\alpha-$neighborhood of 0.

Keywords: α-open set, pseudonorm, locally convex $\alpha-$topological vector space.

Resumen. El objetivo de este artículo es proporcionar una caracterización de funciones α-irresolutas que fueron introducidas en [1]. Si p es el calibre de un subconjunto U de un espacio vectorial topológico que es absorvente y absolutamente convexo, entonces se demuestra que p es α-irresoluto si y solo si U es una α-vecindad de 0.

Palabras claves: α-conjunto abierto, pseudonorma, α-espacio vectorial topológico localmente convexo.

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1. Introduction

A topological vector space (TVS) is a vector space with a topological structure such that the algebraic operations: addition and scalar multiplication, are continuous, see for example Jarchow [15] and Köthe [16], Al-Hawary and Al-Nayef [8, 12]. The theory of topological vector spaces often clarifies results in many branches of functional analysis such as the theory of normed spaces. Let $(X, \mathcal{T})$ be a topological space. A subset $A \subseteq X$ is α-open if $A \subseteq \overline{A^o}^o$, where $\bar{A}$ denotes the closure of A in X and A^o denotes the interior of A. The collection of all α-open sets in $(X,\mathcal{T})$ is denoted by $\alpha O(X)$ and the pair $(X, \alpha O(X))$ is called the *α-topological space* associated with $(X,\mathcal{T})$. We remark that $(X, \alpha O(X))$ is a topological space. A subset U of X is an *α-neighborhood* of a point $x \in X$ if U contains an α-open set that contains x. The set of all α-neighborhoods of x will be denoted by $N_x(X)$.

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Let $(X,\mathcal{T})$ and $(Y,\mathcal{T}^{'})$ be topological spaces. A map $f : X \to Y$ is $\alpha-$*irresolute* if the inverse image of every α-open set in Y is α-open in X, see Maheshwari and Thakur [17] and Takashi [19]. A map $f : X \to Y$ is *pre-α-open* if the image of any α-open set in X is α-open in Y. We refer the reader interested in more details about the preceding notions to Al-Hawary [2, 3, 4, 5, 6, 7, 9, 10, 11], Crossley and Hildebrand [13, 14], Maheshwari and Thakur [17], and Takashi [19].

Our main goal in this paper is to provide a characterization of α-irresolute maps that were introduced in [1]. If p is the gauge of an absolutely convex and absorbent subset U of an $\alpha-$topological vector space, then p is shown to be $\alpha-$irresolute if and only if U is an $\alpha-$neighborhood of 0. Finally, we show that in a locally convex αTVS $(X, \alpha O(X))$, a pseudonorm p is α-irresolute if and only if p is α-irresolute at 0 and the gauge of an absolutely convex and absorbent subset is shown to be a pseudonorm.

We next recall some necessary results:

Lemma 1.1. *[19] Let X be a topological space. Then a subset A of X is α-open if and only if there exists an open set V in X such that $V \subseteq A \subseteq \overline{V}^o$.*

Definition 1.2. [1] Let X be a vector space over the field of real numbers, and let $\mathcal{T}$ be a topology on X such that the addition map $S_X : X \times X \to X$ and the scalar multiplication map $M_X : \mathbb{R} \times X \to X$ are $\alpha-$irresolute. Then $(X, \alpha O(X))$ is called an $\alpha-$*topological vector space (α TVS).*

This concept was shown to be totally distinct from the concept of TVS. The proof of the following theorem follows from the fact that the addition and the scalar multiplication maps are α-irresolute.

Lemma 1.3. *[1] Let $(X, \alpha O(X))$ be an αTVS and $y \in X$. Then*

(a) A subset $U \in N_0(X)$ if and only if $y + U \in N_y(X)$.

(b) If $U \in N_0(X)$, then $tU \in N_0(X)$ for all scalars $t \in \mathbb{R} \setminus \{0\}$.

2. Locally Convex αTVSs

We begin this section by giving an interesting example of an αTVS. For that, we prove the following new result:

Lemma 2.1. *Let $f : X \to Y$ be a continuous and open map. Then f is α-irresolute.*

