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Nonsmooth multiobjective fractional programming with generalized convexity

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Abstract. In this paper we study a class of nonconvex and nondifferentiable multiobjective fractional problems. We use the transformation proposed by Dinkelbach [2] and Jagannathan [4] and we obtain optimality conditions for weakly efficient solutions for these problems. Furthermore, we define a dual problem and we establish some results on duality. To obtain our results, we use a notion of generalized convexity, called KT-invexity. Our paper generalizes the results given by Osuna-Gómez *et al.* in [6], where the authors considered smooth problems.

Resumen. En el artículo estudiamos una clase de problemas fraccionales multiobjetivos no convexos y no diferenciables. Usamos la transformación propuesta por Dinkelbach [2] y Jagannathan [4] y obtenemos condiciones de optimalidad para soluciones débilmente eficientes de dichos problemas. Además, definimos un problema dual y establecemos algunos resultados sobre dualidad. Para lograrlo, utilizamos una noción de convexidad generalizada llamada KT-invexidad. El artículo generaliza los resultados obtenidos por Osuna-Gómez *et al.* en [6], en donde los autores consideran problemas suaves.

Keywords: vector fractional programming, Clarke generalized gradient, duality, weak efficiency, optimality conditions

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1. Introduction

In this work, we will study the following nonlinear and nonconvex multiobjective fractional problem:

$$\begin{aligned} &\text{Minimize} \quad \frac{f(x)}{g(x)} := \left(\frac{f_1(x)}{g_1(x)}, \ldots, \frac{f_p(x)}{g_p(x)}\right), \\ &\text{subject to:} \quad h_j(x) \leq 0, \quad j = 1, \ldots, m, \\ &\qquad\qquad\quad x \in S. \end{aligned} \tag{VFP}$$

where S is a nonempty subset of $\mathbb{R}^n$ and $f_i,\ g_i,\ h_j : \mathbb{R}^n \to \mathbb{R}$, $i = 1, \ldots, p$, $j = 1, \ldots, m$ are locally Lipschitz functions.

We will denote

$$f(x) := (f_1(x), \ldots, f_p(x)),\ g(x) := (g_1(x), \ldots, g_p(x)) \text{ and } h(x) := (h_1(x), \ldots, h_m(x)).$$

We will suppose that $g_i(x) > 0$ for all $x \in S$ and we will denote by

$$X := \{x \in S : h_j(x) \leq 0,\ j = 1, \ldots, m\}$$

the feasible set of (VFP).

A fractional programming problem arises whenever the optimization of ratios, such as performance/cost, income/investiment and cost/time, is required and then, various real-life problems admit this formulation. For more details on applications of fractional programming, we suggest [9] and the references therein.

One of the most known approach used for solving nonlinear fractional programming problems is the called *parametric approach*. Dinkelbach [2] and Jagannathan [4] introduced this approach that was used later by Osuna-Gómez *et al.* in [6] to characterize solutions of a multiobjective fractional problem under generalized convexity and differentiability hypotheses and, also, they established some duality results.

Our main goal in this work is to show that these results can be extended to the nonsmooth problems, whose functions are locally Lipschitz and, to achieve our objective, we use the techniques of nonsmooth analysis [1] and we extend a notion of KT-invexity to the nondifferentiable context.

This paper have the following structure: in Section 2, we remind some results on Nonsmooth Analysis which we will use in the following sections. In Section 3, we establish some optimality conditions for the nonsmooth vector fractional problem and, in Section 4, we apply the previous results to obtain some duality results.

2. Preliminaries

In this section we remind some notions and results from nonsmooth analysis and conditions of optimality for vector problems. The proofs will be ommited and we sugest the references [1], [3] for more details.

The *generalized gradient* of a local Lipschitz function $\phi : \mathbb{R}^n \to \mathbb{R}$ at $\overline{x}$ in the direction d, denoted by $\phi^0(\overline{x}; d)$, is given by

$$\phi^0(\overline{x}; d) = \limsup_{\substack{x \to \overline{x} \\ t \to 0^+}} \frac{\phi(x + td) - \phi(x)}{t}$$

and the *generalized gradient* of ϕ at $\overline{x}$ is given by

$$\partial\phi(\overline{x}) = \{x^* \in \mathbb{R}^n : \phi^0(\overline{x}; v) \geq \langle x^*, v\rangle, \ \forall v \in \mathbb{R}^n\}.$$

Let C be a nonempty subset of $\mathbb{R}^n$ and consider its *distance function*, that is, the function $\delta_C : \mathbb{R}^n \to \mathbb{R}$ defined by

$$\delta_C = \inf_{c \in C} ||x - c||.$$

The distance function is not everywhere differentiable, but is globally Lipschitz.

