Revista Integración
Escuela de Matemáticas
Universidad Industrial de Santander
Vol. 35, No. 1, 2017, pág. 37–50

DOI: http://dx.doi.org/10.18273/revint.v35n1-2017003

A recursive condition for the symmetric nonnegative inverse eigenvalue problem

Elvis Ronald Valero[a*], Exequiel Mallea-Zepeda[a], Eber Lenes[b]

[a] Universidad de Tarapacá, Departamento de Matemática, Arica, Chile.

[b] Universidad del Sinú. Elías Bechara Zainum, Departamento de Matemática, Cartagena, Colombia.

Abstract. In this paper we present a sufficient condition and a necessary condition for *Symmetric Nonnegative Inverse Eigenvalue Problem*. This condition is independent of the existing realizability criteria. This criterion is recursive, that is, it determines whether a list $\Lambda = \{\lambda_1, ..., \lambda_n, \lambda_{n+1}\}$ is realizable by a nonnegative symmetric matrix, if the list $\mu = \{\mu_1, ..., \mu_n\}$ associated to Λ is realizable. This result is easy to program and improves some existing criteria.

Keywords: Inverse problems, eigenvalues, orthogonal matrices, symmetric matrix.

MSC2010: 15A29, 15A18, 15B10, 15A57.

Una condición recursiva para el problema inverso del autovalor para matrices simétricas no negativas

Abstract. En este artículo presentamos una condición suficiente y una condición necesaria para el *Problema Inverso de Autovalores para Matrices Simétricas no Negativas*. Esta condición es independiente de los criterios de realizabilidad existentes. Este criterio es recursivo, es decir determina si una lista $\Lambda = \{\lambda_1, ..., \lambda_n, \lambda_{n+1}\}$ es realizable por una matriz simétrica no negativa, si la lista $\mu = \{\mu_1, ..., \mu_n\}$ asociada a Λ es realizable. Este resultado es fácil de programar y mejora algunos criterios existentes.

Palabras clave: Problemas inversos, autovalores, matrices ortogonales, matrices simétricas.

* E-mail: evalero@uta.cl
Received: 9 March 2017, Accepted: 24 May 2017.
To cite this article: E.R. Valero, E. Mallea-Zepeda, E. Lenes, A recursive condition for the symmetric nonnegative inverse eigenvalue problem, *Rev. Integr. Temas Mat.* 35 (2017), No. 1, 37–50.

1. Introduction

The *nonnegative inverse eigenvalue problem (NIEP)* is the problem of finding necessary and sufficient conditions for a list $\Lambda = \{\lambda_1, \lambda_2, ..., \lambda_n\}$ of complex numbers to be the spectrum of a $n \times n$ nonnegative matrix. The problem of finding necessary and sufficient conditions for a list of real numbers to be spectrum of a nonnegative matrix is called the *real nonnegative inverse eigenvalue problem (RNIEP)*. Particularly, the problem of finding necessary and sufficient conditions for a list of real numbers Λ, to be the spectrum of a nonnegative symmetric matrix is called *symmetric nonnegative inverse eigenvalue problem (SNIEP)*. These problems remain unsolved.

The $NIEP$, $RNIEP$ and $SNIEP$ are completely solved for $n \leq 4$. The $NIEP$ has been solved for $n = 3$ in 1978 by Loewy and London [7], for $n = 4$ was solved by Meehan [9] in 1998, and subsequently independently in a different formulation by Torre-Mayo and others [17] in 2007.

The $SNIEP$ has been solved when $n = 3$ by Fiedler [2] in 1974 , and for $n = 4$ has been solved by Guo [3] in 1996 . The $RNIEP$ and $SNIEP$ are equivalent for $n \leq 4$ [3], but are different otherwise [5]. In 2011 Spector gives a complete solution to $SNIEP$ when $n = 5$ and the trace of nonnegative matrix is zero [16]. Partial results for the $SNIEP$ have been obtained in [1], [2], [6], [8], [10], [11], [12], [13], [15].

This paper is organized as follows: In Section 2 we establish the notation and basic results in relation to $SNIEP$. In Section 3 we present the main results, a sufficient condition and a necessary condition. In Section 4, we show a programming algorithm in relation to the main results. The realizability criteria shown in Section 3 are independent of the criteria presented in [1], [13], and they improve those criteria.

2. Notations and basic results

Throughout this paper we use the following notation: Let $\mathbb{R}^{m\times n}$ the matrix set of order $m\times n$ with entries real numbers, in particular $\mathbb{R}^{n\times n}$ square matrices of order n. We denote $\rho(A)$ be the spectral radius of $A \in \mathbb{R}^{n\times n}$. We say that $A = [a_{ij}] \in \mathbb{R}^{n\times n}$ is nonnegative if $a_{ij} \geq 0$ for all $i, j \in \{1, 2, ..., n\}$. We shall say that $\Lambda = \{\lambda_1, \lambda_2, ..., \lambda_n\}$ is realizable if there exists an nonnegative matrix $A \in \mathbb{R}^{n\times n}$ with spectrum Λ and $\rho(A) = \lambda_1$. If Λ is realizable for A, then it is said that A realiza Λ.

If $\Lambda = \{\lambda_1, \lambda_2, ..., \lambda_{n+1}\}$ and $\mu = \{\mu_1, \mu_2, ..., \mu_n\}$ such that

$$\lambda_1 \geq \mu_1 \geq \lambda_2 \geq ... \geq \mu_{n-1} \geq \lambda_n \geq \mu_n \geq \lambda_{n+1},$$

we define the functions $f(t) = \prod_{k=1}^{n+1}(t - \lambda_k)$, $g(t) = \prod_{k=1}^{n}(t - \mu_k)$.

Show that the vector $\mathbf{y} = \begin{bmatrix} y_1 & y_2 & \cdots & y_n \end{bmatrix}^T$, with

$$y_i^2 = -\frac{f(\mu_i)}{g'(\mu_i)} = -\frac{\prod_{k=1}^{n+1}(\mu_i - \lambda_k)}{\prod_{k=1}^{i-1}(\mu_i - \mu_k) \cdot \prod_{k=i+1}^{n}(\mu_i - \mu_k)}, \qquad (*)$$

is well defined (see [4]).

