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THE ROLE OF LEIBNIZ AND JAKOB BERNOULLI FOR THE DEVELOPMENT OF PROBABILITY THEORY

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RESUMEN

Este artículo consta de dos partes. En la primera se desarrollan las ideas sobre el cálculo de la probabilidad antes de Jakob Bernoulli, fundador de dicha teoría, con el propósito de mostrar los antecedentes de los logros de Bernoulli y enfatizar el papel de Leibnitz. La segunda parte trata de las relaciones entre Leibnitz y Bernoulli y de Bernoulli mismo, especialmente en el tema de la introducción de la probabilidad en Matemáticas.

ABSTRACT

This paper consist therefore of two parts: The first is concerned with the development of the calculus of chance before Bernoulli in order to provide a backgraund for the achievements of Jakob Bernoulli and will emphase especially the role of Leibniz. The second part deals with the relationship between Leibniz and Bernoulli and with Bernoulli himself, particularly with the question how it came about that he introduced probability into mathematics.

Palabras clave: Jakob Bernoulli, Leibnitz, siglo XVII, siglo XVIII, Teoría de la Probabilidad.

First some preliminary remarks:

Jakob Bernoulli is of special interest to me, because he is the founder of a mathematical theory of probability. That is to say that it is mainly due to him that a concept of probability was introduced into a field of mathematics which one could call the calculus of games of chance before Bernoulli. This has another consequence that makes up for a whole programme: The mathematical tools of this calculus should be applied in the whole realm of areas which used a concept of probability. In other words the Bernoullian probability theory should be applied not only to games of chance and mortality questions but also to fields like jurisprudence, medicine, etc.

My paper consists therefore of two parts: The first is concerned with the development of the calculus of chance before Bernoulli in order to provide a background for the achievements of Jakob Bernoulli and will emphasize especially the rolę of Leibniz. The second part deals with the relationship between Leibniz and Bernoulli and Bernoulli himself, particularly with the question how it came about that he introduced probability into mathematics.

I should add that a good deal of this paper consists in a reaction to the presentation of the same topic by Ian Hacking which one can find in his excellent book *The emergence of probability*[1]. Hacking, however, uses a completely different approach from mine. First he is concerned to show that the concepts of probability as distinguished by philosophers like Carnap in our century were already available in the 17th century. Secondly he uses these concepts in order to evaluate the results archieved by e.g. Leibniz and Jakob Bernoulli. This approach has a serious disadvantage: it forces Hacking to confess that Bernoulli's understanding of probability fits nowhere in the pigeon-hole system of modern distinctions and labels of probability. This at least may be understood as a justification for my simpler historical approach.

Whenever one asks why something like a calculus of probabilities arose in the 17th century, one already assumes several things: for instance that before the 17th century it did not exist, and that only then and not later did such a calculus emerge. If one examines the quite impressive secondary literature on the history of probability, one finds that it is by no means a foregone conclusion that there was no calculus of probabilities before the 17th century. Even if one disregards numerous references to qualitative and quantitative inquiries in antiquity and among the Arabs and the Jews, which, rather freely interpreted, seem to suggest the application of a kind of probability-concept or the use of statistical methods, it is nevertheless certain that by the end of the 15th century an attempt was being made in some arithmetic works to solve problems of games of chance by computation. But since similar problems form the major part of the early writings on probability in the 17th century, one may be induced to ask why then a calculus of probabilities did not emerge in the late 15th century.

One could say many things: For example, that these early game-calculations in fact represent one branch of a development which ultimately resulted in a calculus of probabilities. Then why shouldn't one place the origin of the calculus of probabilities before the 17th after all? Quite simply because a suitable *concept* of probability was missing from the earlier computations. Once the calculus of probabilities had been developed, it became obvious that the older studies of games of chance formed a part of the new discipline.

We need not consider the argument that practically all the solutions of problems of games of chance proposed in the 15th and 16th centuries could have been viewed as inexact, and thus at best as approximate, by Pascal and Fermat in the middle of the 17th century, —that is, before the emergence of a calculus of probabilities—.

The assertion that no concept of probability was applied to games of chance up to the middle of the 17th century can mean either that there existed no concept of probability (or none suitable), or that though such a concept existed it was not appllied to games of chance. I consider the latter to be correct, and in this I differ from Hacking, who argues that an appropriate concept of probability was first devised in the 17th century.

I should like to mention that Hacking and I agree on a number of points. For instance, on the significance of the legal tradition and of the practical ("low") sciences: Hacking makes such factors responsible for the emergence of a new concept of probability, suited to a game calculus, while perceive them as bringing about the transfer and quantification of a pre-existent probability-concept.

To defend my thesis I shall first show that such a concept did in fact exist. I shall then explain why it remained impossible to connect the concepts of chance and probability until 17th century. As the final step I shall portray the background which made possible a quantification of the probable.

The antecedent concept I have in mind is not to be understood as an extension of a medieval scholastic view of probability, such as one meets in Thomas Aquinas[2], but is rather a direct adoption of one already fully elaborated in antiquity. "Direct" is to be taken literally, i.e., such pioneers in the quantification of the probable as Leibniz and the authors of the Logic of Port Royal drew not from a continuing tradition, but rather direct from ancient Greek sources, indeed frenquently with verbatim citations. Thus, to answer why no quantification of probability was attained earlier when such a concept had been so long at hand, I will propose that this quantification required a displacement or, better, an extension of this concept's realm of application and that this occurred only in the 17th century.

