Some models of beta-generated distributions with applications in finance

Sarabia Alegría, José María (sarabiaj@unican.es)
Prieto Mendoza, Faustino (faustino.prieto@unican.es)
Jordá Gil, Vanesa (vanesa.jorda@unican.es)
Remuzgo Pérez, Lorena (lorena.remuzgo@unican.es)
Departamento de Economía, Universidad de Cantabria
Avda. de los Castros s/n, 39005 Santander, España

RESUMEN

Las características empíricas de las series de datos financieros han motivado el estudio de clases de distribuciones flexibles que incorporan propiedades tales como la asimetría y el peso de las colas. En este trabajo se propone el uso de algunos modelos de distribuciones generadas por la beta y distribuciones generalizadas generadas por la beta (ver Eugene et al., 2002 y Jones, 2004), para la modelización de datos financieros. En particular, se estudian dos clases de distribuciones t asimétricas, propuestas por Jones y Faddy (2003) y Alexander et al. (2012). La primera familia depende de dos parámetros de forma que controlan la asimetría y el peso de las colas, y la segunda familia incluye un parámetro adicional. Obtenemos expresiones analíticas para la función de distribución, la función de cuantiles y los momentos, así como para algunas cantidades útiles en econometría financiera, incluyendo el valor en riesgo. Se obtienen varias representaciones estocásticas de estas familias en términos de distribuciones estadísticas de uso habitual. Proponemos algunas extensiones multivariantes y estudiamos algunas de sus propiedades. Por último, incluimos una aplicación empírica con datos reales.

Palabras clave: Distribuciones asimétricas; valor en riesgo; extensiones multivariantes.
Área temática: Aspectos Cuantitativos de Problemas Económicos y Empresariales con incertidumbre.

ABSTRACT

Empirical features of many financial data series have motivated the study of flexible classes of distributions which can incorporate properties such as skewness and fat-tailedness. In this paper we propose the use of some models of beta-generated and generalized beta-generated distributions (see Eugene et al., 2002 and Jones, 2004), for modelling financial data. In particular, we study two classes of skew t distributions, proposed by Jones and Faddy (2003) and Alexander et al. (2012). The first family depends on two shape parameters which control the skewness and the tail weight, and the second family includes an extra parameter. We obtain analytical expressions for the cumulative distribution function, quantile function and moments, and some quantities useful in financial econometrics, including the value at risk. We provide several stochastic representations for these families in terms of usual distributions functions. We also propose some multivariate extensions and we explore some of their properties. Finally, and empirical application with real data is provided.

1 INTRODUCTION

Empirical features of many financial data series have motivated the study of flexible classes of distributions which can incorporate properties such as skewness and fat-tailedness.

The student t distribution is used in financial econometrics and risk management to model the conditional asset returns (see Bollerslev, 1987). However, this model does not fully describe the empirical regularities of many financial data. In this sense, there are several proposals of skewed Student's t distributions to model skewness and fat-tail in conditional distributions of financial returns. Some previous model have been proposed by Theodossiou (1998), Jones and Faddy (2003), Azzalini and Capitanio (2003) and Zhu and Galbraith (2010) among others.

In this paper we propose the use of some models of beta-generated and generalized beta-generated distributions (see Eugene et al., 2002 and Jones, 2004), for modelling financial data. These families of distributions have been used extensively in the recent statistical literature about distribution theory. In this research, we study two classes of skew t distributions, proposed by Jones and Faddy (2003) and Alexander et al. (2012). The first family depends on two shape parameters which control the skewness and the tail weight, and the second family includes an extra parameter. We obtain analytical expressions for the cumulative distribution function, quantile function and moments, and some quantities useful in financial econometrics, including the value at risk. We provide several stochastic representations for these families in terms of usual distributions functions. We also propose some multivariate extensions and we explore some of their properties. Finally, and empirical application with real data is provided.

The contents of this paper are as follows. In Section 2 we present some basic properties of the class of the BG distributions. In Section 3 we presents two classes of skew-t distributions. Section 4 we consider some financial risk measures. In Section

5 we introduce some multivariate versions of the two classes of skew t distributions, and some of their properties are studied. Some applications with real data are included in Section 6. Finally, some conclusions are given in Section 7.

2 THE CLASS OF BETA-GENERATED AND GENERALIZED BETA-GENERATED DISTRIBUTIONS

In this section we present basic properties of the class of BG distributions. We begin with an initial baseline probability density function (PDF) $f(x)$, where the corresponding cumulative distribution function (CDF) is represented by $F(x)$. The class of BG distributions is defined in terms of the PDF by $(a, b > 0)$,

$$g_F(x; a, b) = [B(a, b)]^{-1} f(x) F(x)^{a-1} [1 - F(x)]^{b-1}, \tag{1}$$

where $B(a, b) = \Gamma(a)\Gamma(b)/\Gamma(a + b)$ denotes the classical beta function. A random variable X with PDF (1) will be denoted by $X \sim \mathcal{BG}(a, b; F)$. If $a = i$ and $b = n - i + 1$ in (1), we obtain the PDF of the i-th order statistic from F (Jones, 2004). Below, we highlight some representative values of a and b,

- If $a = b = 1$, $g_F = f$.
- If $a = n$ and $b = 1$, we obtain the distribution of the maximum.
- If $a = 1$ and $b = n$, we obtain the distribution of the minimum.
- If $a \neq b$, we obtain a family of skew distributions.