Finally, we present three results, the first two results are due to Horn and Johnson [4], and the third result was presented by Guo [3]. These will be later used for the development of new necessary and sufficient conditions for *SNIEP*.

Theorem 2.1. *Let $\bar{A} \in \mathbb{R}^{n\times n}$ be a given symmetric matrix with eigenvalues $\mu_1, ..., \mu_n$, let $\mathbf{z} \in \mathbb{R}^{n\times 1}$ be a given vector (column matrix), and let α be a given real number. Let $A \in \mathbb{R}^{n+1\times n+1}$ be the symmetric matrix obtained by bordering $\bar{A}$ with $\mathbf{z}$ and α as follows: $A = \begin{bmatrix} \bar{A} & \mathbf{z} \\ \mathbf{z}^T & \alpha \end{bmatrix}$, with eigenvalue $\lambda_1, \lambda_2, ..., \lambda_{n+1}$. If $\lambda_1 \geq \lambda_2 \geq ... \geq \lambda_{n+1}$ and $\mu_1 \geq \mu_2 \geq ... \geq \mu_n$. Then*

$$\lambda_1 \geq \mu_1 \geq \lambda_2 \geq ... \geq \lambda_n \geq \mu_n \geq \lambda_{n+1}.$$

Theorem 2.2. *Let n be a given positive integer, and let $\mu = \{\mu_1, ..., \mu_n\}$ and $\Lambda = \{\lambda_1, \lambda_2, ..., \lambda_{n+1}\}$ be two lists of real numbers arranged in descending order such that*

$$\lambda_1 \geq \mu_1 \geq \lambda_2 \geq ... \geq \lambda_n \geq \mu_n \geq \lambda_{n+1}.$$

Let $D = diag(\mu_1, ..., \mu_n)$. Then there exists a real number α and real vector $\mathbf{y} \in \mathbb{R}^{n\times 1}$ such that $\Lambda = \{\lambda_1, \lambda_2, ..., \lambda_{n+1}\}$ is the set of eigenvalues of the real symmetric matrix

$$A = \begin{bmatrix} D & \mathbf{y} \\ \mathbf{y}^T & a \end{bmatrix}.$$

Theorem 2.3. *The list $\lambda = \{\lambda_1, \lambda_2, \lambda_3, \lambda_4\}$ of real numbers is a realizable symmetric matrix if, and only if $\lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 \geq 0$ and $\lambda_1 \geq |\lambda_i|$ for $i = 2, 3, 4$.*

Observation: The matrix A of Theorem 2.2 is similar to the matrix $\widetilde{A} = \begin{bmatrix} a_{11} & z^T \\ z & D \end{bmatrix}$, which has a spectrum Λ and a main submatrix with spectrum μ. In consequence, the results of the following section can be reformulated in such a way that the matrix that realize Λ has the form of $\widetilde{A}$.

3. Sufficient condition and necessary condition

We consider a list $\Lambda = \{\lambda_1, \lambda_2, ..., \lambda_n\}$ of real numbers, such that $\lambda_1 \geq \lambda_2 \geq ... \geq \lambda_n$. The next result is a sufficient condition for the $SNIEP$ [14].

Theorem 3.1. *Let $\Lambda = \{\lambda_1, \lambda_2, ..., \lambda_{n+1}\}$, $\mu = \{\mu_1, \mu_2, ..., \mu_n\}$ be lists of real numbers, such that $\lambda_1 \geq \mu_1 \geq \lambda_2 \geq \mu_2 \geq ... \geq \mu_n \geq \lambda_{n+1}$, with $\sum_{k=1}^{n+1} \lambda_k - \sum_{k=1}^{n} \mu_k \geq 0$. Moreover let*

P be an orthogonal matrix, $D = diag\{\mu_1, ..., \mu_n\}$ such that $PDP^T \geq 0$, and $\mathbf{y} \in \mathbb{R}^{n+1}$ given as $(*)$ such that $P\mathbf{y} \geq 0$. Then there exists a nonnegative symmetric matrix A with spectrum Λ.

Proof. We define $\mathbf{y} = \begin{bmatrix} y_1 & ... & y_n \end{bmatrix}$ where each component y_i is given as $(*)$, that is:

$$y_i^2 = -\frac{\prod_{k=1}^{n+1}(\mu_i - \lambda_k)}{\prod_{k=1}^{i-1}(\mu_i - \mu_k) \cdot \prod_{k=i+1}^{n}(\mu_i - \mu_k)}.$$

By Theorem 2.1 the matrix $\bar{A} = \begin{bmatrix} D & \mathbf{y} \\ \mathbf{y}^T & \sum_{k=1}^{n+1} \lambda_k - \sum_{k=1}^{n} \mu_k \end{bmatrix}$ has spectrum Λ.

We define the orthogonal matrix $\begin{bmatrix} P & \mathbf{0} \\ \mathbf{0}^T & 1 \end{bmatrix}$, with $\mathbf{0} \in \mathbb{R}^{n\times 1}$. Since $PDP^T \geq 0$, $P\mathbf{y} \geq 0$, then the symmetric matrix

$$\begin{aligned} A &= \begin{bmatrix} P & \mathbf{0} \\ \mathbf{0}^T & 1 \end{bmatrix} \begin{bmatrix} D & \mathbf{y} \\ \mathbf{y}^T & \sum_{k=1}^{n+1} \lambda_k - \sum_{k=1}^{n} \mu_k \end{bmatrix} \begin{bmatrix} P^T & \mathbf{0} \\ \mathbf{0}^T & 1 \end{bmatrix} \\ &= \begin{bmatrix} PDP^T & P\mathbf{y} \\ (P\mathbf{y})^T & \sum_{k=1}^{n+1} \lambda_k - \sum_{k=1}^{n} \mu_k \end{bmatrix} \end{aligned}$$

is nonnegative and with spectrum Λ.

☑

The following example shows that Theorem 3.1 is independent of the realizability criteria established in [13, Lemma 4].

