ALGEBRAIC MODELS FOR PROPER HOMOTOPY TYPES

by

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The aim of this Workshop on Proper Homotopy Theory was to provide an opportunity to present and discuss various approaches to proper homotopy theory. My current work in this area is with Luis Javier Hernández and I have attempted to explain below the 'philosophy' behind our approach. Why 'philosophy'? Simply, because all too seldom do mathematicians put in writing their overall view of a subject and how it may evolve. This theme would seem particularly important given the aim of the Workshop.

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1. Algebraic models for homotopy types.

It is well known that algebraic topology is based on the hope that given a topological problem, by modelling it in a suitable way by algebra, one may find a solution, or at least, find if one exists. The 'ultimate' hope would be to model topological spaces completely by algebra and thus to reduce hard *topological* problems to *algebraic* problems which are hopefully more easily solved. This corresponds to an 'ideal scenario' for homotopy theory.

1.1 Ideal Scenario

If we are to hope for good algebraic models for *proper* homotopy types, it will be important to consider to what extent existing models of homotopy types satisfy 'ideal' conditions. In our ideal world, we have a category, Spaces, of spaces and maps. Spaces may just be topological spaces or the more specialised CW-complexes or simplicial complexes, and the maps may just be continuous or may be cellular or simplicial. We hope for a category, Alg. Models, which at present we do not know anything about, and an algebraic modelling functor,

$$G : \text{Spaces} \longrightarrow \text{Alg. Models.}$$

We will also call G a "test functor".

In Spaces, we have a notion of homotopy between maps and can form a homotopy category that will be denoted Ho(Spaces). (We will not for the moment enquire how this is constructed.) In Alg. Models, we assume there are some 'quasi-isomorphisms' corresponding loosely to the homotopy equivalences in Spaces, and we expect, or at least hope, that G will induce a functor

$$G : \text{Ho(Spaces)} \longrightarrow \text{Ho(Alg. Models)}$$

where Ho(Alg.Models) is obtained by formally inverting these quasi-isomorphisms. Ideally this induced G will be an equivalence of categories.

If Alg. Models and G are going to be useful for solving homotopy problems, the above is not enough. We need to be able to 'do' homotopy theory within Alg. Models is a way that mirrors the 'homotopy theory' of Spaces. Here we have a great difficulty: Although we know what "doing homotopy theory" means we do not know what is a "homotopy theory". The structures that have been put forward, all model abstractly certain important aspects of homotopy theory but as yet no one abstract theory is decisively the 'right' one. (Until recently this could have been compared adversely with homology theory, where we thought we knew what was happening, at least in the homological algebra context, but Grothendieck [1983] has raised the question of 'what is homological algebra?' by asking for a full theory of derived categories. It is remarkable that his proposed solution is the homological analogue of that proposed by Heller [1988] as an answer to the question 'what is homotopy theory?' Heller's ideas are briefly considered below.)

The basic tools needed to 'do homotopy theory' are constructions such as mapping cylinders, fibration sequences, etc. These constructions take as initial data a map, or a square of spaces and this suggests that, more generally, for each small category, I, the functor

$$G^I : (\text{Spaces})^I \longrightarrow (\text{Alg. Models})^I$$

should induce an equivalence of homotopy categories of I-indexed diagrams:

$$\text{Ho}((\text{Spaces})^I) \longrightarrow \text{Ho}((\text{Alg. Models})^I)$$

and if $\alpha : I \to J$, then the right and left adjoints of

$$\text{Ho}((\text{Spaces})^{\alpha}) : \text{Ho}((\text{Spaces})^J) \longrightarrow \text{Ho}((\text{Spaces})^I)$$

(technically left and right homotopy Kan extensions) should have analogues in Alg. Models. The structure suggested for Alg. Models has thus to be more or less like that of a homotopy theory in Heller's sense (see Heller [1988] for details). He defines a homotopy theory to consist of the assignment of a category, T(I), to each small category I so that if $\alpha : I \to J$, then α induces $T(\alpha) : T(J) \to T(I)$, and this has both left and right adjoints. This structure has of course to satisfy various axioms. To understand the importance of his ideas, consider, I arbitary, J to be the single map category often called [0], the ordinal category corresponding to the one point ordinal, and α to be the unique functor. Then the left and right adjoints of $T(\alpha)$ are the homotopy colimit and homotopy limit functors respectively. As all such constructions as the suspension, loop space, mapping cylinder, mapping cone etc. as well as more complicated classifying space and bar/cobar constructions are describable as homotopy colimits or limits, the importance of the above structure should begin to be clear.

1.2 Additional desirable features

I. Calculability.

The algebraic modelling functor G will not be much good unless for a reasonably rich class of spaces, we can hope to calculate G(X). Moreover we would hope to be able to calculate [X, Y] by calculating [G(X), G(Y)], again for a reasonably rich class of spaces, X and Y.

How is one to calculate such a G(X)? One possible piece of machinery would be a van-Kampen theorem. This should say (again ideally):

If U, V are open, and $X = U \cup V$, then

$$\begin{array}{ccc} G(U\cap V) & \longrightarrow & G(U) \\ \downarrow & & \downarrow \\ G(V) & \longrightarrow & G(X) \end{array}$$

is a pushout in Alg. Models. To understand this sort of theorem better, we will briefly look at known existing cases of a van Kampen type theorem. This will at the same time serve to introduce various algebraic structure for later use.

a) Classical form (Groups) of van Kampen's Theorem.