Proof. For every α-open subset A of Y, by Lemma 1.1, there exists an open set V in Y such that $V \subseteq A \subseteq \overline{V}^o$. Thus $f^{-1}(V) \subseteq f^{-1}(A) \subseteq f^{-1}(\overline{V}^o)$. As f is continuous, $f^{-1}(V)$ is open in X. We show that $f^{-1}(A) \subseteq \overline{f^{-1}(V)}^o$ by showing that $f^{-1}(\overline{V}^o) \subseteq \overline{f^{-1}(V)}^o$. For every $x \in f^{-1}(\overline{V}^o)$, $f(x) \in \overline{V}^o$ and so there exists an open set U in Y such that $f(x) \in U \subseteq \overline{V}$. Now for every $y \in f^{-1}(U)$

and for every open subset W of X such that $y \in W$, $f(y) \in f(W)$ which is open in Y as f is an open map and as $f(y) \in \bar{V}$, $V \cap f(W) \neq \varnothing$. Hence there exists $z \in V \cap f(W)$ and $w \in W$ such that $z = f(w)$, and so $w \in f^{-1}(V) \cap W$. That is $y \in \overline{f^{-1}(V)}$ and so $f^{-1}(U) \subseteq \overline{f^{-1}(V)}$, hence $x \in \overline{f^{-1}(V)}^{o}$.Therefore by Lemma 1.1, $f^{-1}(A)$ is $\alpha-$open. □

Example 2.2. Consider $X = \mathbb{R}$ with the usual topology $\mathcal{T}_u$. Then clearly $S_{\mathbb{R}}$ and $M_{\mathbb{R}}$ are continuous and open, by the Open Mapping Theorem. Hence by Lemma 2.1, are $\alpha-$irresolute maps and consequently $(X, \alpha O(\mathbb{R}))$ is an α TVS.

The proof of the preceding Lemma can also be obtained from the following straightforward result:

Lemma 2.3. *If $f : X \to Y$ is an continuous and open map then for each $B \subseteq Y$, $f^{-1}(\overline{B}^{o}) \subseteq \overline{f^{-1}(B)}^{o}$.*

With this result, the proof of Lemma 2.1 is reduced as follows:

Proof. For every α-open subset A of Y , by Lemma 1.1, there exists an open set V in Y such that $V \subseteq A \subseteq \overline{V}^{o}$. Thus $f^{-1}(V) \subseteq f^{-1}(A) \subseteq f^{-1}(\overline{V}^{o})$. As f is continuous, $f^{-1}(V)$ is open in X and since f is also open, then by Corollary 2.3, $f^{-1}(\overline{V}^{o}) \subseteq \overline{f^{-1}(V)}^{o}$ and so $f^{-1}(V) \subseteq f^{-1}(A) \subseteq \overline{f^{-1}(V)}^{o}$. Therefore by Lemma 1.1, $f^{-1}(A)$ is α-open. □

In this section, we introduce the notion of locally convex αTVSs. Moreover, we give a necessary and sufficient condition, in terms of convex α-neighborhoods of 0, for an αTVS to be locally convex.

Definition 2.4. An αTVS $(X, \alpha(X))$ is locally convex if for all $x \in X$, every $S \in N_x(X)$ contains a convex $U \in N_x(X)$.

Theorem 2.5. *An αTVS $(X, \alpha(X))$ is locally convex if and only if every $S \in N_0(X)$ contains a convex $U \in N_0(X)$.*

Proof. The sufficiency part is trivial. Let $S \in N_x(X)$. Then by part (a) of Lemma 1.3, $S - x \in N_0(X)$ and by assumption, there exists a convex $U \in N_0(X)$ such that $U \subseteq S - x$. Hence by part (a) of Lemma 1.3 again, $U + x \in N_x(X)$. As $U + x \subseteq S$ and as $U + x$ is convex, $(X, \alpha O(X))$ is a locally convex αTVS. □

Example 2.6. Consider $\mathbb{R}$ with the usual topology $\mathcal{T}_u$. Then clearly $S_{\mathbb{R}}$ and $M_{\mathbb{R}}$ are continuous and open, by the Open Mapping Theorem. Hence by Lemma 1.1, they are $\alpha-$irresolute maps and consequently $(\mathbb{R}, \alpha O(\mathbb{R}))$ is an αTVS. Since every interval in $\mathbb{R}$ is convex, $(\mathbb{R}, \alpha O(\mathbb{R}))$ is locally convex.

3. Pseudonorm and α-irresolute Maps

Let A be a subset of a vector space X. Recall that A is called *balanced* if $tA \subseteq A$ for $|t| \leq 1$, *absorbing* if for every $x \in X$, there exists $\epsilon > 0$ such that $tx \in A$ for $|t| < \epsilon$ and *absolutely convex* if it is both convex and balanced. Next, the definition of pseudonorm is given.

Definition 3.1. Let X be a vector space over $\mathbb{R}$. A non-negative real-valued function p defined on X is a *pseudonorm* if it satisfies the following two conditions:

(i) $p(\alpha x) = |\alpha| p(x)$, for all $x \in X$ and $\alpha \in \mathbb{R}$;

(ii) $p(x + y) \leq p(x) + p(y)$, for all $x, y \in X$.