Example 3.2. *Let $\Lambda = \{6, 1, 1, -4, -4\}$. For this list consider $\mu = \{4, 1, -1, -4\}$, $a = 2$. The matrix*

$$B = Pdiag\{4, 1, -1, 4\}P^T \geq 0,$$

where

$$P = \begin{bmatrix} \frac{2}{15}\sqrt{15} & -\frac{1}{30}\sqrt{210} & -\frac{1}{30}\sqrt{210} & \frac{2}{15}\sqrt{15} \\ \frac{8}{45}\sqrt{15} & -\frac{1}{90}\sqrt{210} & \frac{1}{90}\sqrt{210} & -\frac{8}{45}\sqrt{15} \\ \frac{1}{90}\sqrt{210} & \frac{8}{45}\sqrt{15} & -\frac{8}{45}\sqrt{15} & -\frac{1}{90}\sqrt{210} \\ \frac{1}{30}\sqrt{210} & \frac{2}{15}\sqrt{15} & \frac{2}{15}\sqrt{15} & \frac{1}{30}\sqrt{210} \end{bmatrix},$$

have spectrum μ.

We define $y = (\sqrt{\frac{48}{5}}, 0, -\sqrt{\frac{42}{5}}, 0)^T$; *then, because of Theorem 3.1 the matrix*

$$A = \begin{bmatrix} P & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} diag(4,1,-1,4) & y \\ y^T & 0 \end{bmatrix} \begin{bmatrix} P^T & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 3 & 0 & 0 & 3 \\ 3 & 0 & 0 & \frac{2}{3}\sqrt{14} & \frac{5}{3} \\ 0 & 0 & 0 & \frac{4}{3} & \frac{2}{3}\sqrt{14} \\ 0 & \frac{2}{3}\sqrt{14} & \frac{4}{3} & 0 & 0 \\ 3 & \frac{5}{3} & \frac{2}{3}\sqrt{14} & 0 & 0 \end{bmatrix}$$

is nonnegative with spectrum Λ.

The next example show that the Theorem 3.1 is independent of the realizability criteria shown in the [1, Theorem 3.4] and [13, Theorem 6].

Example 3.3. *We consider the list* $\Lambda = \{3+\sqrt{10}, 1, 1, 3-\sqrt{10}, -4, -4\}$. *For this list there exist a list* $\mu = \{6, 1, 1, -4, -4\}$ *and the orthogonal matrix*

$$P = \begin{bmatrix} \frac{\sqrt{7}}{5} & -\frac{\sqrt{10}}{10} & -\frac{\sqrt{14}}{10} & -\frac{\sqrt{10}}{5} & \frac{\sqrt{2}}{5} \\ \frac{\sqrt{2}}{5} & \frac{2\sqrt{35}}{15} & \frac{2}{15} & -\frac{\sqrt{35}}{15} & -\frac{2\sqrt{7}}{15} \\ \frac{\sqrt{7}}{5} & -\frac{2\sqrt{10}}{15} & \frac{\sqrt{14}}{15} & \frac{\sqrt{10}}{15} & -\frac{7\sqrt{2}}{15} \\ \frac{\sqrt{2}}{5} & 0 & \frac{4}{5} & 0 & \frac{\sqrt{7}}{5} \\ \frac{\sqrt{7}}{5} & \frac{\sqrt{10}}{5} & -\frac{\sqrt{14}}{10} & \frac{\sqrt{10}}{5} & \frac{\sqrt{2}}{5} \end{bmatrix},$$

such that for $\boldsymbol{y} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \end{bmatrix}$, *defined as (∗) we have*

$$P\boldsymbol{y} = \begin{bmatrix} \frac{1}{5}\sqrt{7} & \frac{1}{5}\sqrt{2} & \frac{1}{5}\sqrt{7} & \frac{1}{5}\sqrt{2} & \frac{1}{5}\sqrt{7} \end{bmatrix}^T \geq 0.$$

Thus, by Theorem 3.1, with $D = diag\{6, 1, 1, -4, -4\}$, *the matrix*

$$A = \begin{bmatrix} PDP^T & P\boldsymbol{y} \\ (P\boldsymbol{y})^T & \sum_{i=1}^{6}\lambda_i - \sum_{i=1}^{5}\mu_i \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 3 & 0 & 3 & \frac{1}{5}\sqrt{7} \\ 0 & 0 & 0 & \frac{4}{3} & \frac{2}{3}\sqrt{14} & \frac{1}{5}\sqrt{2} \\ 3 & 0 & 0 & \frac{2}{3}\sqrt{14} & \frac{5}{3} & \frac{1}{5}\sqrt{7} \\ 0 & \frac{4}{3} & \frac{2}{3}\sqrt{14} & 0 & 0 & \frac{1}{5}\sqrt{2} \\ 3 & \frac{2}{3}\sqrt{14} & \frac{5}{3} & 0 & 0 & \frac{1}{5}\sqrt{7} \\ \frac{1}{5}\sqrt{7} & \frac{1}{5}\sqrt{2} & \frac{1}{5}\sqrt{7} & \frac{1}{5}\sqrt{2} & \frac{1}{5}\sqrt{7} & 0 \end{bmatrix} \geq 0,$$

has desired eigenvalues.

These examples show that Theorem 3.1 improves the realizability criteria presented in [1, Theorem 3.4] and [13, Lemma 4 and Theorem 6], in the sense that the lists $\{6, 1, 1, -4, -4\}$, $\{3 + \sqrt{10}, 1, 1, 3 - \sqrt{10}, -4, -4\}$ are not realizable by those criteria.

The next result shows a necessary condition for the $SNIEP$.

Theorem 3.4. *Let $\Lambda = \{\lambda_1, \lambda_2, ..., \lambda_{n+1}\}$ realizable for a nonnegative symmetric matrix A. Then there exists:*

1. *A list of real number $\mu = \{\mu_1, ..., \mu_n\}$, such that $\lambda_1 \geq \mu_1 \geq \lambda_2 \geq \mu_2 \geq ... \geq \mu_n \geq \lambda_{n+1}$.*
2. *A orthogonal matrix Q and $\mathbf{b} \in \mathbb{R}^{n\times 1}$ such that $Q\mathbf{b} \geq 0$, and $QDQ^T \geq 0$, where $D = diag\{\mu_1, ..., \mu_n\}$.*

Also it holds $\sum_{k=1}^{n+1}\lambda_k - \sum_{k=1}^{n}\mu_k \geq 0$.