We are given a pointed topological space (X, x_0) with non empty open subsets U, V such that $U \cup V = X$ and U, V and $U \cap V$ are arcwise connected with $x_0 \in U \cap V$, then

$$\begin{array}{ccc} \pi_1(U \cap V) & \xrightarrow{j_{U*}} & \pi_1(U) \\ {\scriptstyle j_{V*}}\downarrow & & \downarrow \\ \pi_1(V) & \longrightarrow & \pi_1(X) \end{array}$$

is a pushout of groups. (c.f. Massey [1967]). Of interest for its computational interpretation, the version favoured by Crowell and Fox [1953] shows how one can build a presentation of the group $\pi_1(X)$ given presentations of the other groups and information about the two homomorphisms j_{U*} and j_{V*}. If we fix notation so that if we write

$$G \cong (X : R)$$

we will mean that X is a set, R is a subset of the free group, F(X), and denoting by <<R>> the normal closure of R in F(X), there is an isomorphism, $G \cong F(X)/\langle\langle R \rangle\rangle$. The combinatorial group theoretic version is based on the fact that if

$$\begin{array}{ccc} G_0 & \xrightarrow{\theta_1} & G_1 \\ {\scriptstyle \theta_2}\downarrow & & \downarrow \\ G_2 & \longrightarrow & G \end{array}$$

is a pushout of groups, and if we are given presentations $G_0 \cong (Z : T)$, $G_1 \cong (X : R)$, $G_2 \cong (Y : S)$ then G has presentation

$$G \cong (X \cup Y : R \cup S \cup \{\theta_1(z) = \theta_2(z) : z \in Z\})$$

Here we are abusing notation as $\theta_1(z)$ and $\theta_2(z)$ are not defined! Of course we are meaning that $\theta_1(z)$ should be a chosen word/element in F(X) which represents the image under θ_1 of the generator z, (so writing $\theta_1(z) \in F(X)$ is much simpler and leads to no problems, as should be clear). Similarly for $\theta_2(z) \in F(Y)$.

The combinatorial description is important not only because of the possibility of calculations but also because it starts to bridge the gap between the 'geometry' and the algebra, in this case, the group theory. Of course given a space X that can be built up by attaching cells, this provides an efficient tool for calculating $\pi_1(X)$ and vice versa, given a presentation $\mathcal{P} = (X : R)$ of a group G, we can build a complex $K(\mathcal{P})$ using the data in $\mathcal{P}$, such that $\pi_1(K(P))$ is isomorphic to G. This provides the basis for a lot of combinatorial group theory, the cohomology theory of groups and the study of the ends of groups. This interconnectivity of the various areas is worth remembering for when, later, we look at proper homotopy theory. For instance is there an analogue of the construction, $K(\mathcal{P})$, in the proper context?

Of course the above uses only π_1 and this only models a tiny part of the homotopy type, or, if you prefer, provides an accurate model only for a small class of homotopy types.

b) Classical form (Groupoids) (c.f. Brown [1967] and [1988])

The restriction on arcwise connectedness of U, V and $U \cap V$ is unnecessary if instead of groups, one uses groupoids. This also allows one to model non-connected spaces. One small but important point is that the group based version will not allow you to calculate $\pi_1(S^1)$, but this can be easily read off from the groupoid version.

The main tool is the relative fundamental groupoid, Π_1XX^0; here X is a space, X^0 is a collection of base points (possibly all of X) and Π_1XX^0 is the set of fixed end point homotopy classes of maps α from $I = [0, 1]$ to X in which $\alpha(0), \alpha(1) \in X^0$.

The statement of the van Kampen theorem in its groupoid form is similar to that in the group form, but $U \cap V$ need only have one base point in each arcwise connected component, instead of being itself arcwise connected.

The combinatorial group theory has its analogue here and in fact the combinatorial and geometric features of the algebra are much nearer the surface, (see Brown [1988] or Cohen [1989]).

c) Crossed module version (Brown-Higgins [1978])

The groupoid van Kampen theorem still only tells you about the very lowest levels of the homotopy type of a space. To obtain information on the next level, one needs to work with crossed modules. We replace the spaces X, U, V by (multiply pointed) pairs (X, X^1) etc., the fundamental groupoid by a fundamental crossed module. This is the structure given by

$$\{\pi_2(X, X^1, p) : p \in X^0\} \xrightarrow{\partial} \Pi_1 X^1 X^0$$

where X^0 is, again, the set of base points and $\Pi_1 X^1 X^0$ is the fundamental groupoid of the 1-skeleton, X^1, based at X^0. This structure is best known in the case, $X^0 = \{x_0\}$, that is of a single base point. Here it is simply the boundary map

$$\partial : \pi_2(X, X^1, x_0) \longrightarrow \pi_1(X^1, x_0).$$

This is a crossed module of groups. Since the relative π_2 consists of homotopy classes of maps of squares having three edges at x_0 and the last in X^1, there is a combinatorial interpretation of this structure as well. For instance if $X = K(\mathcal{P})$ filtered by skeleta, then this crossed module encodes information about the identities amongst the relations of the presentation, $\mathcal{P}$ (c.f. Brown-Hubschmann [1982]). (For a first introduction to crossed modules see Hilton's book, [1966], on homotopy theory which contains an introduction to many of the ideas of J H C Whitehead)

Crossed modules (preferably, of groupoids) satisfy a van Kampen theorem. If the space X is a CW-complex filtered by skeleta, then the fundamental crossed module determines the 2-type of X (see later for the meaning of the n-types of spaces.)

d) Crossed complex versions (Brown-Higgins [1981])

The crossed module of a (multiply pointed) pair is still only giving us information in dimensions 0, 1 and 2. Whitehead [1949] provided a model for much more of the homotopy type of a CW-complex however. His model, called by him "homotopy system" consisted of the relative homotopy groups and boundary maps, together with the action of π_1. The essence of his construction (in a groupoid version) is what is now known as a crossed complex (c.f. Brown and Higgins [1981]) or a crossed chain complex (in the reduced case c.f. Baues [1989] and [1991]).