Parameters a and b control the tailweight of the distribution. Specifically, the a parameter controls left-hand tailweight and the b parameter controls the right-hand tailweight of the distribution. On the other hand, if $a = b$ yields a symmetric sub-family, with a controlling tailweight. In this sense, the BG distribution accommodates several kind of tails. For example (see Jones, 2004),

- Potential tails: If $f \sim x^{-(\alpha+1)}$ and $\alpha > 0$, when $x \to \infty$ $g_F \sim x^{-b\alpha-1}$,
- Exponential tails: If $f \sim e^{-\alpha x}$ and $\beta > 0$, then $g_F \sim e^{-b\beta x}$ if $x \to \infty$

The CDF associated to (1) is,

$$G_F(x; a, b) = I_{F(x)}(a, b),$$

where $I_{F(x)}(\cdot, \cdot)$ denotes the incomplete beta ratio.

If $B \sim \mathcal{B}e(a, b)$ represents the classical beta distribution, a simple stochastic representation of (1) is,

$$X = F^{-1}(B). \tag{2}$$

This representation (2) permits a direct simulation of the values of a random variable with PDF (1), which can be also used for generating multivariate versions of the BG distribution. The raw moments of a BG distribution can be obtained by,

$$E[X^r] = E[\{F^{-1}(B)\}^r], \quad r > 0.$$

An important number of new classes of distributions have been proposed using this methodology.

Some extensions of this family have been proposed by Alexander and Sarabia (2010), Alexander et al. (2012) and Cordeiro and de Castro (2011).

The PDF of the generalized beta-generated is given by,

$$g_F(x; a, b, c) = c[B(a, b)]^{-1} f(x) F(x)^{ac-1} [1 - F(x)^c]^{b-1}. \tag{3}$$

3 TWO CLASSES OF SKEW-t DISTRIBUTIONS

We consider two classes of skew t distributions based on distributions (1) and (3). If we take the baseline CDF,

$$F_X(x;a,b) = \frac{1}{2}\left(1 + \frac{x}{\sqrt{a+b+x^2}}\right),$$

and we substitute in (1) and (3) we obtain the PDF,

$$f_{T_1}(x;a,b) = k_1\left(1 + \frac{x}{\sqrt{a+b+x^2}}\right)^{a+1/2}\left(1 - \frac{x}{\sqrt{a+b+x^2}}\right)^{b+1/2}, \quad (4)$$

where $k_1 = \frac{1}{B(a,b)\sqrt{a+b}2^{a+b-1}}$ and

$$f_{T_2}(x;a,b,c) = k_2\frac{1}{(a+b+x^2)^{3/2}}\left(1 + \frac{x}{\sqrt{a+b+x^2}}\right)^{ac-1}\left(1 - \frac{1}{2^c}\left(1 + \frac{x}{\sqrt{a+b+x^2}}\right)^c\right)^{b-1}, \quad (5)$$

with $k_2 = \frac{c(a+b)}{B(a,b)2^{ac}}$, respectively. The class (4) was proposed by Jones and Faddy (2003) and the class (5) was considered by Alexander et al. (2012).

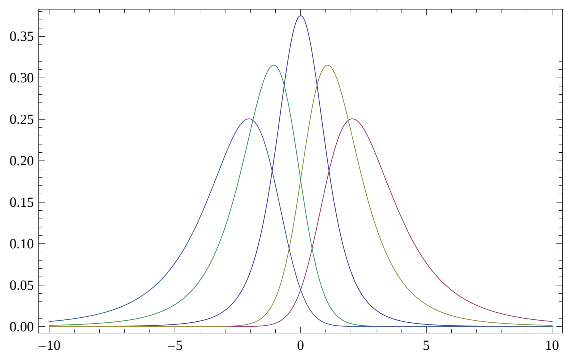

Figure 1: Graphics of the probability density function (Equation (4)) of the skew-t with for (a,b) = (2,2), (5,2), (8,2), (2,5) and (2,8).

3.1 Basic properties of the univariate Skew t

If $a = b$ in (4) we obtain a classical Student t distribution with $2a$ degrees of freedom and the same for (5) taking $a = b$ and $c = 1$.