Proof. Let A be a nonnegative symmetric matrix with eigenvalues Λ. Without loss of generality we can assume that $A = \begin{bmatrix} B & \mathbf{z} \\ \mathbf{z}^T & \alpha \end{bmatrix}$, with $B \in \mathbb{R}^{n\times n}$, $\mathbf{z} \in \mathbb{R}^{n\times 1}$. Let $\mu = \sigma(B) = \{\mu_1, ..., \mu_n\}$. By Theorem 2.1 we get that

$$\lambda_1 \geq \mu_1 \geq \lambda_2 \geq \mu_2 \geq ... \geq \mu_n \geq \lambda_{n+1}.$$

Since B is a nonnegative symmetric matrix, there exits a orthogonal matrix Q such that $B = QDQ^T \geq 0$. Let $\mathbf{b} = Q^T\mathbf{z}$. Since $\mathbf{z} \geq 0$ then we have that $Q\mathbf{b} = Q(Q^T\mathbf{z}) = \mathbf{z} \geq 0$.

The matrix

$$\hat{A} = \begin{bmatrix} D & Q^T\mathbf{z} \\ (Q^T\mathbf{z})^T & \alpha \end{bmatrix}$$

is similar to A, then $\sigma(\hat{A}) = \{\lambda_1, ..., \lambda_{n+1}\}$. Since $\hat{A}$ is a symmetric matrix, then eigenvalues and the diagonal entries satisfy the following relation:

$$\sum_{k=1}^{n+1} \lambda_k \quad = \quad \sum_{k=1}^{n} \mu_k + \alpha;$$

so,

$$\sum_{k=1}^{n+1} \lambda_k - \sum_{k=1}^{n} \mu_k = \alpha \geq 0,$$

therefore $\sum_{k=1}^{n+1} \lambda_k - \sum_{k=1}^{n} \mu_k \geq 0.$ ☑

Note that if $P\mathbf{y} = \mathbf{z}$, with P given in Theorem 3.1, $\mathbf{y}$ defined as $(*)$ and $\mathbf{z}$ given in Theorem 3.4, then Theorem 3.1 and Theorem 3.4 establish necessary and sufficient conditions for $SNIEP$.

Let $\Lambda = \{\lambda_1, ..., \lambda_n\}$ a list of real numbers. If $n = 2, 3$, it is easy to see that $P\mathbf{y} = \mathbf{z}$. For $n = 4$, with some ideas presented in [3], we consider the following cases:

1. If $\lambda_1 \geq \lambda_2 \geq \lambda_3 \geq \lambda_4 > 0$, then the symmetric matrix

$$A_1 = diag\{\lambda_1, \lambda_2, \lambda_3, \lambda_4\}$$

   has eigenvalues Λ. In this case we have $\mu = \{\lambda_1, \lambda_2, \lambda_3\}$, $P\mathbf{y} = \mathbf{z}$ with $P = I \in \mathbb{R}^{3\times 3}$, and $\mathbf{y} = \mathbf{z} = \begin{bmatrix} 0 & 0 & 0 \end{bmatrix}^T$.

2. If $\lambda_1 \geq \lambda_2 \geq \lambda_3 \geq 0 > \lambda_4$, then the symmetric matrix

$$A_2 = \begin{bmatrix} B_1 & O \\ O & B_2 \end{bmatrix},$$

   with

$$B_1 = \begin{bmatrix} \frac{\lambda_1+\lambda_4}{2} & \frac{\lambda_1-\lambda_4}{2} \\ \frac{\lambda_1-\lambda_4}{2} & \frac{\lambda_1+\lambda_4}{2} \end{bmatrix}, B_2 = \begin{bmatrix} \lambda_2 & 0 \\ 0 & \lambda_3 \end{bmatrix} \text{ and } O = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix},$$

   has spectrum Λ. In this case we have $\mu = \{\lambda_1, \lambda_2, \lambda_4\}$, $P\mathbf{y} = \mathbf{z}$, with

$$P = \begin{bmatrix} \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ 0 & 1 & 0 \end{bmatrix} \text{ and } \mathbf{y} = \mathbf{z} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

3. If $\lambda_1 \geq \lambda_2 \geq 0 > \lambda_3 \geq \lambda_4$ and $\lambda_2 \geq |\lambda_3|$, the matrix

$$A = \begin{bmatrix} \frac{\lambda_2+\lambda_3}{2} & \frac{\lambda_2-\lambda_3}{2} & 0 & 0 \\ \frac{\lambda_2-\lambda_3}{2} & \frac{\lambda_2+\lambda_3}{2} & 0 & 0 \\ 0 & 0 & \frac{\lambda_1+\lambda_4}{2} & \frac{\lambda_1-\lambda_4}{2} \\ 0 & 0 & \frac{\lambda_1-\lambda_4}{2} & \frac{\lambda_1+\lambda_4}{2} \end{bmatrix}$$

has spectrum Λ. In this case we have

$$\mu = \{\lambda_2, \lambda_3, \frac{\lambda_1+\lambda_4}{2}\}, \; P = \begin{bmatrix} \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ 0 & 1 & 0 \end{bmatrix},$$

$$\mathbf{y} = \begin{bmatrix} 0 \\ \frac{\lambda_1-\lambda_4}{2} \\ 0 \end{bmatrix} \text{ and } \mathbf{z} = \begin{bmatrix} 0 \\ 0 \\ \frac{\lambda_1-\lambda_4}{2} \end{bmatrix},$$

and so, $P\mathbf{y} = \mathbf{z}$.

The case $\lambda_1 \geq \lambda_2 \geq 0 > \lambda_3 \geq \lambda_4$ and $\lambda_2 < |\lambda_3|$ is analogous to the previous case.