Let $\overline{x} \in C$. A vector $d \in \mathbb{R}^n$ is said to be tangent to C at $\overline{x}$ if $\delta_C^0(\overline{x}; d) = 0$. The set of tangent vectors to C at $\overline{x}$ is a closed convex cone in $\mathbb{R}^n$, called the *tangent cone* to C at $\overline{x}$ and will be denoted by $T_C(\overline{x})$.

By polarity, we define the *normal cone* to C at $\overline{x}$:

$$N_C(\overline{x}) := \{\xi \in \mathbb{R}^n : \langle \xi, v\rangle \leq 0, \ \forall v \in T_C(\overline{x})\}.$$

We remind that $N_C(\overline{x})$ is a closed convex cone.

It can be proved that if $\phi : \mathbb{R}^n \to \mathbb{R}$ is a locally Lipschitz funtion and $x_0 \in C$ is a local minimizer of f on C, then

$$0 \in \partial\phi(x_0) + N_C(x_0). \tag{1}$$

Note that (1) is equivalent to

$$\phi^0(x_0; v) \geq 0, \ \forall v \in T_C(x_0),$$

and, in this case, x_0 is called a *stationary point* of ϕ at C.

We will adopt the following convention for inequalities between vectors: let $x = (x_1, \ldots, x_n)$, $y = (y_1, \ldots, y_n) \in \mathbb{R}^n$:

$$\begin{aligned} x &< y \iff x_i < y_i, \ \forall i = 1, \ldots, n, \\ x &\leqq y \iff x_i \leq y_i, \ \forall i = 1, \ldots, n, \\ x &\leq y \iff x \leqq y \ \text{ and } \ x \neq y. \end{aligned}$$

In a similar way, we define the inequalities $>$, $\geqq$ and $\geq$.

Now, we consider the following general multiobjective optimization problem:

$$\begin{aligned} &\text{Minimize } \phi(x) = (\phi_1(x), \ldots, \phi_p(x)), \\ &\text{subject to: } x \in F, \end{aligned} \tag{P$_0$}$$

where $\phi_i : \mathbb{R}^n \to \mathbb{R}$, $i = 1, \ldots, p$ are given functions and $F \subset \mathbb{R}^n$ is a nonempty set.

We say that $x_0 \in F$ is a *weakly efficient minimizer* (maximizer) of (P_0) if there is not $x \in F$ such that $\phi(x) < \phi(x_0)$ (respec. $\phi(x) > \phi(x_0)$.)

Now, we consider the following particular case: $F = \{x \in S : \beta_j(x) \leq 0,\ j = 1, \ldots, m\}$, where S is a nonempty closed set of $\mathbb{R}^n$.

Necessary conditions for weakly efficiency are given by the following proposition:

Proposition 2.1. *If $x_0 \in F$ is a weakly efficient minimizer of (P_0), then there exist $\mu \in \mathbb{R}^p, \lambda \in \mathbb{R}^m$, such that*

$$\begin{aligned} &0 \in \partial\Big(\textstyle\sum_{i=1}^{p} \mu_i \phi_i + \sum_{j=1}^{m} \lambda_j \beta_j \Big)(x_0) + N_S(x_0), \\ &(\mu, \lambda) \geq 0, \\ &\langle \lambda, \beta(x) \rangle = 0. \end{aligned}$$

In order to operationalize the determination of the weakly efficient minimizers of (P_0), we should to relate it with a more familiar problem. So, Geoffrion [3] characterized the solutions of the multiobjective problems in terms of optimal solutions of appropriate scalar problem. He considered the weighting problem defined by:

$$\begin{aligned} &\text{Minimize} \quad \textstyle\sum_{i=1}^{p} w_i \phi_i(x), \\ &\text{subject to:} \quad x \in F, \end{aligned} \tag{P(w)}$$

where $w \in \mathbb{R}^p_+ \backslash \{0\}$. The following proposition establishes a relation between the solutions of (P_0) and (P(w)).