This answer will consist of three steps. The first of which deals with probability as understood in antiquity and first of all by Aristotle. The second step shows that this concept of probability was not applied to games of chance. In the third step it will be shown why and how in the person of Jakob Bernoulli the probability concept was combined with an already existing calculus of chances.

As a starting point for a probability concept in antiquity we can take Aristotle. Aristotle uses different terms in order to express the probable. The most important is the term "endoxos". By "endoxos" he denoted the range of a belief dependent on the state of information of the subject; for Aristotle this precursor of a later concept of subjective probability was significant above all in dialectical debate and particularly in rhetoric, so important for political and juristic decision-making. (It is interesting that these areas of application of the probability concept became of explicit concern in this same context once again in the 17th & 18th centuries). In contrast to Hacking, who views Aristotle and his followers as irrelevant because of their great chronological remoteness, one can show that the understanding of probability in the 17th century was linked directly with that of antiquity. For instance Leibniz refers not only to Aristotle's *Topics*, where the meaning of "endoxos" is explained[3], but also directly to the phrase "hös epi to poly"[4], which Aristotle uses in anticipation of an understanding of probability independent of the subject and which for a given starting situation describes the subsequent situation that as a general rule is to be expected.

Aristotle took this concept of the general rule, that is, of that which occurs in most or the majority of cases, from the field of medicine. For example, in the first book of the Hippocratic tract *De Morbis* one finds repeatedly the formulation that a sickness of such-and-such a kind will "in most cases" end in death. Aristotle links the notion of the general rule with "endoxos" because the subjective expectation of an event which occurs as a rule is greater than that of the exception to the rule. It is decisive for the realm of applicability of the Aristotelian understanding of probability that, of the three realms of[1] that which always and therefore neccessarily happens[2] that which happens in most cases or as a rule and[3] that which happens by pure chance, only the first two are accesible to scientific research. Since one may further show that inn Aristotle's view games of chance belong to the third realm, it follows that, according to his dogma, probability, taken as a scientific notion cannot be applied to them. The impact of this in the 17th century can be seen in Johannes Kepler, who never appllies the concept of "probabilitas" in his deliberations on dice and the realm of chance[5].

Up to now we have considered an understanding of probability only as it appears in the thought of Aristotle. But we have yet to examine the elaboration of this Aristotelian understanding by the Sceptics of the Middle Academy, in particular by Carneades. From the writings of Sextus Empiricus one learns that Carneades distinguished three levels of the probable as the only possible landmarks for one's decisions in practical life and that each of these levels embraces a continuos interval of intensities[6].

To be sure, virtually nothing of Sceptical thought passed into the scholastic tradition. But from the middle of the 16th century the writings of Sextus Empiricus became available again, in printed form. Furthermore, and above all in view of the then current theological controversies, the need for coming to terms with Scepticism had long since become clear[7].

In particular one can discover a familiarity with Sceptical ideas taken from a reading of Sextus Empiricus in Leibniz and in the Logic of Port Royal. Hence, a concept of probability suited to the quantification of the probable achieved in the second half of the century could have been derived directly from the Sceptical tradition.

At the same time one sees that, as long as the Aristotelian dogma of the impossibility of any scientific treatment of chance could not be included within the realm of "probabilis"; and moreover that any attempt to mathematize problems about games of chance within the framework of the established scientific domain was inconceivable. The mathematization of games of chance had to be achieved in an environment freed from the tabus of scholastic science. Such an environment was defined by the *maestri d'abaco* (reckoning masters), who above all satisfied the need for training in the economic and commercial spheres of the late middle ages, a training not available within the universities.

From the above it is clear that an attempt to mathematize games of chance could be undertaken only in a community which found itself outside the dominion of the Aristotelian dogma; that is, it either knew nothing of the alleged impossibility of treating chance by scientific means, or else it could afford to ignore it.

That the economic circles from the 14th century on formed an environment wich dared to ignore or even to break with traditional values can be shown by another example. In the course of the economic expansion of the late Middle Ages and early modern period the idea that money is sterile was overthrown. This notion staunchly defended by theologians, was weakened by among other things the establishment of games of chance in commercial circles and by the attempt to view determinate situations in games of chance as analogous to risk-taking ventures in trade. One example of this is the structural equivalence between the problem of dividing stakes when a game must be interrupted (problem of points) and the division of gains and losses in the so-called "commenda". (Contracts like the "commenda" completely disregarded the Church's insistence on the sterility of money). Games of chance, which of course have always existed, were spread far and wide from the 14th century on by the travelling merchants, and came to be seen as temporally and spatially condensed representations of commercial activity. Above all one saw an essential parallel in the chance redistribution, of an investment (stake) among the participants. It is not surprising, then, that the determination of profit shares that was a matter of course in high-risk trading ventures, particularly in the sea trade, was transferred to gaming situations, in which risk enters in the form of the unknown outcome of a chance event.

Besides the need to supervise affairs of trade, through the fullest possible quantification of all relevant factors it was necessary to consider the dimension of *time*, especially in speculative business. The planning individual is of particular significance for the early forms of commercial forecasting, out of which there developed a general interest in future events. Connected with this new interest is the goal of controlling the future[8].

In the aftermath the efforts to solve problems of games of chance in the Italian arithmetics up to the middle of the 16th century were interrupted, largely because the Counter Reformation re-emphasized traditional moral values and not only prohibited games of chance but also regarded them as snares of the devil.