4. If $\lambda_1 \geq 0 > \lambda_2 \geq \lambda_3 \geq \lambda_4$, the matrix $\begin{bmatrix} A_1 & \mathbf{z} \\ \mathbf{z}^T & 0 \end{bmatrix}$ has eigenvalues Λ, with

$$\mathbf{z} = \begin{bmatrix} \sqrt{\frac{-\lambda_1\lambda_2(\lambda_1+\lambda_2)}{2(\lambda_1+\lambda_2-\lambda_3)}} & \sqrt{\frac{-\lambda_1\lambda_2(\lambda_1+\lambda_2)}{2(\lambda_1+\lambda_2-\lambda_3)}} & \sqrt{\frac{\lambda_1\lambda_2\lambda_3}{\lambda_1+\lambda_2-\lambda_3}} \end{bmatrix}^T,$$

$$A_1 = \begin{bmatrix} \frac{\lambda_1+\lambda_2+\lambda_3+\lambda_4}{2} & \frac{\lambda_1+\lambda_2+\lambda_3-\lambda_4}{2} & \sqrt{\frac{-\lambda_3(\lambda_1+\lambda_2)}{2}} \\ \frac{\lambda_1+\lambda_2+\lambda_3-\lambda_4}{2} & \frac{\lambda_1+\lambda_2+\lambda_3+\lambda_4}{2} & \sqrt{\frac{-\lambda_3(\lambda_1+\lambda_2)}{2}} \\ \sqrt{\frac{-\lambda_3(\lambda_1+\lambda_2)}{2}} & \sqrt{\frac{-\lambda_3(\lambda_1+\lambda_2)}{2}} & 0 \end{bmatrix}.$$

In this case we have $\mu = \{\lambda_1+\lambda_2, \lambda_3, \lambda_4\}$,

$$P = \begin{bmatrix} \sqrt{\frac{\lambda_1+\lambda_2}{2(\lambda_1+\lambda_2-\lambda_3)}} & -\sqrt{\frac{-\lambda_3}{2(\lambda_1+\lambda_2-\lambda_3)}} & -\frac{1}{\sqrt{2}} \\ \sqrt{\frac{\lambda_1+\lambda_2}{2(\lambda_1+\lambda_2-\lambda_3)}} & -\sqrt{\frac{-\lambda_3}{2(\lambda_1+\lambda_2-\lambda_3)}} & \frac{1}{\sqrt{2}} \\ \sqrt{\frac{-\lambda_3}{\lambda_1+\lambda_2-\lambda_3}} & \sqrt{\frac{\lambda_1+\lambda_2}{\lambda_1+\lambda_2-\lambda_3}} & 0 \end{bmatrix}, \; \mathbf{y} = \begin{bmatrix} \sqrt{-\lambda_1\lambda_2} \\ 0 \\ 0 \end{bmatrix},$$

and it holds that $P\mathbf{y} = \mathbf{z}$.

We conjecture that the Theorem 3.4 and Theorem 3.1 establish a necessary and sufficient condition for $SNIEP$ when $n \geq 5$, that is, it holds $P\mathbf{y} = \mathbf{z}$, with P given as in Theorem 3.1, $\mathbf{y}$ given as $(*)$ and $\mathbf{z}$ given as in the Theorem 3.4.

The $SNIEP$ for a list $\Lambda = \{\lambda_1, \lambda_2, ..., \lambda_n\}$ is equivalent to find an orthogonal $n-$dimensional matrix P, such that $PD_\Lambda P^T \geq 0$, where $D_\Lambda = diag\{\lambda_1, \lambda_2, ..., \lambda_n\}$.

We consider the following orthogonal matrices:

1.
$$P_1 = \begin{bmatrix} \frac{1}{\sqrt{2}} & 0 & 0 & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & 0 & 0 & \frac{1}{\sqrt{2}} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}, P_2 = \begin{bmatrix} 0 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & 0 & 0 & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & 0 & 0 & \frac{1}{\sqrt{2}} \end{bmatrix}.$$

2. For the list of real numbers $\mu = \{\mu_1, \mu_2, \mu_3, \mu_4\}$ such that $\mu_1 \geq |\lambda_i|$, for $i = 2, 3, 4$, $\mu_3 < 0$ and $\mu_2 <| \mu_3 |$, we define

$$P_3 = \begin{bmatrix} \frac{\sqrt{\mu_1+\mu_3}}{\sqrt{2}\sqrt{\mu_1-\mu_2}} & -\frac{\sqrt{-(\mu_2+\mu_3)}}{\sqrt{2}\sqrt{\mu_1-\mu_2}} & 0 & -\frac{1}{\sqrt{2}} \\ \frac{\sqrt{\mu_1+\mu_3}}{\sqrt{2}\sqrt{\mu_1-\mu_2}} & -\frac{\sqrt{-(\mu_2+\mu_3)}}{\sqrt{2}\sqrt{\mu_1-\mu_2}} & 0 & \frac{1}{\sqrt{2}} \\ \frac{\sqrt{-(\mu_2+\mu_3)}}{\sqrt{2}\sqrt{\mu_1-\mu_2}} & \frac{\sqrt{\mu_1+\mu_3}}{\sqrt{2}\sqrt{\mu_1-\mu_2}} & -\frac{1}{\sqrt{2}} & 0 \\ \frac{\sqrt{-(\mu_2+\mu_3)}}{\sqrt{2}\sqrt{\mu_1-\mu_2}} & \frac{\sqrt{\mu_1+\mu_3}}{\sqrt{2}\sqrt{\mu_1-\mu_2}} & \frac{1}{\sqrt{2}} & 0 \end{bmatrix}.$$