The basic starting point is a filtered space, $\mathbf{X} = \{X^n\}_{n\geq 0}$, typically the skeletal filtration of a CW-complex, but not necessarily as well behaved as that. The structure of a crossed complex consists of a "chain complex"

$$\to C_n \xrightarrow{\partial_{n-1}} C_{n-1} \longrightarrow \dots \xrightarrow{\partial_2} C_2 \xrightarrow{\partial_1} C_1$$

of groupoids over a fixed base X^0, where for $n\geq 2$, the groupoid C_n is a disjoint union of groups $\{C_n(p) : p \in X^0\}$, which are abelian for $n\geq 3$, C_1 is assumed to act on all the C_n such that the ∂_n are compatible with the action and ∂_1 is a crossed module. One or two other conditions are also needed, but we will not give these here, (see Brown-Higgins [1981], Brown-Golasinski [1989], Carrasco and Cegarra [1991], noting that the indexing of the groupoids is sometimes different for the different authors.) If $\mathbf{X}$ is a filtered space, the associated crossed complex has

$$C_n(p) = \pi_n(X^n, X^{n-1}, p) \qquad\qquad p \in X^0,\ n \geq 2$$

$$C_1 = \Pi_1 X^1 X^0$$

with the ∂_n the obvious boundary maps,

$$\partial_n : \pi_n(X^n, X^{n-1}, p) \longrightarrow \pi_{n-1}(X^{n-1}, X^{n-2}, p),$$

from the long exact sequence of the triple (X^n, X^{n-1}, X^{n-2}). These boundary maps have thus a nice geometric interpretation.

This crossed complex associated to $\mathbf{X}$ is often denoted $\pi(\mathbf{X})$. It satisfies a form of van Kampen theorem in that it converts certain colimit diagrams of filtered spaces into colimits diagrams of crossed complexes. (The way in which this yields powerful results is fascinating and the reader is thoroughly recommended to look through those of the papers of Brown and Higgins, listed in the references, that deal with the applications. The proof of their van Kampen theorem is not easy, as it involves a lot of subsidiary concepts: ω-groupoids, T-complexes, etc., that have each a rich structure linking algebra and geometry in interesting ways, however their richness can tend to obscure the simple idea behind the plan of the proof.)

e) Cat^n-Group Version.

In 1984, Loday published a paper (Loday [1984]) in which he introduced a new algebraic 'gadget', the cat^n-group. He gave a proof (which however contained some technical errors) which showed that these cat^n-groups completely modelled n-types (see later). (The corrections to his proof have since been provided by Steiner [1986] with clarification by Gilbert [1987]; see also Porter [1991] for an algebraic proof.)

Loday's cat^n-group functor satisfies a form of van Kampen theorem (see Brown & Loday [1987a], [1987b]) which gives another extension of the crossed module van Kampen theorem.

f) Exact sequences and Spectral sequences.

From fibration sequences in Spaces, we can hope to get (co-)fibration sequences in Alg. Models. In the hands of a skilled operator, the resulting exact sequences can yield revealing information about the spaces, but there is always the problem that exact sequences often give information on an algebraic model only 'up to extension' and extension problems are hard in many algebraic settings. These comments apply equally to spectral sequences.

II Minimality

As Baues points out in [1991], the minimality of a model is extremely useful. By minimality one implies that only essential information is in the model and experience tends to show that this minimality reflects the geometric and combinatorial structure of the space at a deep level. Minimality does not always seem easy to obtain.

1.3 **Problems:**

(i) Often *complete invariants* (i.e. modelling functors) are very difficult to calculate, e.g. if X is connected, then one can model its homotopy type by a simplicial group, G, but this G is typically free in each dimension and extraction of even quite simple invariants can use up a lot of time. The functor that is used to go from the space to the simplicial group factors through the category of simplicial sets and one can reduce the proof of van Kampen type theorem to proving that the singular functor, Sing, has nice properties since the passage from Simp.Sets to Simp.Groups is algebraic, and preserves colimits as it is a left adjoint.

(ii) As complete invariants are difficult to calculate, we can restrict to well chosen but incomplete test functors. Typically these give complete information on a smaller class of spaces, or determine a courser notion of equivalence. For instance, π_1 is a good invariant, but it gives complete information only on the $K(\pi,1)$'s, that is, on those connected spaces with π_1 non-trivial, whilst all the π_i's for $i>1$ are trivial groups, however it does classify <u>all</u> nice spaces up to the weaker equivalence called 1-equivalence. In general for a given test functor, G, this raises the problems (a) of characterizing the corresponding class of spaces which are completely determined up to homotopy type by the values of G on them and (b) finding a geometric interpretation of the weaker notion of G-equivalence i.e. of determining or describing when $f_0, f_1 : X \to Y$ are such that $G(f_0) \simeq G(f_1)$ in the algebraic models, or when G(X) and G(Y) are isomorphic or equivalent for some notion of equivalence within Alg. Models. For instance Loday's cat^n-group functor completely models (n+1)-equivalence, which has a good geometric description. The completely determined homotopy types for the crossed complexes are those given by the J-spaces in the sense of Whitehead [1949].