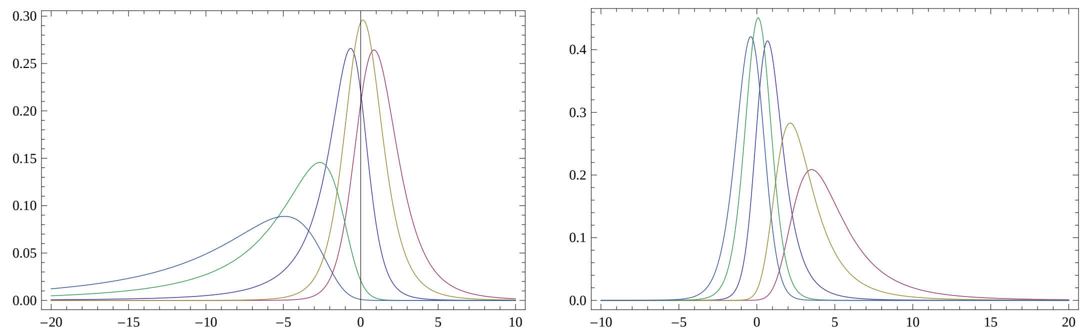

Figure 2: Graphics of the probability density function (Equation (5)) of the skew-t with for (a, b, c) = (2,2,0.5), (8,2,0.5), (5,2,0.5), (2,5,0.5), (2,8,0.5) (left) and (2,2,2), (8,2,2), (5,2,2), (2,5,2) and (2,8,2).

The CDF corresponding to (4) and (5) are given by,

$$F_{t_1}(x; a, b) = I(F_X(x; a, b); a, b), \tag{6}$$

and

$$F_{t_2}(x; a, b, c) = I(F_X^c(x; a, b); a, b),$$

respectively, where $I(x; a, b)$ denotes the incomplete beta ratio function.

The raw moments of (4) are given by,

$$E(X^r) = \frac{(a+b)^{r/2}}{B(a,b)} \sum_{i=0}^{r} \binom{r}{i} 2^{-i} (1-)^i B\left(a - \frac{r}{2}, b - \frac{r}{2}\right),$$

if $a, b > r/2$ (see Jones and Faddy, 2003). For the family (5) we have (Sarabia et al, 2016),

$$E(X^r) = \frac{(a+b)^{r/2}}{B(a,b)} \sum_{j=0}^{r} (-1)^j \binom{r}{j} 2^{-j} \sum_{i=0}^{\infty} \binom{-r/2}{i} (-1)^i B\left(a - \frac{r/2 + j - i}{c}, b\right).$$

3.2 Stochastic representations

There are several alternative stochastic representation for the previous Skew t random variables. If $B \sim \mathcal{B}(a, b)$ is a classical beta random variable, previous random variables can be represented as,

$$T_1(a,b)=\frac{\sqrt{a+b}(2B-1)}{2\sqrt{B(1-B)}},$$

and

$$T_2(a,b,c)=\frac{\sqrt{a+b}(2B^{1/c}-1)}{2\sqrt{B^{1/c}(1-B^{1/c})}},$$

respectively.

The second kind of stochastic representation is in terms of chi-squared random variables. If U_ν represents a chi-squared random variable with 2ν degrees of freedom, we have the alternative representations,

$$T_1(a,b)=\frac{\sqrt{a+b}(U_a-U_b)}{2\sqrt{U_aU_b}}, \tag{7}$$

and

$$T_2(a,b,c)=\frac{\sqrt{a+b}}{2}\frac{2U_a^{1/c}-(U_a+U_b)^{1/c}}{\sqrt{U_a^{1/c}((U_a+U_b)^{1/c}-U_a^{1/c})}}. \tag{8}$$

4 FINANCIAL RISK MEASURES

In this section we provide closed expressions for the value at risk, for the two classes of skew t distributions. The value at risk measures of the skew t (4) and (5) are given by (see Sarabia et al., 2016),

$$\mathrm{VaR}_{T_1}[p;a,b]=\frac{\sqrt{a+b}(2\mathrm{VaR}_B[p;a,b]-1)}{2\sqrt{\mathrm{VaR}_B[p;a,b](1-\mathrm{VaR}_B[p;a,b])}}, \tag{9}$$

and

$$\mathrm{VaR}_{T_2}[p;a,b,c]=\frac{\sqrt{a+b}(2\mathrm{VaR}_B^{1/c}[p;a,b]-1)}{2\sqrt{\mathrm{VaR}_B^{1/c}[p;a,b](1-\mathrm{VaR}_B^{1/c}[p;a,b])}}, \tag{10}$$

respectively, with $0\le p\le 1$, where $\mathrm{VaR}_B[p;a,b]$ denotes the value at risk of a classical $\mathcal{B}e(a,b)$ distribution. The tail value and risk can be obtained numerically using Formulas (9) and (10).