3. For the list of real numbers $\mu = \{\mu_1, \mu_2, \mu_3, \mu_4\}$ such that $\mu_1 \geq |\lambda_i|$, for $i = 2, 3, 4$ and $\mu_2 < 0$, we define

$$P_4 = \begin{bmatrix} \frac{\sqrt{\mu_1}\sqrt{\mu_1+\mu_2}}{\sqrt{2}\sqrt{\mu_1+\mu_2-\mu_3}\sqrt{\mu_1-\mu_2}} & \frac{-\sqrt{-\mu_2}\sqrt{\mu_1+\mu_2}}{\sqrt{2}\sqrt{\mu_1+\mu_2-\mu_3}\sqrt{\mu_1-\mu_2}} & \frac{-\sqrt{-\mu_3}}{\sqrt{2}\sqrt{\mu_1+\mu_2-\mu_3}} & -\frac{1}{\sqrt{2}} \\ \frac{\sqrt{\mu_1}\sqrt{\mu_1+\mu_2}}{\sqrt{2}\sqrt{\mu_1+\mu_2-\mu_3}\sqrt{\mu_1-\mu_2}} & \frac{-\sqrt{-\mu_2}\sqrt{\mu_1+\mu_2}}{\sqrt{2}\sqrt{\mu_1+\mu_2-\mu_3}\sqrt{\mu_1-\mu_2}} & \frac{-\sqrt{-\mu_3}}{\sqrt{2}\sqrt{\mu_1+\mu_2-\mu_3}} & \frac{1}{\sqrt{2}} \\ \frac{\sqrt{\mu_1}\sqrt{-\mu_3}}{\sqrt{\mu_1+\mu_2-\mu_3}\sqrt{\mu_1-\mu_2}} & \frac{-\sqrt{-\mu_2}\sqrt{-\mu_3}}{\sqrt{\mu_1+\mu_2-\mu_3}\sqrt{\mu_1-\mu_2}} & \frac{\sqrt{\mu_1+\mu_2}}{\sqrt{\mu_1+\mu_2-\mu_3}} & 0 \\ \frac{\sqrt{-\mu_2}}{\sqrt{\mu_1-\mu_2}} & \frac{\sqrt{\mu_1}}{\sqrt{\mu_1-\mu_2}} & 0 & 0 \end{bmatrix}.$$

The following result gives a sufficient condition for realizability of list Λ when $n = 5$, by means of orthogonal matrices.

Corollary 3.5. *Let* $\Lambda = \{\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5\}$, $\mu = \{\mu_1, \mu_2, \mu_3, \mu_4\}$ *lists of real numbers such that* $\lambda_1 \geq \mu_1 \geq \lambda_2 \geq \mu_2 \geq \lambda_3 \geq \mu_3 \geq \lambda_4 \geq \mu_4 \geq \lambda_5$; $\sum_{i=1}^{5} \lambda_i - \sum_{i=1}^{4} \mu_i \geq 0$; $\mu_1 \geq |\lambda_i|$ *for* $i = 2, 3, 4$ *and* $\mathbf{y} \in \mathbb{R}^4$ *defined as* $(*)$. *If one of the following conditions is true:*

1. $P_1\mathbf{y} \geq 0 \wedge \mu_3 \geq 0 > \mu_4$,
2. $P_2\mathbf{y} \geq 0 \wedge \mu_2 \geq 0 > \mu_3 \wedge \mu_2 \geq| \mu_3 |$,
3. $P_3\mathbf{y} \geq 0 \wedge \mu_2 \geq 0 > \mu_3 \wedge \mu_2 <| \mu_3 |$,

4. $P_4\mathbf{y} \geq 0 \wedge \mu_1 \geq 0 > \mu_2$,

then Λ *is the spectrum of a nonnegative symmetric matrix.*

Proof. We recall that $\mathbf{y} \in \mathbb{R}^{4\times 1}$ is defined by

$$y_i = -\frac{\displaystyle\prod_{k=1}^{5}(\mu_i - \lambda_k)}{\displaystyle\prod_{k=1}^{i-1}(\mu_i - \mu_k) \cdot \prod_{k=i+1}^{4}(\mu_i - \mu_k)}.$$

We study the case (1): If $P_1\mathbf{y} \geq 0$, $\mu_3 \geq 0 > \mu_4$.

We define the symmetric matrix

$$A = \begin{bmatrix} \frac{1}{\sqrt{2}} & 0 & 0 & -\frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & 0 & 0 & \frac{1}{\sqrt{2}} & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \mu_1 & 0 & 0 & 0 & y_1 \\ 0 & \mu_2 & 0 & 0 & y_2 \\ 0 & 0 & \mu_3 & 0 & y_3 \\ 0 & 0 & 0 & \mu_4 & y_4 \\ y_1 & y_2 & y_3 & y_4 & a \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix},$$

that is

$$A = \begin{bmatrix} \frac{1}{2}\mu_1 + \frac{1}{2}\mu_4 & \frac{1}{2}\mu_1 - \frac{1}{2}\mu_4 & 0 & 0 & \frac{1}{2}\sqrt{2}y_1 - \frac{1}{2}\sqrt{2}y_4 \\ \frac{1}{2}\mu_1 - \frac{1}{2}\mu_4 & \frac{1}{2}\mu_1 + \frac{1}{2}\mu_4 & 0 & 0 & \frac{1}{2}\sqrt{2}y_1 + \frac{1}{2}\sqrt{2}y_4 \\ 0 & 0 & \mu_2 & 0 & y_2 \\ 0 & 0 & 0 & \mu_3 & y_3 \\ \frac{1}{2}\sqrt{2}y_1 - \frac{1}{2}\sqrt{2}y_4 & \frac{1}{2}\sqrt{2}y_1 + \frac{1}{2}\sqrt{2}y_4 & y_2 & y_3 & a \end{bmatrix},$$

whit $a = \displaystyle\sum_{i=1}^{5}\lambda_i - \sum_{i=1}^{4}\mu_i \geq 0$.

Since, by hypothesis $\mu_1 \geq \mid \mu_i \mid, i = 2,3,4,$, we have $\frac{1}{2}\mu_1 \pm \frac{1}{2}\mu_4 \geq 0$. Moreover, since $\mu_3 \geq 0$, then $\mu_2 \geq 0$. Therefore the matrix A is nonnegative.

Also

$$P_1\mathbf{y} = \begin{bmatrix} \frac{1}{2}\sqrt{2}y_1 - \frac{1}{2}\sqrt{2}y_4 & \frac{1}{2}\sqrt{2}y_1 + \frac{1}{2}\sqrt{2}y_4 & y_3 & y_4 \end{bmatrix}^T;$$

by Theorem 3.1 the nonnegative symmetric matrix A has spectrum Λ.