(iii) The final problem is to do enough with the algebraic models to produce a rich 'homotopy theory'. This is not always easy! For instance what does $G(X \times Y)$ look like in terms of G(X) and G(Y) and possibly other invariants. Classical (incomplete) invariants yield tensor product formulae and results like the Eilenberg-Zilber theorem. What is the analogue in our "ideal scenario" situation or for the "fall back" incomplete invariants? This often holds the key to defining nice homotopy structure in the algebraic models since the homotopy in Spaces is intimately linked with the cylinder $X \times I$ and the various monoid multiplications on the space I (again involving a product). The test functor G should convert $G(X \times I)$ to something like $G(X) \otimes G(I)$ with an as-yet undefined tensor product and the monoid multiplication on I to one on G(I) with respect to this ⊗.

To sum up:-

One hopes for a functor

$$G\colon \text{Spaces} \longrightarrow \text{Alg. Models}$$

which (i) induces an equivalence on homotopy categories

(ii) preserves certain colimits.

At second best, one would like a computable G which gives complete information on as large as possible a subcategory of spaces and that the models reflect the homotopy structure in a nice way.

Problem:

How can we do a similar job for proper homotopy?

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2. Proper homotopy (at ∞ and globally).

2.1 In proper homotopy theory, the interesting spaces are not compact so it is difficult to get invariants of them using maps from spheres, or other compact spaces, as would be the way in classical homotopy theory. Such maps do not tell us about what is happening "far out" towards infinity. Here, of course, the classic example is $\mathbb{R}^2$, where maps with compact domain cannot detect the 'hole' at infinity corresponding to the puncture in the sphere used for a stereographic projection. To gain fuller information one has to use *proper* maps to investigate behavior at infinity and to combine that information with more standard information to obtain "global" invariants to try to gain insight into the nature of the proper homotopy type of the space.

At the moment there are two related but distinct approaches being tried out. One, developed by Ayala, Quintero, and Dominguez here in Spain with Baues and Zobel from Bonn, takes a global view from the start and analyses the similarity between the 'abstract homotopy' of the classical situation and that of the "proper" situation. The second pioneered by Edwards and Hastings embeds the problem of determining proper homotopy type in a wider context namely that of determining prohomotopy types i.e. homotopy types of prospaces, or inverse systems of spaces. This is technically slightly more complicated but allows a reasonable amount of geometric freedom as constructions that seem to be the proper analogues of classical ones may not be that easy to construct as spaces, but they can usually be constructed within the pro-category. Thus one might construct a mapping cylinder within the pro-category even though this may not correspond to any space. This second approach has been exploited by Luis Javier Hernandez and myself in a series of articles and as Hans Baues has talked in this Workshop on the alternative approach. I will concentrate on the "pro"-formulation.

2.2 The basic categories of spaces used are:

Notation

P ~ the proper category of σ compact spaces

CW ~ the proper category of locally finite CW-complexes

SC ~ the proper category of locally finite simplicial complexes.

Ho(A) will denote the corresponding proper homotopy category to the category A.

P_∞ etc. denote the variant of P etc., consisting of the same spaces but with germs of proper maps at ∞ as maps (see Edwards-Hastings [1976])

If $X \in P$, let $\varepsilon(X) = \{(X - C) : C \text{ compact}, C \subset X\} \in \text{pro(Top)}$. Given a proper map or proper map germ $f : X \to Y$, as f is proper, inverse images of compact sets in Y are compact in X, so if D compact in Y, f restricts to a map

$$f_D : X - f^{-1}(D) \to Y - D$$

with $f^{-1}(D)$ compact in X. Of course, this is exactly what is needed for f to induce a "pro-morphism",

$$\varepsilon(f) : \varepsilon(X) \to \varepsilon(Y)$$

in pro(Top) and hence to make ε into a functor

$$\varepsilon : P \to \text{pro(Top)},$$

called the *end functor*.

This functor ε can also be defined on proper map germs, since a map germ $f : X \to Y$, only needs to be defined on the complement of some compact subset of X. We thus also get an end functor

$$\varepsilon : P_\infty \to \text{pro(Top)}$$

which is compatible with the projection from P to P_∞ (c.f. Edwards and Hastings [1976].)

As $\varepsilon(X) = \{cl(X - C) : C \text{ compact in } X\}$, there is a natural promorphism from $\varepsilon(X)$ to X, where here X is being considered as a constant object in pro(Top). This morphism can conveniently be thought of as an object in the comma category (pro(Top), Top) consisting of all maps in pro(Top), with codomain a constant object (N.B. the notation used by Edwards and Hastings here is different and is "non-standard" from a categorical point of view. I, of course, prefer the notation I have used above!)

Assigning $\varepsilon(X) \to X$ to X, etc., gives for each proper map $f:X \to Y$ a diagram

$$\begin{array}{ccc} \varepsilon(X) & \xrightarrow{\varepsilon(f)} & \varepsilon(Y) \\ \downarrow & & \downarrow \\ X & \xrightarrow[f]{} & Y \end{array}$$

which is commutative precisely because $\varepsilon(f)$ consists of restrictions of f. This gives a second functor

$$(\varepsilon, \text{forget}) : P \longrightarrow \text{(pro(Top),Top)}$$

(where the "forget"-part of this forgets the "proper" condition and only "remembers" the continuity of the maps in P).

Edwards and Hastings first embedding results proved:

the functors

$$(\varepsilon,\text{forget}) : P \longrightarrow (\text{pro(Top)},\text{Top})$$

and

$$\varepsilon : P_\infty \longrightarrow \text{pro(Top)}$$

are embeddings.