5 MULTIVARIATE EXTENSIONS

In this section we provide some multivariate versions of the Skew t distributions. The first multivariate version is a natural extension of the formulas (7) and (8). Let $U_i \sim \chi^2_{2\nu_i}$ and $U_0 \sim \chi^2_{2\nu_0}$, $i = 1, 2, \ldots, m$ be $m + 1$ independent chi-square distributions, with $2\nu_i$, $i = 1, 2, \ldots, m$ and $2\nu_0$ degrees of freedom respectively, with $\nu_i, \nu_0 > 0$, $i = 1, 2, \ldots, m$.

The multivariate Skew-t distribution corresponding to first version is defined by the stochastic representation,

$$\left(X_1^{(1)}, \ldots, X_m^{(1)}\right)^\top = \left(\frac{\sqrt{\nu_1+\nu_0}(U_1-U_0)}{2\sqrt{U_1U_0}}, \ldots, \frac{\sqrt{\nu_m+\nu_0}(U_m-U_0)}{2\sqrt{U_mU_0}}\right)^\top. \quad (11)$$

The multivariate Skew-t distribution corresponding to the second version is defined by the stochastic representation,

$$\begin{aligned}&\left(X_1^{(2)}, \ldots, X_m^{(2)}\right)^\top \\ =& \left(\frac{\sqrt{\nu_i+\nu_0}}{2}\frac{2U_{\nu_i}^{1/c}-(U_{\nu_i}+U_{\nu_0})^{1/c}}{\sqrt{U_{\nu_i}^{1/c}((U_{\nu_i}+U_{\nu_0})^{1/c}-U_{\nu_i}^{1/c})}};\ i=1,2,\ldots,m\right)\end{aligned} \quad (12)$$

By construction, the marginal distributions belong to the same family. In the case of (11), the marginal distributions are Skew-t of the first type with parameters (ν_i, ν_0), $i = 1, 2, \ldots, m$. For the second multivariate version (12), the marginal distributions are Skew-t of the second type with parameters (ν_i, ν_0, c), for $i = 1, 2, \ldots, m$.

Figures 3 and 4 show two simulated sample of size 1000 from (11) with linear correlation coefficients 0.485 and 0.756 respectively.

In relation with the dependence structure in (11), we have the following result (see Sarabia el al. 2016).

Theorem 1 *Let consider the multivariate random variable* $\left(X_1^{(1)}, \ldots, X_m^{(1)}\right)^\top$ *de-*

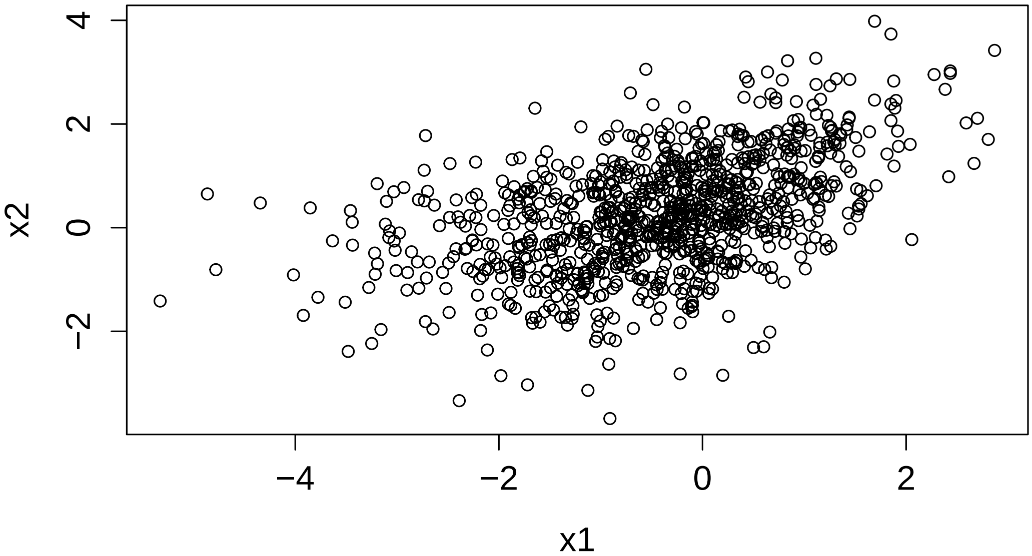

Figure 3: A simulated sample of $n = 1000$ of the bivariate skew-t defined in (11) with $(a_0, a_1) = (5.5, 4.5)$ and $(a_0, a_2) = (5.5, 6.5)$.

fined in (11). Then, the random variables $X_1^{(1)}, \dots, X_m^{(1)}$ *are associated. In consequence, the covariance between pairs of variables is always positive.*

If we want more flexibility for the marginal distributions, we can use the results by Sarabia et al (2014) for multivariate beta-generated distributions. For the first skew-t family, we consider the multivariate distribution,

$$\left(X_1^{(1)}, \dots, X_m^{(1)}\right)^\top = \left(F_i^{-1}\left\{\frac{G_{a_i}}{G_{a_i} + \sum_{j=1}^{i} G_{b_j}}\right\},\ i = 1, 2, \dots, m\right)^\top,$$

where G_a represent a classical gamma distribution with shape parameter a. The marginal distributions are Skew t of the first type with parameters $(a_i, b_1 + \cdots + b_i)$, $i = 1, 2, \dots, m$.