At the French court, the rage for games of chance, imported from Italy, took hold only in the 17th century. In the middle of the century there grew the feeling, at least in Paris court circles, that the tabus of the Church were no longer binding as they had been a century before. The development of the new mathematics by Viète and Descartes seemed at the same time to justify an enthusiastic optimism that all conceivable problems could be solved mathematically. An oral tradition, going back to the attempts of the Italian *maestri d'abaco* of the late 15th and 16th centuries, may have contributed to the extension of this optimism to the realm of chance. This led finally to the well-known successes of Pascal, Fermat and Huygens in solving special problems of games of chance by the methods of the new algebra.

The analogy between trade and games of chance, so obvious at the time, made it easy for Huygens to use the value of expectation as his central concept[9]. But having achieved the numerical determination by means of algebra, he was left facing a problem of justification. After what we have said above it is not surprising that words like "probabilis" or "verisimilis" nowhere appear in Huygens treatise. These concepts, which fall within the tradition of the Aristotelian "endoxos", he uses in an entirely different context, for example in his discussion of the possibility of life on other planets.

Relatively soon after the first publication of Huygens' treatise we encounter an extension of the validity of "probabilis" to chance event like games of chance. This took place in the *Ars Cogictandi*, better known as the Logic of Port Royal[10] in the 1660's. In the very last chapter of this book the chances for specific events are equated with the ratios of the associated degrees of probability. This belongs to a discussion about judging the certainty of occurrence of events.

While in the 15th chapter the *Ars Cogictandi* discusses the evaluation of our belief in past events, in the sixteenth chapter it takes up the belief in future contingent events.

The attendant observation that, in the decisions of daily life, only the gain hoped for or the loss feared are generally taken into account, without consideration of the *probability* that gain or loss will occur, motivates a discussion of degrees of probability. As a model for the quantitative discrimination of such degrees of probablity the example of a game involving ten players with equal chances of winning and equal contributions to the stakes is used. The winner takes all, i.e., nine others besides his own stake, while the other players lose theirs. This situation is explained as follows: "Thus, a player has nine silvers to hope for, one piece to lose - nine degrees of probability of losing a coin, and only one degree of winning nine"[11]. The only thing new here, in comparison with Huygens' computation of expectation values, is the translation of ratios of chances in random decisions into a ratio of degrees of probability.

To readers of the widely-disseminated, Jansenist *ars cogitandi* this translational equivalence —prompted by the subject of game-calculations, future events with uncertain outcomes— may have suggested that the entire realm to which the concept *probabilis* was applicable could be made accessible to mathematics. The *Ars cogitandi* nevertheless gives us no indication of how one is to evaluate the circumstances relevant in any particular case to an event of interest.

As factors promoting the use of the ratio of chances model in determining the ratio of degrees of probability one might cite the following:

1. A new attitude toward the future as an outgrowth of economic speculation. In the 17th century it had developed into the quest for rational planning; with this was joined the question of the "certainty" with which one could expect he occurrence of future events.

2. A concept of chance much changed from the "blind chance" of the Epicureans and from the Aristotelian concept. A new understanding of chance arose through the notion of being placed in a world determined by divine Providence; it allowed the application of "probabilis" to chance events.

3. The revival of Sceptical ideas, through which the idea of degrees of probability, as had been distinguished in the Middle Academy, were once again brought into currency.

4. The struggle against the probabilism of the Jesuits, with the aim (among other things) of hindering the misuse of authority with the help of quantitative approaches in the realm of probabilities hiterto only qualitatively differentiated. The extension of the applicability of "probabilis" to future chance events suggested the adoptiono, for the purposes of quantification, of the games of chance model which had developed independent of any understanding of probability.

Beyond this by no means insignificant suggestion, the *Ars cogitandi* contains little of consequence for the foundation of a calculus of probability.

It is quite more difficult to understand the role of my compatriote Gottfried Wilhelm Leibniz in the creation of a calculus of probabilities. Leibniz with his encyclopedic mind might be a good witness of what seemed to be contemporary knowledge about the probable.Beyond this purely passive mirroring of the contemporary situation Leibniz had programmatic ideas in which probability played an important role. So one can find passages in the work of Leibniz that establish Leibniz' interest in making the evaluation of degrees of probability in adjudication exact. The significance of jurisprudence as an area for the application of the clculus of probability developed by Jakob Bernoulli is indicated not only in the correspondence of Bernoulli and Leibniz, but also in the dissertation of Niklaus Bernoulli, Jakob's nephew, who sought to apply the findings of his uncle to a series of concrete problems in law. Hacking holds the view that the 1660's were the decisive period in the development of the numerical evaluation of probabilities, to support which he cites Leibniz and the *Ars cogitandi* as principal witnesses. In contrast to his position, I maintain that the decisive program of finding a measure for probability conceived of as a degree or fraction of certainty and erecting upon this measure the first calculus of probabilities was carried out by Jakob Bernoulli, who took from his predecessors essentially little more than an inspiration.

The decisive works of Jakob Bernoulli, dating from the 1680's, applied concrete computation to problems other than the calculation of games of chance and of mortality. The role played by Leibniz must now be considered once again, in order to assess the achievement of Bernoulli in becoming the true creator of the calculus of probabilities.