Proposition 2.2. *Suppose that $\overline{x} \in F$ is a solution of (P(w)), for some $w \geq 0$. Then, $\overline{x}$ is a weakly efficient solution of (P_0).*

We are now in position to estated our results.

3. Optimality conditions

In this section we will establish optimality conditions for (VFP), under assumptions of generalized convexity. The basic idea consists in to attach the intermediate multiobjective problem, by using of an approach due to Dinkelbach [2] and Jagannathan [4]. For each $v = (v_1, \dots, v_p) \in \mathbb{R}^p$, we define the following problem associated to (VFP):

$$\begin{array}{lll} \text{Minimize} & (f_1(x) - v_1 g_1(x), \dots, f_p(x) - v_p g_p(x)), & \\ \text{subject to:} & h_j(x) \leq 0,\ j = 1, \dots, m, & \qquad \text{(VFP(v))} \\ & x \in S. & \end{array}$$

In [6], the next lemma is proved.

Lemma 3.1. *A point $\overline{x} \in X$ is a weakly efficient minimizer of (VFP) if and only if $\overline{x}$ is a weakly efficient minimizer of (VFP($\overline{v}$)), with $\overline{v} = \dfrac{f(\overline{x})}{g(\overline{x})}$.*

Then, to estate optimality conditions for (VFP) we will consider some hypothesis of generalized convexity.

In [5] Martin define a class of scalar nonlinear programming problem that later was called KT-invex problems, that has the following property: the problem is KT-invex if only if all points that satisfy the Karush-Kuhn-Tucker conditions are minimal points. This notion was extended to multiobjective problems by Osuna-Gómez *et al.* [7], where more general results were showed.

To do this, we will need the weighting problem related to (VFP(v)). For each $v \in \mathbb{R}^p$ and $w \in \mathbb{R}^p_+ \backslash \{0\}$ we define:

$$\begin{array}{lll} \text{Minimize} & \sum_{i=1}^p w_i (f_i(x) - v_i g_i(x)), & \\ \text{subject to:} & h_j(x) \leq 0,\ j = 1, \dots, m, & \qquad (\text{VFP}_v(w)) \\ & x \in S. & \end{array}$$

Let $\Phi_{v,i}(x) = f_i(x) - v_i g_i(x)$, $i = 1, \dots, p$. We propose the following KT-invex definition suitable for a nonsmooth fractional multiobjective programming problem.

Definition 3.2. We will say that the problem (VFP(v)) is **KT-invex** on the feasible set with respect to η if for each $x_1, x_2 \in X$, there exists a vector $\eta(x_1, x_2) \in T_S(x_2)$ such that

$$\begin{aligned} \Phi_{v,i}(x_1) - \Phi_{v,i}(x_2) &\geq \Phi^0_{v,i}(x_2; \eta(x_1, x_2)), \quad i = 1, \dots, m, \\ h^0_j(x_2, \eta(x_1, x_2)) &\leq 0, \quad \forall j \in J(x_2), \end{aligned}$$

where $J(x_2) = \{j : h_j(x_2) = 0\}$.

We note that our definition is a little different of the other definitions of KT-invexity (see [7], [8], for instance) because we claim that the vector belong to $T_S(x_2)$. This is done to allow us to consider those problems that have a set of abstract constraints, that is, restrictions that are not of inequality-type. In the absence of these constraints, that is, $S=\mathbb{R}^n$, we have $T_S(x_2)=\mathbb{R}^n$ and the Definition 3.2 coincides with those given by Osuna-Gómez et al. [7] for differentiable problems and coincides with those given by [8] for problems that do not have abstract constraints.

To guarantee that the multiplier associated to the objective function is nonzero, it is necessary that the problem is regular, that is, that the problem satisfies some constraint qualification.