Let X and Y be topological spaces and $f : X \to Y$ be a map. Next, we define what we mean by f is $\alpha-$irresolute at a point $x \in X$ and then show that a map is $\alpha-$irresolute if and only if it is $\alpha-$irresolute at each point $x \in X$.

Definition 3.2. A map $f : X \to Y$ is $\alpha-$irresolute at a point $x \in X$ if for each $V \in N_{f(x)}(Y)$, there exists $U \in N_x(X)$ such that $f(U) \subseteq V$.

We remark that the set U in the preceding Definition may be chosen α-open.

Theorem 3.3. *A map $f : X \to Y$ is $\alpha-$irresolute if and only if f is $\alpha-$irresolute at each point $x \in X$.*

Proof. Let $x \in X$, and let $V \in N_{f(x)}(Y)$. Then there exists an $\alpha-$open set W such that $f(x) \in W \subseteq V$. Hence, $x \in f^{-1}(W) \subseteq f^{-1}(V)$. As f is $\alpha-$irresolute, $U = f^{-1}(W)$ is $\alpha-$open in X. Thus, $U \in N_x(X)$ and clearly $f(U) \subseteq V$.

Conversely, let W be $\alpha-$open in Y. For all $x \in f^{-1}(W)$, $f(x) \in W$ and hence $W \in N_{f(x)}(Y)$. By assumption, there exists an α-open set $U_x \in N_x(X)$ such that $f(U_x) \subseteq W$. Thus, $U_x \subseteq f^{-1}(W)$ and so $\cup\{U_x : x \in f^{-1}(W)\} = f^{-1}(W)$. Now U_x is α-open for all $x \in f^{-1}(W)$ and thus, $f^{-1}(W) = \cup\{U_x : x \in f^{-1}(W)\}$ is α-open. □

Corollary 3.4. *In a locally convex αTVS $(X, \alpha O(X))$, a pseudonorm p is α-irresolute if and only if p is α-irresolute at* 0.

Proof. If p is α-irresolute, then by Theorem 3.3, p is α-irresolute at 0. Conversely, suppose p is α-irresolute at 0, and let $x \in X$ and $V \in N_{p(x)}(\mathbb{R})$. Then by part (a) of Lemma 1.3, $V - p(x) \in N_0(\mathbb{R}) = N_{p(0)}(\mathbb{R})$ and thus $(-\epsilon, \epsilon) \subseteq V - p(x)$ for some $\epsilon > 0$. By assumption, there exists $U \in N_0(X)$ such that $p(U) \subseteq (-\epsilon, \epsilon)$ and as $p(y) \geq 0$ for all $y \in U$, $p(U) \subseteq [0, \epsilon)$. Then by part (a) of Lemma 1.3, $U+x \in N_x(X)$. For all $y \in U$, $0 \leq p(x+y) \leq p(x)+p(y) \leq p(x)+\epsilon$, $p(x + y) \in [p(x), p(x) + \epsilon)$. Therefore $p(U + x) \subseteq V$. □

For the following definition, see for example Robertson and Robertson [18].

Definition 3.5. Let A be an absolutely convex subset of a vector space X. Then the functional defined by $p(x) = \inf\{\lambda : \lambda > 0, x \in \lambda A\}$ is called the gauge of A.

Lemma 3.6. *[20] In a vector space X, the gauge of an absolutely convex and absorbent subset is a pseudonorm.*

Now, we are ready to prove our main result in which we characterize absolutely convex and absorbent α-neighborhoods of 0 in terms of their α-irresolute gauges.

Theorem 3.7. *Let p be a gauge of an absolutely convex and absorbent subset U of an αTVS. Then p is α-irresolute if and only if U is an α-neighborhood of 0.*

Proof. If p is α-irresolute, then as $(-1, 1)$ is an α-open in $\mathbb{R}$, $V = \{x : p(x) < 1\} = p^{-1}((-1, 1))$ is an α-open subset of X. Thus as $V \subseteq U$, $U \in N_0(X)$. Conversely, if $U \in N_0(X)$ and $\epsilon > 0$, then by part (b) of Lemma 1.3 , $V = \epsilon U \in N_0(X)$ and $p(x) < \epsilon$ for all $x \in V$. Thus, $p(V) \subseteq (-\epsilon, \epsilon)$. Hence, p is α-irresolute at 0. By Lemma 3.6, p is a pseudonorm and by Corollary 3.4, p is α-irresolute at each $x \in X$. Therefore by Theorem 3.3, p is α-irresolute. □

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