The proper homotopy structure on P or P_∞ is equally well reflected by homotopy structure in (pro(Top),Top) and pro(Top). Edwards and Hastings [1976] produced a neat Quillen model category structure on both of these categories. We will not need all of that structure in detail, but recall that the weak equivalences in these categories are generated by the "level weak equivalences" that is maps of the form

$$\{f_i : X_i \rightarrow Y_i | i \in I\}$$

for which each f_i is a weak equivalence in Top, (see also Porter [1988]).

The second set of embedding results of Edwards and Hastings are that *the previous embeddings induce embeddings*

$$\text{Ho}(P) \longrightarrow \text{Ho}(\text{pro(Top)},\text{Top})$$

and

$$\text{Ho}(P_\infty) \longrightarrow \text{Ho}(\text{pro(Top)}).$$

and similarly for pairs, n-ads etc. of spaces.

The constructions of Ho(pro(Top)) and Ho(pro(Top),Top) from Top are functorial so could be applied to any category, C. This suggests together with our discussion in section 1, a method of attack for studying proper homotopy theory:

1. Pick your favourite algebraic test-functor

$$G : \text{Top} \longrightarrow \text{Alg. Models}$$

2. Study the structure of

pro(Alg.Models) and (pro(Alg.Models),Alg.Models)

and try to investigate the analogues of homotopy constructions in these categories, e.g. to form Ho(pro(Alg.Models),Alg.Models) and to "do homotopy" there.

3. Use the composites

$$P \longrightarrow (\text{pro(Top)},\text{Top}) \xrightarrow{G} (\text{pro(Alg.Models)},\text{Alg.Models})$$

$$P_{\infty} \longrightarrow \text{pro(Top)} \xrightarrow{G} \text{pro(Alg.Models)}$$

to "encode" proper homotopy theory into "pro-algebra".

2.3 Critique:

(i) Inverse systems of algebras are useful, but are an acquired taste! Some people do not like them. If the algebras are finite, or "finite dimensional", e.g. made up of finite groups or Artinian modules, then one can take inverse limits without disturbing information and be left with profinite algebras of the same type. These are topological algebras with a certain link between the topology and the algebra - loosely speaking the "normal subalgebras" determine a system of open neighborhoods of the identity. This can, in some cases, be useful as it replaces a collection of interacting algebraic models by a single topologised one - but the process of analysis of the result is in many aspects bound to be equivalent in difficulty to that of analysing the original system. The limiting process will destroy information if the finiteness or compactness condition is not present and in any case the limiting process is not that geometrical as limits and homotopy mix badly.

(ii) On the positive side, there are geometrically motivated constructions that yield "algebraic models" from the "pro-algebraic" ones. For instance, the alternative approach to proper homotopy theory used by Baues, et al, often uses "favourite test functors" somewhat of form Ho(P)(S,-), where S is some space in P. For instance S might be simply a string of circles (see later) or may be much more complicated. To aid in the study of such a functor we can throw the problem into (pro(Alg.Models),Alg.Models) and compare this Ho(P)(S,X) with the corresponding hom-set, $[G(\varepsilon S),G(\varepsilon X)]$, in Ho(pro(Alg.Models),Alg.Models). Any natural structure in Ho(P)(S,X) will be geometrically realised by some corresponding dual homotopy structure in S, just as in ordinary homotopy theory the n-spheres are cogroups "up to homotopy". The pro-algebraic model $G(\varepsilon S)$ will therefore have a rich algebraic structure and a homotopy "costructure". This structure will then yield on $[G\varepsilon(S), G\varepsilon(X)]$ a natural structure of the same type as the original one, reflecting the structure Ho(P)[S, X] to a greater or less extent depending on whether or not G is or is not an embedding in some relevant range of dimensions.

This may seem vague so let us take a simple example. Suppose the original G used is π_1. This of course needs pointed spaces, so we assume X is supplied with a "base ray" $* : [0,\infty) \to X$. Then, using a "pairs" version of the Edwards-Hastings embedding, we get $\varepsilon(*) : \varepsilon[0,\infty) \to \varepsilon(X)$ in pro(Top), similarly in (pro(Top),Top). Within Ho(pro(Top)), $\varepsilon[0,\infty)$ is isomorphic to the constant system with "value" a single point, hence we get a pointed object in Ho(pro(Top)) and similarly in Ho(pro(Top),Top). (We will for simplicity restrict attention to P_∞, and hence to pro(Top) and Ho(pro(Top)).) We can thus apply π_1 to $\varepsilon(X)$ without difficulty basing the loops at the relevant base points given by the map $\varepsilon[0,\infty) \to \varepsilon(X)$, to get a fundamental pro-group $\pi_1(\varepsilon(X),*)$.

The alternative approach might look at a space S with $S = [0,\infty) \cup \bigsqcup_{i \in \mathbb{N}} S^1_i$ where S^1_i is a circle attached at the point $i \in \mathbb{N} \subset [0,\infty)$ and then look at $Ho(P^{pairs}_\infty)((S,[0,\infty)),(X,*))$, the proper pointed homotopy germs from $(S,[0,\infty))$ to $(X,*)$. Following the plan that was sketched out earlier, we would look at $G\varepsilon(S)$. As G is taken to be π_1, this is the progroup $\pi_1\varepsilon(S)$ with at index n, $F(x_n,x_{n+1}, \dots)$, a free group on a set $\{x_n,x_{n+1}, \dots\}$ of elements, where from $\pi_1\varepsilon(S)_{n+1}$ to $\pi_1\varepsilon(S)_n$ the map is that induced by the inclusion of $\{x_{n+1}, \dots\}$ into $\{x_n,x_{n+1}, \dots\}$. The set $Ho(P^{pairs}_\infty)((S,[0,\infty)),(X,*))$ has a group structure inherited from the level wise H-cogroup structure on S i.e. $S^1 \to S^1 \vee S^1$ induces, at each i, a comultiplication 'up to homotopy'