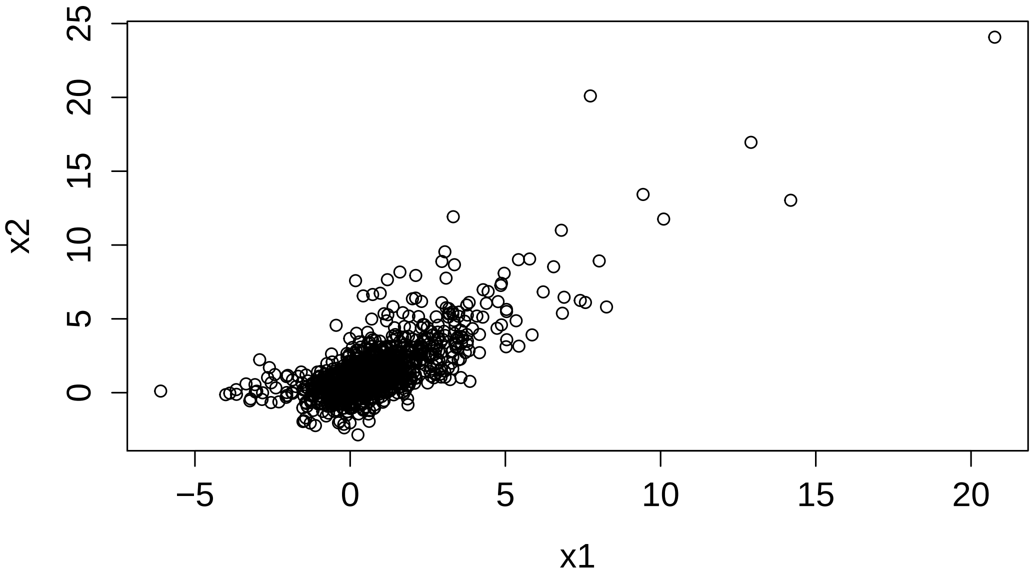

Figure 4: A simulated sample of $n = 1000$ of the bivariate skew-t defined in (11) with $(a_0, a_1) = (1.5, 2.5)$ and $(a_0, a_2) = (1.5, 3.5)$.

6 EMPIRICAL APPLICATION IN FINANCE

In this section we include an application with financial data. We have considered daily stock-returns data, from 1st January 2015 to 31st December 2015 for five companies of the Spanish value-weighted index IBEX 35: Amadeus (IT solutions to tourism industry); BBVA (global financial services); Mapfre (insurance market); Repsol (energy sector); and Telefónica (information and communications technology services).

Some relevant information about the data sets used are included in Table 1. For each company, we have included the sample size n, the maximum and the minimum daily stock-return in the period considered, the sample mean and standard deviation and the corresponding skewness and kurtosis. In particular, it can be shown that the empirical distribution is negatively skewed in four of the five companies considered and positively skewed in the remaining one.

Table 1

Some relevant information about the datasets considered.

Stock	Amadeus	BBVA	Mapfre	Repsol	Telefónica
Sample size (n)	261	261	261	261	261
Maximum daily return	0.046286	0.040975	0.050847	0.073466	0.062264
Minimum daily return	-0.097367	-0.060703	-0.067901	-0.0877323	-0.051563
Mean	0.000900	-0.000452	-0.000623	-0.001416	-0.000408
Standard Deviation	0.014601	0.016249	0.015942	0.021349	0.016301
Skewness	-1.163797	-0.465779	-0.723655	-0.166165	0.130372
Kurtosis	10.292160	3.824688	4.873980	5.435928	4.422885

We have worked with standardized data by subtracting the sample mean and dividing by the sample standard deviation. Then, we have fitted by maximum likelihood both models considered: the univariate Skew t distribution with two parameters (t1), with PDF defined in Eq. (4), and the univariate Skew t distribution with three parameters (t2), with PDF given by Eq. (5). Then, we have compared two models by using the Bayesian information criterion (BIC) considered by Schwarz (1978) and defined as follows,

$$BIC = \log L - \frac{1}{2} d \log n,$$

where $\log L$ is the log-likelihood of the model evaluated at the maximum likelihood estimates, d is the number of parameters and n is the sample size. The model chosen is that with largest BIC value. Finally, we have checked graphically the adequacy of both models to the data by comparing the theoretical CDF of both models defined in Eq.(6), with the corresponding empirical CDF given by the plotting position formula (Castillo et al. 2005) defined as,

$$F_n(x_i) \approx (n+1)^{-1} \sum_{j=1}^{n} I_{[x_j \le x_i]}.$$

Table 2 shows the BIC statistics obtained, for the two selected models. It can be observed that the three parameter model presents the largest values of BIC statistics in all the five stocks considered.