I will thus seek to answer the following questions:

1. Were Leibniz' ideas on probability novel?

2. Why was he interested in the evaluation of probabilities?

3. What examples of concrete numerical evaluation of degrees of probability are to be found in Leibniz' works?

One should first note that Leibniz intended to devise a new logic for determinig different degrees of the probable even before his stay in Paris, that is, at a very early stage of his scientific development. For instance in a letter of 1670, Leibniz is urged to work out his announced *Doctrina de gradibus probabilitatis*[12]; this establishes that Leibniz had in mind the program of setting out a more comprehensive *Doctrina* quite early.

Concerning the question of Leibniz' sources for his understanding of probability and of the distinction of degrees of probability, one may observe that for the then usual understanding of *probabilitas* he turns directly to Aristotle, whom he frequently cites, especially in connection with the *Topics*. Of greater interest here, however, are his references to the Middle Academy and the Sceptics, for these suggest that Leibniz owed to his study of the Sceptics' ideas (as available from Sextus Empiricus) his distinction of degrees of probability. It is clear fron the *Nouveaux Essais*[13] that Leibniz, in his distinction among the different degrees of probability, was able to specify concretely only qualitative differences, not levels which might be conceived quantitatively.

Here, as in the examples from civil law which follow, Leibniz is thinking not of a numerical gradation of the probable, but of a qualitative ordering, corresponding to the view of the Sceptics,and indeed they too had entertained the notion of a continuum of possibilities of modality. His comment in the same paragraph on the situation in medicine confirms this.

From this we can already see that Leibniz' understanding of probability was linked directly with that of the ancients and that his motives for attempting to evaluate degrees of probability grew chiefly out of his study of law. It remains to inquire whether he did not after all go on to make quantitative trials in this field. To this end, we may consider two questions: (1) What was Leibniz' own judgment of such efforts? (2) Did Leibniz give examples of evaluations of probabilities?.

Of interest in connection with the first question is a passage in a letter Leibniz wrote to Gabriel Wagner (1698), in which he says that in determining the "gradus probabilitatis", one ought to pay attention to indications or signs that do not make up for a complete proof as in medicine where a set of symptoms does not demonstrate completely the existence of a certain illness[14]. These indications should not be counted but rather be weighted despite the non existence of an appropriate balance. However, the best approximation for the construction of such a balance has been provided by jurisprudence according to Leibniz.

From this passage it is clear that he regarded the problem of evaluating degrees of probability as no mere enumeration say of the circumstances favoring the occurrence of an event, but rather as a weighting that allows for the diversity of these circumstances. This does not mean that he excludes the usual enumeration of "equipossible" cases, an done in deriving the ratios of chances in computing games of chance; on the contrary an example in another passage, having to do with the relative evaluation of events, namely, of obtaining a 9 or a 7 by the cast of two dice, shows how such a weighting can succeed, at least in computing games of chance.

It would now be interesting to see in what way, if at all, Leibniz succeeded in making such evaluations in the area of law. Hacking has already examined his treatment of "jus conditionale" which, in contrast with "jus nullum" (assigned the value 0) and "jus purum" (assigned the value 1), is given a fractional value between 0 and 1. Unfortunately, examples in which some such fractional value is actually assigned in a concrete instance of law are absent from the extant juristic writings of Leibniz.

All this demonstrates that Leibniz failed to carry out his own program of evaluation concretely, both in his published works and in his posthumous and unpublished papers. He held essentially qualitative distinctions among degrees of probability, as was then common in the practice of law and medicine. But he mantained that his program could be implemented by a mathematician who studied systematically and in detail problems of games of chance and of games in general.

Leibniz lived to hear of such a mathematician, who independent of him and without knowledge of his program had worked on the application of an "ars conjectandi", as this mathematician called the numerical calculation of probabilities, to "civic, moral and economic matters". Thgis man was Jakob Bernoulli's death. An assessment by Leibniz of Bernoulli's achievement indicates clearly who, in Leibniz' own view, had founded the calculus of probabilities and thereby made possible, to use one of Leibniz' own phrases a new logic that treats of degrees of probability (*logique qui traiteroit des degrés de probabilité*)[15].