We will use the following notion of regularity. Let $\overline{v}\in\mathbb{R}^p$ be a fixed vector. We say that (VFP($\overline{v}$)) satisfies the *constraint qualification* at $\overline{x}\in X$ if exists a vector $v_0\in T_S(\overline{x})$ such that

$$h_j^0(\overline{x};v_0)<0,\quad \forall j\in J(x_2).$$

Theorem 3.3. *We assume that $\overline{x}$ is a weakly efficient minimizer of (VFP) and the problem (VFP($\overline{v}$)) satisfies the constraint qualification in $\overline{x}$. Suppose that $\overline{v}=\dfrac{f(\overline{x})}{g(\overline{x})}$. Then, $\overline{x}$ is a solution of the weighting problem (VFP$_{\overline{v}}(w)$), for some $w\in\mathbb{R}^p_+\backslash\{0\}$.*

Proof. Let $\overline{x}$ be a weakly efficient minimizer of (VFP). Then, it follows from Lemma 3.1 that $\overline{x}$ is a weakly efficient minimizer of (VFP($\overline{v}$)). Therefore, by applying Proposition 2.1 we have that there exists a nonzero pair $(\theta,\lambda)\in\mathbb{R}^p_+\times\mathbb{R}^m_+$ such that

$$\begin{aligned}\Big[\sum_{i=1}^p\theta_i(f_i-\overline{v}_ig_i)+\sum_{j=1}^m\lambda_jh_j\Big]^0(\overline{x};v)\geq 0,&\qquad \forall v\in T_S(\overline{x}),\\ \lambda_jh_j(\overline{x})=0,&\qquad j=1,\dots,m.\end{aligned}$$

In particular, for each feasible point $x\in X$ we have

$$\Big[\sum_{i=1}^p\theta_i(f_i-\overline{v}_ig_i)\Big]^0(\overline{x};\eta(x,\overline{x}))+\sum_{j=1}^m\lambda_jh_j^0(\overline{x};\eta(x,\overline{x}))\geq 0, \tag{2}$$

and so,

$$\begin{aligned}\Big[\sum_{i=1}^p\theta_i(f_i-\overline{v}_ig_i)\Big]^0(\overline{x};\eta(x,\overline{x}))&\geq-\sum_{j=1}^m\lambda_jh_j^0(\overline{x};\eta(x,\overline{x}))=\\ &=-\sum_{j\in J(\overline{x})}\lambda_jh_j^0(\overline{x};\eta(x,\overline{x}))\geq 0,\end{aligned} \tag{3}$$

and from KT-invexity of (VFP($\overline{v}$)) we obtain

$$\sum_{i=1}^p[\theta_i(f_i(x)-\overline{v}_ig_i(x)]-\sum_{i=1}^p[\theta_i(f_i(\overline{x})-\overline{v}_ig_i(\overline{x})]\geq\sum_{i=1}^p\theta_i(f_i-\overline{v}_ig_i)^0(\overline{x};\eta(x,\overline{x}))\geq 0 \tag{4}$$

for each point x feasible of (VFP).

We claim that $\theta \neq 0$. In effect, if $\theta = 0$, then $\lambda \neq 0$ and the next inequality follows from (2)

$$\sum_{j \in J(\overline{x})} \lambda_j h_j^0(\overline{x}; v) \geq 0, \ \forall v \in T_X(\overline{x}). \tag{5}$$

But $\lambda \geq 0$ and $h_j^0(\overline{x}; v_0) < 0, \forall j \in J(\overline{x})$ and then $\sum_{j \in J(\overline{x})} \lambda_j h_j^0(\overline{x}; v_0) < 0$, that contradicts (5).

Then, it is sufficient to take $w = \theta$. The equation (4) guarantees that $\overline{x}$ is solution of $(\mathrm{VFP}_{\bar{v}}(w))$ ☑

As a straightaway consequence of Proposition 2.1 and Lemma 3.1, we have:

Theorem 3.4. *Let $\overline{x}$ be a weakly efficient minimizer of (VFP) and assume that $\overline{v} = \dfrac{f(\overline{x})}{g(\overline{x})}$. Then, there exists $(\overline{\lambda}, \overline{\mu}) \geq 0$ such that*

$$\Big[\sum_{i=1}^{p} \overline{\lambda}_i (f_i - \overline{v}_i g_i) + \sum_{j=1}^{m} \overline{\mu}_j h_j\Big]^0 (\overline{x}; v) \geq 0, \quad \forall v \in T_S(\overline{x}),$$
$$\overline{\mu}_j h_j(\overline{x}) = 0, \quad j = 1, \ldots, m.$$