Table 2

BIC statistics for both candidate models, fitted by maximum likelihood to dataset (standardized). Larger values indicate better fitted models.

Stock	Amadeus	BBVA	Mapfre	Repsol	Telefónica
Skew t distribution (2 parameters)	-366.1991	-374.3572	-372.2886	-369.8228	-372.8480
Skew t distribution (3 parameters)	-361.4669	-372.9159	-368.0314	-364.7018	-371.5093

Tables 3 and 4 show the parameter estimates and their corresponding standard errors, for the two models studied. We can observe that in two of the five stocks considered (Amadeus and Repsol) the parameters are significant for both models and are not significant in some cases for the remaining three stocks (BBVA, Mapfre and Telefonica). In addition, the parameters estimates ($\hat{a}$ and $\hat{b}$) are similar in the case of the model with two parameters.

Table 3

Parameter estimates from Skew t model with two parameters, to stardardized datasets, by maximum likelihood (standard errors in parenthesis).

Stock	Amadeus	BBVA	Mapfre	Repsol	Telefónica
$\hat{a}$	6.194309	10.773980	7.271484	5.009976	7.083988
	(2.378890)	(8.474473)	(3.684818)	(1.980271)	(3.810294)
$\hat{b}$	6.171897	10.76088	7.250156	5.005015	7.086958
	(2.378415)	(8.477441)	(3.686769)	(1.980433)	(3.810390)

Table 4

Parameter estimates from Skew t model with two parameters, to stardardized datasets, by maximum likelihood (standard errors in parenthesis).

Stock	Amadeus	BBVA	Mapfre	Repsol	Telefónica
$\hat{a}$	1.050617	0.935678	0.8685684	0.808572	1.120804
	(0.443072)	(0.407211)	(0.329048)	(0.363157)	(0.650834)
$\hat{b}$	5.126098	7.144007	6.217545	2.998354	3.497796
	(2.091549)	(5.104088)	(3.184219)	(0.917090)	(1.110353)
$\hat{c}$	2.973896	3.653026	3.617017	2.721519	2.331579
	(0.761879)	(0.733523)	(0.668503)	(0.698227)	(0.774010)

Figures 5 and 6 show (as a graphical model validation) the plots obtained with the theoretical CDF for both models (left: t1 model with 2 parameters; right: t2 model with 3 parameters) and the corresponding empirical CDF, for the five stocks considered. It can be observed that the t2 model, with three parameter, presents the best fit, for all the stocks considered.

We have also calculated the value at risk at 95% confidence level for the five stocks considered for both Skew t models (VaR_{T1} and VaR_{T2}), by Eqs. (9) and (10). Table 5 shows the results obtained. It can be concluded that the Skew t model with two parameters (t1 model) provides higher VaR values than the second Skew t model with three paramenters (t2 model), in all five stocks considered.