Otherwise it is similarly derived. ☑

Example 3.6. *Consider the list* $\Lambda = \{6, 3, 3, -5, -5\}$. *If the list* Λ *is realizable, then there exists a realizable list* $\mu = \{\mu_1, \mu_2, \mu_3, \mu_4\}$. *By theorems 2.1, 2.2, 2.3, we have* $6 \geq \mu_1 \geq 5$, $\mu_2 = 3$, $-1 \geq \mu_3 \geq -4$, $\mu_4 = -5$ *and* $4 \geq \mu_1 + \mu_3 \geq 2$.

On the other hand, for Λ *and* μ *define* $\boldsymbol{y} = [y_1\ y_2\ y_3\ y_4]^T$, *with* y_i, $i = 1, 2, 3, 4$, *as* (*). *There is not* P_i *of the Corollary 3.5 such that* $P_i\boldsymbol{y} \geq 0$, *thus by Theorems 3.1, 3.4, the list* $\Lambda = \{6, 3, 3, -5, -5\}$ *is not realizable.*

Observation: In [6] it is shown that this list can not be performed by any criterion.

4. Algorithm

In this section, we will implement an algorithm to decide whether a list $\Lambda = \{\lambda_1, \lambda_2, ..., \lambda_n\}$ is the spectrum of a nonnegative symmetric matrix, by using the results presented in the previous section. We introduce the following notation:

$$\begin{aligned}
\mathbb{O}_n &= \{P \in \mathbb{R}^{n\times n} : PP^T = P^TP = I\}, \\
\mathbb{S}_n &= \{\Lambda = \{\lambda_1, ..., \lambda_n\} : \exists A = A^T, \sigma(A) = \Lambda\}, \\
D_\Lambda &= diag\{\lambda_1, ..., \lambda_n\}, \\
\mathbb{S}_n(\Lambda) &= \{A \geq 0 : A = A^T \geq 0, \sigma(A) = \Lambda\}, \\
\mathbb{O}_n(\Lambda) &= \{P \in \mathbb{O}_n : PD_\Lambda P^T \geq 0\}.
\end{aligned}$$

Note that $\mathbb{S}_n(\Lambda) \neq \emptyset$ if only if $\mathbb{O}_n(\Lambda) \neq \emptyset$, and if $\lambda_n \geq 0$ then $\mathbb{O}_n(\Lambda) \neq \emptyset$. Also $\Lambda \in \mathbb{S}_n$ if and only if $\mathbb{S}_n(\Lambda) \neq \emptyset$ or $\mathbb{O}_n(\Lambda) \neq \emptyset$.

Algorithm

1. Let $\Lambda = \{\lambda_1, \lambda_2, ..., \lambda_{n+1}\}$
2. Let $\mu = \{\mu_1, ..., \mu_n\}$ be such that $\lambda_1 \geq \mu_1 \geq \lambda_2 \geq ... \geq \lambda_n \geq \mu_n \geq \lambda_{n+1}$, $\sum_{i=1}^{n+1} \lambda_i - \sum_{i=1}^{n} \mu_i \geq 0$ and $\sum_{i=1}^{n} \mu_i \geq 0$.
3. If $\mu \in \mathbb{S}_n$, define $\mathbf{y} = \begin{bmatrix} y_1 & ... & y_n \end{bmatrix}^T$ as (*). If else return step 2.
4. Let $P \in \mathbb{O}_n(\mu)$.
5. If $P\mathbf{y} \geq 0$, define

$$A = \begin{bmatrix} PD_\mu P^T & P\mathbf{y} \\ (P\mathbf{y})^T & \sum_{i=1}^{n+1} \lambda_i - \sum_{i=1}^{n} \mu_i \end{bmatrix} \in \mathbb{S}_n(\Lambda);$$

   if else, return to step 4.

Naturally there exists several ways to select the μ_i, as well as several ways to determine if $\mu \in \mathbb{S}_n$. For the case $n = 5$, the selection of μ is limited, since for $n = 4$ there are necessary and sufficient conditions to determine if $\mu \in \mathbb{S}_n$.

The following example shows that the algorithm presented is recursive.

Example 4.1. *We consider the list* $\Lambda = \{9, 1, -1, -2, -6\}$.

We select $\mu = \{6, -1, -2, -3\}$. *To show that* μ *is realizable, select the list* $\nu = \{3, -1, -2\}$.

For the list ν, *we define* $\mathbf{x}^T = \left[\begin{array}{ccc} 3\sqrt{2} & 0 & 0 \end{array}\right]$, *and the orthogonal matrix*

$$P_1 = \begin{bmatrix} \frac{1}{10}\sqrt{30} & -\frac{1}{\sqrt{2}} & -\frac{\sqrt{5}}{5} \\ \frac{1}{10}\sqrt{30} & \frac{1}{\sqrt{2}} & -\frac{\sqrt{5}}{5} \\ \frac{1}{5}\sqrt{10} & 0 & \frac{}{5}\sqrt{15} \end{bmatrix}.$$

Thus the matrix

$$\begin{aligned} A_1 &= \begin{bmatrix} P_1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} diag\{3, -1, -2\} & \mathbf{x} \\ \mathbf{x}^T & 0 \end{bmatrix} \begin{bmatrix} P_1^T & 0 \\ 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 0 & 1 & \frac{1}{2}\sqrt{2}\sqrt{6} & \frac{3}{10}\sqrt{2}\sqrt{5}\sqrt{6} \\ 1 & 0 & \frac{1}{2}\sqrt{2}\sqrt{6} & \frac{3}{10}\sqrt{2}\sqrt{5}\sqrt{6} \\ \frac{1}{2}\sqrt{2}\sqrt{6} & \frac{1}{2}\sqrt{2}\sqrt{6} & 0 & \frac{6}{5}\sqrt{5} \\ \frac{3}{10}\sqrt{2}\sqrt{5}\sqrt{6} & \frac{3}{10}\sqrt{2}\sqrt{5}\sqrt{6} & \frac{6}{5}\sqrt{5} & 0 \end{bmatrix} \\ &= \begin{bmatrix} 0 & 1.0 & 1.7321 & 2.3238 \\ 1.0 & 0 & 1.7321 & 2.3238 \\ 1.7321 & 1.7321 & 0 & 2.6833 \\ 2.3238 & 2.3238 & 2.6833 & 0 \end{bmatrix} \end{aligned}$$

it is nonnegative symmetrical with spectrum μ.