One more time, we observe that, in the previous theorem we have $\overline{\lambda} \neq 0$, under regularity conditions. In effect:

Theorem 3.5. *Let $\overline{x}$ be a weakly efficient minimizer of (VFP) and assume that $\overline{v} = \dfrac{f(\overline{x})}{g(\overline{x})}$. Furthermore, suppose that (VFP) satisfies a constraint qualification in $\overline{x}$. Then, there exists $\overline{\lambda} \in \mathbb{R}^p_+ \backslash \{0\}, \overline{\mu} \in \mathbb{R}^m_+$ such that*

$$\Big[\sum_{i=1}^{p} \overline{\lambda}_i (f_i - \overline{v}_i g_i) + \sum_{j=1}^{m} \overline{\mu}_j h_j\Big]^0 (\overline{x}; v) \geq 0, \quad \forall v \in T_S(\overline{x}) \tag{6}$$
$$\overline{\mu}_j h_j(\overline{x}) = 0, \quad j = 1, \ldots, m.$$

The proof is very similar to the proof of Theorem 3.3.

The reciprocal of above theorem is true, under KT-invexity hypothesis.

Theorem 3.6. *Suppose that $\overline{x} \in X$ is such that it satisfies (6) with $\overline{\lambda} \in \mathbb{R}^p_+ \backslash \{0\}$ and $\overline{\mu} \in \mathbb{R}^m_+$. Let $\overline{v} = \dfrac{f(\overline{x})}{g(\overline{x})}$ and suppose that (VFP($\overline{v}$)) is KT-invex on the feasible set. Then, $\overline{x}$ is a weakly efficient minimizer of (VFP).*

Proof. Suppose that $\overline{x}$ is not a weakly efficient minimizer of (VFP). From Lemma 3.1, we have that $\overline{x}$ is not a weakly efficient minimizer of (VFP($\overline{v}$)). Then, there exists a feasible point x such that

$$f_i(x) - \overline{v}_i g_i(x) < f_i(\overline{x}) - \overline{v}_i g_i(\overline{x}) = 0, \forall i = 1, \ldots, m. \tag{7}$$

On the other hand, (6) implies

$$\Big[\sum_{i=1}^{p} \overline{\lambda}_i (f_i - \overline{v}_i g_i) + \sum_{j=1}^{m} \overline{\mu}_j h_j\Big]^0 (\overline{x}; \eta(x, \overline{x})) \geq 0,$$

and hence

$$\Big[\sum_{i=1}^{p} \overline{\lambda}_i (f_i - \overline{v}_i g_i)\Big]^0 (\overline{x}; \eta(x, \overline{x})) + (\sum_{j=1}^{m} \overline{\mu}_j h_j)^0 (\overline{x}; \eta(x, \overline{x}) \geq 0. \tag{8}$$

Since (VFP($\overline{v}$)) is KT-invex, we have $h_j^0(\overline{x}; \eta(x, \overline{x}) \leq 0$ and then

$$\sum_{j=1}^{m} \overline{\lambda}_j h_j^0 (\overline{x}; \eta(x, \overline{x}) \leq 0.$$

From the last two inequalities, we can conclude that

$$\Big[\sum_{i=1}^{p} \overline{\lambda}_i (f_i - \overline{v}_i g_i)\Big]^0 (\overline{x}; \eta(x, \overline{x})) \geq 0. \tag{9}$$

But, from KT-invexity we obtain

$$\Big[\sum_{i=1}^{p} \overline{\lambda}_i (f_i(x) - \overline{v}_i g_i(x)\Big] - \Big[\sum_{i=1}^{p} \overline{\lambda}_i (f_i(\overline{x}) - \overline{v}_i g_i(\overline{x})\Big] \geq 0. \tag{10}$$

On the other hand, (7) implies

$$\Big[\sum_{i=1}^{p} \overline{\lambda}_i (f_i(x) - \overline{v}_i g_i(x)\Big] - \Big[\sum_{i=1}^{p} \overline{\lambda}_i (f_i(\overline{x}) - \overline{v}_i g_i(\overline{x})\Big] < 0,$$

and this contradicts (10). Hence, $\overline{x}$ is a weakly efficient solution (VFP). ☑

4. Duality

In this Section, we will formulate a dual model for (VFP). To do this, we based on those models proposed by Jagannathan [4] and Schaible [9]. We will establish some results of duality for the vector fractional problem (VFP).