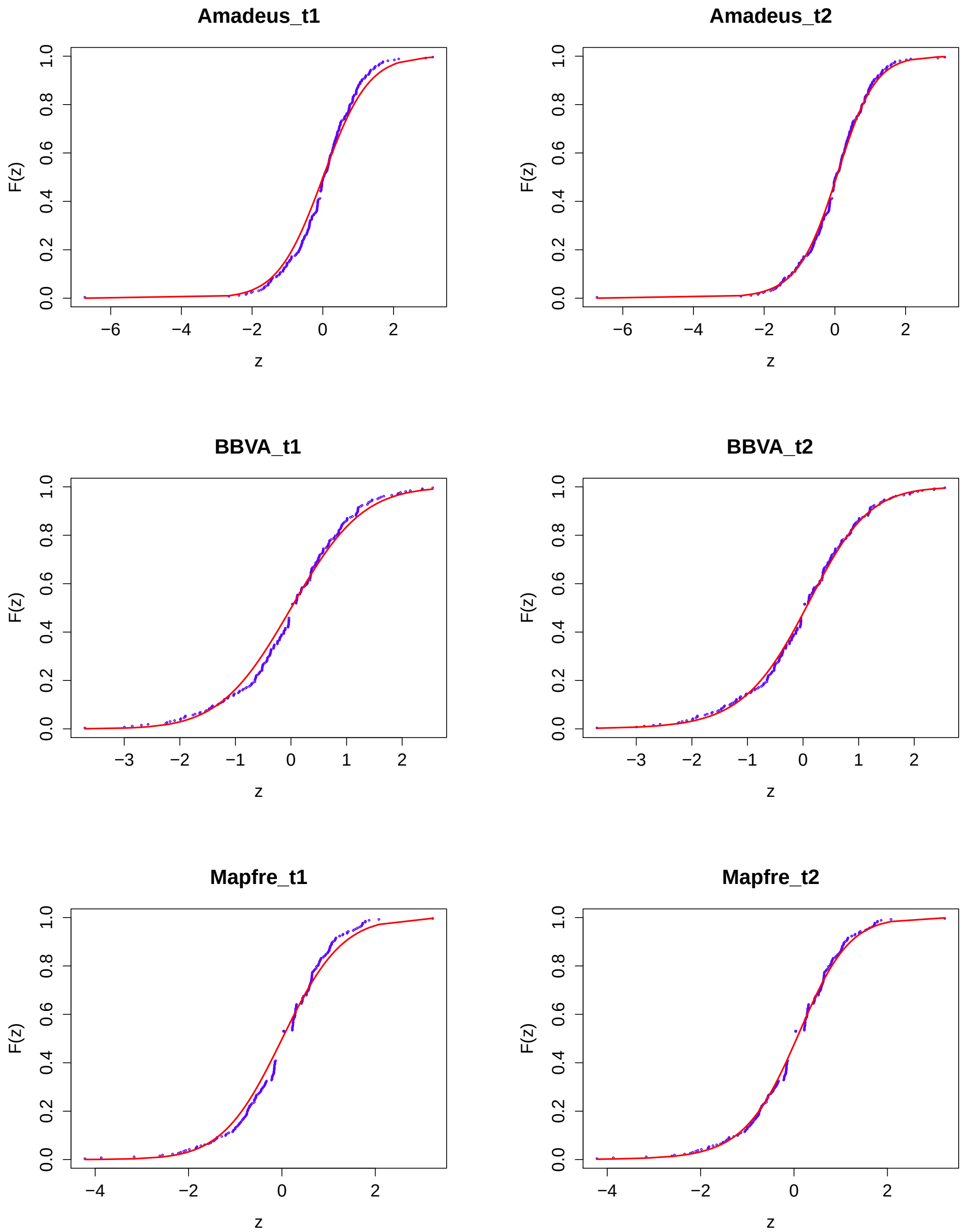

Figure 5: Plots of the theoretical CDFs of the Skew t models (Left: t1, model with two parameter, Right: t2, model with three parameters) and the empirical CDF. Stocks: Amadeus; BBVA; Mapfre.

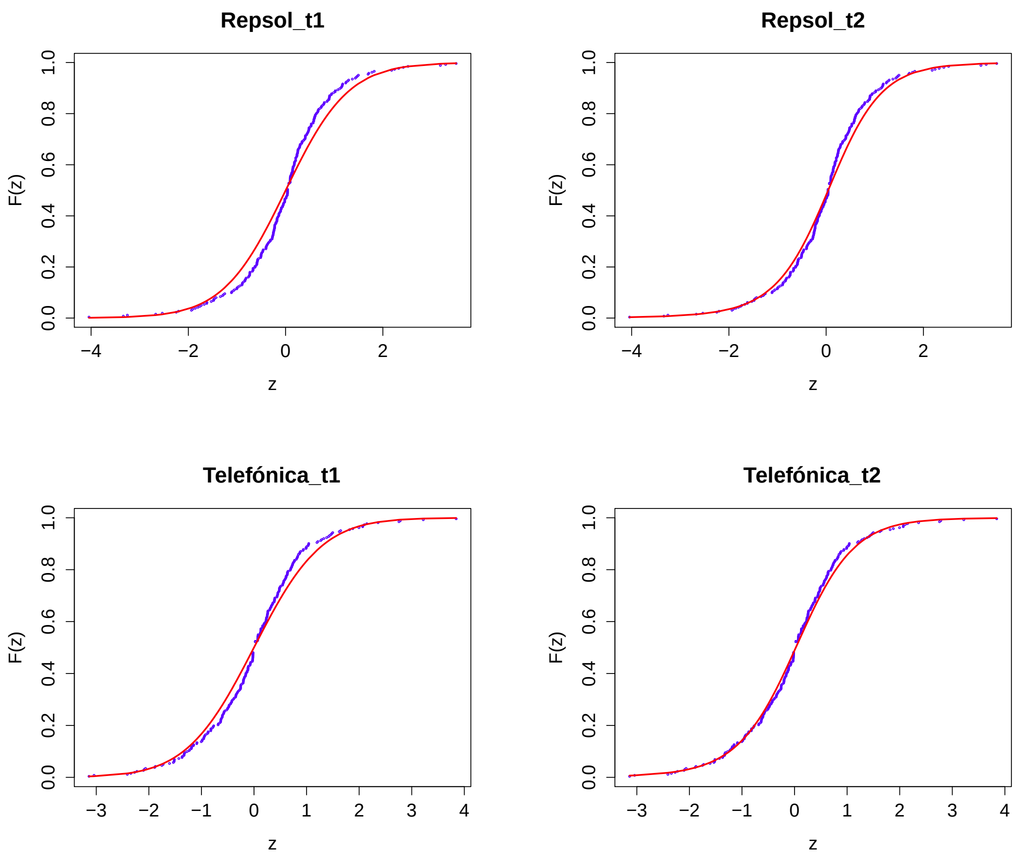

Figure 6: Plots of the theoretical CDFs of the Skew t models (Left: t1 model with two parameter, Right: t2 model with three parameters) and the empirical CDF. Stocks: Repsol; Telefónica.