For the list μ, *we define* $\mathbf{y} = \left[\begin{array}{cccc} 2\sqrt{5} & 0 & 0 & 4 \end{array}\right]^T$ *and the orthogonal matrix*

$$P = \begin{bmatrix} \frac{1}{5}\sqrt{5} & -\frac{1}{\sqrt{2}} & -\frac{1}{5}\sqrt{5} & -\frac{1}{30}\sqrt{90} \\ \frac{1}{5}\sqrt{5} & \frac{1}{\sqrt{2}} & -\frac{1}{5}\sqrt{5} & -\frac{1}{30}\sqrt{90} \\ \frac{2}{15}\sqrt{15} & 0 & \frac{1}{5}\sqrt{15} & -\frac{1}{15}\sqrt{30} \\ \frac{1}{3}\sqrt{3} & 0 & 0 & \frac{1}{3}\sqrt{6} \end{bmatrix}.$$

Thus the matrix

$$A = \begin{bmatrix} P & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} diag\{6,-1,-2,-3\} & \mathbf{y} \\ \mathbf{y}^T & 0 \end{bmatrix} \begin{bmatrix} P^T & 0 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 1 & \frac{1}{2}\sqrt{12} & \frac{3}{10}\sqrt{60} & 2-\frac{2\sqrt{90}}{15} \\ 1 & 0 & \frac{1}{2}\sqrt{12} & \frac{3}{10}\sqrt{60} & 2-\frac{2\sqrt{90}}{15} \\ \frac{1}{2}\sqrt{12} & \frac{1}{2}\sqrt{12} & 0 & \frac{6}{5}\sqrt{5} & \frac{4}{3}\sqrt{3}-\frac{4\sqrt{30}}{15} \\ \frac{3}{10}\sqrt{60} & \frac{3}{10}\sqrt{60} & \frac{6}{5}\sqrt{5} & 0 & \frac{4}{3}\sqrt{6}+\frac{2\sqrt{15}}{3} \\ 2-\frac{2}{15}\sqrt{90} & 2-\frac{2}{15}\sqrt{90} & \frac{4}{3}\sqrt{3}-\frac{4}{15}\sqrt{30} & \frac{4}{3}\sqrt{6}+\frac{2\sqrt{15}}{3} & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 1.0 & 1.7321 & 2.3238 & 0.73509 \\ 1.0 & 0 & 1.7321 & 2.3238 & 0.73509 \\ 1.7321 & 1.7321 & 0 & 2.6833 & 0.84881 \\ 2.3238 & 2.3238 & 2.6833 & 0 & 5.8480 \\ 0.73509 & 0.73509 & 0.84881 & 5.8480 & 1.0 \end{bmatrix}$$

it is nonnegative symmetrical with spectrum $\Lambda = \{9, 1, -1, -2, -6\}$.

Acknowledgement

This work was supported by Proyecto UTA-Mayor, 4735-16 of the Universidad de Tarapacá, Arica-Chile.

References

[1] Ellard R. and Smigoc H., "Connecting sufficient conditions for the Symmetric Nonnegative Inverse Eigenvalues Problem", *Linear Algebra Appl.* 498 (2016), 521–552.

[2] Fiedler M., "Eigenvalues of Nonnegative Symmetric Matrices", *Linear Algebra Appl.* 9 (1974), 119–142.

[3] Guo W., "An Inverse Eigenvalues Problem for Nonnegative Matrices", *Linear Algebra Appl.* 249 (1996), No. 1-3, 67–78.

[4] Horn A. and Johnson C.R., *Matrix analysis*, Cambridge University Press, Cambridge, 1990. Corrected reprint of the 1985 original.

[5] Johnson C.R., Laffey T.J. and Loewy R., "The real and the symmetric nonnegative inverse eigenvalue problems are different", *Proc. Amer. Math. Soc.* 124 (1996), No. 12, 3647–3651.

[6] Johnson C.R., Marijuán C. and Pisonero M., "Ruling out certain 5-spectra for the symmetric nonnegative inverse eigenvalue problem", *Linear Algebra Appl.* 512 (2017), 129–135.

[7] Loewy R. and London D., "A note on an inverse problem for nonnegative matrices", *Linear Multilinear Algebra* 6 (1978), No. 1, 83–90.

[8] Loewy R. and McDonald J.J., "The symmetric nonnegative inverse eigenvalue problem for 5×5 matrices", *Linear Algebra Appl.* 393 (2004), 275–298.

[9] Meehan M.E., "Some results on the matrix spectra", Thesis (Ph.D.), National University of Ireland, Dublin, 1998.

[10] McDonald J.J. and Neumann M., "The Soules approach to the inverse eigenvalues problem for nonnegative symmetric matrices of order $n \leq 5$", in *Contemp. Math.* 259, Amer. Math. Soc.(2000), 387–407.

[11] Radwan N., "An Inverse eigenvalue problem for symmetric and normal matrices", *Linear Algebra Appl.* 248 (1996), 101–109.

[12] Soto R.L., "A family of realizability criteria for the real and symmetric nonnegative inverse eigenvalue problem", *Numer. Linear Algebra Appl.* 20 (2013), No. 2, 336–348.

[13] Soto R.L., "Realizability criterion for the symmetric nonnegative inverse eigenvalue problem", *Linear Algebra Appl.* 416 (2006), No. 2-3, 783–794.

[14] Soto R.L. and Valero E., "On Symmetric Nonnegative matrices with Prescribed Spectrum", *International Mathematical Forum* 9 (2014), No. 24, 1161–1176.

[15] Soules G.,"Constructing symmetric nonnegative matrices", *Linear Multilinear Algebra* 13 (1983), No. 3, 241–251.

[16] Spector O., "A characterization of trace zero symmetric nonnegative 5×5 matrices", *Linear Algebra Appl.* 434 (2011), No. 4, 1000–1017.

[17] Torre-Mayo J., Abril-Raymundo M.R., Alarcia-Estévez E., Marijuán C., and Pisonero M., "The nonnegative inverse eigenvalue problem from the coefficients of the characteristic polynomial. ELB digraphs", *Linear Algebra Appl.* 426 (2007), No. 2-3, 729–773.