In the light of these statements it seems to be worthwhile to look at Leibniz' possible influence on Jakob Bernoulli's conceptions of probability and especially on the fourth book of the *Ars conjectandi*[16]. Leibniz exchange of ideas with Jakob Bernoulli concerning mathematization of probabilities begins with a letter from Leibniz of April 1703[17]. In his postscript Leibniz remarked that he had heard of Bernoulli's involvement with a *doctrina de aestimandis probabilitatibus*. In this well-known prelude to a longer discussion of the topic, it is interesting to find briefly expressed Leibniz hope, corresponding to his early program, that someone like Bernoulli would treat mathematically the different kinds of games in which beautiful examples of such a *doctrina* could be found. That means that at least by this time it had become natural for Leibniz to regard the mathematical treatment of games of chance as a part of a theory of probability estimation. The secondary literature has justifiably emphasized that Jakob Bernoulli, who had worked for many years on questions of games of chance and probability[18] calculations, took the postscript as an invitation to communicate his most cherished ideas. This is all the more understandable as Jakob Bernoulli had long sought in vain for a suitable correspondent in this area of study and Leibniz appeared to be able to fulfill that role. This explains too, why Bernoulli already in his answer presented the central problem of his research as well as his most important result, his main theorem. Bernoulli was concerned with the determination of such probabilities as that a young man of age twenty will survive a man of sixty. Bernoulli called these probabilities *a posteriori*, because they can be determined only in retrospect on the basis of numerous observations of the occurrence of a relevant event. Bernoulli's main theorem was supposed to establish that with an increasing number of observations the estimated value of the probability approaches the true value, at least with probability. That Jakob Bernoulli knew of Leibniz' activities in the field, is demonstrated by his request that Leibniz should send him juridical material to which one could, in Leibniz' judgement, apply *a posteriori* determination of probabilities. At the same time Bernoulli was interested in obtaining the assessment[19] of Jan de Witt, Raadspensionaris of Holland, in which the advantage of buying and selling life annuities, was determined on the basis of hypotheses about life expectations at different ages. In his answer Leibniz[20] emphasized at first the extraordinary utility of the *aestimandae probabilitates* only to add inmediately the qualification that in the area of jurisprudence and politics, which was so important for Bernoulli's program, no such extended calculations were usually required, since an enumeration of the relevant conditions would suffice. Considering the request of Bernoulli, this implies that Leibniz was not in the position to offer concrete juridical material to which the methods of Bernoulli's probability theory could be applied. It is relatively certain that if Leibniz had found non-trivial evaluations in the realm of conditional right, he did not remember them in 1703. Leibniz tried to shake Jakob Bernoulli's self-confidence which was founded above all on his discovery of the main theorem. Against the possibility of attaining a better approximation to a sought-after probability with an increasing number of observations, Leibniz suggested that contingent events, here identified with dependence on infinitely many conditions, could not be determined by a finite number of experiments. As a foundation Leibniz added that to be sure, nature has her conventions, which follow from the permanent repetition of causes. That this holds only as a rule which permits exemptions, is expressed by the classical greek term *hos epi to poly*. In this sense Bernoulli's presupposition of the absolute determinability of a probability *a posteriori* seemed already questionable, because it implied the invariance of such a probability with time. For Leibniz the appearance of new diseases could change the probability of survival of a twenty year old relative to a sixty year old. Leibniz attempted to lend greater weight to his objection through the example of determining the orbits of comets, these were always found under the assumption that the orbit was a conic section. But if this presupposition is dismissed, then there would be infinitely many diferent curves that fit the observations. Bernoulli was understandably not particularly pleased with Leibniz's objections. In a letter of 1704[21] he emphasized that the mere enumeration of conditions in law did not suffice; rather, calculations were required just as for games of chance. Bernoulli referred to problems of insurance, life annuities, marriage contracts, *praesumptiones* and others. He put off until later supplying Leibniz with an illustration of such calculations perhaps because Leibniz had disappointed him regarding the requested juridical material. Anew he attempted to clarify his main theorem, using the example of an urn containing white and black stones in the ratio of 2:1. In this case Bernoulli claimed to be able to determine exactly the number of draws (with replacement) for which it would be ten times, a hundred times, a thousand times, etc., more probable that the ratio of white to black stones found by drawing would fall inside rather than outside a given interval about the true value, for example, 199/100, 201/100. Although Bernoulli could only prove this assertion for probabilities from *a priori*, as in the urn model, he was convinced that he had also shown with his main theorem, the solubility of the reverse problem, namely, the determination of unknown, *a posteriori* probabilities. This false conclusion becomes understandable through Bernoulli's implication that it would make no difference for the behaviour of the observed ratio whether the person drawing the stones knew the true ratio or not. The possibility that two urnes containing different ratios of white to black stones would yield the same ratio for an equal number of draws, appeared conceivable to Bernoulli only for a small number of draws, while for a large number such a result would be excluded by the "moral certainty" *secured* through the main theorem.

In this way of thinking Bernoulli saw no problem with applying the urn model to human mortality, with the stones corresponding to diseases with which a person can be taken ill[22].

However, he was prepared to concede that with the data then available the life expectancy of the antediluvians could not be found. For him it was only important, to be able to determine the validity of the approximation, since in any concrete case the data would be only finite, and he proceeded on the assumption that the probability to be determined would remain stable over a sufficiently long time. Leibniz objections did not hit on the non applicability of the main theorem to the reverse problem; they were concerned with the applicability of the urn model to areas like human mortality. Jakob Bernoulli's research program was not affected by these objections. This research program stood firm after Bernoulli's discovery of his main theorem in 1689.

After all Leibniz was not convinced that an increase in the number of observations would in all cases improve the certainty of the attained result. Leibniz acknowledged that for pure mental games and games of chance one could calculate the chance of winning even though with some difficulty,while in most cases, on the basis purely of reflection, one could determine only who has the better position. There were, certainly, inventive players who without calculating made their decisions as in military matters and in medicine on the basis of a multitude of judgements. Leibniz appreciated this way of thinking as an *ars*.

To what degree Leibniz stimulated Bernoulli's treatment of the qualitative evaluation of probabilities, as presented in the fourth part of the *Ars conjectandi* may no longer be determined[23]. This correspondence, untimely ended by Jakob Bernoulli's death, shows that Leibniz could fulfill Bernoulli's desire for a congenial correspondent only in part.