$$S^1_i \longrightarrow S^1_i \vee S^1_i$$

and hence a comultiplication 'up to proper homotopy'

$$S \longrightarrow [0,\infty) \cup \bigsqcup_{i \geq 0} S^1_i \vee S^1_i \simeq S \sqcup_{[0,\infty)} S$$

which gives the group structure. Writing $\underline{\pi}_1(X,*)$ in place of $H_0(P^{pairs}_\infty)[(S,[0,\infty)),(X,*)]$, we have that $\underline{\pi}_1(X,*)$ is a group, but more interestingly there is a natural action of $\underline{\pi}_1(S,[0,\infty))$ on $\underline{\pi}_1(X,*)$ given by composition within the category $H_0(P^{pairs}_\infty)$. The analogous action of $\pi_1(S^1)$ on $\pi_1(X)$ is usually ignored as it is given by power mappings and hence is not an additional operation but is already given by the group structure. Here in the proper context, $\underline{\pi}_1(S)$ gives us a lot of extra structure, both algebraic and geometric via the simple geometric structure of S. One of my points earlier was that if a general G can be interpreted in geometric terms then analogous algebraic/geometric structure will be present in G(X). Note that $\underline{\pi}_1(X,*)$ is related to $\text{pro}(\text{Groups})(\pi_1(\varepsilon(S)),\pi_1(\varepsilon(X)))$ (omitting base points for simplicity of notation) as in general $\text{Ho}(P^{pairs}_\infty)((X,*)(Y,*))$ is related, via the function induced by the functor π_1, to $\text{pro}(\text{Groups})(\pi_1\varepsilon(X),\pi_1\varepsilon(Y))$. In fact this gives an isomorphism between $\underline{\pi}_1(X)$ and $\text{pro}(\text{Groups})(\pi_1\varepsilon(S),\pi_1\varepsilon(X))$. This provides the basis for another way of producing a single object from an inverse system.

The lesson to be learned from the example above would seem to be that if possible the analysing machinery used to attack objects should reflect geometric structure.

(iii) Another means of extracting information from $\varepsilon(X)$ is by applying a homotopy limit functor before applying G. This avoids the need to handle inverse systems of algebraic models, but has its limitations. For instance using the nth homotopy group functor π_n, we get

$$\begin{aligned}\pi_n(\text{holim}\,\varepsilon(X)) &= [S^n,\text{holim}\,\varepsilon(X)] \\ &= [k(S^n),\varepsilon(X)]\end{aligned}$$

where $k(S^n)$ is the constant system with 'value' S^n,

$$= Ho(P_{\infty}^{pairs})(S^n \times [0,\infty),X)$$

i.e. proper homotopy classes of maps from the half-infinite cylinder $S^n \times [0,\infty)$ to X that map $\{1\} \times [0,\infty)$ to $*$. This again provides a means of extracting informa·'on from $\varepsilon(X)$. (A comparison of the methods used in (ii) and here in (iii) gave the basic idea to Porter [1982], c.f. [1984], [1987], and hence led to Hernández and Porter [1988a] and [1988b].)

(iv) Several important classes of spaces are defined via a specification of $\varepsilon(X)$. For these, the methods based on the Edwards-Hastings embedding seem very natural. In particular it is worth noting that the Whitehead manifolds, which are examples of contractible 3-manifolds which are not homeomorphic to $\mathbb{R}^3$, offer some very hard examples on which readers may try out their favourite test functor, (see Whitehead [1939], Hempel [1976] and also McMillan [1962]).

2.4 **Problems:**

a) Take your favourite test functor: G : Spaces ⟶ Alg.Models and
(i) interpret it geometrically - if possible, then
(ii) apply it to holimε(X) or a suitable variant of this - then interpret the results.

b) The homotopy groups of holimε(X) are linked via a short exact "Milnor" sequence with the limit groups $\lim\{\pi_n(cl(X - C))\}$ and $\lim^{(1)}\pi_{n+1}(cl(X - C))$ This helps both in interpretation and calculation. Find similar interpretations for your favourite, G.

3. Partial and complete models for proper homotopy type (at ∞).

(I will only model "at ∞" for brevity of exposition.)

3.1 Prosimplicial groupoids.

Given any space, X, we write S(X) for the singular complex of X, i.e. $S(X)_n = \text{Spaces}(\Delta^n, X)$. This is a simplicial set, and S gives a functor from Spaces to Simp.Sets (with an adjoint, $|\ \ |$, geometric realisation), which induces an equivalence of homotopy categories. (Variants of S can be used with the category of arbitrary spaces replaced by CW or SC and in which the maps used to define S are cellular or simplicial. We will denote these variants by s.) Simplicial sets are hardly algebraic models although the combinatorial information they encode does have algebraic content, but we can still apply our general 'plan of action' to get

$$_{*}P_{\infty} \xrightarrow{\ \varepsilon\ } \text{proTop} \xrightarrow{\ S\ } \text{proSimp.Sets}$$

which induces

$$\text{Ho}(P_{\infty}) \longrightarrow \text{Ho}(\text{proSimp.Sets}).$$

From simplicial sets, there are various directions we can go. Probably the best is towards simplicial groupoids as Dwyer and Kan [1984] constructed an adjoint pair

$$\text{Simp.Sets} \underset{\bar{W}}{\overset{G}{\rightleftarrows}} \text{Simp.Gpds}_*$$

extending Kan's own functor from *connected* simplicial sets to simplicial groups (c.f. Curtis [1971], which is still a useful reference for simplicial sets). The simplicial groupoids concerned are not just simplicial objects in the category of groupoids, as they must satisfy an additional condition, namely that the simplicial set of objects for the groupoids is constant. (This is the intended meaning of the * included in the notation Simp.Gpds_*.) The construction of G(K) for a simplicial set K illustrates this. The groupoid $G(K)_n$ in dimension n is a groupoid with object set $\{\bar{x} : x \in K_0\}$ i.e. is independent of n, and with arrows generated by all

$$\bar{y} : \overline{d_1 d_2 \dots d_{n+1} y} \longrightarrow \overline{d_0 d_2 \dots d_{n+1} y},\ y \in K_{n+1}$$

with relations

$$\overline{s_0 z} = id_{\overline{d_1 \dots d_n z}} \text{ for all } z \in K_n.$$