We define the dual problem[1] of (VFP) by

$$\begin{array}{ll} \text{Maximize} & v = (v_1, \dots, v_p), \\ \text{subject to:} & 0 \in \partial\Big(\sum_{i=1}^{p} \lambda_i(f_i - v_i g_i) + \sum_{j=1}^{m} \mu_j h_j\Big) + N_S(u), \\ & \sum_{i=1}^{p} \lambda_i(f_i(u) - v_i g_i(u)) \geq 0, \\ & u \in S,\ \lambda \geq 0,\ \mu \geqq 0. \end{array} \tag{DF}$$

We will denote by Y the set of all feasible solutions for (VFP).

Now, we will prove some duality results for the pair of problems (VFP) and (DF).

Theorem 4.1 (Weak duality). *Let $x \in X$ and $(u, \lambda, \mu, v) \in Y$ be given. If the problem (VFP(v)) is KT-invex, then*

$$\frac{f(x)}{g(x)} \not< v.$$

Proof. Since (VFP($\overline{v}$)) is KT-invex,

$$\sum_{i=1}^{p} \lambda_i(f_i(x) - f_i(x)) \geq \sum_{i=1}^{p} \lambda_i(f_i(u) - f_i(u)) + \sum_{i=1}^{p} \lambda_i(f_i - v_i g_i)^0(u; \eta(x, u)). \tag{11}$$

Because (u, λ, μ, v) is feasible for (DF), we have

$$\sum_{i=1}^{p} \lambda_i(f_i(u) - f_i(u)) \geq 0. \tag{12}$$

From (11) and (12) we obtain

$$\sum_{i=1}^{p} \lambda_i(f_i(x) - f_i(x)) \geq \sum_{i=1}^{p} \lambda_i(f_i - v_i g_i)^0(u; \eta(x, u)).$$

Since (u, λ, μ, v) is feasible for (DF), we have

$$0 \in \partial\Big(\sum_{i=1}^{p} \lambda_i(f_i - v_i g_i\Big) + \sum_{j=1}^{m} \mu_j h_j) + N_S(u),$$

that is,

$$0 \leq \Big(\sum_{i=1}^{p} \lambda_i(f_i - v_i g_i) + \sum_{j=1}^{m} \mu_j h_j\Big)^0(u; w), \forall w \in T_S(u).$$

In particular, by taking $w = \eta(x, u)$,

$$0 \leq \Big(\sum_{i=1}^{p} \lambda_i(f_i - v_i g_i) + \sum_{j=1}^{m} \mu_j h_j\Big)^0(u; \eta(x, u)).$$

[1]In this problem by "maximize" we mean "find the (weakly) efficient maximizer" of (DF).

Hence,

$$0 \leq \sum_{i=1}^{p} \lambda_i (f_i - v_i g_i)^0 (u; \eta(x,u)) + \sum_{j=1}^{m} \mu_j {h_j}^0 (u; \eta(x,u)). \tag{13}$$

Furthermore, (13) and KT-invexity imply,

$$\begin{aligned} &\sum_{i=1}^{p} \lambda_i (f_i(x) - v_i g_i(x)) - \sum_{i=1}^{p} \lambda_i (f_i(u) - v_i g_i(u)) \geq \\ &\geq \sum_{i=1}^{p} \lambda_i (f_i - v_i g_i)^0 (x; \eta(x,u)) \geq - \sum_{j=1}^{m} \mu_j {h_j}^0 (u; \eta(x,u)) = \\ &\qquad\qquad = - \sum_{j \in J(u)} \mu_j {h_j}^0 (u; \eta(x,u)) \geq 0. \end{aligned} \tag{14}$$

Now we will suppose that $\dfrac{f(x)}{g(x)} < v$. Then $f_i(x) < v_i g_i(x)$ and this implies

$$\sum_{i=1}^{p} \lambda_i (f_i(x) - v_i g_i(x)) < 0,$$

and it contradicts (14). ☑

Theorem 4.2 (Strong duality). *Suppose that (VFP(v)) is KT-invex for each $v \in \mathbb{R}^p$ such that there exists (u, λ, μ) satisfying $(u, \lambda, \mu, v) \in Y$. Moreover, suppose that $\overline{x} \in X$ is a weakly efficient solution for (VFP) and that the constraint qualification is verified at $\overline{x}$. Then, there exists $(\overline{\lambda}, \overline{\mu}, \overline{v})$ such that it is a weakly efficient maximizer of (DF).*