With this remark we can turn to Jakob Bernoulli himself and begin with his main work, the *Ars conjectandi.*

Bernoulli left the *Ars conjectandi* as an unfinished manuscript, whose content in its most important sections went back to preliminary studies he had done in the 1680's. These early studies are now available, with the publication of the relevant passages of his scientific diary, the *Meditationes*[24]. One can now reconstruct the origins of his various ideas on probability. One can identify with certainty among his sources, first, the tract by Huygens, which was reprinted in the first part of the *Ars conjectandi* with Bernoulli's annotations; the *Ars cogitandi*; and the combinatorial investigations by Pascal in his *Triangle arithmétique*. One can see how Bernoulli, beginning from the notion, contained in the Logic of Port Royal, of the identification of ratios of chances with the ratio of degrees of probability, developed the classical concept of probability, still current far into the 19th century, and how he established as its measure a generalization of Huygens' determination of expectation, namely, the ratio of favorable to possible cases. At the same time he became the first to set down the prerequisites for consciously formulating a program for the mathematization of all the fields of application subject to "probabilis". Bernoulli himself sought to execute this program, but his premature death prevented him. His nephew, Niklaus Bernoulli, who was especially close to him in the last years, took up the work of applying the *Ars conjectandi* to the study of law, where in constrast with Leibniz' efforts concrete instances of law were treated numerically.

A key passage for the transformation in the conception of probabilities and of probability involves the treatment of a problem of law, on which Bernoulli worked, according to the *Meditationes*, in 1685-86. It has to do with a marriage contract, which, assuming that the couple is blessed with children and that the wife dies before the husband, will govern the division of their common property between the father and the children. A distinction is made among the possibilities that both, one or neither of the fathers of the bridal couple, alive at the time of the conclusion of the contract, die and leave their states to their children. Only the property which has become the common property of the couple is to be regulated, and so such distinctions need to be drawn. The portion of the groom will be larger if he has already entered into his inheritance, smaller if not, unless both fathers have died. The bride's father objects to his initial proposal; this induces Titius to make a second proposal, according to which he will receive the same portion of the common property regardless of what happens to the fathers.

On this basis Bernoulli poses the question: which suggestion would be more favorable for the children? To this end he has to make assumptions about the possible order of death of the three people involved, the two fathers and the bride, Caja. He first assumes that all six possible orders have equal weight. But this assumption does not satisfy him, since the youth of the bride has not been taken into account. Thus, he assumes that for every two instances —e.g., diseases, symptoms or events— which might bring about the death of either father, there is only one which threatens Caja with death. There are thus five cases in all, each equally likely to take its victim first. Since Caja is affected by only one of these, while the two fathers are affected by four, here situation is evaluated as one-fifth of certainty of her being first to die, that is, "one probability, five of which make the entire certainty"[25].

Here Bernoulli uses the plural "porobabilities", where these are equated sith the no more precisely distinguished individual cases; this usage does not permit the conception of "probability" as "degree of certainty" which is observed in the next stage. Aided by Huygens' formula for determining expectation, Bernoulli then derives a certainty of 4/15, written 4/15 c (where "c" stands for "certitudo"), or 4 probabilities our of 15, that Caja will die second, and finally 8/15 c that she will die third. (We will not here describe the further hypotheses which Bernoulli employs to weight the various orders to death).

It is interesting that at this time Bernoulli appears not to have had knowledge of either the *Observations* of Graunt (1662) or the *Waerdye* (Estimate) of de Witt (1671)[26]. Yet at the end of his treatment of the marriage contract between Titius and Caja he proposes extensive investigations on human mortality. Here he speaks also of the degree of probability:

"Generally in civic and moral affaires (things) are to be understood, in which we of course know that the one thing is more probable, better or more advisable than another; but by what degree of probability or goodness they exceed others we determine only according to probability, not exactly. The surest way of estimating probabilities in these cases is not *a priori*, that is by cause, but *a posteriori*, that is, from the frequently observed event in similar examples. In our example, if in the course of many years it had been observed that twice as many old men died as young girls, all of whom were of the same age and constitution as our young girls and old men, we would conclude then that there was one case which threatens the young girl whith death and two cases which threaten the old man"[27].

Just before this passage Bernoulli had introduced the fundamental distinction between determination *a priori* and *a posteriori*:

"The reason, that in card and dice games, which are governed solely by chance, the expectation can be precisely and scientifically determined, is that we can perceive accurately and clearly the number of cases in which gain or loss must follow infallibly and that these cases behave indifferently and can each occur with equal facility or when one is more probable than another we can at least determine scientifically by how much it is the more probable. But what mortal, I pray you, counts the number of cases, diseases or other circumstances to which now the old men, now the young men are made subject, and knows whether or not these will be overtaken by death, and determines how much more probable it is that one will be taken unawares than another, since all of these depend on causes that are completely hidden and beyond our knowledge"[28].

One should regard the somewhat pessimistic conclusion of this passage in the light of the passage which follows it and has been cited above, in which the execution of probabilities *a posteriori* is explained.

How does Bernoulli intend to carry out the determination of probabilities *a posteriori*? In particular, does he suppose that the reliability of the derived values will increase with the number of observations? Bernoulli sought to answer these questions by means of what he called his golden theorem (theorema aureum), the first version of which appeared about 1689. In the time between this version and his earlier treatment of the problem of the marriage conotract, he came to avoid using the plural "probabilities" in the sense of the different equipossible cases in games of chance. In formulating the theorem he used "probabilitas" only in the singular, in the sense of the degree of certainty with which the occurrence of an event may be expected. What he establishes in this theorem is that as the number of observations of a repeatable event increases, so too does the probability that the relative frequency of ocurrence of a possible outcome will lie in the vicinity of the probability of this outcome. Only much later did it become clear that he did and could prove this theorem only for the relative frequency of events of known *a priori* probability, but not for those of unknown probability. He understood this theorem as a justification for adopting relative frequencies determined through observation as estimates of probabilities which could not be given *a priori*. At the same time the fundamental theorem served as the essential foundation of Bernoulli's program to extend the realm of application of numerically determinable probabilities.