Face and degeneracies are induced by those of K,

$\delta_0 \bar{x} = (\overline{d_1 x}).(\overline{d_0 x})^{-1}$

$\delta_i \bar{x} = \overline{d_{i+1} x}$ $\qquad i \geq 1$

$\sigma_i \bar{x} = \overline{s_{i+1} x}$ $\qquad i \geq 0$

(We will not look at $\bar{W}$ here.) Note that if K is a reduced simplicial set, i.e. K_0 is a singleton set, then G(K) is a simplicial group.

Theorem (Dwyer and Kan)

There is a Quillen model category structure on Simp.Gpds$_*$ *such that* $(G, \bar{W})$ *induces an equivalence of categories*

$$\text{Ho(Simp.Sets)} \underset{\bar{W}}{\overset{G}{\rightleftarrows}} \text{Ho(Simp.Gpds}_*)$$

Thus GS will be a good test functor taking Alg.Models to be these simplicial groupoids. On the negative side, S(X) is an enormous simplicial set even for simple spaces such as X = [0,1] and GS(X) will thus be a huge simplicial groupoid. Can we reduce it in size? We will examine this question shortly.

Applying the "en." or "pro" method would give us a prosimplicial groupoid, however we can start reducing the amount of unnecessary information that we carry around by assuming X is a (σ-compact) simplicial complex. This is not much of a restriction for us as many interesting spaces have this form, but it enables us to write $X = \cup K(n)$ where each $K(n)$ is compact and $K(n) \subset \mathrm{int}K(n+1)$, then to take for each n, $X - K(n)$ and apply s (not S) i.e. we use only the singular simplicies $\sigma : \Delta^m \rightarrow X - K(n)$ which are simplicial maps. Since locally finite infinite simplicial complexes satisfy a simplicial approximation theorem, $s(X - K(n))$ and $S(X - K(n))$ are homotopy equivalent. We can thus look at $Gs(\varepsilon X)$ in pro(Simp.Gpds$_*$) and obtain an embedding

$$\mathrm{Ho}(SC_\infty) \longrightarrow \mathrm{Ho}(\mathrm{pro}(\mathrm{Simp.Gpds}_*))$$

This would seem to reduce the task of finding algebraic invariants for ends to an (algebraic) analysis of these pro-simplicial groupoids. Of course that analysis is by no means easy!

How is $Gs\varepsilon(X)$ related to other invariants?

0) $\pi_0(Gs\varepsilon(X))$ is a profinite set whose inverse limit is the space of Freudenthal ends of X.

1) From any G in Simp.Gpds$_*$, one can form a Moore complex NG., whose homology groups are the homotopy groups of G.
In particular,

$$H_n(NGs\varepsilon(X)) = \{\pi_{n+1}(\mathrm{cl}(X - K(n)), p(i)) : p(i) \in (X - K(i)_0\}$$

an inverse system of families of homotopy groups, for $n \geq 1$ and for $n = 0$, an inverse system of fundamental groupoids based at the vertices of the $X - K(i)$.

2) There is a functor (first noticed by Carrasco and Cegarra for simplicial groups, then adapted for groupoids by Ehlers and Porter)

$$C : \mathrm{Simp.Gpds}_* \longrightarrow \mathrm{Crs}_*$$

where Crs$_*$ is the category of crossed complexes. Here $(C(G),\partial.)$ is a crossed complex (over a groupoid), in which

$$C(G)_n = \frac{NG_n}{(NG_n \cap D_n)d_0(NG_{n+1} \cap D_{n+1})}$$

where D_n is the subgroupoid generated by the degenerate elements in G_n.

If we apply this to $Gs\varepsilon(X)$, this gives $\pi RS\varepsilon(X)$ up to isomorphism in $H_0(\text{proCrs}_*)$ (see Hernández-Porter [1991] for $\pi RS\varepsilon(X)$ and related theory.) The nth truncation of this invariant is a full invariant on what we have called the J_n-spaces, again see the above cited paper.

3) Given n>0, there is a functor

$$M(-,n) : \text{Simp.Gpds}_* \longrightarrow \text{Crs}^n\text{-Gpds}$$

that is, crossed n-squares of groupoids. If we denote by $Ho_n(C)$ the quotient category, obtained by formally inverting the n-equivalences in C then M(–,n) induces

$$Ho_n(\text{Simp.Gpds}_*) \xrightarrow{\cong} Ho_{n+1}(\text{Crs}^n\text{-Gpds})$$

(some details still need to be double checked here.)

Presumably there is an embedding

$$Ho_{n+1}(SC_{0,\infty}) \longrightarrow Ho_{n+1}(\text{proCrs}^n\text{-Gpds})$$

but the details of the argument still need to be checked.

4) There would seem to be a common generalisation of 2 and 3. The resulting notion of n-crossed complex has a functorial construction starting with simplicial groupoids so again should yield a rich theory modelling a large class of spaces and again feeding us new algebraic models for the end.