Proof. Let $\overline{v} = \dfrac{f(\overline{x})}{g(\overline{x})}$. If $\overline{x}$ is a weakly efficient minimizer of (VFP), then the Theorem 3.4 implies that there exist $\overline{\lambda} \geq 0$ and $\overline{\mu} \geqq 0$ such that

$$\Big[\sum_{i=1}^{p} \overline{\lambda}_i (f_i - \overline{v}_i g_i) + \sum_{j=1}^{m} \overline{\mu}_j h_j\Big]^0 (\overline{x}; v) \geq 0, \quad \forall v \in T_S(\overline{x});$$

$$\sum_{j=1}^{m} \overline{\mu}_j h_j(\overline{x}) = 0.$$

Note that the last inequality is equivalent to

$$0 \in \partial \Big(\sum_{i=1}^{p} \overline{\lambda}_i (f_i - \overline{v}_i g_i) + \sum_{j=1}^{m} \overline{\mu}_j h_j \Big)(\overline{x}) + N_S(\overline{x}) \tag{15}$$

and

$$f_i(\overline{x}) - \overline{v}_i g_i(\overline{x}) = 0, \quad i = 1, \ldots, m. \tag{16}$$

Hence, (15), (16) imply $(\overline{x}, \overline{\lambda}, \overline{\mu}, \overline{v}) \in Y$. Suppose that $(\overline{x}, \overline{\lambda}, \overline{\mu}, \overline{v})$ is not a weakly efficient maximizer of (DF). Hence, there exists (x, λ, μ, v) such that

$$v_i > \overline{v}_i, \quad i = 1, \ldots, m,$$

that is

$$\frac{f_i(\overline{x})}{g_i(\overline{x})} < v_i, \quad i = 1, \ldots, m,$$

which contradicts Theorem 4.1. ☑

Theorem 4.3 (Inverse duality). *Let $(\overline{x}, \overline{\lambda}, \overline{\mu}, \overline{v}) \in Y$ and (VFP($\overline{v}$)) is a KT-invex problem. If $\overline{v} = \dfrac{f(\overline{x})}{g(\overline{x})}$ and $\overline{x} \in X$, then $\overline{x}$ is a weakly efficient minimizer of (VFP). Moreover, if for each $(u, \lambda, \mu, v) \in Y$ the problem (VFP(v)) is KT-invex on the feasilble set, then $(\overline{x}, \overline{\lambda}, \overline{\mu}, \overline{v})$ is a weakly efficient maximizer of (DF).*

Proof. Let $(\overline{u}, \overline{\lambda}, \overline{\mu}, \overline{v}) \in Y$ be given. If $\overline{x}$ is not a weakly efficient minimizer of (VFP) then, from Lemma 3.1, this point cannot be a weakly efficient minimizer of (VFP($\overline{v}$)). Hence, there exists $x \in X$ such that

$$f_i(x) - \overline{v}_i g_i(x) < f_i(\overline{x}) - \overline{v}_i g_i(\overline{x}) = 0, \ \forall i = 1, \ldots, m,$$

or equivalently,

$$\frac{f(x)}{g(x)} < \overline{v}.$$

But it contradicts the Theorem 4.1.

Now, we will prove the second affirmation. Suppose that $(\overline{u}, \overline{\lambda}, \overline{\mu}, \overline{v})$ is not a weakly efficient maximizer of (DF). Then, there exists $(u, \lambda, \mu, v) \in Y$ such that

$$v_i > \frac{f_i(\overline{x})}{g_i(\overline{x})}, \forall i = 1, \ldots, m,$$

and it, again, contradicts the Theorem 4.1. ☑

Conclusion: In this paper we obtained necessary and sufficient conditions for weakly efficiency to nonsmooth vector fractional problems. To establish our results, we used the parametric approach proposed by Jagannathan [4] and we employed a notion of KT-invexity, generalized to the nonsmooth context. Our results extends those obtained by Osuna-Gómez *et al.* [6]. Also, we established some duality results for these problems, that generalizes those obtained in [6].

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