The fundamental theorem was rigorously proved by Bernoulli in the *Meditationes*. There it is presented in a form that holds for arbitrary initial probabilities and that is terminologically oriented toward calculating games of chance:

"It is possible to carry out so many observations that, with any given high degree of probability, it will be more probable that (the ratio) of games won by both sides will lie within any given narrow limits rather than outside them"[29].

A last essential point for the research program of the "calculus of probabilities", the new discipline Bernoulli prepared for in the *Meditationes* and formulated in the *Ars conjectandi*, was a new concept of chance. To be sure, the latter work had assumed an understanding of the contingent that on the one hand permitted the application of the probable to chance events, and on the other hand merely asserted the compatibility of this concept of chance with divine Providence, without explaining this further. Bernoulli sought to close these gaps in the *Ars conjectandi*:

"'Contingent (in the sense 'free'; insofar as it depends on the will of a rational creature, and in the sense 'fortuitous' and 'casual' insofar as it depends on a chance event or on fortune) is that which could not be, become or have been... Contingency does not always entirely exclude necessity, as far as secondary causes, as I shall make clear from examples"[30].

These examples make clear that Bernoulli never thought of events as occurring indeterminately. He was convinced that through a more precise knowledge of the parameters affecting the motion of a die, for instance, it would be possible to specify in advance the result of the throw. In similar fashion he viewed changes in weather as a determinate process, just as the occurrences of astronomical events are. Chance, in his view and later in the view of Laplace, was reduced to a subjective lack of information. Thus, depending on the state of their information, an event may be described by one person as chance, but by another as necessary. With this anticipation of Laplacian determinism Bernoulli appears to solve the problem of the connection between chance and divine Providence. The entire realm of events which are described in daily life as uncertain or contingent in their outcome is such, he claims, merely because of incomplete information: nevertheless, these too fall within the field of the concept "probabilitas". Bernoulli's program to mathematize as much of this realm as possible with the aid of the classical measure of probability occupied researchers throughout the 18th century and into the second half of the 19th.

ABREVIATIONS

AE. *Acta Eruditorum*, Leipzig (from 1682).

JBW. *Die Werke von Jakob Bernoulli* (ed. Naturforschende Gesellschaft in Basel, Basel from 1969).

LMG. G.W. Leibniz, *Mathematische Shriften* (ed. C.I. Gerhardt) vols. I-VII (Berlin/Halle. 1849-63).

LNE. G.W. Leibniz, *Nouveaux Essais sur l'entendement humain* (ed. W.w. Engelhardt and H.H. Holz, two vols., Frankfurt/Main, 1961).

LPG. G.W. Leibniz, *Die philosophischen Schriften und Briefe* (ed. C.I. Gerhardt) vol. I-VII (Berlin. 1875-90).

LSSB. G.W. Leibniz, *Sämtliche Schriften und Briefe* (Darmstadt/Leipzig/Berlin, from. 1923).

PT. *Philosophical Transactions of the Royal Society in London* (from 1665).

NOTES

1 Hacking. I. *The Emergence of Probability: A Philosophical Study of Early Ideas of Probability*. Cambridge, 1975.

2 Cf. Edmund F. Byrne, *Probability and Opinion: A study in the Medieval Presuppositions of Post-Medieval Theories of Probability*. The Hague, 1968, p. 188.

3 Cf. *L.N.E.*, vol. II, pp. 265-267.

4 See Leibniz' letter to James Bernoulli of December 3, 1703; cf. L.M.G., vol. II, p. 84.

5 Cf. Ivo Schneider, 'Wahrscheinlichkeit und Zufall bei Kepler', *Philosophia Naturalis* 16. (1976), 40-63.

6 Cf. Ivo Schneider, 'Contributions of the sceptic philosophers Arcesilas and Carneades to the development of an inductive logic compared with the Jaina-Logic, *Indian Journal for the History of Science* 12 (1977), 173-80. The relevant contributions of Aristotle are also reviewed in this article.

7 Cf. Richard H. Popkin. *The History of Scepticism: From Erasmus to Descartes*, New York, 1964.

8 For the connection between the economic development of the late Middle Ages and the computational treatment of games of chance, see Ivo Schneider and Karin Reich, 'Die wirtschaftliche Entwicklung des Mittelalters im Spiegel der arithmetischen Aufgabensammlungen und ihrer Nachfolger, der Rechenbucher des 15. und 16. Jahrhunderts', in *Aus dem Antiquariat*, June 1978, pp. 217-229; see also Ivo Schneider, 'Die Mathematisierung der Vorhersage Kunftiger Ereignisse in der Wahrscheinlichkeitstheorie vom 17, bis zum 19. Jahrhundert', in *Berichte zur Wissenschaftsgeschichte* 2 (1979), pp. 101-112.

9 See Christiaan Huygens, "De ratiociniis in ludo aleae", in: Frans van Schooten, *Exercitationum mathematicarum libri quinque*, Leyden, 1657, p. 517-534.