These last three all are given by left adjoints and hence preserve pullbacks. Because of this the question as to whether or not they satisfy a van Kampen type theorem (which would then extend to the pro category and hence possibly back to $SC_{\sigma,\infty}$ reduces to one about simplicial groupoids. It would seem likely that GS : Top $\rightarrow$ Simp.Gpds$_*$ does not satisfy a van Kampen Theorem, but that some geometrically inspired quotient of GS may. If this is the case then we may hope for van Kampen theorems for simplicial groupoids, which would then raise interesting questions about possible van Kampen theorems for SC_σ based proper homotopy invariants.

We just touched on n-type and its proper analogue above. Within CW, each n-type has a representative with trivial homotopy groups above dimension n and each CW-complex can be built up functorially in Ho(CW) as a limit of spaces of this type (i.e. has a Postnikov resolution). The analogue of this in Simp.Sets holds functorially using the coskeleton construction and analogues of this thus work in proSimp.Sets and (pro(Simp.Sets),Simp.Sets). Can this be done for locally finite simplicial complexes?

Within his tree* based theory, Baues can use general abstract homotopy methods to obtain such a result, but this depends strongly on working under and over the base point, so may not give exactly what we need here.

Explicitly we want a subcategory

$$SC(n)_\infty \longrightarrow SC_\infty$$

such that

$$Ho(SC(n)_\infty) \xrightarrow{\cong} Ho_n(SC_\infty)$$

together with an explicit functorial construction of an associated space in $SC(n)_\infty$ for each X in SC_∞. There is, of course, a similar problem globally. One of the doubts in the unbased case is "how should one measure n-type?" that is, "which of the possible proper homotopy groups should one use?" All the candidates are "based" at trees, and so correspond to the connected analogue in the usual setting. Can we use the simplicial groupoid method to obtain a proper analogue of the fundamental groupoid which, although uncalculable, will be a better theoretical tool?

This same query comes also in another situation. The classical construction of the fundamental groupoid is closely related to that of the universal covering space for "locally nice" spaces. If X is, say, a connected CW-complex, $\Pi_1 X$ is obtained from the set of paths X^I by quotienting out by the equivalence relation of "homotopic relative to end points", but of course X^I is a space so we could give $\Pi_1 X$ the quotient topology and consider it as a topological groupoid. Taking any $x_0 \in X$ (remember X is assumed to be connected) we consider just those path classes that start at x_0. This gives a subspace, $\tilde{X}$, of $\Pi_1 X$ with the "other end point" map giving a continuous $p : \tilde{X} \to X$. Of course, $(\tilde{X},p)$ is the universal covering space of X. Thus we can think of $\Pi_1 X$ as being made up of copies of $(\tilde{X},p)$ one for each choice of $x_0 \in X$. Another of our "favourite" constructions above is related to the universal covering space. If X is a CW-complex, we have mentioned the crossed complex $\pi(X)$ of the filtered space, X, i.e. X filtered by its skeletal filtration. There is a functor from the category of chain complexes of modules to that of crossed complexes. This functor has a left adjoint which is a sort of "relative abelianisation". When applied to $\pi(X)$ one obtains the "chains on $\tilde{X}$" chain complex as a complex of $\Pi_1 X$-modules (see Brown-Higgins [1990]).

If we apply these constructions to $\varepsilon(X)$ for X in SC_∞, then we clearly have potentially important information - but how can we interpret it? It there a proper analogue of a universal covering space? Not just something in pro(Top) or (pro(Top),Top), but an "honest space". If not can we use $\Pi_1\varepsilon(X)$ and $\pi(\varepsilon X)$ as substitutes, identifying their properties and abstracting to analogues of (non-universal) covering spaces, within these algebraic models?

A final question within this area is whether there is a geometric notion classified by subobjects of $\underline{\pi}_1(X)$, or for that matter of any of the other candidates for a proper fundamental group(oid). There is a view which says that the use of paths to obtain homotopy information on spaces is misplaced since paths are only useful if the space is "locally nice". From such a view point, $\pi_1(X)$ or $\Pi_1 X$ is the "classifier" for covering spaces, and the van Kampen theorem is about gluing together categories of covering spaces. This view is wide enough to allow Galois theory to be interpreted as being about coverings and a fundamental group(oid) (c.f. Grothendieck's SGA1 [1971] or Douady and Douady [1977]). Perhaps we should look to the methods of algebraic geometry to see if there is not some way of interpreting the end $\varepsilon(X)$ as a germ of a space and to use sheaf-theory to give geometric meaning to the proper fundamental group(oid)s. The work of Goldman (Thesis Yale, late 1960's) mentioned by Larry Siebenmann at the Workshop, may indicate a possible way forward in this. Even if this can be resolved, there remains the problem of the higher homotopy invariants of $\varepsilon(X)$ for instance the n-type. As it is far from clear even in the compact classical situation, what the n-type classifies (see "Pursuing Stacks": Grothendieck [1983] or the discussion in my own "Abstract Homotopy Theory", Bressanone 1991, Porter [1992], for a discussion of some of the many facets of this area.) What does seem fairly clear is that from this viewpoint the higher $\pi_n(X)$ or for our purposes $\underline{\pi}_n(X)$ or $\pi_n(\varepsilon(X))$ do not classify very much. This area, relatively untrodden even in the classical case, is more or less completely unexplored in the proper homotopy context, but as it would seem to connect up with algebras of continuous functions, it may be a way in which proper homotopy theory will influence or at least interact with areas other than that of geometric topology.

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