Table 5

Value at risk (VaR), at 95% confidence level, for the five stocks considered, from both Skew t models, VaR_{T1} and VaR_{T2}.

Stock	Amadeus	BBVA	Mapfre	Repsol	Telefónica
VAR_{T1}	-0.024941	-0.028328	-0.028521	-0.040059	-0.029110
VAR_{T2}	-0.023089	-0.028179	-0.027794	-0.038330	-0.028029

7 CONCLUSIONS

In this paper we have proposed the use of some models of beta-generated and generalized beta-generated distributions (see Eugene et al., 2002 and Jones, 2004), for modelling financial data. We have studied two classes of skew t distributions, proposed by Jones and Faddy (2003) and Alexander et al. (2012). The first family depends on two shape parameters which control the skewness and the tail weight, and the second family includes an extra parameter. We have obtained analytical expressions for the cumulative distribution function, quantile function and moments, and some quantities useful in financial econometrics, including the value at risks, and we have provided several stochastic representations for these families in terms of usual distributions functions. We have proposed some multivariate extensions and we have explored some of their properties. Finally, and empirical application with real data have been provided.

Acknowledgements. The authors gratefully acknowledge financial support from the Programa Estatal de Fomento de la Investigación Científica y Técnica de Excelencia, Spanish Ministry of Economy and Competitiveness, ECO2013-48326-C2-2-P. In addition, this work is part of the Research Project APIE 1/2015-17: “New methods for the empirical analysis of financial markets” of the Santander Financial Institute (SANFI) of UCEIF Foundation resolved by the University of Cantabria and funded with sponsorship from Banco Santander.

8 REFERENCES

- ALEXANDER, C., CORDEIRO, G.M., ORTEGA, E.M.M., SARABIA, J.M. (2012). “Generalized beta-generated distributions”. Computational Statistics and Data Analysis 56, 1880-1897.

- ALEXANDER, C., SARABIA, J.M. (2010). "Generalized beta-generated distributions". ICMA Centre Discussion Papers in Finance DP2010-09, ICMA Centre. The University of Reading, Witheknights, PO Box 242, Reading RG6 6BA, UK.

- AZZALINI, A., CAPITANIO, A. (2003). "Distributions generated by perturbation of symmetry with emphasis on a multivariate skew t distribution". Journal of the Royal Statistical Society B 65, 367ñ389.

- BOLLERSLEV, T. (1987). "A conditional heteroskedastic time series model for speculative prices and rates of return". Review of Economics and Statistics 69, 542-547.

- CASTILLO, E., HADI, A.S., BALAKRISHNAN , N., SARABIA, J.M. (2005). "Extreme value and related models with applications in engineering and science". John Wiley & Sons.

- CORDEIRO, G.M., DE CASTRO, M. (2011). "A new family of generalized distributions". Journal of Statistical Computation and Simulation, 81, 883-893.

- EUGENE, N., LEE, C., FAMOYE, F. (2002). "The beta-normal distribution and its applications". Communications in Statistics, Theory and Methods, 31, 497-512.

- JONES, M.C. (2004). "Families of distributions arising from distributions of order statistics". Test 13, 1-43.

- JONES, M.C., FADDY, M.J. (2003). "A skew extension of the t-distribution, with applications". Journal of the Royal Statistical Society. Series B 65, 159-174.

- SARABIA, J.M., PRIETO, F., JORDA, V. (2014). "Bivariate Beta-Generated Distributions with Applications to Well-Being Data". Journal of Statistical Distributions and Applications 2014, 1-15.

- SARABIA, J.M., PRIETO, F., JORDA, V., REMUZGO, L. (2016). "Beta-Generated Distributions with Applications in Finance". Working Paper, Department of Economics, University of Cantabria.

- SCHWARZ, G. (1978). "Estimating the dimension of a model". The Annals of Statistics 5, 461-464.

- THEODOSSIOU, P. (1998). "Financial data and the skewed generalized t distribution". Management Science 44, 1650-1661.

- ZHU, D., GALBRAITH, J.W. (2010). "A generalized asymmetric Student-t distribution with application to financial econometrics". Journal of Econometrics 157, 297-305.