10 Antoine Arnauld and Pierre Nicole, *La logique ou l'art de penser*, Paris, 1662 and later: The *Art of Thinking* (transl. by J. Dickoff and P. James), Indianapolis-N.Y.-Kansas City 1964, p. 354.

11 Cf. *The Art of Thinking*. p. 355.

12 Cf. *Leibniz' Samtliche Schriften und Briefe* , ser. II, vol. I, p. 72.

13 Cf. *L.N.E.*, vol. II. pp. 268-273, where Leibniz draws attention to his controversy with the sceptic Foucher, see also vol. II, pp. 508-514.

14 Cf. *L.P.G.*, ed. C.J. Gerhardt, Berlín, 1890, vol. VII, p. 521.

15 Leibniz, *Philosophische Schriften*, vol. III. p. 570.

16 Leibniz' impact has been emphasized especially by Corrado Gini. "Gedanken zum Theorem von Bernoulli", *Schweizerische Zeitschrift für Volkswirtschaft und Statistik*, 82, (1946), 401-413, and by Ian Hacking see note 1.

17 *LMG*. vol. III/1, pp. 62-71, especially p. 71.

18 See the relevant sections of the *Meditationes* published in *JBW*, vol. III. pp. 21-89.

19 Jan de Witt. *Waerdye van Lyf-Renten naer Proportie van Los-Renten* ('s Graven-Hage 1671; reprint in *JBW*, vol. III, pp. 327-350). A detailed description of the role de Witt's tract played for Leibniz, partly represented in the correspondence with James Bernouilli can be found in Jurt-R Biermann, and Margot Faak. 'G.W. Leibniz und die Berechnung der Sterbewahrscheinlichkeit bei J. de Witt", *Forschungen und Fortschritte*, 33, 168-173. For an evaluation of the correspondence between Leibniz and James Bernoulli see K. Kohli, "Aus dem Briefwechel zwischen Leibniz und Jakob Bernoulli", *JBW*, vol. III, pp. 509-513.

20 *LMG*, vol III/I, pp. 79-86, especially p. 83f.

21 *LMG*, vol. III/I, pp. 86-89, esp. pp. 87-89.

22 See *Ars Conjectandi*, p. 226.

23 For this area of James Bernouilli's interests, especially the occurrence of non-additive probabilities, see Glenn Shafer. "Non-Additive Probabilities sin the Work of Bernoulli and Lambert" *Archive for history of exact sciences*, 19, (1978), 309-370.

24 *JBW.*, vol. III, Basel, 1975, pp. 21-89.

25 "...id quod valet 1/5 certitudinis mortis primae seu unam probabilitatem, quarum 5 faciunt omnimodam certitudinem". Cf. *JBW*, vol. III, p. 43.

26 See John Graunt, *Natural and Political Observations mentioned in a following Index, and made upon the Bills of Mortality...*, London, 1662 and frequently thereafter; for Jan de Witt, see note 19. It remains surprising that Bernoulli, who from 1703 to his death in 1705 continually pressed Leibniz to send him a copy of de Witt's work, nowhere mention the ultimately much more illuminating publication of Halley, which should have been easly accessible to him. See Edmond Halley, 'An estimate of the degrees of the mortality of mankind...', *Philosophical Transactions*. No 196, 1693, pp. 596-610 and No. 198, pp. 654-656.

27 Quod et in genere de civilibus et moralibus intelligendum, ubi plerunque unum altero probabilius, satius aut consultius quidem esse novimus, at quot gradibus probabilitatis aut bonitatis antecellet, probabiliter tantum, non accuratè determinamus. Tutissima probabilitates aestimandi via in istis est non à priori, seu causâ, sed à posteriori seu ab eventu in similibus exemplis multoties observato. Quemadmodum in nostro exemplo, si plurimorum annorum decursu observatum fuisset, duplò plures senes quam juvenculas, ejusdem et aetatis et temperamenti cum nostris juvenculis et senibus, mortuos esse; coneluderemus, unum esse casum, qui juvenculae, et duos, qui seni mortem minantur. *JBW*, vol. III, pp. 46f.

28 Quod enim in sortilegis et ludis, quos sola gubernat sors, expectatio praecisè et scientificè determinari possit, causa est, quia accuratè et clarè percipimus numerum casuum, ad quos infallibiliter sequi debet lucrum aut damnum, et quod hi casus indifferenter se habeant et aequè facilè evenire possint, aut saltèm si unus altero sit probabilior, scientificè definire possumus quanto sit probabilior. At quis mortalium obsecro numerum casuum, morborum sc. aliorumve accidentium, quibus obnoxii tum senes tum juvenes, dinumeret, sciatque illos infallibiliter excipi à morte nec ne, determinetque quantò quis altero probabilius grassari possit, cùm haec omnia dependeant à causis omninò occultis et à cognitione nostrâ remotis. *Ibid.*, p. 46.

29 Possible est, tot observationes instituere, ut datá quâvis probabilitate probabilius sit, ut numeri ludorum ab utroque victorum intrà datos limites quantumcunque arctos cadant, quàm extrà illos. *Ibid.*, p. 17.

30 Contingens (tam *liberum*, quod ab arbitrio creaturae rationalis: quam *fortuitum* et *casuale*, quod à casu vel fortuna dependet) est id, quod posset non esse, fore aut fuisse;... nec enim contingentia semper omnem necessitatem, etiam quoad causas secundas, excludit; quod exemplis declaro. *Ars conjectandi*, Basel, 1713, p. 212.